Introduction to Methods of Applied Mathematics
or
Advanced Mathematical Methods for Scientists and Engineers

Sean Mauch

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Preface

During the summer before my final undergraduate year at Caltech I set out to write a math text unlike any other, namely, one written by me. In that respect I have succeeded beautifully. Unfortunately, the text is neither complete nor polished. I have a "Warnings and Disclaimers" section below that is a little amusing, and an appendix on probability that I feel concisesly captures the essence of the subject. However, all the material in between is in some stage of development. I am currently working to improve and expand this text.

This text is freely available from my web set. Currently I'm at http://www.its.caltech.edu/~sean. I post new versions a couple of times a year.

2.1 Advice to Teachers

If you have something worth saying, write it down.

2.2 Acknowledgments

I would like to thank Professor Saffman for advising me on this project and the Caltech SURF program for providing the funding for me to write the first edition of this book.

2.3 Warnings and Disclaimers

- This book is a work in progress. It contains quite a few mistakes and typos. I would greatly appreciate your constructive criticism. You can reach me at 'sean@its.caltech.edu'.
- Reading this book impairs your ability to drive a car or operate machinery.
- This book has been found to cause drowsiness in laboratory animals.
- This book contains twenty-three times the US RDA of fiber.
- Caution: FLAMMABLE - Do not read while smoking or near a fire.
- If infection, rash, or irritation develops, discontinue use and consult a physician.
- Warning: For external use only. Use only as directed. Intentional misuse by deliberately concentrating contents can be harmful or fatal. KEEP OUT OF REACH OF CHILDREN.
- In the unlikely event of a water landing do not use this book as a flotation device.
- The material in this text is fiction; any resemblance to real theorems, living or dead, is purely coincidental.
- This is by far the most amusing section of this book.
- Finding the typos and mistakes in this book is left as an exercise for the reader. (Eye ewes a spelling chequer from thyme too thyme, sew their should knot bee two many misspellings. Though I ain't so sure the grammar's too good.)
- The theorems and methods in this text are subject to change without notice.
- This is a chain book. If you do not make seven copies and distribute them to your friends within ten days of obtaining this text you will suffer great misfortune and other nastiness.
- The surgeon general has determined that excessive studying is detrimental to your social life.

- This text has been buffered for your protection and ribbed for your pleasure.
- Stop reading this rubbish and get back to work!

2.4 Suggested Use

This text is well suited to the student, professional or lay-person. It makes a superb gift. This text has a boquet that is light and fruity, with some earthy undertones. It is ideal with dinner or as an apertif. Bon apetit!

2.5 About the Title

The title is only making light of naming conventions in the sciences and is not an insult to engineers. If you want to find a good math text to learn a subject, look for books with "Introduction" and "Elementary" in the title. If it is an "Intermediate" text it will be incomprehensible. If it is "Advanced" then not only will it be incomprehensible, it will have low production qualities, i.e. a crappy typewriter font, no graphics and no examples. There is an exception to this rule when the title also contains the word "Scientists" or "Engineers". Then an advanced book may be quite suitable for actually learning the material.

Part I

Algebra

Chapter 3

Sets and Functions

3.1 Sets

Definition. A *set* is a collection of objects. We call the objects, *elements*. A set is denoted by listing the elements between braces. For example: $\{e, i, \pi, 1\}$. We use ellipses to indicate patterns. The set of positive integers is $\{1, 2, 3, \dots\}$. We also denote a sets with the notation $\{x|\text{conditions on } x\}$ for sets that are more easily described than enumerated. This is read as "the set of elements x such that x satisfies … ". $x \in S$ is the notation for "x is an element of the set S." To express the opposite we have $x \notin S$ for "x is not an element of the set S."

Examples. We have notations for denoting some of the commonly encountered sets.

- $\emptyset = \{\}$ is the *empty set*, the set containing no elements.
- $\mathbb{Z} = \{\dots, -1, 0, 1 \dots\}$ is the set of *integers*. (Z is for "Zahlen", the German word for "number".)
- $\mathbb{Q} = \{p/q | p, q \in \mathbb{Z}, q \neq 0\}$ is the set of *rational numbers*. (Q is for quotient.)
- $\mathbb{R} = \{x | x = a_1 a_2 \cdots a_n.b_1 b_2 \cdots\}$ is the set of *real numbers*, i.e. the set of numbers with decimal expansions.

- $\mathbb{C} = \{a + ib | a, b \in \mathbb{R}, i^2 = -1\}$ is the set of *complex numbers*. i is the square root of -1. (If you haven't seen complex numbers before, don't dismay. We'll cover them later.)

- $\mathbb{Z}^+$, $\mathbb{Q}^+$ and $\mathbb{R}^+$ are the sets of positive integers, rationals and reals, respectively. For example, $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

- $\mathbb{Z}^{0+}$, $\mathbb{Q}^{0+}$ and $\mathbb{R}^{0+}$ are the sets of non-negative integers, rationals and reals, respectively. For example, $\mathbb{Z}^{0+} = \{0, 1, 2, \ldots\}$.

- $(a \ldots b)$ denotes an *open interval* on the real axis. $(a \ldots b) \equiv \{x | x \in \mathbb{R}, a < x < b\}$

- We use brackets to denote the *closed interval*. $[a \ldots b] \equiv \{x | x \in \mathbb{R}, a \leq x \leq b\}$

The *cardinality* or *order* of a set S is denoted $|S|$. For finite sets, the cardinality is the number of elements in the set. The *Cartesian product* of two sets is the set of ordered pairs:

$$X \times Y \equiv \{(x, y) | x \in X, y \in Y\}.$$

The Cartesian product of n sets is the set of ordered n-tuples:

$$X_1 \times X_2 \times \cdots \times X_n \equiv \{(x_1, x_2, \ldots, x_n) | x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n\}.$$

Equality. Two sets S and T are *equal* if each element of S is an element of T and vice versa. This is denoted, $S = T$. Inequality is $S \neq T$, of course. S is a *subset* of T, $S \subseteq T$, if every element of S is an element of T. S is a *proper subset* of T, $S \subset T$, if $S \subseteq T$ and $S \neq T$. For example: The empty set is a subset of every set, $\emptyset \subseteq S$. The rational numbers are a proper subset of the real numbers, $\mathbb{Q} \subset \mathbb{R}$.

Operations. The *union* of two sets, $S \cup T$, is the set whose elements are in either of the two sets. The union of n sets,

$$\cup_{j=1}^{n} S_j \equiv S_1 \cup S_2 \cup \cdots \cup S_n$$

is the set whose elements are in any of the sets S_j. The *intersection* of two sets, $S \cap T$, is the set whose elements are in both of the two sets. In other words, the intersection of two sets in the set of elements that the two sets have in common. The intersection of n sets,

$$\cap_{j=1}^{n} S_j \equiv S_1 \cap S_2 \cap \cdots \cap S_n$$

is the set whose elements are in all of the sets S_j. If two sets have no elements in common, $S \cap T = \emptyset$, then the sets are *disjoint*. If $T \subseteq S$, then the *difference* between S and T, $S \setminus T$, is the set of elements in S which are not in T.

$$S \setminus T \equiv \{x | x \in S, x \notin T\}$$

The difference of sets is also denoted $S - T$.

Properties. The following properties are easily verified from the above definitions.

- $S \cup \emptyset = S$, $S \cap \emptyset = \emptyset$, $S \setminus \emptyset = S$, $S \setminus S = \emptyset$.
- Commutative. $S \cup T = T \cup S$, $S \cap T = T \cap S$.
- Associative. $(S \cup T) \cup U = S \cup (T \cup U) = S \cup T \cup U$, $(S \cap T) \cap U = S \cap (T \cap U) = S \cap T \cap U$.
- Distributive. $S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$, $S \cap (T \cup U) = (S \cap T) \cup (S \cap U)$.

3.2 Single Valued Functions

Single-Valued Functions. A *single-valued function* or *single-valued mapping* is a mapping of the elements $x \in X$ into elements $y \in Y$. This is expressed notationally as $f : X \to Y$ or $X \xrightarrow{f} Y$. If such a function is well-defined, then for each $x \in X$ there exists a unique element of y such that $f(x) = y$. The set X is the *domain* of the function, Y is the *codomain*, (not to be confused with the *range*, which we introduce shortly). To denote the value of a function on a particular element we can use any of the notations: $f(x) = y$, $f : x \mapsto y$ or simply $x \mapsto y$. f is the *identity map* on X if $f(x) = x$ for all $x \in X$.

Let $f : X \to Y$. The *range* or *image* of f is

$$f(X) = \{y | y = f(x) \text{ for some } x \in X\}.$$

The range is a subset of the codomain. For each $Z \subseteq Y$, the *inverse image* of Z is defined:

$$f^{-1}(Z) \equiv \{x \in X | f(x) = z \text{ for some } z \in Z\}.$$

Examples.

- Finite polynomials and the exponential function are examples of single valued functions which map real numbers to real numbers.
- The *greatest integer function*, $\lfloor \cdot \rfloor$, is a mapping from $\mathbb{R}$ to $\mathbb{Z}$. $\lfloor x \rfloor$ in the greatest integer less than or equal to x. Likewise, the *least integer function*, $\lceil x \rceil$, is the least integer greater than or equal to x.

The -jectives. A function is *injective* if for each $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$. In other words, for each x in the domain there is a unique $y = f(x)$ in the range. f is *surjective* if for each y in the codomain, there is an x such that $y = f(x)$. If a function is both injective and surjective, then it is *bijective*. A bijective function is also called a *one-to-one mapping*.

Examples.

- The exponential function $y = \mathrm{e}^x$ is a bijective function, (one-to-one mapping), that maps $\mathbb{R}$ to $\mathbb{R}^+$. ($\mathbb{R}$ is the set of real numbers; $\mathbb{R}^+$ is the set of positive real numbers.)
- $f(x) = x^2$ is a bijection from $\mathbb{R}^+$ to $\mathbb{R}^+$. f is not injective from $\mathbb{R}$ to $\mathbb{R}^+$. For each positive y in the range, there are two values of x such that $y = x^2$.
- $f(x) = \sin x$ is not injective from $\mathbb{R}$ to $[-1..1]$. For each $y \in [-1, 1]$ there exists an infinite number of values of x such that $y = \sin x$.

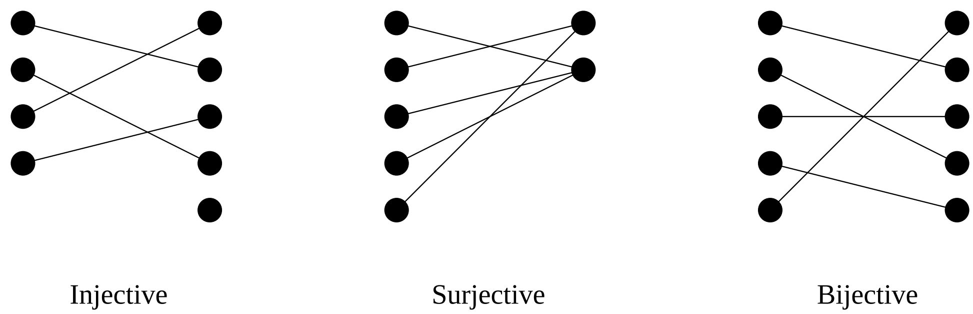


Figure 3.1: Depictions of Injective, Surjective and Bijective Functions

3.3 Inverses and Multi-Valued Functions

If $y = f(x)$, then we can write $x = f^{-1}(y)$ where f^{-1} is the inverse of f. If $y = f(x)$ is a one-to-one function, then $f^{-1}(y)$ is also a one-to-one function. In this case, $x = f^{-1}(f(x)) = f(f^{-1}(x))$ for values of x where both $f(x)$ and $f^{-1}(x)$ are defined. For example $\log x$, which maps $\mathbb{R}^+$ to $\mathbb{R}$ is the inverse of e^x. $x = \mathrm{e}^{\log x} = \log(\mathrm{e}^x)$ for all $x \in \mathbb{R}^+$. (Note the $x \in \mathbb{R}^+$ ensures that $\log x$ is defined.)

If $y = f(x)$ is a many-to-one function, then $x = f^{-1}(y)$ is a one-to-many function. $f^{-1}(y)$ is a multi-valued function. We have $x = f(f^{-1}(x))$ for values of x where $f^{-1}(x)$ is defined, however $x \neq f^{-1}(f(x))$. There are diagrams showing one-to-one, many-to-one and one-to-many functions in Figure 3.2.

Example 3.3.1 $y = x^2$, a many-to-one function has the inverse $x = y^{1/2}$. For each positive y, there are two values of x such that $x = y^{1/2}$. $y = x^2$ and $y = x^{1/2}$ are graphed in Figure 3.3.

We say that there are two **branches** of $y = x^{1/2}$: the positive and the negative branch. We denote the positive branch as $y = \sqrt{x}$; the negative branch is $y = -\sqrt{x}$. We call $\sqrt{x}$ the **principal branch** of $x^{1/2}$. Note that $\sqrt{x}$ is a one-to-one function. Finally, $x = (x^{1/2})^2$ since $(\pm\sqrt{x})^2 = x$, but $x \neq (x^2)^{1/2}$ since $(x^2)^{1/2} = \pm x$. $y = \sqrt{x}$ is

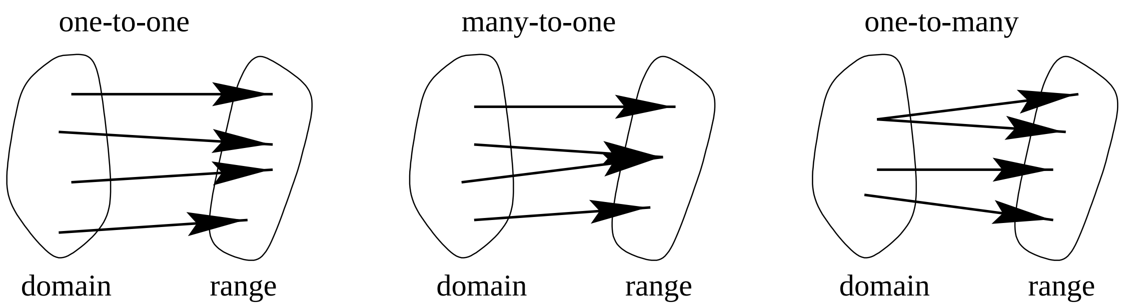


Figure 3.2: Diagrams of One-To-One, Many-To-One and One-To-Many Functions

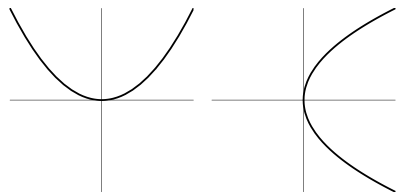


Figure 3.3: $y = x^2$ and $y = x^{1/2}$

graphed in Figure 3.4.

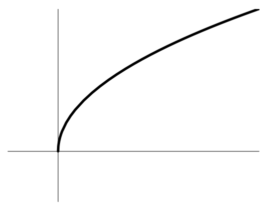


Figure 3.4: $y = \sqrt{x}$

Now consider the many-to-one function $y = \sin x$. The inverse is $x = \arcsin y$. For each $y \in [-1, 1]$ there are an infinite number of values x such that $x = \arcsin y$. In Figure 3.5 is a graph of $y = \sin x$ and a graph of a few branches of $y = \arcsin x$.

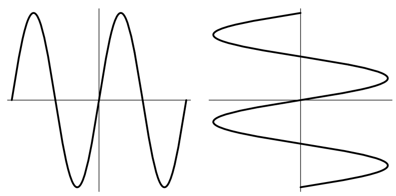


Figure 3.5: $y = \sin x$ and $y = \arcsin x$

Example 3.3.2 $\arcsin x$ has an infinite number of branches. We will denote the principal branch by $\operatorname{Arcsin} x$ which maps $[-1, 1]$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Note that $x = \sin(\arcsin x)$, but $x \neq \arcsin(\sin x)$. $y = \operatorname{Arcsin} x$ in Figure 3.6.

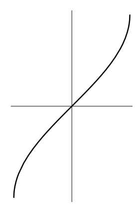


Figure 3.6: $y = \operatorname{Arcsin} x$

Example 3.3.3 Consider $1^{1/3}$. Since x^3 is a one-to-one function, $x^{1/3}$ is a single-valued function. (See Figure 3.7.) $1^{1/3} = 1$.

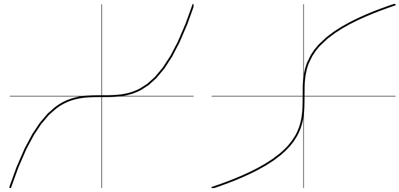


Figure 3.7: $y = x^3$ and $y = x^{1/3}$

Example 3.3.4 Consider $\arccos(1/2)$. $\cos x$ and a few branches of $\arccos x$ are graphed in Figure 3.8. $\cos x = 1/2$

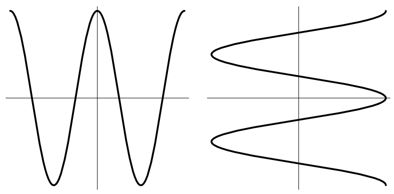


Figure 3.8: $y = \cos x$ and $y = \arccos x$

has the two solutions $x = \pm\pi/3$ in the range $x \in [-\pi, \pi]$. Since $\cos(x + \pi) = -\cos x$,

$$\arccos(1/2) = \{\pm\pi/3 + n\pi\}.$$

3.4 Transforming Equations

We must take care in applying functions to equations. It is always safe to apply a one-to-one function to an equation, (provided it is defined for that domain). For example, we can apply $y = x^3$ or $y = \mathrm{e}^x$ to the equation $x = 1$. The equations $x^3 = 1$ and $\mathrm{e}^x = e$ have the unique solution $x = 1$.

If we apply a many-to-one function to an equation, we may introduce spurious solutions. Applying $y = x^2$ and $y = \sin x$ to the equation $x = \frac{\pi}{2}$ results in $x^2 = \frac{\pi^2}{4}$ and $\sin x = 1$. The former equation has the two solutions $x = \pm\frac{\pi}{2}$; the latter has the infinite number of solutions $x = \frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$.

We do not generally apply a one-to-many function to both sides of an equation as this rarely is useful. Consider the equation

$$\sin^2 x = 1.$$

Applying the function $f(x) = x^{1/2}$ to the equation would not get us anywhere

$$(\sin^2 x)^{1/2} = 1^{1/2}.$$

Since $(\sin^2 x)^{1/2} \neq \sin x$, we cannot simplify the left side of the equation. Instead we could use the definition of $f(x) = x^{1/2}$ as the inverse of the x^2 function to obtain

$$\sin x = 1^{1/2} = \pm 1.$$

Then we could use the definition of arcsin as the inverse of sin to get

$$x = \arcsin(\pm 1).$$

$x = \arcsin(1)$ has the solutions $x = \pi/2 + 2n\pi$ and $x = \arcsin(-1)$ has the solutions $x = -\pi/2 + 2n\pi$. Thus

$$x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}.$$

Note that we cannot just apply arcsin to both sides of the equation as $\arcsin(\sin x) \neq x$.

Chapter 4

Vectors

4.1 Vectors

4.1.1 Scalars and Vectors

A *vector* is a quantity having both a magnitude and a direction. Examples of vector quantities are velocity, force and position. One can represent a vector in n-dimensional space with an arrow whose initial point is at the origin, (Figure 4.1). The magnitude is the length of the vector. Typographically, variables representing vectors are often written in capital letters, bold face or with a vector over-line, $A, \mathbf{a}, \vec{a}$. The magnitude of a vector is denoted $|\mathbf{a}|$.

A *scalar* has only a magnitude. Examples of scalar quantities are mass, time and speed.

Vector Algebra. Two vectors are equal if they have the same magnitude and direction. The negative of a vector, denoted $-\mathbf{a}$, is a vector of the same magnitude as $\mathbf{a}$ but in the opposite direction. We add two vectors $\mathbf{a}$ and $\mathbf{b}$ by placing the tail of $\mathbf{b}$ at the head of $\mathbf{a}$ and defining $\mathbf{a} + \mathbf{b}$ to be the vector with tail at the origin and head at the head of $\mathbf{b}$. (See Figure 4.2.)

The difference, $\mathbf{a} - \mathbf{b}$, is defined as the sum of $\mathbf{a}$ and the negative of $\mathbf{b}$, $\mathbf{a} + (-\mathbf{b})$. The result of multiplying $\mathbf{a}$ by a scalar α is a vector of magnitude $|\alpha|\,|\mathbf{a}|$ with the same/opposite direction if α is positive/negative. (See Figure 4.2.)

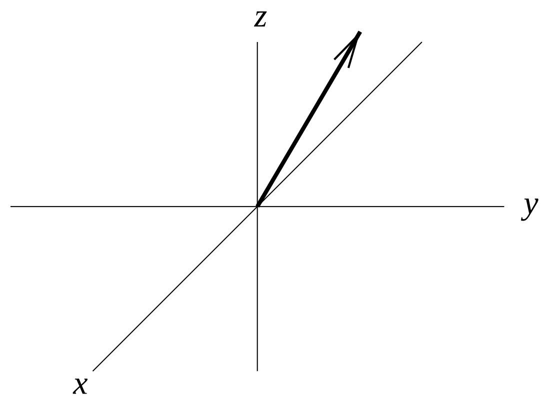


Figure 4.1: Graphical Representation of a Vector in Three Dimensions

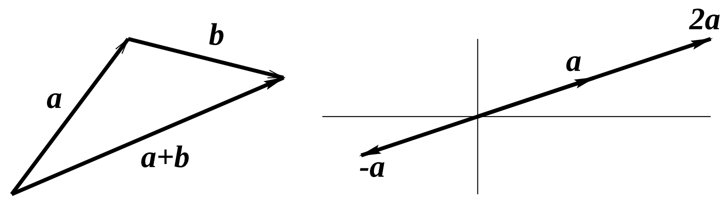


Figure 4.2: Vector Arithmetic

Here are the properties of adding vectors and multiplying them by a scalar. They are evident from geometric considerations.

$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$	$\alpha\mathbf{a}=\mathbf{a}\alpha$	commutative laws
$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$	$\alpha(\beta\mathbf{a})=(\alpha\beta)\mathbf{a}$	associative laws
$\alpha(\mathbf{a}+\mathbf{b})=\alpha\mathbf{a}+\alpha\mathbf{b}$	$(\alpha+\beta)\mathbf{a}=\alpha\mathbf{a}+\beta\mathbf{a}$	distributive laws

Zero and Unit Vectors. The additive identity element for vectors is the *zero vector* or *null vector*. This is a vector of magnitude zero which is denoted as $\mathbf{0}$. A *unit vector* is a vector of magnitude one. If $\mathbf{a}$ is nonzero then $\mathbf{a}/|\mathbf{a}|$ is a unit vector in the direction of $\mathbf{a}$. Unit vectors are often denoted with a caret over-line, $\hat{\mathbf{n}}$.

Rectangular Unit Vectors. In n dimensional Cartesian space, $\mathbb{R}^n$, the unit vectors in the directions of the coordinates axes are $\mathbf{e}_1, \ldots \mathbf{e}_n$. These are called the *rectangular unit vectors*. To cut down on subscripts, the unit vectors in three dimensional space are often denoted with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. (Figure 4.3).

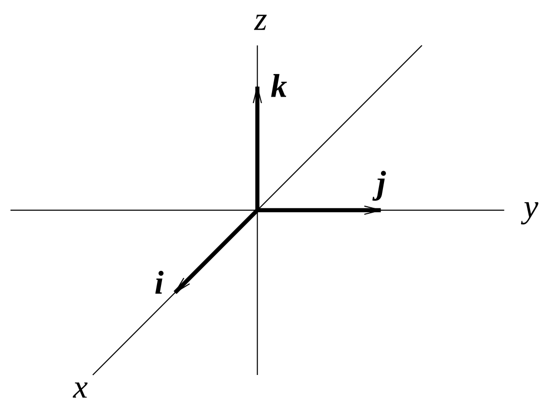


Figure 4.3: Rectangular Unit Vectors

Components of a Vector. Consider a vector $\mathbf{a}$ with tail at the origin and head having the Cartesian coordinates $(a_1, \ldots, a_n)$. We can represent this vector as the sum of n *rectangular component vectors*, $\mathbf{a} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$. (See Figure 4.4.) Another notation for the vector $\mathbf{a}$ is $\langle a_1, \ldots, a_n\rangle$. By the Pythagorean theorem, the magnitude of the vector $\mathbf{a}$ is $|\mathbf{a}| = \sqrt{a_1^2 + \cdots + a_n^2}$.

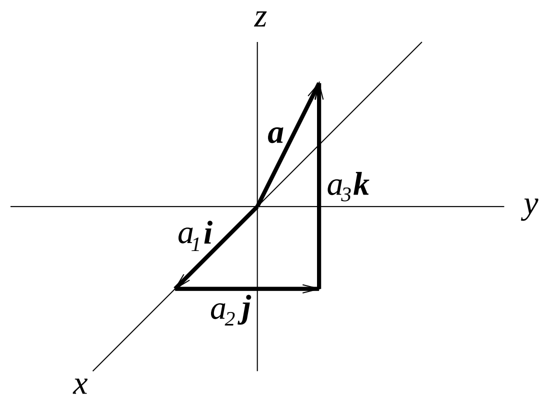


Figure 4.4: Components of a Vector

4.1.2 The Kronecker Delta and Einstein Summation Convention

The Kronecker Delta tensor is defined

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

This notation will be useful in our work with vectors.

Consider writing a vector in terms of its rectangular components. Instead of using ellipses: $\mathbf{a} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$, we could write the expression as a sum: $\mathbf{a} = \sum_{i=1}^{n} a_i\mathbf{e}_i$. We can shorten this notation by leaving out the sum: $\mathbf{a} = a_i\mathbf{e}_i$, where it is understood that whenever an index is repeated in a term we sum over that index from 1 to n. This is the *Einstein summation convention*. A repeated index is called a *summation index* or a *dummy index*. Other indices can take any value from 1 to n and are called *free indices*.

Example 4.1.1 Consider the matrix equation: $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$. We can write out the matrix and vectors explicitly.

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

This takes much less space when we use the summation convention.

$$a_{ij} x_j = b_i$$

Here j is a summation index and i is a free index.

4.1.3 The Dot and Cross Product

Dot Product. The *dot product* or *scalar product* of two vectors is defined,

$$\mathbf{a} \cdot \mathbf{b} \equiv |\mathbf{a}||\mathbf{b}| \cos\theta,$$

where θ is the angle from $\mathbf{a}$ to $\mathbf{b}$. From this definition one can derive the following properties:

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, commutative.
- $\alpha(\mathbf{a} \cdot \mathbf{b}) = (\alpha\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\alpha\mathbf{b})$, associativity of scalar multiplication.
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$, distributive.
- $\mathbf{e}_i \mathbf{e}_j = \delta_{ij}$. In three dimension, this is

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \qquad \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0.$$

- $\mathbf{a} \cdot \mathbf{b} = a_i b_i \equiv a_1 b_1 + \cdots + a_n b_n$, dot product in terms of rectangular components.
- If $\mathbf{a} \cdot \mathbf{b} = 0$ then either $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, (perpendicular), or one of $\mathbf{a}$ and $\mathbf{b}$ are zero.

The Angle Between Two Vectors. We can use the dot product to find the angle between two vectors, $\mathbf{a}$ and $\mathbf{b}$. From the definition of the dot product,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.$$

If the vectors are nonzero, then

$$\theta = \arccos \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right).$$

Example 4.1.2 What is the angle between $\mathbf{i}$ and $\mathbf{i} + \mathbf{j}$?

$$\begin{aligned} \theta &= \arccos \left(\frac{\mathbf{i} \cdot (\mathbf{i} + \mathbf{j})}{|\mathbf{i}||\mathbf{i} + \mathbf{j}|} \right) \\ &= \arccos \left(\frac{1}{\sqrt{2}} \right) \\ &= \frac{\pi}{4}. \end{aligned}$$

Parametric Equation of a Line. Consider a line that passes through the point $\mathbf{a}$ and is parallel to the vector $\mathbf{t}$, (tangent). A parametric equation of the line is

$$\mathbf{x} = \mathbf{a} + u\mathbf{t}, \quad u \in \mathbb{R}.$$

Implicit Equation of a Line. Consider a line that passes through the point $\mathbf{a}$ and is normal, (orthogonal, perpendicular), to the vector $\mathbf{n}$. All the lines that are normal to $\mathbf{n}$ have the property that $\mathbf{x} \cdot \mathbf{n}$ is a constant, where $\mathbf{x}$ is any point on the line. (See Figure 4.5.) $\mathbf{x} \cdot \mathbf{n} = 0$ is the line that is normal to $\mathbf{n}$ and passes through the origin. The line that is normal to $\mathbf{n}$ and passes through the point $\mathbf{a}$ is

$$\mathbf{x} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}.$$

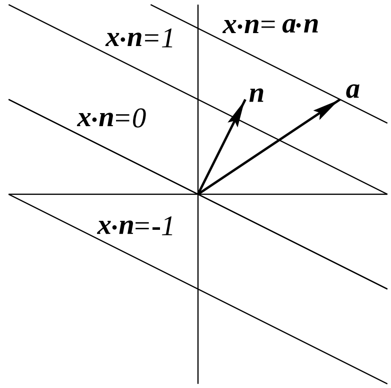


Figure 4.5: Equation for a Line

The normal to a line determines an orientation of the line. The normal points in the direction that is above the line. A point $\mathbf{b}$ is (above/on/below) the line if $(\mathbf{b} - \mathbf{a}) \cdot \mathbf{n}$ is (positive/zero/negative). The signed distance of a point $\mathbf{b}$ from the line $\mathbf{x} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ is

$$(\mathbf{b} - \mathbf{a}) \cdot \frac{\mathbf{n}}{|\mathbf{n}|}.$$

Implicit Equation of a Hyperplane. A hyperplane in $\mathbb{R}^n$ is an $n - 1$ dimensional "sheet" which passes through a given point and is normal to a given direction. In $\mathbb{R}^3$ we call this a plane. Consider a hyperplane that passes through the point $\mathbf{a}$ and is normal to the vector $\mathbf{n}$. All the hyperplanes that are normal to $\mathbf{n}$ have the property that $\mathbf{x} \cdot \mathbf{n}$ is a constant, where $\mathbf{x}$ is any point in the hyperplane. $\mathbf{x} \cdot \mathbf{n} = 0$ is the hyperplane that is normal to $\mathbf{n}$ and passes through the origin. The hyperplane that is normal to $\mathbf{n}$ and passes through the point $\mathbf{a}$ is

$$\mathbf{x} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}.$$

The normal determines an orientation of the hyperplane. The normal points in the direction that is above the hyperplane. A point $\mathbf{b}$ is (above/on/below) the hyperplane if $(\mathbf{b} - \mathbf{a}) \cdot \mathbf{n}$ is (positive/zero/negative). The signed distance of a point $\mathbf{b}$ from the hyperplane $\mathbf{x} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ is

$$(\mathbf{b} - \mathbf{a}) \cdot \frac{\mathbf{n}}{|\mathbf{n}|}.$$

Right and Left-Handed Coordinate Systems. Consider a rectangular coordinate system in two dimensions. Angles are measured from the positive x axis in the direction of the positive y axis. There are two ways of labeling the axes. (See Figure 4.6.) In one the angle increases in the counterclockwise direction and in the other the angle increases in the clockwise direction. The former is the familiar Cartesian coordinate system.

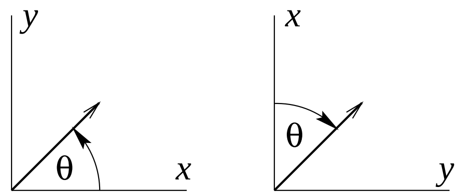


Figure 4.6: There are Two Ways of Labeling the Axes in Two Dimensions.

There are also two ways of labeling the axes in a three-dimensional rectangular coordinate system. These are called right-handed and left-handed coordinate systems. See Figure 4.7. Any other labelling of the axes could be rotated into one of these configurations. The right-handed system is the one that is used by default. If you put your right thumb in the direction of the z axis in a right-handed coordinate system, then your fingers curl in the direction from the x axis to the y axis.

Cross Product. The *cross product* or *vector product* is defined,

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \mathbf{n},$$

where θ is the angle from $\mathbf{a}$ to $\mathbf{b}$ and $\mathbf{n}$ is a unit vector that is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ and in the direction such that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{n}$ form a right-handed system.

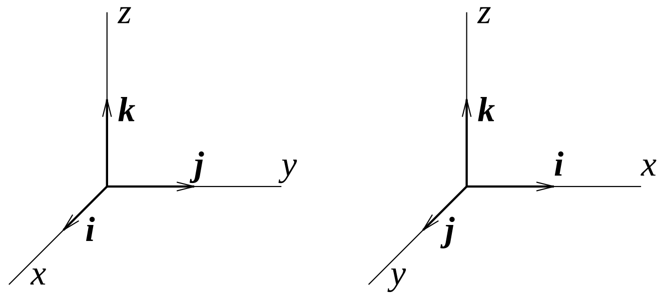


Figure 4.7: Right and Left Handed Coordinate Systems

You can visualize the direction of $\mathbf{a} \times \mathbf{b}$ by applying the *right hand rule*. Curl the fingers of your right hand in the direction from $\mathbf{a}$ to $\mathbf{b}$. Your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$. **Warning**: Unless you are a lefty, get in the habit of putting down your pencil before applying the right hand rule.

The dot and cross products behave a little differently. First note that unlike the dot product, the cross product is not commutative. The magnitudes of $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ are the same, but their directions are opposite. (See Figure 4.8.)

Let

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \; \mathbf{n} \quad \text{and} \quad \mathbf{b} \times \mathbf{a} = |\mathbf{b}||\mathbf{a}| \sin\phi \; \mathbf{m}.$$

The angle from $\mathbf{a}$ to $\mathbf{b}$ is the same as the angle from $\mathbf{b}$ to $\mathbf{a}$. Since $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ and $\{\mathbf{b}, \mathbf{a}, \mathbf{m}\}$ are right-handed systems, $\mathbf{m}$ points in the opposite direction as $\mathbf{n}$. Since $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ we say that the cross product is anti-commutative.

Next we note that since

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta,$$

the magnitude of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram defined by the two vectors. (See Figure 4.9.) The area of the triangle defined by two vectors is then $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

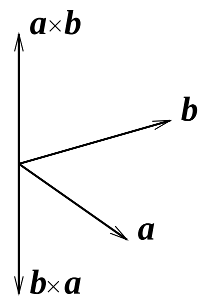


Figure 4.8: The Cross Product is Anti-Commutative.

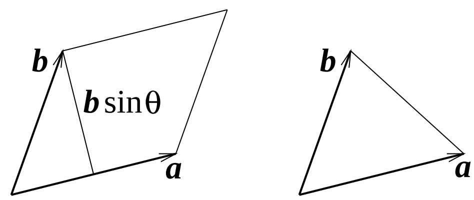


Figure 4.9: The Parallelogram and the Triangle Defined by Two Vectors

From the definition of the cross product, one can derive the following properties:

- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, anti-commutative.
- $\alpha(\mathbf{a} \times \mathbf{b}) = (\alpha\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\alpha\mathbf{b})$, associativity of scalar multiplication.
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$, distributive.
- $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. The cross product is not associative.
- $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$.

- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.
- $$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

  cross product in terms of rectangular components.
- If $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$ then either $\mathbf{a}$ and $\mathbf{b}$ are parallel or one of $\mathbf{a}$ or $\mathbf{b}$ is zero.

Scalar Triple Product. Consider the volume of the parallelopiped defined by three vectors. (See Figure 4.10.) The area of the base is $||\mathbf{b}||\mathbf{c}| \sin\theta|$, where θ is the angle between $\mathbf{b}$ and $\mathbf{c}$. The height is $|\mathbf{a}| \cos\phi$, where ϕ is the angle between $\mathbf{b} \times \mathbf{c}$ and $\mathbf{a}$. Thus the volume of the parallelopiped is $|\mathbf{a}||\mathbf{b}||\mathbf{c}| \sin\theta \cos\phi$.

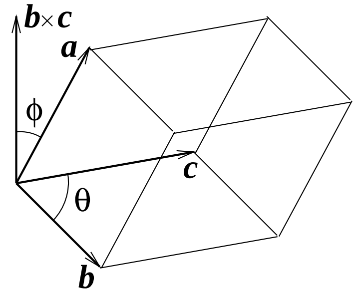


Figure 4.10: The Parallelopiped Defined by Three Vectors

Note that

$$\begin{aligned} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| &= |\mathbf{a} \cdot (|\mathbf{b}||\mathbf{c}| \sin\theta \ \mathbf{n})| \\ &= ||\mathbf{a}||\mathbf{b}||\mathbf{c}| \sin\theta \cos\phi| \,. \end{aligned}$$

Thus $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is the volume of the parallelopiped. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is the volume or the negative of the volume depending on whether $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a right or left-handed system.

Note that parentheses are unnecessary in $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$. There is only one way to interpret the expression. If you did the dot product first then you would be left with the cross product of a scalar and a vector which is meaningless. $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ is called the *scalar triple product*.

Plane Defined by Three Points. Three points which are not collinear define a plane. Consider a plane that passes through the three points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. One way of expressing that the point $\mathbf{x}$ lies in the plane is that the vectors $\mathbf{x} - \mathbf{a}$, $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ are coplanar. (See Figure 4.11.) If the vectors are coplanar, then the parallelopiped defined by these three vectors will have zero volume. We can express this in an equation using the scalar triple product,

$$(\mathbf{x} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = 0.$$

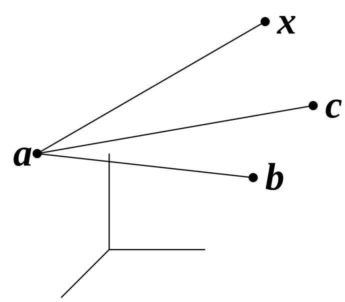


Figure 4.11: Three Points Define a Plane.

4.2 Sets of Vectors in n Dimensions

Orthogonality. Consider two n-dimensional vectors

$$\mathbf{x} = (x_1, x_2, \dots, x_n), \qquad \mathbf{y} = (y_1, y_2, \dots, y_n).$$

The inner product of these vectors can be defined

$$\langle \mathbf{x} | \mathbf{y} \rangle \equiv \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

The vectors are orthogonal if $\mathbf{x} \cdot \mathbf{y} = 0$. The norm of a vector is the length of the vector generalized to n dimensions.

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$$

Consider a set of vectors

$$\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}.$$

If each pair of vectors in the set is orthogonal, then the set is orthogonal.

$$\mathbf{x}_i \cdot \mathbf{x}_j = 0 \quad \text{if } i \neq j$$

If in addition each vector in the set has norm 1, then the set is orthonormal.

$$\mathbf{x}_i \cdot \mathbf{x}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Here δ_{ij} is known as the Kronecker delta function.

Completeness. A set of n, n-dimensional vectors

$$\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$$

is complete if any n-dimensional vector can be written as a linear combination of the vectors in the set. That is, any vector $\mathbf{y}$ can be written

$$\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{x}_i.$$

Taking the inner product of each side of this equation with $\mathbf{x}_m$,

$$\begin{aligned}
\mathbf{y} \cdot \mathbf{x}_m &= \left(\sum_{i=1}^{n} c_i \mathbf{x}_i \right) \cdot \mathbf{x}_m \\
&= \sum_{i=1}^{n} c_i \mathbf{x}_i \cdot \mathbf{x}_m \\
&= c_m \mathbf{x}_m \cdot \mathbf{x}_m \\
c_m &= \frac{\mathbf{y} \cdot \mathbf{x}_m}{\|\mathbf{x}_m\|^2}
\end{aligned}$$

Thus $\mathbf{y}$ has the expansion

$$\mathbf{y} = \sum_{i=1}^{n} \frac{\mathbf{y} \cdot \mathbf{x}_i}{\|\mathbf{x}_i\|^2} \mathbf{x}_i.$$

If in addition the set is orthonormal, then

$$\mathbf{y} = \sum_{i=1}^{n} (\mathbf{y} \cdot \mathbf{x}_i) \mathbf{x}_i.$$

4.3 Exercises

The Dot and Cross Product

Exercise 4.1
Prove the distributive law for the dot product,

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$

Exercise 4.2
Prove that

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i \equiv a_1 b_1 + \cdots + a_n b_n.$$

Exercise 4.3
What is the angle between the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + 3\mathbf{j}$?

Exercise 4.4
Prove the distributive law for the cross product,

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b}.$$

Exercise 4.5
Show that

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Exercise 4.6
What is the area of the quadrilateral with vertices at $(1,1)$, $(4,2)$, $(3,7)$ and $(2,3)$?

Exercise 4.7
What is the volume of the tetrahedron with vertices at $(1,1,0)$, $(3,2,1)$, $(2,4,1)$ and $(1,2,5)$?

Exercise 4.8
What is the equation of the plane that passes through the points $(1,2,3)$, $(2,3,1)$ and $(3,1,2)$? What is the distance from the point $(2,3,5)$ to the plane?

4.4 Hints

The Dot and Cross Product

Hint 4.1
First prove the distributive law when the first vector is of unit length,

$$\mathbf{n} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{n} \cdot \mathbf{b} + \mathbf{n} \cdot \mathbf{c}.$$

Then all the quantities in the equation are projections onto the unit vector $\mathbf{n}$ and you can use geometry.

Hint 4.2
First prove that the dot product of a rectangular unit vector with itself is one and the dot product of two distinct rectangular unit vectors is zero. Then write $\mathbf{a}$ and $\mathbf{b}$ in rectangular components and use the distributive law.

Hint 4.3
Use $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$.

Hint 4.4
First consider the case that both $\mathbf{b}$ and $\mathbf{c}$ are orthogonal to $\mathbf{a}$. Prove the distributive law in this case from geometric considerations.

Next consider two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$. We can write $\mathbf{b} = \mathbf{b}_\perp + \mathbf{b}_\|$ where $\mathbf{b}_\perp$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}_\|$ is parallel to $\mathbf{a}$. Show that

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}_\perp.$$

Finally prove the distributive law for arbitrary $\mathbf{b}$ and $\mathbf{c}$.

Hint 4.5
Write the vectors in their rectangular components and use,

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j},$$

and,

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0.$$

Hint 4.6
The quadrilateral is composed of two triangles. The area of a triangle defined by the two vectors $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}|\mathbf{a} \cdot \mathbf{b}|$.

Hint 4.7
Justify that the area of a tetrahedron determined by three vectors is one sixth the area of the parallelogram determined by those three vectors. The area of a parallelogram determined by three vectors is the magnitude of the scalar triple product of the vectors: $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$.

Hint 4.8
The equation of a line that is orthogonal to $\mathbf{a}$ and passes through the point $\mathbf{b}$ is $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{b}$. The distance of a point $\mathbf{c}$ from the plane is

$$\left| (\mathbf{c} - \mathbf{b}) \cdot \frac{\mathbf{a}}{|\mathbf{a}|} \right|$$

4.5 Solutions

The Dot and Cross Product

Solution 4.1
First we prove the distributive law when the first vector is of unit length, i.e.,

$$\mathbf{n} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{n} \cdot \mathbf{b} + \mathbf{n} \cdot \mathbf{c}. \tag{4.1}$$

From Figure 4.12 we see that the projection of the vector $\mathbf{b} + \mathbf{c}$ onto $\mathbf{n}$ is equal to the sum of the projections $\mathbf{b} \cdot \mathbf{n}$ and $\mathbf{c} \cdot \mathbf{n}$.

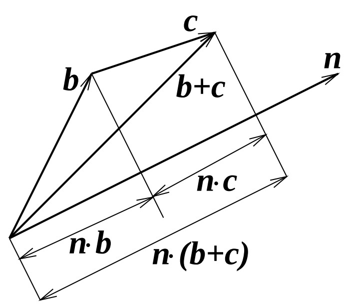


Figure 4.12: The Distributive Law for the Dot Product

Now we extend the result to the case when the first vector has arbitrary length. We define $\mathbf{a} = |\mathbf{a}|\mathbf{n}$ and multiply Equation 4.1 by the scalar, $|\mathbf{a}|$.

$$|\mathbf{a}|\mathbf{n} \cdot (\mathbf{b} + \mathbf{c}) = |\mathbf{a}|\mathbf{n} \cdot \mathbf{b} + |\mathbf{a}|\mathbf{n} \cdot \mathbf{c}$$
$$\boxed{\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.}$$

Solution 4.2
First note that

$$\mathbf{e}_i \cdot \mathbf{e}_i = |\mathbf{e}_i||\mathbf{e}_i| \cos(0) = 1.$$

Then note that that dot product of any two distinct rectangular unit vectors is zero because they are orthogonal. Now we write $\mathbf{a}$ and $\mathbf{b}$ in terms of their rectangular components and use the distributive law.

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= a_i \mathbf{e}_i \cdot b_j \mathbf{e}_j \\ &= a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j \\ &= a_i b_j \delta_{ij} \\ &= a_i b_i\end{aligned}$$

Solution 4.3
Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, we have

$$\theta = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$$

when $\mathbf{a}$ and $\mathbf{b}$ are nonzero.

$$\theta = \arccos\left(\frac{(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + 3\mathbf{j})}{|\mathbf{i} + \mathbf{j}||\mathbf{i} + 3\mathbf{j}|}\right) = \arccos\left(\frac{4}{\sqrt{2}\sqrt{10}}\right) = \arccos\left(\frac{2\sqrt{5}}{5}\right) \approx 0.463648$$

Solution 4.4
First consider the case that both $\mathbf{b}$ and $\mathbf{c}$ are orthogonal to $\mathbf{a}$. $\mathbf{b} + \mathbf{c}$ is the diagonal of the parallelogram defined by $\mathbf{b}$ and $\mathbf{c}$, (see Figure 4.13). Since $\mathbf{a}$ is orthogonal to each of these vectors, taking the cross product of $\mathbf{a}$ with these vectors has the effect of rotating the vectors through $\pi/2$ radians about $\mathbf{a}$ and multiplying their length by $|\mathbf{a}|$. Note that $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$ is the diagonal of the parallelogram defined by $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times \mathbf{c}$. Thus we see that the distributive law holds when $\mathbf{a}$ is orthogonal to both $\mathbf{b}$ and $\mathbf{c}$,

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

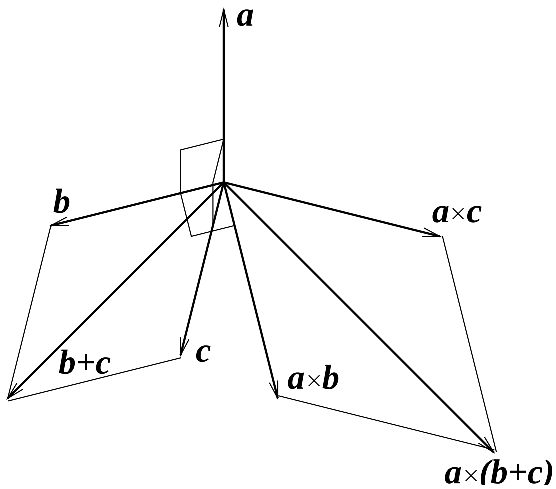


Figure 4.13: The Distributive Law for the Cross Product

Now consider two arbitrary vectors **a** and **b**. We can write $\mathbf{b} = \mathbf{b}_\perp + \mathbf{b}_\parallel$ where $\mathbf{b}_\perp$ is orthogonal to **a** and $\mathbf{b}_\parallel$ is parallel to **a**, (see Figure 4.14).

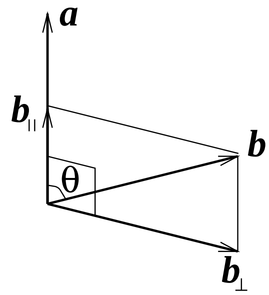


Figure 4.14: The Vector **b** Written as a Sum of Components Orthogonal and Parallel to **a**

By the definition of the cross product,

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \ \mathbf{n}.$$

Note that

$$|\mathbf{b}_\perp| = |\mathbf{b}| \sin\theta,$$

and that $\mathbf{a} \times \mathbf{b}_\perp$ is a vector in the same direction as $\mathbf{a} \times \mathbf{b}$. Thus we see that

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \sin\theta \ \mathbf{n} \\
&= |\mathbf{a}|(\sin\theta|\mathbf{b}|)\mathbf{n} \\
&= |\mathbf{a}||\mathbf{b}_\perp|\mathbf{n} \qquad\qquad\qquad\qquad = |\mathbf{a}||\mathbf{b}_\perp| \sin(\pi/2)\mathbf{n}
\end{aligned}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}_\perp.$$

Now we are prepared to prove the distributive law for arbitrary $\mathbf{b}$ and $\mathbf{c}$.

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times (\mathbf{b}_\perp + \mathbf{b}_\| + \mathbf{c}_\perp + \mathbf{c}_\|) \\
&= \mathbf{a} \times ((\mathbf{b} + \mathbf{c})_\perp + (\mathbf{b} + \mathbf{c})_\|) \\
&= \mathbf{a} \times ((\mathbf{b} + \mathbf{c})_\perp) \\
&= \mathbf{a} \times \mathbf{b}_\perp + \mathbf{a} \times \mathbf{c}_\perp \\
&= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}
\end{aligned}$$

$$\boxed{\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}}$$

Solution 4.5
We know that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j},$$

and that

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0.$$

Now we write **a** and **b** in terms of their rectangular components and use the distributive law to expand the cross product.

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_2\mathbf{j} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_3\mathbf{k} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1 b_2 \mathbf{k} + a_1 b_3(-\mathbf{j}) + a_2 b_1(-\mathbf{k}) + a_2 b_3 \mathbf{i} + a_3 b_1 \mathbf{j} + a_3 b_2(-\mathbf{i}) \\
&= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}
\end{aligned}$$

Next we evaluate the determinant.

$$\begin{aligned}
\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} &= \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\
&= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}
\end{aligned}$$

Thus we see that,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Solution 4.6

The area area of the quadrilateral is the area of two triangles. The first triangle is defined by the vector from $(1,1)$ to $(4,2)$ and the vector from $(1,1)$ to $(2,3)$. The second triangle is defined by the vector from $(3,7)$ to $(4,2)$ and the vector from $(3,7)$ to $(2,3)$. (See Figure 4.15.) The area of a triangle defined by the two vectors **a** and **b** is $\frac{1}{2}|\mathbf{a} \cdot \mathbf{b}|$. The area of the quadrilateral is then,

$$\frac{1}{2}|(3\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + 2\mathbf{j})| + \frac{1}{2}|(\mathbf{i} - 5\mathbf{j}) \cdot (-\mathbf{i} - 4\mathbf{j})| = \frac{1}{2}(5) + \frac{1}{2}(19) = 12.$$

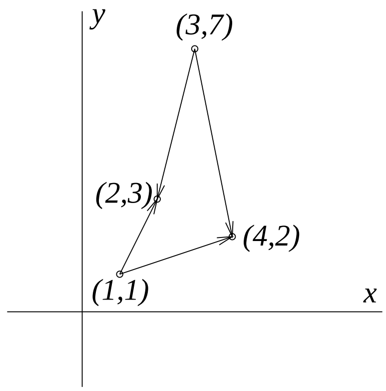


Figure 4.15: Quadrilateral

Solution 4.7
The tetrahedron is determined by the three vectors with tail at $(1, 1, 0)$ and heads at $(3, 2, 1)$, $(2, 4, 1)$ and $(1, 2, 5)$. These are $\langle 2, 1, 1\rangle$, $\langle 1, 3, 1\rangle$ and $\langle 0, 1, 5\rangle$. The area of the tetrahedron is one sixth the area of the parallelogram determined by these vectors. (This is because the area of a pyramid is $\frac{1}{3}$(base)(height). The base of the tetrahedron is half the area of the parallelogram and the heights are the same. $\frac{1}{2}\frac{1}{3} = \frac{1}{6}$) Thus the area of a tetrahedron determined by three vectors is $\frac{1}{6}|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|$. The area of the tetrahedron is

$$\frac{1}{6} |\langle 2, 1, 1\rangle \cdot \langle 1, 3, 1\rangle \times \langle 1, 2, 5\rangle| = \frac{1}{6} |\langle 2, 1, 1\rangle \cdot \langle 13, -4, -1\rangle| = \frac{7}{2}$$

Solution 4.8
The two vectors with tails at $(1, 2, 3)$ and heads at $(2, 3, 1)$ and $(3, 1, 2)$ are parallel to the plane. Taking the cross product of these two vectors gives us a vector that is orthogonal to the plane.

$$\langle 1, 1, -2\rangle \times \langle 2, -1, -1\rangle = \langle -3, -3, -3\rangle$$

We see that the plane is orthogonal to the vector $\langle 1, 1, 1\rangle$ and passes through the point $(1, 2, 3)$. The equation of the plane is

$$\langle 1,1,1\rangle \cdot \langle x,y,z\rangle = \langle 1,1,1\rangle \cdot \langle 1,2,3\rangle,$$

$$x+y+z=6.$$

Consider the vector with tail at $(1,2,3)$ and head at $(2,3,5)$. The magnitude of the dot product of this vector with the unit normal vector gives the distance from the plane.

$$\left|\langle 1,1,2\rangle \cdot \frac{\langle 1,1,1\rangle}{|\langle 1,1,1\rangle|}\right| = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

Part II

Calculus

Chapter 5

Differential Calculus

5.1 Limits of Functions

Definition of a Limit. If the value of the function $y(x)$ gets arbitrarily close to η as x approaches the point ξ, then wee say that the limit of the function as x approaches ξ is equal to η. This is written:

$$\lim_{x \to \xi} y(x) = \eta$$

To make the notion of "arbitrarily close" precise: for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|y(x) - \eta| < \epsilon$ for all $0 < |x - \xi| < \delta$. That is, there is an interval surrounding the point $x = \xi$ for which the function is within ϵ of η. See Figure 5.1. Note that the interval surrounding $x = \xi$ is a deleted neighborhood, that is it does not contain the point $x = \xi$. Thus the value function at $x = \xi$ need not be equal to η for the limit to exist. Indeed the function need not even be defined at $x = \xi$.

To prove that a function has a limit at a point ξ we first bound $|y(x) - \eta|$ in terms of δ for values of x satisfying $0 < |x - \xi| < \delta$. Denote this upper bound by $u(\delta)$. Then for an arbitrary $\epsilon > 0$, we determine a $\delta > 0$ such that the the upper bound $u(\delta)$ and hence $|y(x) - \eta|$ is less than ϵ.

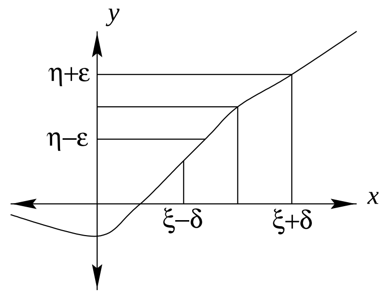


Figure 5.1: The δ neighborhood of $x = \xi$ such that $|y(x) - \eta| < \epsilon$.

Example 5.1.1 Show that

$$\lim_{x \to 1} x^2 = 1.$$

Consider any $\epsilon > 0$. We need to show that there exists a $\delta > 0$ such that $|x^2 - 1| < \epsilon$ for all $|x - 1| < \delta$. First we obtain a bound on $|x^2 - 1|$.

$$\begin{aligned}
|x^2 - 1| &= |(x - 1)(x + 1)| \\
&= |x - 1||x + 1| \\
&< \delta|x + 1| \\
&= \delta|(x - 1) + 2| \\
&< \delta(\delta + 2)
\end{aligned}$$

Now we choose a positive δ such that,

$$\delta(\delta + 2) = \epsilon.$$

We see that

$$\delta = \sqrt{1 + \epsilon} - 1,$$

is positive and satisfies the criterion that $|x^2 - 1| < \epsilon$ for all $0 < |x - 1| < \delta$. Thus the limit exists.

Note that the value of the function $y(\xi)$ need not be equal to $\lim_{x \to \xi} y(x)$. This is illustrated in Example 5.1.2.

Example 5.1.2 Consider the function

$$y(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Z}, \\ 0 & \text{for } x \notin \mathbb{Z}. \end{cases}$$

For what values of ξ does $\lim_{x \to \xi} y(x)$ exist?

First consider $\xi \notin \mathbb{Z}$. Then there exists an open neighborhood $a < \xi < b$ around ξ such that $y(x)$ is identically zero for $x \in (a, b)$. Then trivially, $\lim_{x \to \xi} y(x) = 0$.

Now consider $\xi \in \mathbb{Z}$. Consider any $\epsilon > 0$. Then if $|x - \xi| < 1$ then $|y(x) - 0| = 0 < \epsilon$. Thus we see that $\lim_{x \to \xi} y(x) = 0$.

Thus, regardless of the value of ξ, $\lim_{x \to \xi} y(x) = 0$.

Left and Right Limits. With the notation $\lim_{x \to \xi^+} y(x)$ we denote the right limit of $y(x)$. This is the limit as x approaches ξ from above. Mathematically: $\lim_{x \to \xi^+}$ exists if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|y(x) - \eta| < \epsilon$ for all $0 < \xi - x < \delta$. The left limit $\lim_{x \to \xi^-} y(x)$ is defined analogously.

Example 5.1.3 Consider the function, $\frac{\sin x}{|x|}$, defined for $x \neq 0$. (See Figure 5.2.) The left and right limits exist as x approaches zero.

$$\lim_{x \to 0^+} \frac{\sin x}{|x|} = 1, \qquad \lim_{x \to 0^-} \frac{\sin x}{|x|} = -1$$

However the limit,

$$\lim_{x \to 0} \frac{\sin x}{|x|},$$

does not exist.

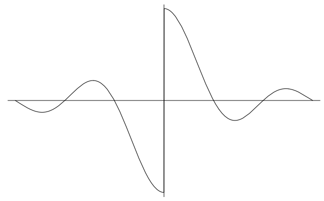


Figure 5.2: Plot of $\sin(x)/|x|$.

Properties of Limits. Let $\lim_{x\to\xi} u(x)$ and $\lim_{x\to\xi} v(x)$ exist.

- $\lim_{x\to\xi} (au(x) + bv(x)) = a \lim_{x\to\xi} u(x) + b \lim_{x\to\xi} v(x)$.
- $\lim_{x\to\xi} (u(x)v(x)) = (\lim_{x\to\xi} u(x)) (\lim_{x\to\xi} v(x))$.
- $\lim_{x\to\xi} \left(\frac{u(x)}{v(x)}\right) = \frac{\lim_{x\to\xi} u(x)}{\lim_{x\to\xi} v(x)}$ if $\lim_{x\to\xi} v(x) \neq 0$.

Example 5.1.4 Prove that if $\lim_{x\to\xi} u(x) = \mu$ and $\lim_{x\to\xi} v(x) = \nu$ exist then

$$\lim_{x\to\xi} (u(x)v(x)) = \left(\lim_{x\to\xi} u(x)\right) \left(\lim_{x\to\xi} v(x)\right).$$

Assume that μ and ν are nonzero. (The cases where one or both are zero are similar and simpler.)

$$\begin{aligned} |u(x)v(x) - \mu\nu| &= |uv - (u + \mu - u)\nu| \\ &= |u(v - \nu) + (u - \mu)\nu| \\ &= |u||v - \nu| + |u - \mu||\nu| \end{aligned}$$

A sufficient condition for $|u(x)v(x) - \mu\nu| < \epsilon$ is

$$|u - \mu| < \frac{\epsilon}{2|\nu|} \quad \text{and} \quad |v - \nu| < \frac{\epsilon}{2\left(|\mu| + \frac{\epsilon}{2|\nu|}\right)}.$$

Since the two right sides of the inequalities are positive, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that the first inequality is satisfied for all $|x - \xi| < \delta_1$ and the second inequality is satisfied for all $|x - \xi| < \delta_2$. By choosing δ to be the smaller of δ_1 and δ_2 we see that

$$|u(x)v(x) - \mu\nu| < \epsilon \text{ for all } |x - \xi| < \delta.$$

Thus

$$\lim_{x \to \xi} (u(x)v(x)) = \left(\lim_{x \to \xi} u(x)\right) \left(\lim_{x \to \xi} v(x)\right) = \mu\nu.$$

Result 5.1.1 Definition of a Limit. The statement:

$$\lim_{x \to \xi} y(x) = \eta$$

means that $y(x)$ gets arbitrarily close to η as x approaches ξ. For any $\epsilon > 0$ there exists a $\delta > 0$ such that $|y(x) - \eta| < \epsilon$ for all x in the neighborhood $0 < |x - \xi| < \delta$. The left and right limits,

$$\lim_{x \to \xi^-} y(x) = \eta \quad \text{and} \quad \lim_{x \to \xi^+} y(x) = \eta$$

denote the limiting value as x approaches ξ respectively from below and above. The neighborhoods are respectively $-\delta < x - \xi < 0$ and $0 < x - \xi < \delta$.

Properties of Limits. Let $\lim_{x \to \xi} u(x)$ and $\lim_{x \to \xi} v(x)$ exist.

- $\lim_{x \to \xi} (au(x) + bv(x)) = a \lim_{x \to \xi} u(x) + b \lim_{x \to \xi} v(x)$.
- $\lim_{x \to \xi} (u(x)v(x)) = (\lim_{x \to \xi} u(x)) (\lim_{x \to \xi} v(x))$.
- $\lim_{x \to \xi} \left(\frac{u(x)}{v(x)} \right) = \frac{\lim_{x \to \xi} u(x)}{\lim_{x \to \xi} v(x)}$ if $\lim_{x \to \xi} v(x) \neq 0$.

5.2 Continuous Functions

Definition of Continuity. A function $y(x)$ is said to be *continuous at* $x = \xi$ if the value of the function is equal to its limit, that is, $\lim_{x \to \xi} y(x) = y(\xi)$. Note that this one condition is actually the three conditions: $y(\xi)$ is defined, $\lim_{x \to \xi} y(x)$ exists and $\lim_{x \to \xi} y(x) = y(\xi)$. A function is *continuous* if it is continuous at each point in its domain. A function is *continuous on the closed interval* $[a, b]$ if the function is continuous for each point

$x \in (a, b)$ and $\lim_{x \to a^+} y(x) = y(a)$ and $\lim_{x \to b_-} y(x) = y(b)$.

Discontinuous Functions. If a function is not continuous at a point it is called *discontinuous* at that point. If $\lim_{x \to \xi} y(x)$ exists but is not equal to $y(\xi)$, then the function has a *removable discontinuity*. It is thus named because we could define a continuous function

$$z(x) = \begin{cases} y(x) & \text{for } x \neq \xi, \\ \lim_{x \to \xi} y(x) & \text{for } x = \xi, \end{cases}$$

to remove the discontinuity. If both the left and right limit of a function at a point exist, but are not equal, then the function has a *jump discontinuity* at that point. If either the left or right limit of a function does not exist, then the function is said to have an *infinite discontinuity* at that point.

Example 5.2.1 $\frac{\sin x}{x}$ has a removable discontinuity at $x = 0$. The Heaviside function,

$$H(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1/2 & \text{for } x = 0, \\ 1 & \text{for } x > 0, \end{cases}$$

has a jump discontinuity at $x = 0$. $\frac{1}{x}$ has an infinite discontinuity at $x = 0$. See Figure 5.3.

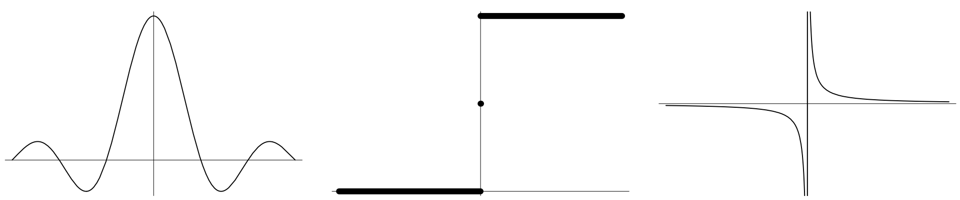


Figure 5.3: A Removable discontinuity, a Jump Discontinuity and an Infinite Discontinuity

Properties of Continuous Functions.

Arithmetic. If $u(x)$ and $v(x)$ are continuous at $x = \xi$ then $u(x) \pm v(x)$ and $u(x)v(x)$ are continuous at $x = \xi$. $\frac{u(x)}{v(x)}$ is continuous at $x = \xi$ if $v(\xi) \neq 0$.

Function Composition. If $u(x)$ is continuous at $x = \xi$ and $v(x)$ is continuous at $x = \mu = u(\xi)$ then $u(v(x))$ is continuous at $x = \xi$. The composition of continuous functions is a continuous function.

Boundedness. A function which is continuous on a closed interval is bounded in that closed interval.

Nonzero in a Neighborhood. If $y(\xi) \neq 0$ then there exists a neighborhood $(\xi - \epsilon, \xi + \epsilon)$, $\epsilon > 0$ of the point ξ such that $y(x) \neq 0$ for $x \in (\xi - \epsilon, \xi + \epsilon)$.

Intermediate Value Theorem. Let $u(x)$ be continuous on $[a, b]$. If $u(a) \leq \mu \leq u(b)$ then there exists $\xi \in [a, b]$ such that $u(\xi) = \mu$. This is known as the *intermediate value theorem*. A corollary of this is that if $u(a)$ and $u(b)$ are of opposite sign then $u(x)$ has at least one zero on the interval (a, b).

Maxima and Minima. If $u(x)$ is continuous on $[a, b]$ then $u(x)$ has a maximum and a minimum on $[a, b]$. That is, there is at least one point $\xi \in [a, b]$ such that $u(\xi) \geq u(x)$ for all $x \in [a, b]$ and there is at least one point $\eta \in [a, b]$ such that $u(\eta) \leq u(x)$ for all $x \in [a, b]$.

Piecewise Continuous Functions. A function is *piecewise continuous* on an interval if the function is bounded on the interval and the interval can be divided into a finite number of intervals on each of which the function is continuous. For example, the greatest integer function, $\lfloor x \rfloor$, is piecewise continuous. ($\lfloor x \rfloor$ is defined to the the greatest integer less than or equal to x.) See Figure 5.4 for graphs of two piecewise continuous functions.

Uniform Continuity. Consider a function $f(x)$ that is continuous on an interval. This means that for any point ξ in the interval and any positive ϵ there exists a $\delta > 0$ such that $|f(x) - f(\xi)| < \epsilon$ for all $0 < |x - \xi| < \delta$. In general, this value of δ depends on both ξ and ϵ. If δ can be chosen so it is a function of ϵ alone and independent of ξ then the function is said to be *uniformly continuous* on the interval. A sufficient condition for uniform continuity is that the function is continuous on a closed interval.

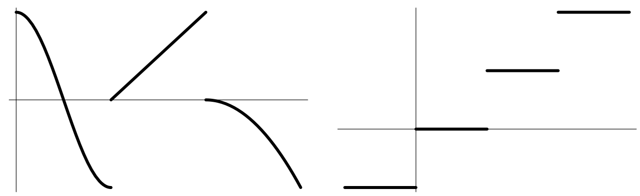


Figure 5.4: Piecewise Continuous Functions

5.3 The Derivative

Consider a function $y(x)$ on the interval $(x \dots x + \Delta x)$ for some $\Delta x > 0$. We define the increment $\Delta y = y(x+\Delta x) - y(x)$. The *average rate of change*, (average velocity), of the function on the interval is $\frac{\Delta y}{\Delta x}$. The average rate of change is the slope of the secant line that passes through the points $(x, y(x))$ and $(x + \Delta x, y(x + \Delta x))$. See Figure 5.5.

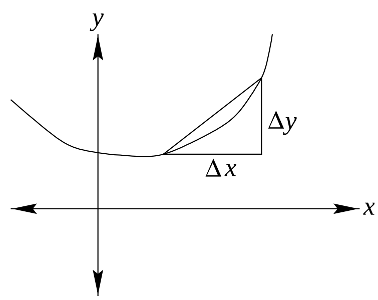


Figure 5.5: The increments Δx and Δy.

If the slope of the secant line has a limit as Δx approaches zero then we call this slope the *derivative* or *instantaneous rate of change* of the function at the point x. We denote the derivative by $\frac{\mathrm{d}y}{\mathrm{d}x}$, which is a nice notation as the derivative is the limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} \equiv \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}.$$

Δx may approach zero from below or above. It is common to denote the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $\frac{\mathrm{d}}{\mathrm{d}x} y$, $y'(x)$, y' or Dy.

A function is said to be *differentiable* at a point if the derivative exists there. Note that differentiability implies continuity, but not vice versa.

Example 5.3.1 Consider the derivative of $y(x) = x^2$ at the point $x = 1$.

$$\begin{aligned} y'(1) &\equiv \lim_{\Delta x \to 0} \frac{y(1 + \Delta x) - y(1)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(1 + \Delta x)^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2 + \Delta x) \\ &= 2 \end{aligned}$$

Figure 5.6 shows the secant lines approaching the tangent line as Δx approaches zero from above and below.

Example 5.3.2 We can compute the derivative of $y(x) = x^2$ at an arbitrary point x.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2x + \Delta x) \\ &= 2x \end{aligned}$$

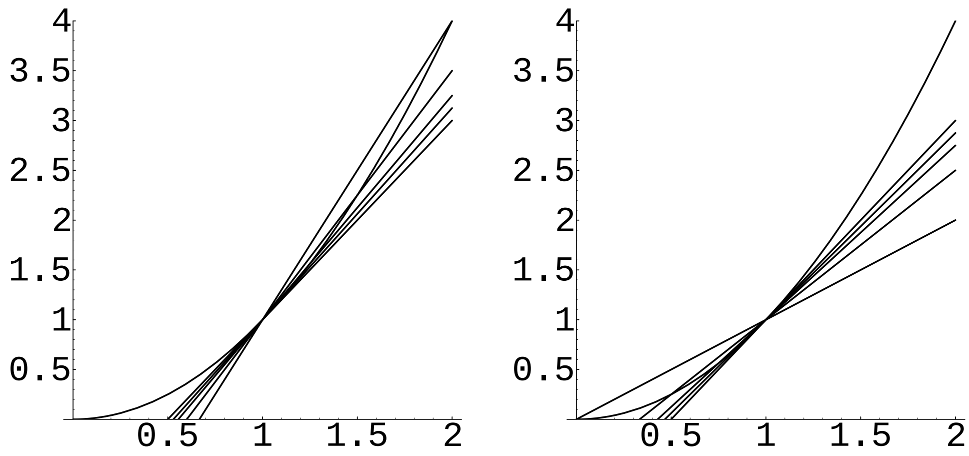


Figure 5.6: Secant lines and the tangent to x^2 at $x = 1$.

Properties. Let $u(x)$ and $v(x)$ be differentiable. Let a and b be constants. Some fundamental properties of derivatives are:

$$\frac{\mathrm{d}}{\mathrm{d}x}(au + bv) = a\frac{\mathrm{d}u}{\mathrm{d}x} + b\frac{\mathrm{d}v}{\mathrm{d}x} \qquad \text{Linearity}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = \frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x} \qquad \text{Product Rule}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \qquad \text{Quotient Rule}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(u^a) = au^{a-1}\frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{Power Rule}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(u(v(x))) = \frac{\mathrm{d}u}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}x} = u'(v(x))v'(x) \qquad \text{Chain Rule}$$

These can be proved by using the definition of differentiation.

Example 5.3.3 Prove the quotient rule for derivatives.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) &= \lim_{\Delta x\to 0}\frac{\frac{u(x+\Delta x)}{v(x+\Delta x)}-\frac{u(x)}{v(x)}}{\Delta x} \\
&= \lim_{\Delta x\to 0}\frac{u(x+\Delta x)v(x)-u(x)v(x+\Delta x)}{\Delta x v(x)v(x+\Delta x)} \\
&= \lim_{\Delta x\to 0}\frac{u(x+\Delta x)v(x)-u(x)v(x)-u(x)v(x+\Delta x)+u(x)v(x)}{\Delta x v(x)v(x)} \\
&= \lim_{\Delta x\to 0}\frac{(u(x+\Delta x)-u(x))v(x)-u(x)(v(x+\Delta x)-v(x))}{\Delta x v^2(x)} \\
&= \frac{\lim_{\Delta x\to 0}\frac{u(x+\Delta x)-u(x)}{\Delta x}v(x)-u(x)\lim_{\Delta x\to 0}\frac{v(x+\Delta x)-v(x)}{\Delta x}}{v^2(x)} \\
&= \frac{v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}
\end{aligned}
$$

Trigonometric Functions. Some derivatives of trigonometric functions are:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \cos x & \frac{\mathrm{d}}{\mathrm{d}x}\arcsin x &= \frac{1}{(1-x^2)^{1/2}} \\
\frac{\mathrm{d}}{\mathrm{d}x}\cos x &= -\sin x & \frac{\mathrm{d}}{\mathrm{d}x}\arccos x &= -\frac{1}{(1-x^2)^{1/2}} \\
\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \frac{1}{\cos^2 x} & \frac{\mathrm{d}}{\mathrm{d}x}\arctan x &= \frac{1}{1+x^2} \\
\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x &= \mathrm{e}^x & \frac{\mathrm{d}}{\mathrm{d}x}\log x &= \frac{1}{x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\sinh x &= \cosh x & \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcsinh} x &= \frac{1}{(x^2+1)^{1/2}} \\
\frac{\mathrm{d}}{\mathrm{d}x}\cosh x &= \sinh x & \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccosh} x &= \frac{1}{(x^2-1)^{1/2}} \\
\frac{\mathrm{d}}{\mathrm{d}x}\tanh x &= \frac{1}{\cosh^2 x} & \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arctanh} x &= \frac{1}{1-x^2}
\end{aligned}
$$

Example 5.3.4 We can evaluate the derivative of x^x by using the identity $a^b = \mathrm{e}^{b \log a}$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}x^x &= \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{x \log x} \\
&= \mathrm{e}^{x \log x}\frac{\mathrm{d}}{\mathrm{d}x}(x \log x) \\
&= x^x(1 \cdot \log x + x\frac{1}{x}) \\
&= x^x(1 + \log x)
\end{aligned}
$$

Inverse Functions. If we have a function $y(x)$, we can consider x as a function of y, $x(y)$. For example, if $y(x) = 8x^3$ then $x(y) = 2\sqrt[3]{y}$; if $y(x) = \frac{x+2}{x+1}$ then $x(y) = \frac{2-y}{y-1}$. The derivative of an inverse function is

$$\frac{\mathrm{d}}{\mathrm{d}y} x(y) = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}.$$

Example 5.3.5 The inverse function of $y(x) = \mathrm{e}^x$ is $x(y) = \log y$. We can obtain the derivative of the logarithm from the derivative of the exponential. The derivative of the exponential is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x.$$

Thus the derivative of the logarithm is

$$\frac{\mathrm{d}}{\mathrm{d}y} \log y = \frac{\mathrm{d}}{\mathrm{d}y} x(y) = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}} = \frac{1}{\mathrm{e}^x} = \frac{1}{y}.$$

5.4 Implicit Differentiation

An *explicitly defined* function has the form $y = f(x)$. A *implicitly defined* function has the form $f(x, y) = 0$. A few examples of implicit functions are $x^2 + y^2 - 1 = 0$ and $x + y + \sin(xy) = 0$. Often it is not possible to write an implicit equation in explicit form. This is true of the latter example above. One can calculate the derivative of $y(x)$ in terms of x and y even when $y(x)$ is defined by an implicit equation.

Example 5.4.1 Consider the implicit equation

$$x^2 - xy - y^2 = 1.$$

This implicit equation can be solved for the dependent variable.

$$y(x) = \frac{1}{2}\left(-x \pm \sqrt{5x^2 - 4}\right).$$

We can differentiate this expression to obtain

$$y' = \frac{1}{2}\left(-1 \pm \frac{5x}{\sqrt{5x^2-4}}\right).$$

One can obtain the same result without first solving for y. If we differentiate the implicit equation, we obtain

$$2x - y - x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

We can solve this equation for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x-y}{x+2y}$$

We can differentiate this expression to obtain the second derivative of y.

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{(x+2y)(2-y') - (2x-y)(1+2y')}{(x+2y)^2} \\ &= \frac{5(y-xy')}{(x+2y)^2}\end{aligned}$$

Substitute in the expression for y'.

$$= -\frac{10(x^2-xy-y^2)}{(x+2y)^2}$$

Use the original implicit equation.

$$= -\frac{10}{(x+2y)^2}$$

5.5 Maxima and Minima

A differentiable function is *increasing* where $f'(x) > 0$, *decreasing* where $f'(x) < 0$ and *stationary* where $f'(x) = 0$.

A function $f(x)$ has a *relative maxima* at a point $x = \xi$ if there exists a neighborhood around ξ such that $f(x) \leq f(\xi)$ for $x \in (x - \delta, x + \delta)$, $\delta > 0$. The *relative minima* is defined analogously. Note that this definition does not require that the function be differentiable, or even continuous. We refer to relative maxima and minima collectively are *relative extrema*.

Relative Extrema and Stationary Points. If $f(x)$ is differentiable and $f(\xi)$ is a relative extrema then $x = \xi$ is a stationary point, $f'(\xi) = 0$. We can prove this using left and right limits. Assume that $f(\xi)$ is a relative maxima. Then there is a neighborhood $(x - \delta, x + \delta)$, $\delta > 0$ for which $f(x) \leq f(\xi)$. Since $f(x)$ is differentiable the derivative at $x = \xi$,

$$f'(\xi) = \lim_{\Delta x \to 0} \frac{f(\xi + \Delta x) - f(\xi)}{\Delta x},$$

exists. This in turn means that the left and right limits exist and are equal. Since $f(x) \leq f(\xi)$ for $\xi - \delta < x < \xi$ the left limit is non-positive,

$$f'(\xi) = \lim_{\Delta x \to 0^-} \frac{f(\xi + \Delta x) - f(\xi)}{\Delta x} \leq 0.$$

Since $f(x) \leq f(\xi)$ for $\xi < x < \xi + \delta$ the right limit is nonnegative,

$$f'(\xi) = \lim_{\Delta x \to 0^+} \frac{f(\xi + \Delta x) - f(\xi)}{\Delta x} \geq 0.$$

Thus we have $0 \leq f'(\xi) \leq 0$ which implies that $f'(\xi) = 0$.

It is not true that all stationary points are relative extrema. That is, $f'(\xi) = 0$ does not imply that $x = \xi$ is an extrema. Consider the function $f(x) = x^3$. $x = 0$ is a stationary point since $f'(x) = x^2$, $f'(0) = 0$. However, $x = 0$ is neither a relative maxima nor a relative minima.

It is also not true that all relative extrema are stationary points. Consider the function $f(x) = |x|$. The point $x = 0$ is a relative minima, but the derivative at that point is undefined.

First Derivative Test. Let $f(x)$ be differentiable and $f'(\xi) = 0$.

- If $f'(x)$ changes sign from positive to negative as we pass through $x = \xi$ then the point is a relative maxima.
- If $f'(x)$ changes sign from negative to positive as we pass through $x = \xi$ then the point is a relative minima.
- If $f'(x)$ is not identically zero in a neighborhood of $x = \xi$ and it does not change sign as we pass through the point then $x = \xi$ is not a relative extrema.

Example 5.5.1 Consider $y = x^2$ and the point $x = 0$. The function is differentiable. The derivative, $y' = 2x$, vanishes at $x = 0$. Since $y'(x)$ is negative for $x < 0$ and positive for $x > 0$, the point $x = 0$ is a relative minima. See Figure 5.7.

Example 5.5.2 Consider $y = \cos x$ and the point $x = 0$. The function is differentiable. The derivative, $y' = -\sin x$ is positive for $-\pi < x < 0$ and negative for $0 < x < \pi$. Since the sign of y' goes from positive to negative, $x = 0$ is a relative maxima. See Figure 5.7.

Example 5.5.3 Consider $y = x^3$ and the point $x = 0$. The function is differentiable. The derivative, $y' = 3x^2$ is positive for $x < 0$ and positive for $0 < x$. Since y' is not identically zero and the sign of y' does not change, $x = 0$ is not a relative extrema. See Figure 5.7.

Concavity. If the portion of a curve in some neighborhood of a point lies above the tangent line through that point, the curve is said to be *concave upward*. If it lies below the tangent it is *concave downward*. If a function is twice differentiable then $f''(x) > 0$ where it is concave upward and $f''(x) < 0$ where it is concave downward. Note that $f''(x) > 0$ is a sufficient, but not a necessary condition for a curve to be concave upward at a point. A

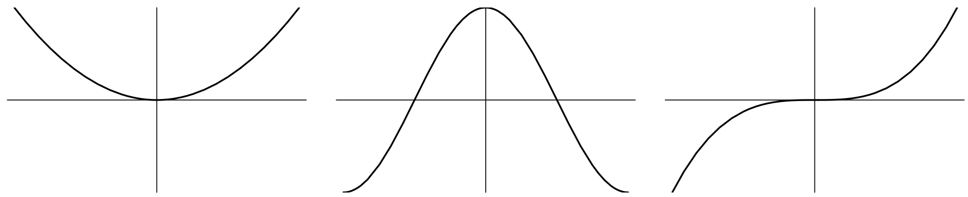


Figure 5.7: Graphs of x^2, $\cos x$ and x^3.

curve may be concave upward at a point where the second derivative vanishes. A point where the curve changes concavity is called a *point of inflection*. At such a point the second derivative vanishes, $f''(x) = 0$. For twice continuously differentiable functions, $f''(x) = 0$ is a necessary but not a sufficient condition for an inflection point. The second derivative may vanish at places which are not inflection points. See Figure 5.8.

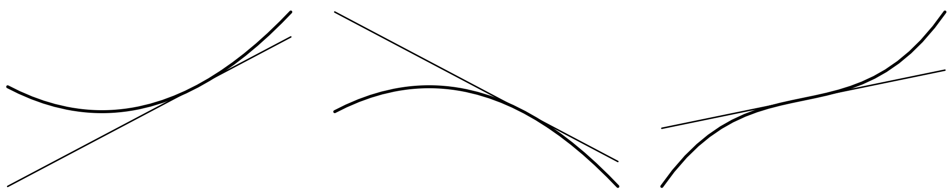


Figure 5.8: Concave Upward, Concave Downward and an Inflection Point.

Second Derivative Test. Let $f(x)$ be twice differentiable and let $x = \xi$ be a stationary point, $f'(\xi) = 0$.

- If $f''(\xi) < 0$ then the point is a relative maxima.
- If $f''(\xi) > 0$ then the point is a relative minima.
- If $f''(\xi) = 0$ then the test fails.

Example 5.5.4 Consider the function $f(x) = \cos x$ and the point $x = 0$. The derivatives of the function are $f'(x) = -\sin x$, $f''(x) = -\cos x$. The point $x = 0$ is a stationary point, $f'(0) = -\sin(0) = 0$. Since the second derivative is negative there, $f''(0) = -\cos(0) = -1$, the point is a relative maxima.

Example 5.5.5 Consider the function $f(x) = x^4$ and the point $x = 0$. The derivatives of the function are $f'(x) = 4x^3$, $f''(x) = 12x^2$. The point $x = 0$ is a stationary point. Since the second derivative also vanishes at that point the second derivative test fails. One must use the first derivative test to determine that $x = 0$ is a relative minima.

5.6 Mean Value Theorems

Rolle's Theorem. If $f(x)$ is continuous in $[a, b]$, differentiable in (a, b) and $f(a) = f(b) = 0$ then there exists a point $\xi \in (a, b)$ such that $f'(\xi) = 0$. See Figure 5.9.

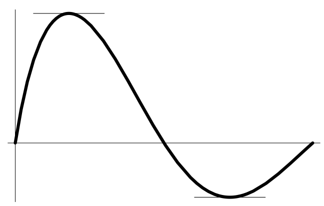


Figure 5.9: Rolle's Theorem.

To prove this we consider two cases. First we have the trivial case that $f(x) \equiv 0$. If $f(x)$ is not identically zero then continuity implies that it must have a nonzero relative maxima or minima in (a, b). Let $x = \xi$ be one of these relative extrema. Since $f(x)$ is differentiable, $x = \xi$ must be a stationary point, $f'(\xi) = 0$.

Theorem of the Mean. If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) then there exists a point $x = \xi$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

That is, there is a point where the instantaneous velocity is equal to the average velocity on the interval.

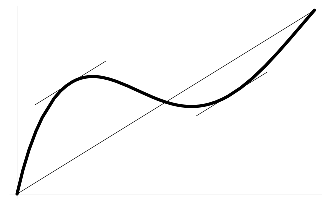


Figure 5.10: Theorem of the Mean.

We prove this theorem by applying Rolle's theorem. Consider the new function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Note that $g(a) = g(b) = 0$, so it satisfies the conditions of Rolle's theorem. There is a point $x = \xi$ such that $g'(\xi) = 0$. We differentiate the expression for $g(x)$ and substitute in $x = \xi$ to obtain the result.

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

$$g'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0$$

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

Generalized Theorem of the Mean. If $f(x)$ and $g(x)$ are continuous in $[a,b]$ and differentiable in (a,b), then there exists a point $x=\xi$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

We have assumed that $g(a) \neq g(b)$ so that the denominator does not vanish and that $f'(x)$ and $g'(x)$ are not simultaneously zero which would produce an indeterminate form. Note that this theorem reduces to the regular theorem of the mean when $g(x)=x$. The proof of the theorem is similar to that for the theorem of the mean.

Taylor's Theorem of the Mean. If $f(x)$ is $n+1$ times continuously differentiable in (a,b) then there exists a point $x=\xi \in (a,b)$ such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^n}{n!}f^{(n)}(a) + \frac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi). \tag{5.1}$$

For the case $n=0$, the formula is

$$f(b) = f(a) + (b-a)f'(\xi),$$

which is just a rearrangement of the terms in the theorem of the mean,

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}.$$

5.6.1 Application: Using Taylor's Theorem to Approximate Functions.

One can use Taylor's theorem to approximate functions with polynomials. Consider an infinitely differentiable function $f(x)$ and a point $x=a$. Substituting x for b into Equation 5.1 we obtain,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi).$$

If the last term in the sum is small then we can approximate our function with an n^{th} order polynomial.

$$f(x) \approx f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a)$$

The last term in Equation 5.6.1 is called the remainder or the error term,

$$R_n = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi).$$

Since the function is infinitely differentiable, $f^{(n+1)}(\xi)$ exists and is bounded. Therefore we note that the error must vanish as $x \to 0$ because of the $(x-a)^{n+1}$ factor. We therefore suspect that our approximation would be a good one if x is close to a. Also note that $n!$ eventually grows faster than $(x-a)^n$,

$$\lim_{n\to\infty} \frac{(x-a)^n}{n!} = 0.$$

So if the derivative term, $f^{(n+1)}(\xi)$, does not grow to quickly, the error for a certain value of x will get smaller with increasing n and the polynomial will become a better approximation of the function. (It is also possible that the derivative factor grows very quickly and the approximation gets worse with increasing n.)

Example 5.6.1 Consider the function $f(x) = \mathrm{e}^x$. We want a polynomial approximation of this function near the point $x = 0$. Since the derivative of e^x is e^x, the value of all the derivatives at $x = 0$ is $f^{(n)}(0) = \mathrm{e}^0 = 1$. Taylor's theorem thus states that

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}\mathrm{e}^{\xi},$$

for some $\xi \in (0, x)$. The first few polynomial approximations of the exponent about the point $x = 0$ are

$$\begin{aligned}
f_1(x) &= 1 \\
f_2(x) &= 1 + x \\
f_3(x) &= 1 + x + \frac{x^2}{2} \\
f_4(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}
\end{aligned}$$

The four approximations are graphed in Figure 5.11.

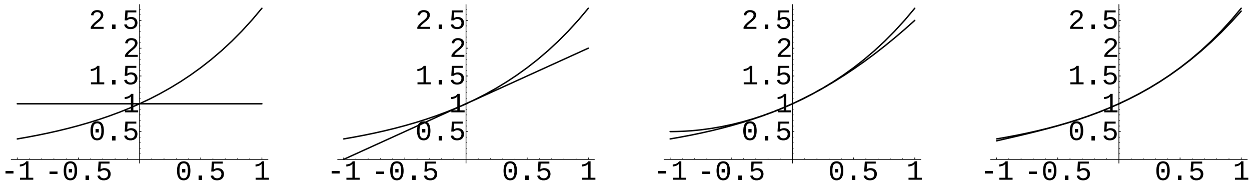


Figure 5.11: Four Finite Taylor Series Approximations of e^x

Note that for the range of x we are looking at, the approximations become more accurate as the number of terms increases.

Example 5.6.2 Consider the function $f(x) = \cos x$. We want a polynomial approximation of this function near the point $x = 0$. The first few derivatives of f are

$$
\begin{aligned}
f(x) &= \cos x \\
f'(x) &= -\sin x \\
f''(x) &= -\cos x \\
f'''(x) &= \sin x \\
f^{(4)}(x) &= \cos x
\end{aligned}
$$

It's easy to pick out the pattern here,

$$
f^{(n)}(x) = \begin{cases} (-1)^{n/2} \cos x & \text{for even } n, \\ (-1)^{(n+1)/2} \sin x & \text{for odd } n. \end{cases}
$$

Since $\cos(0) = 1$ and $\sin(0) = 0$ the n-term approximation of the cosine is,

$$
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^{2(n-1)} \frac{x^{2(n-1)}}{(2(n-1))!} + \frac{x^{2n}}{(2n)!} \cos \xi.
$$

Here are graphs of the one, two, three and four term approximations.

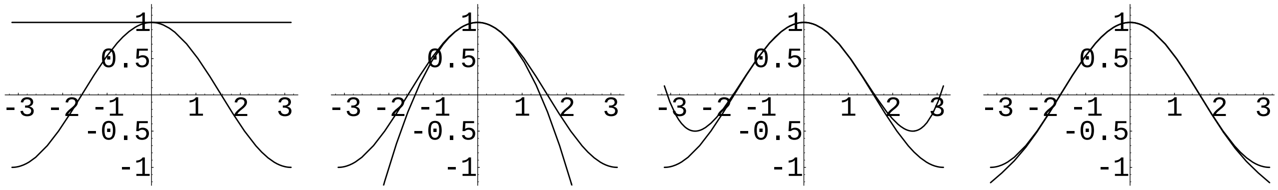


Figure 5.12: Taylor Series Approximations of $\cos x$

Note that for the range of x we are looking at, the approximations become more accurate as the number of terms increases. Consider the ten term approximation of the cosine about $x = 0$,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots - \frac{x^{18}}{18!} + \frac{x^{20}}{20!} \cos \xi.$$

Note that for any value of ξ, $|\cos \xi| \leq 1$. Therefore the absolute value of the error term satisfies,

$$|R| = \left| \frac{x^{20}}{20!} \cos \xi \right| \leq \frac{|x|^{20}}{20!}.$$

$x^{20}/20!$ is plotted in Figure 5.13.

Note that the error is very small for $x < 6$, fairly small but non-negligible for $x \approx 7$ and large for $x > 8$. The ten term approximation of the cosine, plotted below, behaves just we would predict.

The error is very small until it becomes non-negligible at $x \approx 7$ and large at $x \approx 8$.

Example 5.6.3 Consider the function $f(x) = \log x$. We want a polynomial approximation of this function near

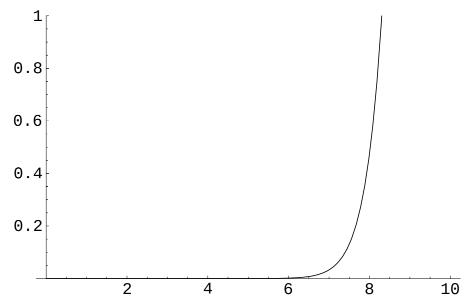


Figure 5.13: Plot of $x^{20}/20!$.

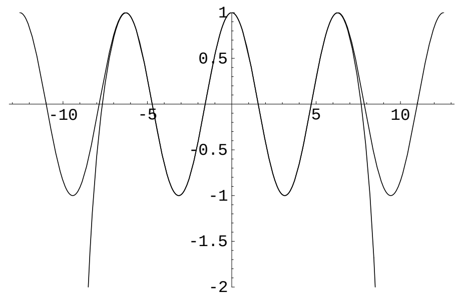


Figure 5.14: Ten Term Taylor Series Approximation of $\cos x$

the point $x = 1$. The first few derivatives of f are

$$f(x) = \log x$$
$$f'(x) = \frac{1}{x}$$
$$f''(x) = -\frac{1}{x^2}$$
$$f'''(x) = \frac{2}{x^3}$$
$$f^{(4)}(x) = -\frac{3}{x^4}$$

The derivatives evaluated at $x = 1$ are

$$f(0) = 0, \qquad f^{(n)}(0) = (-1)^{n-1}(n-1)!, \text{ for } n \geq 1.$$

By Taylor's theorem of the mean we have,

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + (-1)^n \frac{(x-1)^{n+1}}{n+1}\frac{1}{\xi^{n+1}}.$$

Below are plots of the 2, 4, 10 and 50 term approximations.

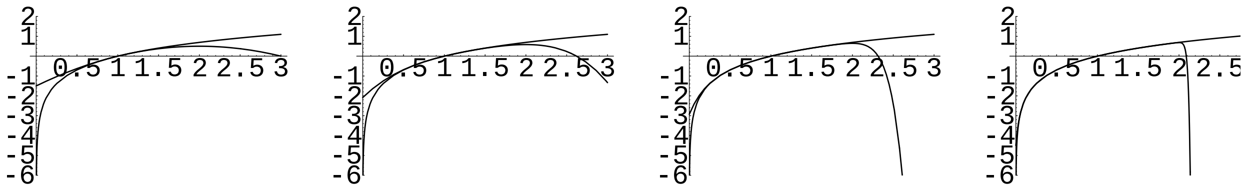


Figure 5.15: The 2, 4, 10 and 50 Term Approximations of $\log x$

Note that the approximation gets better on the interval $(0, 2)$ and worse outside this interval as the number of terms increases. The Taylor series converges to $\log x$ only on this interval.

5.6.2 Application: Finite Difference Schemes

Example 5.6.4 Suppose you sample a function at the discrete points $n\Delta x$, $n \in \mathbb{Z}$. In Figure 5.16 we sample the function $f(x) = \sin x$ on the interval $[-4, 4]$ with $\Delta x = 1/4$ and plot the data points.

We wish to approximate the derivative of the function on the grid points using only the value of the function on those discrete points. From the definition of the derivative, one is lead to the formula

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}. \tag{5.2}$$

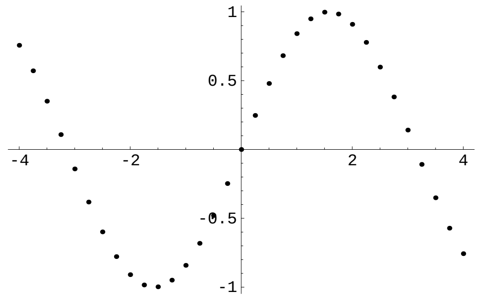


Figure 5.16: Sampling of $\sin x$

Taylor's theorem states that

$$f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(\xi).$$

Substituting this expression into our formula for approximating the derivative we obtain

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(\xi) - f(x)}{\Delta x} = f'(x) + \frac{\Delta x}{2} f''(\xi).$$

Thus we see that the error in our approximation of the first derivative is $\frac{\Delta x}{2} f''(\xi)$. Since the error has a linear factor of Δx, we call this a first order accurate method. Equation 5.2 is called the *forward difference scheme* for calculating the first derivative. Figure 5.17 shows a plot of the value of this scheme for the function $f(x) = \sin x$ and $\Delta x = 1/4$. The first derivative of the function $f'(x) = \cos x$ is shown for comparison.

Another scheme for approximating the first derivative is the *centered difference scheme*,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}.$$

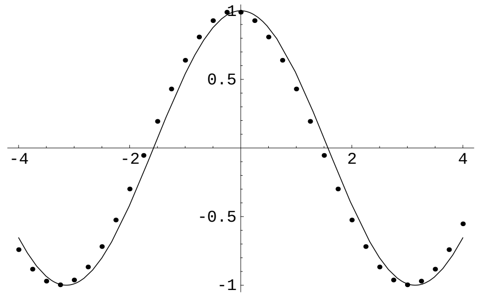


Figure 5.17: The Forward Difference Scheme Approximation of the Derivative

Expanding the numerator using Taylor's theorem,

$$
\begin{aligned}
&\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \\
&\quad = \frac{f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{6} f'''(\xi) - f(x) + \Delta x f'(x) - \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{6} f'''(\eta)}{2\Delta x} \\
&\quad = f'(x) + \frac{\Delta x^2}{12}(f'''(\xi) + f'''(\eta)).
\end{aligned}
$$

The error in the approximation is quadratic in Δx. Therefore this is a second order accurate scheme. Below is a plot of the derivative of the function and the value of this scheme for the function $f(x) = \sin x$ and $\Delta x = 1/4$.

Notice how the centered difference scheme gives a better approximation of the derivative than the forward difference scheme.

5.7 L'Hospital's Rule

Some singularities are easy to diagnose. Consider the function $\frac{\cos x}{x}$ at the point $x = 0$. The function evaluates to $\frac{1}{0}$ and is thus discontinuous at that point. Since the numerator and denominator are continuous functions and

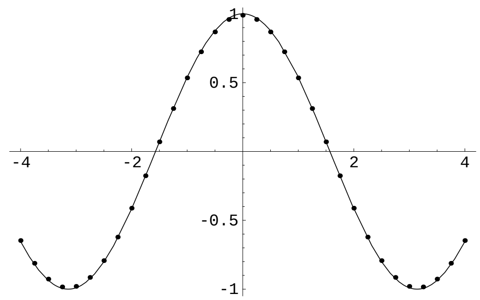


Figure 5.18: Centered Difference Scheme Approximation of the Derivative

the denominator vanishes while the numerator does not, the left and right limits as $x \to 0$ do not exist. Thus the function has an infinite discontinuity at the point $x = 0$. More generally, a function which is composed of continuous functions and evaluates to $\frac{a}{0}$ at a point where $a \neq 0$ must have an infinite discontinuity there.

Other singularities require more analysis to diagnose. Consider the functions $\frac{\sin x}{x}$, $\frac{\sin x}{|x|}$ and $\frac{\sin x}{1-\cos x}$ at the point $x = 0$. All three functions evaluate to $\frac{0}{0}$ at that point, but have different kinds of singularities. The first has a removable discontinuity, the second has a finite discontinuity and the third has an infinite discontinuity. See Figure 5.19.

An expression that evaluates to $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 1^∞, 0^0 or ∞^0 is called an *indeterminate*. A function $f(x)$ which is indeterminate at the point $x = \xi$ is singular at that point. The singularity may be a removable discontinuity, a finite discontinuity or an infinite discontinuity depending on the behavior of the function around that point. If $\lim_{x \to \xi} f(x)$ exists, then the function has a removable discontinuity. If the limit does not exist, but the left and right limits do exist, then the function has a finite discontinuity. If either the left or right limit does not exist then the function has an infinite discontinuity.

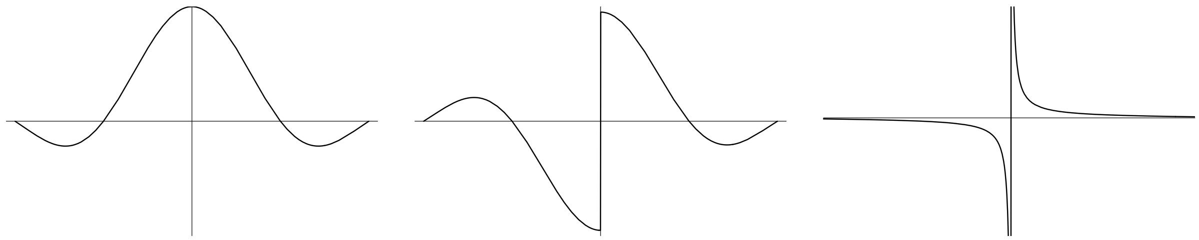


Figure 5.19: The functions $\frac{\sin x}{x}$, $\frac{\sin x}{|x|}$ and $\frac{\sin x}{1-\cos x}$.

L'Hospital's Rule. Let $f(x)$ and $g(x)$ be differentiable and $f(\xi) = g(\xi) = 0$. Further, let $g(x)$ be nonzero in a deleted neighborhood of $x = \xi$, ($g(x) \neq 0$ for $x \in 0 < |x - \xi| < \delta$). Then

$$\lim_{x \to \xi} \frac{f(x)}{g(x)} = \lim_{x \to \xi} \frac{f'(x)}{g'(x)}.$$

To prove this, we note that $f(\xi) = g(\xi) = 0$ and apply the generalized theorem of the mean. Note that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(\xi)}{g(x) - g(\xi)} = \frac{f'(\eta)}{g'(\eta)}$$

for some η between ξ and x. Thus

$$\lim_{x \to \xi} \frac{f(x)}{g(x)} = \lim_{\eta \to \xi} \frac{f'(\eta)}{g'(\eta)} = \lim_{x \to \xi} \frac{f'(x)}{g'(x)}$$

provided that the limits exist.

L'Hospital's Rule is also applicable when both functions tend to infinity instead of zero or when the limit point, ξ, is at infinity. It is also valid for one-sided limits.

L'Hospital's rule is directly applicable to the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Example 5.7.1 Consider the three functions $\frac{\sin x}{x}$, $\frac{\sin x}{|x|}$ and $\frac{\sin x}{1-\cos x}$ at the point $x = 0$.

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

Thus $\frac{\sin x}{x}$ has a removable discontinuity at $x = 0$.

$$\lim_{x \to 0^+} \frac{\sin x}{|x|} = \lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0^-} \frac{\sin x}{|x|} = \lim_{x \to 0^-} \frac{\sin x}{-x} = -1$$

Thus $\frac{\sin x}{|x|}$ has a finite discontinuity at $x = 0$.

$$\lim_{x \to 0} \frac{\sin x}{1 - \cos x} = \lim_{x \to 0} \frac{\cos x}{\sin x} = \frac{1}{0} = \infty$$

Thus $\frac{\sin x}{1-\cos x}$ has an infinite discontinuity at $x = 0$.

Example 5.7.2 Let a and d be nonzero.

$$\begin{aligned} \lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f} &= \lim_{x \to \infty} \frac{2ax + b}{2dx + e} \\ &= \lim_{x \to \infty} \frac{2a}{2d} \\ &= \frac{a}{d} \end{aligned}$$

Example 5.7.3 Consider

$$\lim_{x \to 0} \frac{\cos x - 1}{x \sin x}.$$

This limit is an indeterminate of the form $\frac{0}{0}$. Applying L'Hospital's rule we see that limit is equal to

$$\lim_{x \to 0} \frac{-\sin x}{x \cos x + \sin x}.$$

This limit is again an indeterminate of the form $\frac{0}{0}$. We apply L'Hospital's rule again.

$$\lim_{x \to 0} \frac{-\cos x}{-x \sin x + 2 \cos x} = -\frac{1}{2}$$

Thus the value of the original limit is $-\frac{1}{2}$. We could also obtain this result by expanding the functions in Taylor series.

$$\begin{aligned}
\lim_{x \to 0} \frac{\cos x - 1}{x \sin x} &= \lim_{x \to 0} \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) - 1}{x\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)} \\
&= \lim_{x \to 0} \frac{-\frac{x^2}{2} + \frac{x^4}{24} - \cdots}{x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \cdots} \\
&= \lim_{x \to 0} \frac{-\frac{1}{2} + \frac{x^2}{24} - \cdots}{1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots} \\
&= -\frac{1}{2}
\end{aligned}$$

We can apply L'Hospital's Rule to the indeterminate forms $0 \cdot \infty$ and $\infty - \infty$ by rewriting the expression in a different form, (perhaps putting the expression over a common denominator). If at first you don't succeed, try, try again. You may have to apply L'Hospital's rule several times to evaluate a limit.

Example 5.7.4

$$\begin{aligned}
\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) &= \lim_{x\to 0}\frac{x\cos x - \sin x}{x\sin x}\\
&= \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}\\
&= \lim_{x\to 0}\frac{-x\sin x}{\sin x + x\cos x}\\
&= \lim_{x\to 0}\frac{-x\cos x - \sin x}{\cos x + \cos x - x\sin x}\\
&= 0
\end{aligned}$$

You can apply L'Hospital's rule to the indeterminate forms 1^∞, 0^0 or ∞^0 by taking the logarithm of the expression.

Example 5.7.5 Consider the limit,

$$\lim_{x\to 0} x^x,$$

which gives us the indeterminate form 0^0. The logarithm of the expression is

$$\log(x^x) = x\log x.$$

As $x \to 0$ we now have the indeterminate form $0\cdot\infty$. By rewriting the expression, we can apply L'Hospital's rule.

$$\begin{aligned}
\lim_{x\to 0}\frac{\log x}{1/x} &= \lim_{x\to 0}\frac{1/x}{-1/x^2}\\
&= \lim_{x\to 0}(-x)\\
&= 0
\end{aligned}$$

Thus the original limit is

$$\lim_{x\to 0} x^x = \mathrm{e}^0 = 1.$$

5.8 Exercises

Limits and Continuity

Exercise 5.1
Does

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

exist?

Exercise 5.2
Is the function $\sin(1/x)$ continuous in the open interval $(0, 1)$? Is there a value of a such that the function defined by

$$f(x) = \begin{cases} \sin(1/x) & \text{for } x \neq 0, \\ a & \text{for } x = 0 \end{cases}$$

is continuous on the closed interval $[0, 1]$?

Exercise 5.3
Is the function $\sin(1/x)$ uniformly continuous in the open interval $(0, 1)$?

Exercise 5.4
Are the functions $\sqrt{x}$ and $\frac{1}{x}$ uniformly continuous on the interval $(0, 1)$?

Exercise 5.5
Prove that a function which is continuous on a closed interval is uniformly continuous on that interval.

Definition of Differentiation

Exercise 5.6 (mathematica/calculus/differential/definition.nb)
Use the definition of differentiation to prove the following identities where $f(x)$ and $g(x)$ are differentiable functions and n is a positive integer.

a. $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$, (I suggest that you use Newton's binomial formula.)

b. $\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f}{\mathrm{d}x}$

c. $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$. (You'll need to use some trig identities.)

d. $\frac{\mathrm{d}}{\mathrm{d}x}(f(g(x))) = f'(g(x))g'(x)$

Rules of Differentiation

Exercise 5.7 (mathematica/calculus/differential/rules.nb)
Find the first derivatives of the following:

a. $x\sin(\cos x)$

b. $f(\cos(g(x)))$

c. $\frac{1}{f(\log x)}$

d. x^{x^x}

e. $|x|\sin|x|$

Exercise 5.8 (mathematica/calculus/differential/rules.nb)
Using

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\tan x = \frac{1}{\cos^2 x}$$

find the derivatives of $\arcsin x$ and $\arctan x$.

Implicit Differentiation

Exercise 5.9 (mathematica/calculus/differential/implicit.nb)
Find $y'(x)$, given that $x^2 + y^2 = 1$. What is $y'(1/2)$?

Exercise 5.10 (mathematica/calculus/differential/implicit.nb)
Find $y'(x)$ and $y''(x)$, given that $x^2 - xy + y^2 = 3$.

Maxima and Minima

Exercise 5.11 (mathematica/calculus/differential/maxima.nb)
Identify any maxima and minima of the following functions.

a. $f(x) = x(12 - 2x)^2$.

b. $f(x) = (x - 2)^{2/3}$.

Exercise 5.12 (mathematica/calculus/differential/maxima.nb)
A cylindrical container with a circular base and an open top is to hold 64 cm^3. Find its dimensions so that the surface area of the cup is a minimum.

Mean Value Theorems

Exercise 5.13
Prove the generalized theorem of the mean. If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b), then there exists a point $x = \xi$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Assume that $g(a) \neq g(b)$ so that the denominator does not vanish and that $f'(x)$ and $g'(x)$ are not simultaneously zero which would produce an indeterminate form.

Exercise 5.14 (mathematica/calculus/differential/taylor.nb)
Find a polynomial approximation of $\sin x$ on the interval $[-1, 1]$ that has a maximum error of $\frac{1}{1000}$. Don't use any more terms that you need to. Prove the error bound. Use your polynomial to approximate $\sin 1$.

Exercise 5.15 (mathematica/calculus/differential/taylor.nb)
You use the formula $\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{\Delta x^2}$ to approximate $f''(x)$. What is the error in this approximation?

Exercise 5.16
The formulas $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ and $\frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}$ are first and second order accurate schemes for approximating the first derivative $f'(x)$. Find a couple other schemes that have successively higher orders of accuracy. Would these higher order schemes actually give a better approximation of $f'(x)$? Remember that Δx is small, but not infinitesimal.

L'Hospital's Rule

Exercise 5.17 (mathematica/calculus/differential/lhospitals.nb)
Evaluate the following limits.

a. $\lim_{x \to 0} \frac{x - \sin x}{x^3}$

b. $\lim_{x \to 0} \left(\csc x - \frac{1}{x}\right)$

c. $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x$

d. $\lim_{x \to 0} \left(\csc^2 x - \frac{1}{x^2}\right)$. (First evaluate using L'Hospital's rule then using a Taylor series expansion. You will find that the latter method is more convenient.)

Exercise 5.18 (mathematica/calculus/differential/lhospitals.nb)
Evaluate the following limits,

$$\lim_{x \to \infty} x^{a/x}, \qquad \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{bx},$$

where a and b are constants.

5.9 Hints

Limits and Continuity

Hint 5.1
Apply the ϵ, δ definition of a limit.

Hint 5.2
The composition of continuous functions is continuous. Apply the definition of continuity and look at the point $x = 0$.

Hint 5.3
Note that for $x_1 = \frac{1}{(n-1/2)\pi}$ and $x_2 = \frac{1}{(n+1/2)\pi}$ where $n \in \mathbb{Z}$ we have $|\sin(1/x_1) - \sin(1/x_2)| = 2$.

Hint 5.4
Note that the function $\sqrt{x+\delta} - \sqrt{x}$ is a decreasing function of x and an increasing function of δ for positive x and δ. Bound this function for fixed δ.

Consider any positive δ and ϵ. For what values of x is

$$\frac{1}{x} - \frac{1}{x+\delta} > \epsilon.$$

Hint 5.5
Let the function $f(x)$ be continuous on a closed interval. Consider the function

$$e(x, \delta) = \sup_{|\xi - x| < \delta} |f(\xi) - f(x)|.$$

Bound $e(x, \delta)$ with a function of δ alone.

Definition of Differentiation

Hint 5.6

a. Newton's binomial formula is

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = a^n + a^{n-1}b + \frac{n(n-1)}{2} a^{n-2} b^2 + \cdots + nab^{n-1} + b^n.$$

Recall that the binomial coefficient is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

b. Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \right]$$

and

$$g(x)f'(x) + f(x)g'(x) = g(x) \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] + f(x) \lim_{\Delta x \to 0} \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right].$$

Fill in the blank.

c. First prove that

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

and

$$\lim_{\theta \to 0} \left[\frac{\cos\theta - 1}{\theta} \right] = 0.$$

d. Let $u = g(x)$. Consider a nonzero increment Δx, which induces the increments Δu and Δf. By definition,

$$\Delta f = f(u + \Delta u) - f(u), \qquad \Delta u = g(x + \Delta x) - g(x),$$

and $\Delta f, \Delta u \to 0$ as $\Delta x \to 0$. If $\Delta u \neq 0$ then we have

$$\epsilon = \frac{\Delta f}{\Delta u} - \frac{\mathrm{d}f}{\mathrm{d}u} \to 0 \quad \text{as} \quad \Delta u \to 0.$$

If $\Delta u = 0$ for some values of Δx then Δf also vanishes and we define $\epsilon = 0$ for theses values. In either case,

$$\Delta y = \frac{\mathrm{d}f}{\mathrm{d}u} \Delta u + \epsilon \Delta u.$$

Continue from here.

Rules of Differentiation

Hint 5.7

a. Use the product rule and the chain rule.

b. Use the chain rule.

c. Use the quotient rule and the chain rule.

d. Use the identity $a^b = \mathrm{e}^{b \log a}$.

e. For $x > 0$, the expression is $x \sin x$; for $x < 0$, the expression is $(-x) \sin(-x) = x \sin x$. Do both cases.

Hint 5.8

Use that $x'(y) = 1/y'(x)$ and the identities $\cos x = (1 - \sin^2 x)^{1/2}$ and $\cos(\arctan x) = \frac{1}{(1+x^2)^{1/2}}$.

Implicit Differentiation

Hint 5.9
Differentiating the equation

$$x^2 + [y(x)]^2 = 1$$

yields

$$2x + 2y(x)y'(x) = 0.$$

Solve this equation for $y'(x)$ and write $y(x)$ in terms of x.

Hint 5.10
Differentiate the equation and solve for $y'(x)$ in terms of x and $y(x)$. Differentiate the expression for $y'(x)$ to obtain $y''(x)$. You'll use that

$$x^2 - xy(x) + [y(x)]^2 = 3$$

Maxima and Minima

Hint 5.11

a. Use the second derivative test.

b. The function is not differentiable at the point $x = 2$ so you can't use a derivative test at that point.

Hint 5.12
Let r be the radius and h the height of the cylinder. The volume of the cup is $\pi r^2 h = 64$. The radius and height are related by $h = \frac{64}{\pi r^2}$. The surface area of the cup is $f(r) = \pi r^2 + 2\pi r h = \pi r^2 + \frac{128}{r}$. Use the second derivative test to find the minimum of $f(r)$.

Mean Value Theorems

Hint 5.13
The proof is analogous to the proof of the theorem of the mean.

Hint 5.14
The first few terms in the Taylor series of $\sin(x)$ about $x = 0$ are

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} + \cdots.$$

When determining the error, use the fact that $|\cos x_0| \leq 1$ and $|x^n| \leq 1$ for $x \in [-1, 1]$.

Hint 5.15
The terms in the approximation have the Taylor series,

$$f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{6} f'''(x) + \frac{\Delta x^4}{24} f''''(x_1),$$
$$f(x - \Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) - \frac{\Delta x^3}{6} f'''(x) + \frac{\Delta x^4}{24} f''''(x_2),$$

where $x \leq x_1 \leq x + \Delta x$ and $x - \Delta x \leq x_2 \leq x$.

Hint 5.16

L'Hospital's Rule

Hint 5.17

a. Apply L'Hospital's rule three times.

b. You can write the expression as

$$\frac{x - \sin x}{x \sin x}.$$

c. Find the limit of the logarithm of the expression.

d. It takes four successive applications of L'Hospital's rule to evaluate the limit.

For the Taylor series expansion method,

$$\csc^2 x - \frac{1}{x^2} = \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \frac{x^2 - (x - x^3/6 + O(x^5))^2}{x^2 (x + O(x^3))^2}$$

Hint 5.18
To evaluate the limits use the identity $a^b = e^{b \log a}$ and then apply L'Hospital's rule.

5.10 Solutions

Limits and Continuity

Solution 5.1
Note that in any open neighborhood of zero, $(-\delta, \delta)$, the function $\sin(1/x)$ takes on all values in the interval $[-1, 1]$. Thus if we choose a positive ϵ such that $\epsilon < 1$ then there is no value of η for which $|\sin(1/x) - \eta| < \epsilon$ for all $x \in (-\epsilon, \epsilon)$. Thus the limit does not exist.

Solution 5.2
Since $\frac{1}{x}$ is continuous in the interval $(0, 1)$ and the function $\sin(x)$ is continuous everywhere, the composition $\sin(1/x)$ is continuous in the interval $(0, 1)$.

Since $\lim_{x\to 0} \sin(1/x)$ does not exist, there is no way of defining $\sin(1/x)$ at $x = 0$ to produce a function that is continuous in $[0, 1]$.

Solution 5.3
Note that for $x_1 = \frac{1}{(n-1/2)\pi}$ and $x_2 = \frac{1}{(n+1/2)\pi}$ where $n \in \mathbb{Z}$ we have $|\sin(1/x_1) - \sin(1/x_2)| = 2$. Thus for any $0 < \epsilon < 2$ there is no value of $\delta > 0$ such that $|\sin(1/x_1) - \sin(1/x_2)| < \epsilon$ for all $x_1, x_2 \in (0, 1)$ and $|x_1 - x_2| < \delta$. Thus $\sin(1/x)$ is not uniformly continuous in the open interval $(0, 1)$.

Solution 5.4
First consider the function $\sqrt{x}$. Note that the function $\sqrt{x + \delta} - \sqrt{x}$ is a decreasing function of x and an increasing function of δ for positive x and δ. Thus for any fixed δ, the maximum value of $\sqrt{x + \delta} - \sqrt{x}$ is bounded by $\sqrt{\delta}$. Therefore on the interval $(0, 1)$, a sufficient condition for $|\sqrt{x} - \sqrt{\xi}| < \epsilon$ is $|x - \xi| < \epsilon^2$. The function $\sqrt{x}$ is uniformly continuous on the interval $(0, 1)$.

Consider any positive δ and ϵ. Note that

$$\frac{1}{x} - \frac{1}{x + \delta} > \epsilon$$

for

$$x < \frac{1}{2}\left(\sqrt{\delta^2 + \frac{4\delta}{\epsilon}} - \delta\right).$$

Thus there is no value of δ such that

$$\left|\frac{1}{x} - \frac{1}{\xi}\right| < \epsilon$$

for all $|x - \xi| < \delta$. The function $\frac{1}{x}$ is not uniformly continuous on the interval $(0, 1)$.

Solution 5.5
Let the function $f(x)$ be continuous on a closed interval. Consider the function

$$e(x, \delta) = \sup_{|\xi - x| < \delta} |f(\xi) - f(x)|.$$

Since $f(x)$ is continuous, $e(x, \delta)$ is a continuous function of x on the same closed interval. Since continuous functions on closed intervals are bounded, there is a continuous, increasing function $\epsilon(\delta)$ satisfying,

$$e(x, \delta) \leq \epsilon(\delta),$$

for all x in the closed interval. Since $\epsilon(\delta)$ is continuous and increasing, it has an inverse $\delta(\epsilon)$. Now note that $|f(x) - f(\xi)| < \epsilon$ for all x and ξ in the closed interval satisfying $|x - \xi| < \delta(\epsilon)$. Thus the function is uniformly continuous in the closed interval.

Definition of Differentiation

Solution 5.6

a.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x^n) &= \lim_{\Delta x\to 0}\left[\frac{(x+\Delta x)^n - x^n}{\Delta x}\right]\\ &= \lim_{\Delta x\to 0}\left[\frac{\left(x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}\Delta x^2 + \cdots + \Delta x^n\right) - x^n}{\Delta x}\right]\\ &= \lim_{\Delta x\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + \cdots + \Delta x^{n-1}\right]\\ &= nx^{n-1}\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}}$$

b.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) &= \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}\right]\\ &= \lim_{\Delta x\to 0}\left[\frac{[f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x)] + [f(x)g(x+\Delta x) - f(x)g(x)]}{\Delta x}\right]\\ &= \lim_{\Delta x\to 0}[g(x+\Delta x)]\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right] + f(x)\lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]\\ &= g(x)f'(x) + f(x)g'(x)\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)}$$

c. Consider a right triangle with hypotenuse of length 1 in the first quadrant of the plane. Label the vertices A, B, C, in clockwise order, starting with the vertex at the origin. The angle of A is θ. The length of a circular arc of radius $\cos\theta$ that connects C to the hypotenuse is $\theta\cos\theta$. The length of the side BC is $\sin\theta$. The length of a circular arc of radius 1 that connects B to the x axis is θ. (See Figure 5.20.)

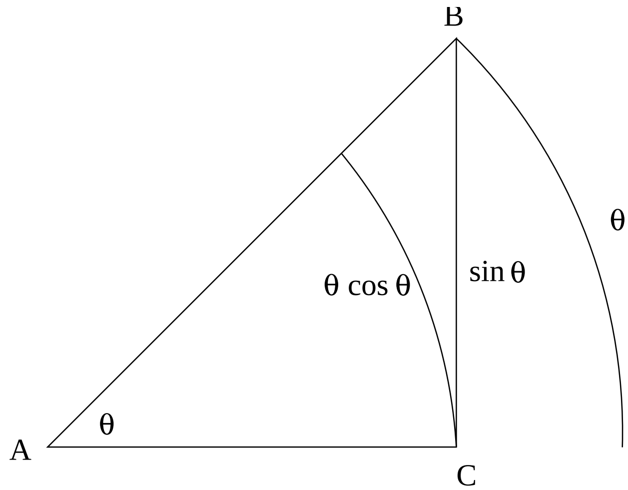


Figure 5.20:

Considering the length of these three curves gives us the inequality:

$$\theta\cos\theta \leq \sin\theta \leq \theta.$$

Dividing by θ,

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

Taking the limit as $\theta \to 0$ gives us

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

One more little tidbit we'll need to know is

$$\begin{aligned}
\lim_{\theta\to 0}\left[\frac{\cos\theta-1}{\theta}\right] &= \lim_{\theta\to 0}\left[\frac{\cos\theta-1}{\theta}\frac{\cos\theta+1}{\cos\theta+1}\right] \\
&= \lim_{\theta\to 0}\left[\frac{\cos^2\theta-1}{\theta(\cos\theta+1)}\right] \\
&= \lim_{\theta\to 0}\left[\frac{-\sin^2\theta}{\theta(\cos\theta+1)}\right] \\
&= \lim_{\theta\to 0}\left[\frac{-\sin\theta}{\theta}\right]\lim_{\theta\to 0}\left[\frac{\sin\theta}{(\cos\theta+1)}\right] \\
&= (-1)\left(\frac{0}{2}\right) \\
&= 0.
\end{aligned}$$

Now we're ready to find the derivative of $\sin x$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \lim_{\Delta x\to 0}\left[\frac{\sin(x+\Delta x)-\sin x}{\Delta x}\right] \\
&= \lim_{\Delta x\to 0}\left[\frac{\cos x\sin\Delta x+\sin x\cos\Delta x-\sin x}{\Delta x}\right] \\
&= \cos x\lim_{\Delta x\to 0}\left[\frac{\sin\Delta x}{\Delta x}\right]+\sin x\lim_{\Delta x\to 0}\left[\frac{\cos\Delta x-1}{\Delta x}\right] \\
&= \cos x
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)=\cos x}$$

d. Let $u=g(x)$. Consider a nonzero increment Δx, which induces the increments Δu and Δf. By definition,

$$\Delta f = f(u+\Delta u)-f(u), \qquad \Delta u = g(x+\Delta x)-g(x),$$

and $\Delta f, \Delta u \to 0$ as $\Delta x \to 0$. If $\Delta u \neq 0$ then we have

$$\epsilon = \frac{\Delta f}{\Delta u} - \frac{\mathrm{d}f}{\mathrm{d}u} \to 0 \quad \text{as} \quad \Delta u \to 0.$$

If $\Delta u = 0$ for some values of Δx then Δf also vanishes and we define $\epsilon = 0$ for theses values. In either case,

$$\Delta y = \frac{\mathrm{d}f}{\mathrm{d}u}\Delta u + \epsilon \Delta u.$$

We divide this equation by Δx and take the limit as $\Delta x \to 0$.

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\frac{\mathrm{d}f}{\mathrm{d}u}\frac{\Delta u}{\Delta x} + \epsilon \frac{\Delta u}{\Delta x} \right) \\
&= \left(\frac{\mathrm{d}f}{\mathrm{d}u} \right) \left(\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \right) + \left(\lim_{\Delta x \to 0} \epsilon \right) \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \right) \\
&= \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} + (0) \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right) \\
&= \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}
\end{aligned}$$

Thus we see that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(f(g(x))) = f'(g(x))g'(x).}$$

Rules of Differentiation

Solution 5.7

a.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[x\sin(\cos x)] &= \frac{\mathrm{d}}{\mathrm{d}x}[x]\sin(\cos x) + x\frac{\mathrm{d}}{\mathrm{d}x}[\sin(\cos x)] \\ &= \sin(\cos x) + x\cos(\cos x)\frac{\mathrm{d}}{\mathrm{d}x}[\cos x] \\ &= \sin(\cos x) - x\cos(\cos x)\sin x\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[x\sin(\cos x)] = \sin(\cos x) - x\cos(\cos x)\sin x}$$

b.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[f(\cos(g(x)))] &= f'(\cos(g(x)))\frac{\mathrm{d}}{\mathrm{d}x}[\cos(g(x))] \\ &= -f'(\cos(g(x)))\sin(g(x))\frac{\mathrm{d}}{\mathrm{d}x}[g(x)] \\ &= -f'(\cos(g(x)))\sin(g(x))g'(x)\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[f(\cos(g(x)))] = -f'(\cos(g(x)))\sin(g(x))g'(x)}$$

c.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{f(\log x)}\right] &= -\frac{\frac{\mathrm{d}}{\mathrm{d}x}[f(\log x)]}{[f(\log x)]^2} \\ &= -\frac{f'(\log x)\frac{\mathrm{d}}{\mathrm{d}x}[\log x]}{[f(\log x)]^2} \\ &= -\frac{f'(\log x)}{x[f(\log x)]^2}\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{f(\log x)}\right] = -\frac{f'(\log x)}{x[f(\log x)]^2}}$$

d. First we write the expression in terms exponentials and logarithms,

$$x^{x^x} = x^{\exp(x \log x)} = \exp(\exp(x \log x) \log x).$$

Then we differentiate using the chain rule and the product rule.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \exp(\exp(x \log x) \log x) &= \exp(\exp(x \log x) \log x) \frac{\mathrm{d}}{\mathrm{d}x}(\exp(x \log x) \log x) \\
&= x^{x^x} \left(\exp(x \log x) \frac{\mathrm{d}}{\mathrm{d}x}(x \log x) \log x + \exp(x \log x) \frac{1}{x} \right) \\
&= x^{x^x} \left(x^x (\log x + x \frac{1}{x}) \log x + x^{-1} \exp(x \log x) \right) \\
&= x^{x^x} \left(x^x (\log x + 1) \log x + x^{-1} x^x \right) \\
&= x^{x^x + x} \left(x^{-1} + \log x + \log^2 x \right)
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} x^{x^x} = x^{x^x + x} \left(x^{-1} + \log x + \log^2 x \right)}$$

e. For $x > 0$, the expression is $x \sin x$; for $x < 0$, the expression is $(-x) \sin(-x) = x \sin x$. Thus we see that

$$|x| \sin |x| = x \sin x.$$

The first derivative of this is

$$\sin x + x \cos x.$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(|x| \sin |x|) = \sin x + x \cos x}$$

Solution 5.8
Let $y(x) = \sin x$. Then $y'(x) = \cos x$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y} \arcsin y &= \frac{1}{y'(x)} \\
&= \frac{1}{\cos x} \\
&= \frac{1}{(1 - \sin^2 x)^{1/2}} \\
&= \frac{1}{(1 - y^2)^{1/2}}
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \arcsin x = \frac{1}{(1 - x^2)^{1/2}}}$$

Let $y(x) = \tan x$. Then $y'(x) = 1/\cos^2 x$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y} \arctan y &= \frac{1}{y'(x)} \\
&= \cos^2 x \\
&= \cos^2(\arctan y) \\
&= \left(\frac{1}{(1 + y^2)^{1/2}} \right) \\
&= \frac{1}{1 + y^2}
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \arctan x = \frac{1}{1 + x^2}}$$

Implicit Differentiation

Solution 5.9
Differentiating the equation

$$x^2 + [y(x)]^2 = 1$$

yields

$$2x + 2y(x)y'(x) = 0.$$

We can solve this equation for $y'(x)$.

$$y'(x) = -\frac{x}{y(x)}$$

To find $y'(1/2)$ we need to find $y(x)$ in terms of x.

$$y(x) = \pm\sqrt{1 - x^2}$$

Thus $y'(x)$ is

$$y'(x) = \pm\frac{x}{\sqrt{1 - x^2}}.$$

$y'(1/2)$ can have the two values:

$$\boxed{y'\left(\frac{1}{2}\right) = \pm\frac{1}{\sqrt{3}}.}$$

Solution 5.10
Differentiating the equation

$$x^2 - xy(x) + [y(x)]^2 = 3$$

yields

$$2x - y(x) - xy'(x) + 2y(x)y'(x) = 0.$$

Solving this equation for $y'(x)$

$$\boxed{y'(x) = \frac{y(x) - 2x}{2y(x) - x}.}$$

Now we differentiate $y'(x)$ to get $y''(x)$.

$$y''(x) = \frac{(y'(x) - 2)(2y(x) - x) - (y(x) - 2x)(2y'(x) - 1)}{(2y(x) - x)^2},$$

$$y''(x) = 3\frac{xy'(x) - y(x)}{(2y(x) - x)^2},$$

$$y''(x) = 3\frac{x\frac{y(x)-2x}{2y(x)-x} - y(x)}{(2y(x) - x)^2},$$

$$y''(x) = 3\frac{x(y(x) - 2x) - y(x)(2y(x) - x)}{(2y(x) - x)^3},$$

$$y''(x) = -6\frac{x^2 - xy(x) + [y(x)]^2}{(2y(x) - x)^3},$$

$$\boxed{y''(x) = \frac{-18}{(2y(x) - x)^3},}$$

Maxima and Minima

Solution 5.11

a.

$$\begin{aligned} f'(x) &= (12 - 2x)^2 + 2x(12 - 2x)(-2) \\ &= 4(x - 6)^2 + 8x(x - 6) \\ &= 12(x - 2)(x - 6) \end{aligned}$$

There are critical points at $x = 2$ and $x = 6$.

$$f''(x) = 12(x - 2) + 12(x - 6) = 24(x - 4)$$

Since $f''(2) = -48 < 0$, $x = 2$ is a local maximum. Since $f''(6) = 48 > 0$, $x = 6$ is a local minimum.

b.

$$f'(x) = \frac{2}{3}(x - 2)^{-1/3}$$

The first derivative exists and is nonzero for $x \neq 2$. At $x = 2$, the derivative does not exist and thus $x = 2$ is a critical point. For $x < 2$, $f'(x) < 0$ and for $x > 2$, $f'(x) > 0$. $x = 2$ is a local minimum.

Solution 5.12

Let r be the radius and h the height of the cylinder. The volume of the cup is $\pi r^2 h = 64$. The radius and height are related by $h = \frac{64}{\pi r^2}$. The surface area of the cup is $f(r) = \pi r^2 + 2\pi r h = \pi r^2 + \frac{128}{r}$. The first derivative of the surface area is $f'(r) = 2\pi r - \frac{128}{r^2}$. Finding the zeros of $f'(r)$,

$$2\pi r - \frac{128}{r^2} = 0,$$

$$2\pi r^3 - 128 = 0,$$

$$r = \frac{4}{\sqrt[3]{\pi}}.$$

The second derivative of the surface area is $f''(r) = 2\pi + \frac{256}{r^3}$. Since $f''(\frac{4}{\sqrt[3]{\pi}}) = 6\pi$, $r = \frac{4}{\sqrt[3]{\pi}}$ is a local minimum of $f(r)$. Since this is the only critical point for $r > 0$, it must be a global minimum.

The cup has a radius of $\frac{4}{\sqrt[3]{\pi}}$ cm and a height of $\frac{4}{\sqrt[3]{\pi}}$.

Mean Value Theorems

Solution 5.13

We define the function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Note that $h(x)$ is differentiable and that $h(a) = h(b) = 0$. Thus $h(x)$ satisfies the conditions of Rolle's theorem and there exists a point $\xi \in (a, b)$ such that

$$h'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(\xi) = 0,$$

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Solution 5.14

The first few terms in the Taylor series of $\sin(x)$ about $x = 0$ are

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} + \cdots.$$

The seventh derivative of $\sin x$ is $-\cos x$. Thus we have that

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{\cos x_0}{5040} x^7,$$

where $0 \leq x_0 \leq x$. Since we are considering $x \in [-1, 1]$ and $-1 \leq \cos(x_0) \leq 1$, the approximation

$$\boxed{\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}}$$

has a maximum error of $\frac{1}{5040} \approx 0.000198$. Using this polynomial to approximate $\sin(1)$,

$$\boxed{1 - \frac{1^3}{6} + \frac{1^5}{120} \approx 0.841667.}$$

To see that this has the required accuracy,

$$\sin(1) \approx 0.841471.$$

Solution 5.15

Expanding the terms in the approximation in Taylor series,

$$f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{6} f'''(x) + \frac{\Delta x^4}{24} f''''(x_1),$$
$$f(x - \Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) - \frac{\Delta x^3}{6} f'''(x) + \frac{\Delta x^4}{24} f''''(x_2),$$

where $x \leq x_1 \leq x + \Delta x$ and $x - \Delta x \leq x_2 \leq x$. Substituting the expansions into the formula,

$$\frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{\Delta x^2} = f''(x) + \frac{\Delta x^2}{24}[f''''(x_1) + f''''(x_2)].$$

Thus the error in the approximation is

$$\boxed{\frac{\Delta x^2}{24}[f''''(x_1) + f''''(x_2)].}$$

Solution 5.16

L'Hospital's Rule

Solution 5.17

a.

$$\begin{aligned}\lim_{x\to 0}\left[\frac{x-\sin x}{x^3}\right] &= \lim_{x\to 0}\left[\frac{1-\cos x}{3x^2}\right]\\ &= \lim_{x\to 0}\left[\frac{\sin x}{6x}\right]\\ &= \lim_{x\to 0}\left[\frac{\cos x}{6}\right]\\ &= \frac{1}{6}\end{aligned}$$

$$\boxed{\lim_{x\to 0}\left[\frac{x-\sin x}{x^3}\right]=\frac{1}{6}}$$

b.

$$\begin{aligned}
\lim_{x\to 0}\left(\csc x - \frac{1}{x}\right) &= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) \\
&= \lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) \\
&= \lim_{x\to 0}\left(\frac{1-\cos x}{x\cos x + \sin x}\right) \\
&= \lim_{x\to 0}\left(\frac{\sin x}{-x\sin x + \cos x + \cos x}\right) \\
&= \frac{0}{2} \\
&= 0
\end{aligned}$$

$$\boxed{\lim_{x\to 0}\left(\csc x - \frac{1}{x}\right) = 0}$$

c.

$$
\begin{aligned}
\log\left(\lim_{x\to+\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]\right) &= \lim_{x\to+\infty}\left[\log\left(\left(1+\frac{1}{x}\right)^{x}\right)\right] \\
&= \lim_{x\to+\infty}\left[x\log\left(1+\frac{1}{x}\right)\right] \\
&= \lim_{x\to+\infty}\left[\frac{\log\left(1+\frac{1}{x}\right)}{1/x}\right] \\
&= \lim_{x\to+\infty}\left[\frac{\left(1+\frac{1}{x}\right)^{-1}\left(-\frac{1}{x^2}\right)}{-1/x^2}\right] \\
&= \lim_{x\to+\infty}\left[\left(1+\frac{1}{x}\right)^{-1}\right] \\
&= 1
\end{aligned}
$$

Thus we have

$$
\boxed{\lim_{x\to+\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right] = e.}
$$

d. It takes four successive applications of L'Hospital's rule to evaluate the limit.

$$\begin{aligned}
\lim_{x \to 0} \left(\csc^2 x - \frac{1}{x^2} \right) &= \lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} \\
&= \lim_{x \to 0} \frac{2x - 2\cos x \sin x}{2x^2 \cos x \sin x + 2x \sin^2 x} \\
&= \lim_{x \to 0} \frac{2 - 2\cos^2 x + 2\sin^2 x}{2x^2 \cos^2 x + 8x \cos x \sin x + 2\sin^2 x - 2x^2 \sin^2 x} \\
&= \lim_{x \to 0} \frac{8 \cos x \sin x}{12x \cos^2 x + 12 \cos x \sin x - 8x^2 \cos x \sin x - 12x \sin^2 x} \\
&= \lim_{x \to 0} \frac{8\cos^2 x - 8\sin^2 x}{24\cos^2 x - 8x^2 \cos^2 x - 64x \cos x \sin x - 24\sin^2 x + 8x^2 \sin^2 x} \\
&= \frac{1}{3}
\end{aligned}$$

It is easier to use a Taylor series expansion.

$$\begin{aligned}
\lim_{x \to 0} \left(\csc^2 x - \frac{1}{x^2} \right) &= \lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} \\
&= \lim_{x \to 0} \frac{x^2 - (x - x^3/6 + O(x^5))^2}{x^2 (x + O(x^3))^2} \\
&= \lim_{x \to 0} \frac{x^2 - (x^2 - x^4/3 + O(x^6))}{x^4 + O(x^6)} \\
&= \lim_{x \to 0} \left(\frac{1}{3} + O(x^2) \right) \\
&= \frac{1}{3}
\end{aligned}$$

Solution 5.18

To evaluate the first limit, we use the identity $a^b = \mathrm{e}^{b \log a}$ and then apply L'Hospital's rule.

$$\begin{aligned}
\lim_{x \to \infty} x^{a/x} &= \lim_{x \to \infty} \mathrm{e}^{\frac{a \log x}{x}} \\
&= \exp\left(\lim_{x \to \infty} \frac{a \log x}{x} \right) \\
&= \exp\left(\lim_{x \to \infty} \frac{a/x}{1} \right) \\
&= \mathrm{e}^0
\end{aligned}$$

$$\boxed{\lim_{x \to \infty} x^{a/x} = 1}$$

We use the same method to evaluate the second limit.

$$\begin{aligned}
\lim_{x \to \infty} \left(1 + \frac{a}{x} \right)^{bx} &= \lim_{x \to \infty} \exp\left(bx \log\left(1 + \frac{a}{x} \right) \right) \\
&= \exp\left(\lim_{x \to \infty} bx \log\left(1 + \frac{a}{x} \right) \right) \\
&= \exp\left(\lim_{x \to \infty} b \frac{\log(1 + a/x)}{1/x} \right) \\
&= \exp\left(\lim_{x \to \infty} b \frac{\frac{-a/x^2}{1+a/x}}{-1/x^2} \right) \\
&= \exp\left(\lim_{x \to \infty} b \frac{a}{1 + a/x} \right)
\end{aligned}$$

$$\boxed{\lim_{x \to \infty} \left(1 + \frac{a}{x} \right)^{bx} = \mathrm{e}^{ab}}$$

Chapter 6

Integral Calculus

6.1 The Indefinite Integral

The opposite of a derivative is the *anti-derivative* or the *indefinite integral*. The indefinite integral of a function $f(x)$ is denoted,

$$\int f(x)\,\mathrm{d}x.$$

It is defined by the property that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int f(x)\,\mathrm{d}x = f(x).$$

While a function $f(x)$ has a unique derivative if it is differentiable, it has an infinite number of indefinite integrals, each of which differ by an additive constant.

Zero Slope Implies a Constant Function. If the value of a function's derivative is identically zero, $\frac{\mathrm{d}f}{\mathrm{d}x} = 0$, then the function is a constant, $f(x) = c$. To prove this, we assume that there exists a non-constant differentiable function whose derivative is zero and obtain a contradiction. Let $f(x)$ be such a function. Since $f(x)$ is non-constant, there exist points a and b such that $f(a) \neq f(b)$. By the Mean Value Theorem of differential calculus, there exists a point $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \neq 0,$$

which contradicts that the derivative is everywhere zero.

Indefinite Integrals Differ by an Additive Constant. Suppose that $F(x)$ and $G(x)$ are indefinite integrals of $f(x)$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x) - G(x)) = F'(x) - G'(x) = f(x) - f(x) = 0.$$

Thus we see that $F(x) - G(x) = c$ and the two indefinite integrals must differ by a constant. For example, we have $\int \sin x \,\mathrm{d}x = -\cos x + c$. While every function that can be expressed in terms of elementary functions, (the exponent, logarithm, trigonometric functions, etc.), has a derivative that can be written explicitly in terms of elementary functions, the same is not true of integrals. For example, $\int \sin(\sin x) \,\mathrm{d}x$ cannot be written explicitly in terms of elementary functions.

Properties. Since the derivative is linear, so is the indefinite integral. That is,

$$\int (af(x) + bg(x)) \,\mathrm{d}x = a \int f(x) \,\mathrm{d}x + b \int g(x) \,\mathrm{d}x.$$

For each derivative identity there is a corresponding integral identity. Consider the power law identity, $\frac{\mathrm{d}}{\mathrm{d}x}(f(x))^a = a(f(x))^{a-1} f'(x)$. The corresponding integral identity is

$$\int (f(x))^a f'(x) \,\mathrm{d}x = \frac{(f(x))^{a+1}}{a+1} + c, \quad a \neq -1,$$

where we require that $a \neq -1$ to avoid division by zero. From the derivative of a logarithm, $\frac{\mathrm{d}}{\mathrm{d}x} \ln(f(x)) = \frac{f'(x)}{f(x)}$, we obtain,

$$\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \ln |f(x)| + c.$$

Note the absolute value signs. This is because $\frac{\mathrm{d}}{\mathrm{d}x} \ln |x| = \frac{1}{x}$ for $x \neq 0$. In Figure 6.1 is a plot of $\ln |x|$ and $\frac{1}{x}$ to reinforce this.

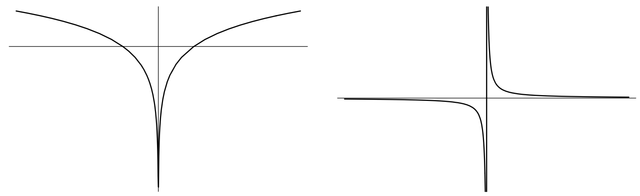


Figure 6.1: Plot of $\ln |x|$ and $1/x$.

Example 6.1.1 Consider

$$I = \int \frac{x}{(x^2+1)^2} \, \mathrm{d}x.$$

We evaluate the integral by choosing $u = x^2 + 1$, $\mathrm{d}u = 2x \, \mathrm{d}x$.

$$\begin{aligned} I &= \frac{1}{2} \int \frac{2x}{(x^2+1)^2} \, \mathrm{d}x \\ &= \frac{1}{2} \int \frac{\mathrm{d}u}{u^2} \\ &= \frac{1}{2} \frac{-1}{u} \\ &= -\frac{1}{2(x^2+1)}. \end{aligned}$$

Example 6.1.2 Consider

$$I = \int \tan x \,\mathrm{d}x = \int \frac{\sin x}{\cos x} \,\mathrm{d}x.$$

By choosing $f(x) = \cos x$, $f'(x) = -\sin x$, we see that the integral is

$$I = -\int \frac{-\sin x}{\cos x} \,\mathrm{d}x = -\ln|\cos x| + c.$$

Change of Variable. The differential of a function $g(x)$ is $dg = g'(x)\,\mathrm{d}x$. Thus one might suspect that for $\xi = g(x)$,

$$\int f(\xi)\,\mathrm{d}\xi = \int f(g(x))g'(x)\,\mathrm{d}x, \tag{6.1}$$

since $d\xi = dg = g'(x)\,\mathrm{d}x$. This turns out to be true. To prove it we will appeal to the the chain rule for differentiation. Let ξ be a function of x. The chain rule is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(\xi) = f'(\xi)\xi'(x),$$

$$\frac{\mathrm{d}}{\mathrm{d}x} f(\xi) = \frac{\mathrm{d}f}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}x}.$$

We can also write this as

$$\frac{\mathrm{d}f}{\mathrm{d}\xi} = \frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}f}{\mathrm{d}x},$$

or in operator notation,

$$\frac{\mathrm{d}}{\mathrm{d}\xi} = \frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}x}.$$

Now we're ready to start. The derivative of the left side of Equation 6.1 is

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\int f(\xi)\,\mathrm{d}\xi = f(\xi).$$

Next we differentiate the right side,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi}\int f(g(x))g'(x)\,\mathrm{d}x &= \frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}x}\int f(g(x))g'(x)\,\mathrm{d}x \\
&= \frac{\mathrm{d}x}{\mathrm{d}\xi}f(g(x))g'(x) \\
&= \frac{\mathrm{d}x}{\mathrm{d}g}f(g(x))\frac{\mathrm{d}g}{\mathrm{d}x} \\
&= f(g(x)) \\
&= f(\xi)
\end{aligned}$$

to see that it is in fact an identity for $\xi = g(x)$.

Example 6.1.3 Consider

$$\int x\sin(x^2)\,\mathrm{d}x.$$

We choose $\xi = x^2$, $d\xi = 2xdx$ to evaluate the integral.

$$\begin{aligned}
\int x\sin(x^2)\,\mathrm{d}x &= \frac{1}{2}\int \sin(x^2)2x\,\mathrm{d}x \\
&= \frac{1}{2}\int \sin\xi\,\mathrm{d}\xi \\
&= \frac{1}{2}(-\cos\xi) + c \\
&= -\frac{1}{2}\cos(x^2) + c
\end{aligned}$$

Integration by Parts. The product rule for differentiation gives us an identity called integration by parts. We start with the product rule and then integrate both sides of the equation.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(u(x)v(x)) &= u'(x)v(x) + u(x)v'(x) \\
\int (u'(x)v(x) + u(x)v'(x))\,\mathrm{d}x &= u(x)v(x) + c \\
\int u'(x)v(x)\,\mathrm{d}x + \int u(x)v'(x))\,\mathrm{d}x &= u(x)v(x) \\
\int u(x)v'(x))\,\mathrm{d}x &= u(x)v(x) - \int v(x)u'(x)\,\mathrm{d}x
\end{aligned}$$

The theorem is most often written in the form

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

So what is the usefulness of this? Well, it may happen for some integrals and a good choice of u and v that the integral on the right is easier to evaluate than the integral on the left.

Example 6.1.4 Consider $\int x\,\mathrm{e}^x\,\mathrm{d}x$. If we choose $u = x$, $\mathrm{d}v = \mathrm{e}^x\,\mathrm{d}x$ then integration by parts yields

$$\int x\,\mathrm{e}^x\,\mathrm{d}x = x\,\mathrm{e}^x - \int \mathrm{e}^x\,\mathrm{d}x = (x-1)\,\mathrm{e}^x.$$

Now notice what happens when we choose $u = \mathrm{e}^x$, $\mathrm{d}v = x\,\mathrm{d}x$.

$$\int x\,\mathrm{e}^x\,\mathrm{d}x = \frac{1}{2}x^2\,\mathrm{e}^x - \int \frac{1}{2}x^2\,\mathrm{e}^x\,\mathrm{d}x$$

The integral gets harder instead of easier.

When applying integration by parts, one must choose u and $\mathrm{d}v$ wisely. As general rules of thumb:

- Pick u so that u' is simpler than u.
- Pick $\mathrm{d}v$ so that v is not more complicated, (hopefully simpler), than $\mathrm{d}v$.

Also note that you may have to apply integration by parts several times to evaluate some integrals.

6.2 The Definite Integral

6.2.1 Definition

The area bounded by the x axis, the vertical lines $x = a$ and $x = b$ and the function $f(x)$ is denoted with a *definite integral*,

$$\int_a^b f(x)\,\mathrm{d}x.$$

The area is signed, that is, if $f(x)$ is negative, then the area is negative. We measure the area with a divide-and-conquer strategy. First partition the interval (a, b) with $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Note that the area under the curve on the subinterval is approximately the area of a rectangle of base $\Delta x_i = x_{i+1} - x_i$ and height $f(\xi_i)$, where $\xi_i \in [x_i, x_{i+1}]$. If we add up the areas of the rectangles, we get an approximation of the area under the curve. See Figure 6.2

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=0}^{n-1} f(\xi_i)\Delta x_i$$

As the Δx_i's get smaller, we expect the approximation of the area to get better. Let $\Delta x = \max_{0 \leq i \leq n-1} \Delta x_i$. We define the definite integral as the sum of the areas of the rectangles in the limit that $\Delta x \to 0$.

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x \to 0} \sum_{i=0}^{n-1} f(\xi_i)\Delta x_i$$

The integral is defined when the limit exists. This is known as the *Riemann integral* of $f(x)$. $f(x)$ is called the *integrand*.

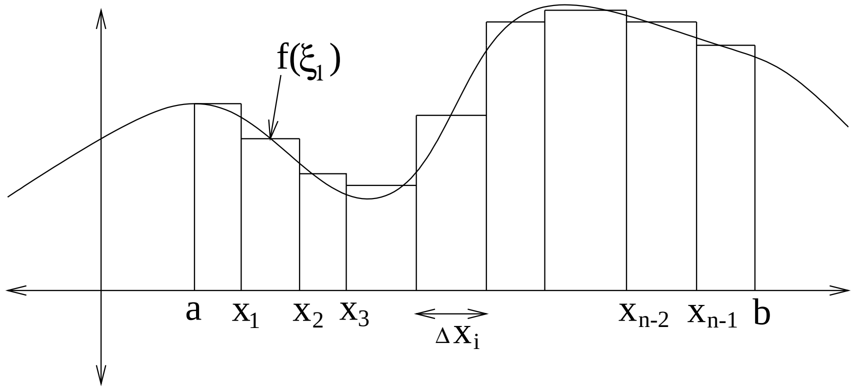


Figure 6.2: Divide-and-Conquer Strategy for Approximating a Definite Integral.

6.2.2 Properties

Linearity and the Basics. Because summation is a linear operator, that is

$$\sum_{i=0}^{n-1}(cf_i + dg_i) = c\sum_{i=0}^{n-1} f_i + d\sum_{i=0}^{n-1} g_i,$$

definite integrals are linear,

$$\int_a^b (cf(x) + dg(x))\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x + d\int_a^b g(x)\,\mathrm{d}x.$$

One can also divide the *range of integration*.

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

We assume that each of the above integrals exist. If $a \leq b$, and we integrate from b to a, then each of the Δx_i will be negative. From this observation, it is clear that

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$$

If we integrate any function from a point a to that same point a, then all the Δx_i are zero and

$$\int_a^a f(x)\,\mathrm{d}x = 0.$$

Bounding the Integral. Recall that if $f_i \leq g_i$, then

$$\sum_{i=0}^{n-1} f_i \leq \sum_{i=0}^{n-1} g_i.$$

Let $m = \min_{x \in [a,b]} f(x)$ and $M = \max_{x \in [a,b]} f(x)$. Then

$$(b-a)m = \sum_{i=0}^{n-1} m\Delta x_i \leq \sum_{i=0}^{n-1} f(\xi_i)\Delta x_i \leq \sum_{i=0}^{n-1} M\Delta x_i = (b-a)M$$

implies that

$$(b-a)m \leq \int_a^b f(x)\,\mathrm{d}x \leq (b-a)M.$$

Since

$$\left|\sum_{i=0}^{n-1} f_i\right| \leq \sum_{i=0}^{n-1} |f_i|,$$

we have

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

Mean Value Theorem of Integral Calculus. Let $f(x)$ be continuous. We know from above that

$$(b-a)m \leq \int_a^b f(x)\,\mathrm{d}x \leq (b-a)M.$$

Therefore there exists a constant $c \in [m, M]$ satisfying

$$\int_a^b f(x)\,\mathrm{d}x = (b-a)c.$$

Since $f(x)$ is continuous, there is a point $\xi \in [a, b]$ such that $f(\xi) = c$. Thus we see that

$$\int_a^b f(x)\,\mathrm{d}x = (b-a)f(\xi),$$

for some $\xi \in [a, b]$.

6.3 The Fundamental Theorem of Integral Calculus

Definite Integrals with Variable Limits of Integration. Consider a to be a constant and x variable, then the function $F(x)$ defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \tag{6.2}$$

is an anti-derivative of $f(x)$, that is $F'(x) = f(x)$. To show this we apply the definition of differentiation and the integral mean value theorem.

$$\begin{aligned}
F'(x) &= \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(t)\,\mathrm{d}t}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(\xi)\Delta x}{\Delta x}, \qquad \xi \in [x, x+\Delta x] \\
&= f(x)
\end{aligned}$$

The Fundamental Theorem of Integral Calculus. Let $F(x)$ be any anti-derivative of $f(x)$. Noting that all anti-derivatives of $f(x)$ differ by a constant and replacing x by b in Equation 6.2, we see that there exists a constant c such that

$$\int_a^b f(x)\,\mathrm{d}x = F(b) + c.$$

Now to find the constant. By plugging in $b = a$,

$$\int_a^a f(x)\,\mathrm{d}x = F(a) + c = 0,$$

we see that $c = -F(a)$. This gives us a result known as the *Fundamental Theorem of Integral Calculus.*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

We introduce the notation

$$[F(x)]_a^b \equiv F(b) - F(a).$$

Example 6.3.1

$$\int_0^{\pi} \sin x \, \mathrm{d}x = [-\cos x]_0^{\pi} = -\cos(\pi) + \cos(0) = 2$$

6.4 Techniques of Integration

6.4.1 Partial Fractions

A proper rational function

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x-a)^n r(x)}$$

Can be written in the form

$$\frac{p(x)}{(x-\alpha)^n r(x)} = \left(\frac{a_0}{(x-\alpha)^n} + \frac{a_1}{(x-\alpha)^{n-1}} + \cdots + \frac{a_{n-1}}{x-\alpha} \right) + (\cdots)$$

where the a_k's are constants and the last ellipses represents the partial fractions expansion of the roots of $r(x)$. The coefficients are

$$a_k = \frac{1}{k!} \frac{d^k}{dx^k} \left(\frac{p(x)}{r(x)} \right) \bigg|_{x=\alpha}.$$

Example 6.4.1 Consider the partial fraction expansion of

$$\frac{1+x+x^2}{(x-1)^3}.$$

The expansion has the form

$$\frac{a_0}{(x-1)^3} + \frac{a_1}{(x-1)^2} + \frac{a_2}{x-1}.$$

The coefficients are

$$
\begin{aligned}
a_0 &= \frac{1}{0!}(1+x+x^2)|_{x=1} = 3, \\
a_1 &= \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}x}(1+x+x^2)|_{x=1} = (1+2x)|_{x=1} = 3, \\
a_2 &= \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(1+x+x^2)|_{x=1} = \frac{1}{2}(2)|_{x=1} = 1.
\end{aligned}
$$

Thus we have

$$
\frac{1+x+x^2}{(x-1)^3} = \frac{3}{(x-1)^3} + \frac{3}{(x-1)^2} + \frac{1}{x-1}.
$$

Example 6.4.2 Suppose we want to evaluate

$$
\int \frac{1+x+x^2}{(x-1)^3}\,\mathrm{d}x.
$$

If we expand the integrand in a partial fraction expansion, then the integral becomes easy.

$$
\begin{aligned}
\int \frac{1+x+x^2}{(x-1)^3}\,\mathrm{d}x. &= \int \left(\frac{3}{(x-1)^3} + \frac{3}{(x-1)^2} + \frac{1}{x-1}\right)\mathrm{d}x \\
&= -\frac{3}{2(x-1)^2} - \frac{3}{(x-1)} + \ln(x-1)
\end{aligned}
$$

Example 6.4.3 Consider the partial fraction expansion of

$$
\frac{1+x+x^2}{x^2(x-1)^2}.
$$

The expansion has the form

$$\frac{a_0}{x^2} + \frac{a_1}{x} + \frac{b_0}{(x-1)^2} + \frac{b_1}{x-1}.$$

The coefficients are

$$\begin{aligned}
a_0 &= \frac{1}{0!}\left(\frac{1+x+x^2}{(x-1)^2}\right)\bigg|_{x=0} = 1, \\
a_1 &= \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+x+x^2}{(x-1)^2}\right)\bigg|_{x=0} = \left(\frac{1+2x}{(x-1)^2} - \frac{2(1+x+x^2)}{(x-1)^3}\right)\bigg|_{x=0} = 3, \\
b_0 &= \frac{1}{0!}\left(\frac{1+x+x^2}{x^2}\right)\bigg|_{x=1} = 3, \\
b_1 &= \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+x+x^2}{x^2}\right)\bigg|_{x=1} = \left(\frac{1+2x}{x^2} - \frac{2(1+x+x^2)}{x^3}\right)\bigg|_{x=1} = -3,
\end{aligned}$$

Thus we have

$$\frac{1+x+x^2}{x^2(x-1)^2} = \frac{1}{x^2} + \frac{3}{x} + \frac{3}{(x-1)^2} - \frac{3}{x-1}.$$

If the rational function has real coefficients and the denominator has complex roots, then you can reduce the work in finding the partial fraction expansion with the following trick: Let α and $\overline{\alpha}$ be complex conjugate pairs of roots of the denominator.

$$\begin{aligned}
\frac{p(x)}{(x-\alpha)^n(x-\overline{\alpha})^n r(x)} = &\left(\frac{a_0}{(x-\alpha)^n} + \frac{a_1}{(x-\alpha)^{n-1}} + \cdots + \frac{a_{n-1}}{x-\alpha}\right) \\
&+ \left(\frac{\overline{a_0}}{(x-\overline{\alpha})^n} + \frac{\overline{a_1}}{(x-\overline{\alpha})^{n-1}} + \cdots + \frac{\overline{a_{n-1}}}{x-\overline{\alpha}}\right) + (\cdots)
\end{aligned}$$

Thus we don't have to calculate the coefficients for the root at $\overline{\alpha}$. We just take the complex conjugate of the coefficients for α.

Example 6.4.4 Consider the partial fraction expansion of

$$\frac{1+x}{x^2+1}.$$

The expansion has the form

$$\frac{a_0}{x-i} + \frac{\overline{a_0}}{x+i}$$

The coefficients are

$$\begin{aligned} a_0 &= \frac{1}{0!}\left(\frac{1+x}{x+i}\right)\bigg|_{x=i} = \frac{1}{2}(1-i), \\ \overline{a_0} &= \overline{\frac{1}{2}(1-i)} = \frac{1}{2}(1+i) \end{aligned}$$

Thus we have

$$\frac{1+x}{x^2+1} = \frac{1-i}{2(x-i)} + \frac{1+i}{2(x+i)}.$$

6.5 Improper Integrals

If the range of integration is infinite or $f(x)$ is discontinuous at some points then $\int_a^b f(x)\,\mathrm{d}x$ is called an *improper integral*.

Discontinuous Functions. If $f(x)$ is continuous on the interval $a \leq x \leq b$ except at the point $x = c$ where $a < c < b$ then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\delta \to 0^+} \int_a^{c-\delta} f(x)\,\mathrm{d}x + \lim_{\epsilon \to 0^+} \int_{c+\epsilon}^b f(x)\,\mathrm{d}x$$

provided that both limits exist.

Example 6.5.1 Consider the integral of $\ln x$ on the interval $[0, 1]$. Since the logarithm has a singularity at $x = 0$, this is an improper integral. We write the integral in terms of a limit and evaluate the limit with L'Hospital's rule.

$$\begin{aligned}
\int_0^1 \ln x \,\mathrm{d}x &= \lim_{\delta \to 0} \int_\delta^1 \ln x \,\mathrm{d}x \\
&= \lim_{\delta \to 0} [x \ln x - x]_\delta^1 \\
&= 1 \ln(1) - 1 - \lim_{\delta \to 0} (\delta \ln \delta - \delta) \\
&= -1 - \lim_{\delta \to 0} (\delta \ln \delta) \\
&= -1 - \lim_{\delta \to 0} \left(\frac{\ln \delta}{1/\delta} \right) \\
&= -1 - \lim_{\delta \to 0} \left(\frac{1/\delta}{-1/\delta^2} \right) \\
&= -1
\end{aligned}$$

Example 6.5.2 Consider the integral of x^a on the range $[0, 1]$. If $a < 0$ then there is a singularity at $x = 0$. First assume that $a \neq -1$.

$$\begin{aligned}
\int_0^1 x^a \,\mathrm{d}x &= \lim_{\delta \to 0^+} \left[\frac{x^{a+1}}{a+1} \right]_\delta^1 \\
&= \frac{1}{a+1} - \lim_{\delta \to 0^+} \frac{\delta^{a+1}}{a+1}
\end{aligned}$$

This limit exists only for $a > -1$. Now consider the case that $a = -1$.

$$\begin{aligned}
\int_0^1 x^{-1} \,\mathrm{d}x &= \lim_{\delta \to 0^+} [\ln x]_\delta^1 \\
&= \ln(0) - \lim_{\delta \to 0^+} \ln \delta
\end{aligned}$$

This limit does not exist. We obtain the result,

$$\int_0^1 x^a \,\mathrm{d}x = \frac{1}{a+1}, \quad \text{for } a > -1.$$

Infinite Limits of Integration. If the range of integration is infinite, say $[a, \infty)$ then we define the integral as

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{\alpha\to\infty} \int_a^\alpha f(x)\,\mathrm{d}x,$$

provided that the limit exists. If the range of integration is $(-\infty, \infty)$ then

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \lim_{\alpha\to-\infty} \int_\alpha^a f(x)\,\mathrm{d}x + \lim_{\beta\to+\infty} \int_a^\beta f(x)\,\mathrm{d}x.$$

Example 6.5.3

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^2}\,\mathrm{d}x &= \int_1^\infty \ln x \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{-1}{x}\right)\mathrm{d}x \\
&= \left[\ln x \frac{-1}{x}\right]_1^\infty - \int_1^\infty \frac{-1}{x}\frac{1}{x}\,\mathrm{d}x \\
&= \lim_{x\to+\infty}\left(-\frac{\ln x}{x}\right) - \left[\frac{1}{x}\right]_1^\infty \\
&= \lim_{x\to+\infty}\left(-\frac{1/x}{1}\right) - \lim_{x\to\infty}\frac{1}{x} + 1 \\
&= 1
\end{aligned}$$

Example 6.5.4 Consider the integral of x^a on $[1, \infty)$. First assume that $a \neq -1$.

$$\begin{aligned}\int_1^{\infty} x^a \,\mathrm{d}x &= \lim_{\beta \to +\infty} \left[\frac{x^{a+1}}{a+1} \right]_1^{\beta} \\ &= \lim_{\beta \to +\infty} \frac{\beta^{a+1}}{a+1} - \frac{1}{a+1}\end{aligned}$$

The limit exists for $\beta < -1$. Now consider the case $a = -1$.

$$\begin{aligned}\int_1^{\infty} x^{-1} \,\mathrm{d}x &= \lim_{\beta \to +\infty} [\ln x]_1^{\beta} \\ &= \lim_{\beta \to +\infty} \ln \beta - \frac{1}{a+1}\end{aligned}$$

This limit does not exist. Thus we have

$$\int_1^{\infty} x^a \,\mathrm{d}x = -\frac{1}{a+1}, \quad \text{for } a < -1.$$

6.6 Exercises

Fundamental Integration Formulas

Exercise 6.1 (mathematica/calculus/integral/fundamental.nb)
Evaluate $\int (2x+3)^{10}\,\mathrm{d}x$.

Exercise 6.2 (mathematica/calculus/integral/fundamental.nb)
Evaluate $\int \frac{(\ln x)^2}{x}\,\mathrm{d}x$.

Exercise 6.3 (mathematica/calculus/integral/fundamental.nb)
Evaluate $\int x\sqrt{x^2+3}\,\mathrm{d}x$.

Exercise 6.4 (mathematica/calculus/integral/fundamental.nb)
Evaluate $\int \frac{\cos x}{\sin x}\,\mathrm{d}x$.

Exercise 6.5 (mathematica/calculus/integral/fundamental.nb)
Evaluate $\int \frac{x^2}{x^3-5}\,\mathrm{d}x$.

Integration by Parts

Exercise 6.6 (mathematica/calculus/integral/parts.nb)
Evaluate $\int x\sin x\,\mathrm{d}x$.

Exercise 6.7 (mathematica/calculus/integral/parts.nb)
Evaluate $\int x^3\,\mathrm{e}^{2x}\,\mathrm{d}x$.

Partial Fractions

Exercise 6.8 (mathematica/calculus/integral/partial.nb)
Evaluate $\int \frac{1}{x^2-4} \, \mathrm{d}x$.

Exercise 6.9 (mathematica/calculus/integral/partial.nb)
Evaluate $\int \frac{x+1}{x^3+x^2-6x} \, \mathrm{d}x$.

Definite Integrals

Exercise 6.10 (mathematica/calculus/integral/definite.nb)
Use the result

$$\int_a^b f(x) \, \mathrm{d}x = \lim_{N\to\infty} \sum_{n=0}^{N-1} f(x_n)\Delta x$$

where $\Delta x = \frac{b-a}{N}$ and $x_n = a + n\Delta x$, to show that

$$\int_0^1 x \, \mathrm{d}x = \frac{1}{2}.$$

Exercise 6.11 (mathematica/calculus/integral/definite.nb)
Evaluate the following integral using integration by parts and the Pythagorean identity. $\int_0^\pi \sin^2 x \, \mathrm{d}x$

Exercise 6.12 (mathematica/calculus/integral/definite.nb)
Prove that

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{g(x)}^{f(x)} h(\xi) \, \mathrm{d}\xi = h(f(x))f'(x) - h(g(x))g'(x).$$

(Don't use the limit definition of differentiation, use the Fundamental Theorem of Integral Calculus.)

Improper Integrals

Exercise 6.13 (mathematica/calculus/integral/improper.nb)
Evaluate $\int_0^4 \frac{1}{(x-1)^2}\,\mathrm{d}x$.

Exercise 6.14 (mathematica/calculus/integral/improper.nb)
Evaluate $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$.

Exercise 6.15 (mathematica/calculus/integral/improper.nb)
Evaluate $\int_0^\infty \frac{1}{x^2+4}\,\mathrm{d}x$.

Taylor Series

Exercise 6.16 (mathematica/calculus/integral/taylor.nb)

a. Show that

$$f(x) = f(0) + \int_0^x f'(x-\xi)\,\mathrm{d}\xi.$$

b. From the above identity show that

$$f(x) = f(0) + xf'(0) + \int_0^x \xi f''(x-\xi)\,\mathrm{d}\xi.$$

c. Using induction, show that

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \cdots + \frac{1}{n!}x^n f^{(n)}(0) + \int_0^x \frac{1}{n!}\xi^n f^{(n+1)}(x-\xi)\,\mathrm{d}\xi.$$

6.7 Hints

Fundamental Integration Formulas

Hint 6.1
Make the change of variables $u = 2x + 3$.

Hint 6.2
Make the change of variables $u = \ln x$.

Hint 6.3
Make the change of variables $u = x^2 + 3$.

Hint 6.4
Make the change of variables $u = \sin x$.

Hint 6.5
Make the change of variables $u = x^3 - 5$.

Integration by Parts

Hint 6.6
Let $u = x$, and $\mathrm{d}v = \sin x \,\mathrm{d}x$.

Hint 6.7
Perform integration by parts three successive times. For the first one let $u = x^3$ and $\mathrm{d}v = \mathrm{e}^{2x} \,\mathrm{d}x$.

Partial Fractions

Hint 6.8
Expanding the integrand in partial fractions,

$$\frac{1}{x^2 - 4} = \frac{1}{(x-2)(x+2)} = \frac{a}{(x-2)} + \frac{b}{(x+2)}$$

$$1 = a(x+2) + b(x-2)$$

Set $x = 2$ and $x = -2$ to solve for a and b.

Hint 6.9
Expanding the integral in partial fractions,

$$\frac{x+1}{x^3 + x^2 - 6x} = \frac{x+1}{x(x-2)(x+3)} = \frac{a}{x} + \frac{b}{x-2} + \frac{c}{x+3}$$

$$x + 1 = a(x-2)(x+3) + bx(x+3) + cx(x-2)$$

Set $x = 0$, $x = 2$ and $x = -3$ to solve for a, b and c.

Definite Integrals

Hint 6.10

$$\begin{aligned}\int_0^1 x \, \mathrm{d}x &= \lim_{N\to\infty} \sum_{n=0}^{N-1} x_n \Delta x \\ &= \lim_{N\to\infty} \sum_{n=0}^{N-1} (n\Delta x)\Delta x\end{aligned}$$

Hint 6.11
Let $u = \sin x$ and $\mathrm{d}v = \sin x \, \mathrm{d}x$. Integration by parts will give you an equation for $\int_0^\pi \sin^2 x \, \mathrm{d}x$.

Hint 6.12
Let $H'(x) = h(x)$ and evaluate the integral in terms of $H(x)$.

Improper Integrals

Hint 6.13

$$\int_0^4 \frac{1}{(x-1)^2} \, \mathrm{d}x = \lim_{\delta \to 0^+} \int_0^{1-\delta} \frac{1}{(x-1)^2} \, \mathrm{d}x + \lim_{\epsilon \to 0^+} \int_{1+\epsilon}^4 \frac{1}{(x-1)^2} \, \mathrm{d}x$$

Hint 6.14

$$\int_0^1 \frac{1}{\sqrt{x}} \, \mathrm{d}x = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{\sqrt{x}} \, \mathrm{d}x$$

Hint 6.15

$$\int \frac{1}{x^2 + a^2} \, \mathrm{d}x = \frac{1}{a} \arctan\left(\frac{x}{a}\right)$$

Taylor Series

Hint 6.16
a. Evaluate the integral.

b. Use integration by parts to evaluate the integral.

c. Use integration by parts with $u = f^{(n+1)}(x - \xi)$ and $\mathrm{d}v = \frac{1}{n!}\xi^n$.

6.8 Solutions

Fundamental Integration Formulas

Solution 6.1

$$\int (2x+3)^{10}\,\mathrm{d}x$$

Let $u = 2x+3$, $g(u) = x = \frac{u-3}{2}$, $g'(u) = \frac{1}{2}$.

$$\begin{aligned}\int (2x+3)^{10}\,\mathrm{d}x &= \int u^{10}\frac{1}{2}\,\mathrm{d}u \\ &= \frac{u^{11}}{11}\frac{1}{2} \\ &= \frac{(2x+3)^{11}}{22}\end{aligned}$$

Solution 6.2

$$\begin{aligned}\int \frac{(\ln x)^2}{x}\,\mathrm{d}x &= \int (\ln x)^2 \frac{\mathrm{d}(\ln x)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \frac{(\ln x)^3}{3}\end{aligned}$$

Solution 6.3

$$\begin{aligned}\int x\sqrt{x^2+3}\,\mathrm{d}x &= \int \sqrt{x^2+3}\frac{1}{2}\frac{\mathrm{d}(x^2)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \frac{1}{2}\frac{(x^2+3)^{3/2}}{3/2} \\ &= \frac{(x^2+3)^{3/2}}{3}\end{aligned}$$

Solution 6.4

$$\begin{aligned}\int \frac{\cos x}{\sin x}\,\mathrm{d}x &= \int \frac{1}{\sin x}\frac{\mathrm{d}(\sin x)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \ln|\sin x|\end{aligned}$$

Solution 6.5

$$\begin{aligned}\int \frac{x^2}{x^3-5}\,\mathrm{d}x &= \int \frac{1}{x^3-5}\frac{1}{3}\frac{\mathrm{d}(x^3)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \frac{1}{3}\ln|x^3-5|\end{aligned}$$

Integration by Parts

Solution 6.6
Let $u = x$, and $\mathrm{d}v = \sin x \,\mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$ and $v = -\cos x$.

$$\begin{aligned}\int x \sin x \,\mathrm{d}x &= -x\cos x + \int \cos x \,\mathrm{d}x \\ &= -x\cos x + \sin x + C\end{aligned}$$

Solution 6.7
Let $u = x^3$ and $\mathrm{d}v = \mathrm{e}^{2x}\,\mathrm{d}x$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$ and $v = \frac{1}{2}\mathrm{e}^{2x}$.

$$\int x^3 \mathrm{e}^{2x}\,\mathrm{d}x = \frac{1}{2}x^3\mathrm{e}^{2x} - \frac{3}{2}\int x^2\mathrm{e}^{2x}\,\mathrm{d}x$$

Let $u = x^2$ and $\mathrm{d}v = \mathrm{e}^{2x}\,\mathrm{d}x$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$ and $v = \frac{1}{2}\mathrm{e}^{2x}$.

$$\int x^3 \mathrm{e}^{2x}\,\mathrm{d}x = \frac{1}{2}x^3\mathrm{e}^{2x} - \frac{3}{2}\left(\frac{1}{2}x^2\mathrm{e}^{2x} - \int x\,\mathrm{e}^{2x}\,\mathrm{d}x\right)$$

$$\int x^3 \mathrm{e}^{2x}\,\mathrm{d}x = \frac{1}{2}x^3\mathrm{e}^{2x} - \frac{3}{4}x^2\mathrm{e}^{2x} + \frac{3}{2}\int x\,\mathrm{e}^{2x}\,\mathrm{d}x$$

Let $u = x$ and $\mathrm{d}v = \mathrm{e}^{2x}\,\mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$ and $v = \frac{1}{2}\mathrm{e}^{2x}$.

$$\int x^3 \mathrm{e}^{2x}\,\mathrm{d}x = \frac{1}{2}x^3\mathrm{e}^{2x} - \frac{3}{4}x^2\mathrm{e}^{2x} + \frac{3}{2}\left(\frac{1}{2}x\,\mathrm{e}^{2x} - \frac{1}{2}\int \mathrm{e}^{2x}\,\mathrm{d}x\right)$$

$$\int x^3 \mathrm{e}^{2x}\,\mathrm{d}x = \frac{1}{2}x^3\mathrm{e}^{2x} - \frac{3}{4}x^2\mathrm{e}^{2x} + \frac{3}{4}x\,\mathrm{e}^{2x} - \frac{3}{8}\mathrm{e}^{2x} + C$$

Partial Fractions

Solution 6.8
Expanding the integrand in partial fractions,

$$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{(x-2)} + \frac{B}{(x+2)}$$

$$1 = A(x+2) + B(x-2)$$

Setting $x = 2$ yields $A = \frac{1}{4}$. Setting $x = -2$ yields $B = -\frac{1}{4}$. Now we can do the integral.

$$\begin{aligned}\int \frac{1}{x^2-4}\,\mathrm{d}x &= \int \left(\frac{1}{4(x-2)} - \frac{1}{4(x+2)}\right)\mathrm{d}x \\ &= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C \\ &= \frac{1}{4}\left|\frac{x-2}{x+2}\right| + C\end{aligned}$$

Solution 6.9
Expanding the integral in partial fractions,

$$\frac{x+1}{x^3+x^2-6x} = \frac{x+1}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$$

$$x+1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2)$$

Setting $x = 0$ yields $A = -\frac{1}{6}$. Setting $x = 2$ yields $B = \frac{3}{10}$. Setting $x = -3$ yields $C = -\frac{2}{15}$.

$$\begin{aligned}\int \frac{x+1}{x^3+x^2-6x}\,\mathrm{d}x &= \int \left(-\frac{1}{6x} + \frac{3}{10(x-2)} - \frac{2}{15(x+3)}\right)\mathrm{d}x \\ &= -\frac{1}{6}\ln|x| + \frac{3}{10}\ln|x-2| - \frac{2}{15}\ln|x+3| + C \\ &= \ln\frac{|x-2|^{3/10}}{|x|^{1/6}|x+3|^{2/15}} + C\end{aligned}$$

Definite Integrals

Solution 6.10

$$
\begin{aligned}
\int_0^1 x \, \mathrm{d}x &= \lim_{N \to \infty} \sum_{n=0}^{N-1} x_n \Delta x \\
&= \lim_{N \to \infty} \sum_{n=0}^{N-1} (n \Delta x) \Delta x \\
&= \lim_{N \to \infty} \Delta x^2 \sum_{n=0}^{N-1} n \\
&= \lim_{N \to \infty} \Delta x^2 \frac{N(N-1)}{2} \\
&= \lim_{N \to \infty} \frac{N(N-1)}{2N^2} \\
&= \frac{1}{2}
\end{aligned}
$$

Solution 6.11
Let $u = \sin x$ and $\mathrm{d}v = \sin x \,\mathrm{d}x$. Then $\mathrm{d}u = \cos x \,\mathrm{d}x$ and $v = -\cos x$.

$$\begin{aligned}
\int_0^\pi \sin^2 x \,\mathrm{d}x &= \big[-\sin x \cos x\big]_0^\pi + \int_0^\pi \cos^2 x \,\mathrm{d}x \\
&= \int_0^\pi \cos^2 x \,\mathrm{d}x \\
&= \int_0^\pi (1 - \sin^2 x) \,\mathrm{d}x \\
&= \pi - \int_0^\pi \sin^2 x \,\mathrm{d}x
\end{aligned}$$

$$2 \int_0^\pi \sin^2 x \,\mathrm{d}x = \pi$$

$$\int_0^\pi \sin^2 x \,\mathrm{d}x = \frac{\pi}{2}$$

Solution 6.12
Let $H'(x) = h(x)$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \int_{g(x)}^{f(x)} h(\xi) \,\mathrm{d}\xi &= \frac{\mathrm{d}}{\mathrm{d}x} \left(H(f(x)) - H(g(x)) \right) \\
&= H'(f(x)) f'(x) - H'(g(x)) g'(x) \\
&= h(f(x)) f'(x) - h(g(x)) g'(x)
\end{aligned}$$

Improper Integrals

Solution 6.13

$$
\begin{aligned}
\int_0^4 \frac{1}{(x-1)^2}\,\mathrm{d}x &= \lim_{\delta\to 0^+}\int_0^{1-\delta}\frac{1}{(x-1)^2}\,\mathrm{d}x + \lim_{\epsilon\to 0^+}\int_{1+\epsilon}^4 \frac{1}{(x-1)^2}\,\mathrm{d}x \\
&= \lim_{\delta\to 0^+}\left[-\frac{1}{x-1}\right]_0^{1-\delta} + \lim_{\epsilon\to 0^+}\left[-\frac{1}{x-1}\right]_{1+\epsilon}^4 \\
&= \lim_{\delta\to 0^+}\left(\frac{1}{\delta}-1\right) + \lim_{\epsilon\to 0^+}\left(-\frac{1}{3}+\frac{1}{\epsilon}\right) \\
&= \infty + \infty
\end{aligned}
$$

The integral diverges.

Solution 6.14

$$
\begin{aligned}
\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x &= \lim_{\epsilon\to 0^+}\int_\epsilon^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x \\
&= \lim_{\epsilon\to 0^+}\left[2\sqrt{x}\right]_\epsilon^1 \\
&= \lim_{\epsilon\to 0^+} 2(1-\sqrt{\epsilon}) \\
&= 2
\end{aligned}
$$

Solution 6.15

$$\begin{aligned}
\int_0^\infty \frac{1}{x^2+4}\,\mathrm{d}x &= \lim_{\alpha\to\infty}\int_0^\alpha \frac{1}{x^2+4}\,\mathrm{d}x \\
&= \lim_{\alpha\to\infty}\left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_0^\alpha \\
&= \frac{1}{2}\left(\frac{\pi}{2}-0\right) \\
&= \frac{\pi}{4}
\end{aligned}$$

Taylor Series

Solution 6.16

a.

$$\begin{aligned}
f(0)+\int_0^x f'(x-\xi)\,\mathrm{d}\xi &= f(0)+[-f(x-\xi)]_0^x \\
&= f(0)-f(0)+f(x) \\
&= f(x)
\end{aligned}$$

b.

$$\begin{aligned}
f(0)+xf'(0)+\int_0^x \xi f''(x-\xi)\,\mathrm{d}\xi &= f(0)+xf'(0)+[-\xi f'(x-\xi)]_0^x-\int_0^x -f'(x-\xi)\,\mathrm{d}\xi \\
&= f(0)+xf'(0)-xf'(0)-[f(x-\xi)]_0^x \\
&= f(0)-f(0)+f(x) \\
&= f(x)
\end{aligned}$$

c. Above we showed that the hypothesis holds for $n = 0$ and $n = 1$. Assume that it holds for some $n = m \geq 0$.

$$
\begin{aligned}
f(x) &= f(0) + x f'(0) + \frac{1}{2} x^2 f''(0) + \cdots + \frac{1}{n!} x^n f^{(n)}(0) + \int_0^x \frac{1}{n!} \xi^n f^{(n+1)}(x - \xi) \, \mathrm{d}\xi \\
&= f(0) + x f'(0) + \frac{1}{2} x^2 f''(0) + \cdots + \frac{1}{n!} x^n f^{(n)}(0) + \left[\frac{1}{(n+1)!} \xi^{n+1} f^{(n+1)}(x - \xi) \right]_0^x \\
&\quad - \int_0^x - \frac{1}{(n+1)!} \xi^{n+1} f^{(n+2)}(x - \xi) \, \mathrm{d}\xi \\
&= f(0) + x f'(0) + \frac{1}{2} x^2 f''(0) + \cdots + \frac{1}{n!} x^n f^{(n)}(0) + \frac{1}{(n+1)!} x^{n+1} f^{(n+1)}(0) \\
&\quad + \int_0^x \frac{1}{(n+1)!} \xi^{n+1} f^{(n+2)}(x - \xi) \, \mathrm{d}\xi
\end{aligned}
$$

This shows that the hypothesis holds for $n = m + 1$. By induction, the hypothesis hold for all $n \geq 0$.

Chapter 7

Vector Calculus

7.1 Vector Functions

Vector-valued Functions. A vector-valued function, $\mathbf{r}(t)$, is a mapping $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^n$ that assigns a vector to each value of t.

$$\mathbf{r}(t) = r_1(t)\mathbf{e}_1 + \cdots + r_n(t)\mathbf{e}_n.$$

An example of a vector-valued function is the position of an object in space as a function of time. The function is continous at a point $t = \tau$ if

$$\lim_{t \to \tau} \mathbf{r}(t) = \mathbf{r}(\tau).$$

This occurs if and only if the component functions are continuous. The function is differentiable if

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \equiv \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

exists. This occurs if and only if the component functions are differentiable.

If $\mathbf{r}(t)$ represents the position of a particle at time t, then the velocity and acceleration of the particle are

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2},$$

respectively. The speed of the particle is $|\mathbf{r}'(t)|$.

Differentiation Formulas. Let $\mathbf{f}(t)$ and $\mathbf{g}(t)$ be vector functions and $a(t)$ be a scalar function. By writing out components you can verify the differentiation formulas:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f} \cdot \mathbf{g}) = \mathbf{f}' \cdot \mathbf{g} + \mathbf{f} \cdot \mathbf{g}'$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f} \times \mathbf{g}) = \mathbf{f}' \times \mathbf{g} + \mathbf{f} \times \mathbf{g}'$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(a\mathbf{f}) = a'\mathbf{f} + a\mathbf{f}'$$

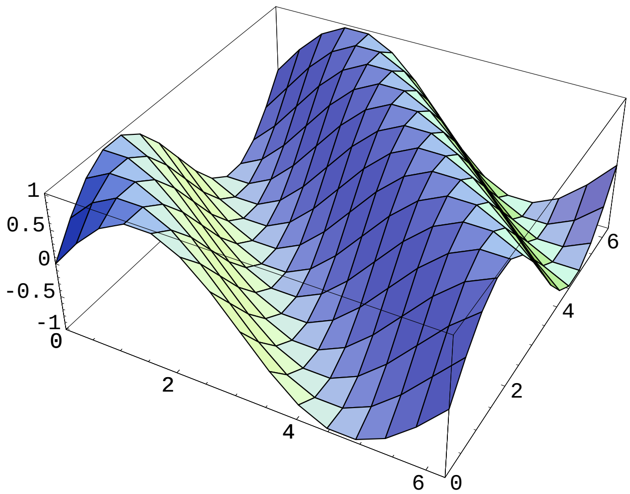
7.2 Gradient, Divergence and Curl

Scalar and Vector Fields. A *scalar field* is a function of position $u(\mathbf{x})$ that assigns a scalar to each point in space. A function that gives the temperature of a material is an example of a scalar field. In two dimensions, you can graph a scalar field as a surface plot, (Figure 7.1), with the vertical axis for the value of the function.

A *vector field* is a function of position $\mathbf{u}(\mathbf{x})$ that assigns a vector to each point in space. Examples of vectors fields are functions that give the acceleration due to gravity or the velocity of a fluid. You can graph a vector field in two or three dimension by drawing vectors at regularly spaced points. (See Figure 7.1 for a vector field in two dimensions.)

Partial Derivatives of Scalar Fields. Consider a scalar field $u(\mathbf{x})$. The *partial derivative* of u with respect to x_k is the derivative of u in which x_k is considered to be a variable and the remaining arguments are considered to be parameters. The partial derivative is denoted $\frac{\partial}{\partial x_k} u(\mathbf{x})$, $\frac{\partial u}{\partial x_k}$ or u_{x_k} and is defined

$$\frac{\partial u}{\partial x_k} \equiv \lim_{\Delta x \to 0} \frac{u(x_1, \ldots, x_k + \Delta x, \ldots, x_n) - u(x_1, \ldots, x_k, \ldots, x_n)}{\Delta x}.$$



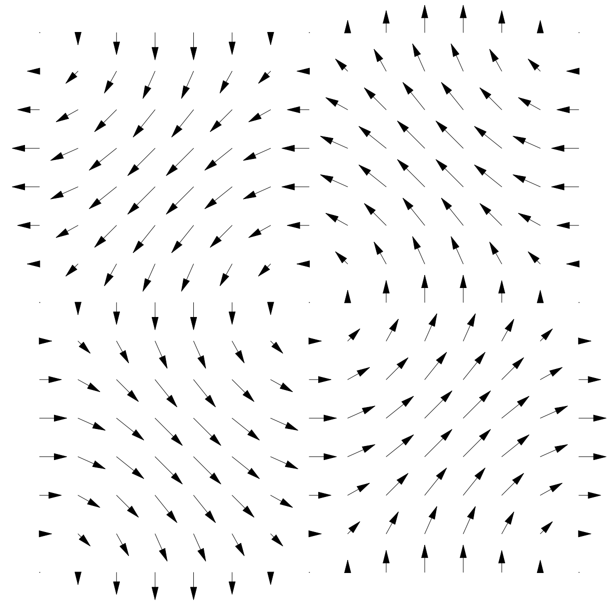


Figure 7.1: A Scalar Field and a Vector Field

Partial derivatives have the same differentiation formulas as ordinary derivatives.

Consider a scalar field in $\mathbb{R}^3$, $u(x, y, z)$. Higher derivatives of u are denoted:

$$
\begin{aligned}
u_{xx} &\equiv \frac{\partial^2 u}{\partial x^2} \equiv \frac{\partial}{\partial x}\frac{\partial u}{\partial x}, \\
u_{xy} &\equiv \frac{\partial^2 u}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\frac{\partial u}{\partial y}, \\
u_{xxyz} &\equiv \frac{\partial^4 u}{\partial x^2 \partial y \partial z} \equiv \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial y}\frac{\partial u}{\partial z}.
\end{aligned}
$$

If u_{xy} and u_{yx} are continuous, then

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

This is referred to as the *equality of mixed partial derivatives.*

Partial Derivatives of Vector Fields. Consider a vector field $\mathbf{u}(\mathbf{x})$. The partial derivative of $\mathbf{u}$ with respect to x_k is denoted $\frac{\partial}{\partial x_k}\mathbf{u}(\mathbf{x})$, $\frac{\partial \mathbf{u}}{\partial x_k}$ or $\mathbf{u}_{x_k}$ and is defined

$$\frac{\partial \mathbf{u}}{\partial x_k} \equiv \lim_{\Delta x \to 0} \frac{\mathbf{u}(x_1, \ldots, x_k + \Delta x, \ldots, x_n) - \mathbf{u}(x_1, \ldots, x_k, \ldots, x_n)}{\Delta x}.$$

Partial derivatives of vector fields have the same differentiation formulas as ordinary derivatives.

Gradient. We introduce the vector differential operator,

$$\nabla \equiv \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_n}\mathbf{e}_n,$$

which is known as *del* or *nabla*. In $\mathbb{R}^3$ it is

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Let $u(\mathbf{x})$ be a differential scalar field. The *gradient* of u is,

$$\nabla u \equiv \frac{\partial u}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial u}{\partial x_n}\mathbf{e}_n,$$

Directional Derivative. Suppose you are standing on some terrain. The slope of the ground in a particular direction is the *directional derivative* of the elevation in that direction. Consider a differentiable scalar field, $u(\mathbf{x})$. The derivative of the function in the direction of the unit vector $\mathbf{a}$ is the rate of change of the function in that direction. Thus the directional derivative, $D_{\mathbf{a}}u$, is defined:

$$\begin{aligned}
D_{\mathbf{a}}u(\mathbf{x}) &= \lim_{\epsilon \to 0} \frac{u(\mathbf{x} + \epsilon\mathbf{a}) - u(\mathbf{x})}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{u(x_1 + \epsilon a_1, \ldots, x_n + \epsilon a_n) - u(x_1, \ldots, x_n)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{(u(\mathbf{x}) + \epsilon a_1 u_{x_1}(\mathbf{x}) + \cdots + \epsilon a_n u_{x_n}(\mathbf{x}) + \mathcal{O}(\epsilon^2)) - u(\mathbf{x})}{\epsilon} \\
&= a_1 u_{x_1}(\mathbf{x}) + \cdots + a_n u_{x_n}(\mathbf{x})
\end{aligned}$$

$$D_{\mathbf{a}}u(\mathbf{x}) = \nabla u(\mathbf{x}) \cdot \mathbf{a}.$$

Tangent to a Surface. The gradient, ∇f, is orthogonal to the surface $f(\mathbf{x}) = 0$. Consider a point $\boldsymbol{\xi}$ on the surface. Let the differential $d\mathbf{r} = dx_1\mathbf{e}_1 + \cdots dx_n\mathbf{e}_n$ lie in the tangent plane at $\boldsymbol{\xi}$. Then

$$df = \frac{\partial f}{\partial x_1}\, dx_1 + \cdots + \frac{\partial f}{\partial x_n}\, dx_n = 0$$

since $f(\mathbf{x}) = 0$ on the surface. Then

$$\begin{aligned}
\nabla f \cdot d\mathbf{r} &= \left(\frac{\partial f}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n \right) \cdot (dx_1\mathbf{e}_1 + \cdots + dx_n\mathbf{e}_n) \\
&= \frac{\partial f}{\partial x_1}\, dx_1 + \cdots + \frac{\partial f}{\partial x_n}\, dx_n \\
&= 0
\end{aligned}$$

Thus ∇f is orthogonal to the tangent plane and hence to the surface.

Example 7.2.1 Consider the paraboloid, $x^2 + y^2 - z = 0$. We want to find the tangent plane to the surface at the point $(1, 1, 2)$. The gradient is

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}.$$

At the point $(1, 1, 2)$ this is

$$\nabla f(1, 1, 2) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

We know a point on the tangent plane, $(1, 1, 2)$, and the normal, $\nabla f(1, 1, 2)$. The equation of the plane is

$$\nabla f(1, 1, 2) \cdot (x, y, z) = \nabla f(1, 1, 2) \cdot (1, 1, 2)$$
$$\boxed{2x + 2y - z = 2}$$

The gradient of the function $f(\mathbf{x}) = 0$, $\nabla f(\mathbf{x})$, is in the direction of the maximum directional derivative. The magnitude of the gradient, $|\nabla f(\mathbf{x})|$, is the value of the directional derivative in that direction. To derive this, note that

$$D_{\mathbf{a}} f = \nabla f \cdot \mathbf{a} = |\nabla f| \cos \theta,$$

where θ is the angle between ∇f and $\mathbf{a}$. $D_{\mathbf{a}} f$ is maximum when $\theta = 0$, i.e. when $\mathbf{a}$ is the same direction as ∇f. In this direction, $D_{\mathbf{a}} f = |\nabla f|$. To use the elevation example, ∇f points in the uphill direction and $|\nabla f|$ is the uphill slope.

Example 7.2.2 Suppose that the two surfaces $f(\mathbf{x}) = 0$ and $g(\mathbf{x}) = 0$ intersect at the point $\mathbf{x} = \boldsymbol{\xi}$. What is the angle between their tangent planes at that point? First we note that the angle between the tangent planes is by definition the angle between their normals. These normals are in the direction of $\nabla f(\boldsymbol{\xi})$ and $\nabla g(\boldsymbol{\xi})$. (We assume these are nonzero.) The angle, θ, between the tangent planes to the surfaces is

$$\boxed{\theta = \arccos\left(\frac{\nabla f(\boldsymbol{\xi}) \cdot \nabla g(\boldsymbol{\xi})}{|\nabla f(\boldsymbol{\xi})|\, |\nabla g(\boldsymbol{\xi})|}\right).}$$

Example 7.2.3 Let u be the distance from the origin:

$$u(\mathbf{x}) = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_i x_i}.$$

In three dimensions, this is

$$u(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

The gradient of u, $\nabla(\mathbf{x})$, is a unit vector in the direction of $\mathbf{x}$. The gradient is:

$$\nabla u(\mathbf{x}) = \left\langle \frac{x_1}{\sqrt{\mathbf{x} \cdot \mathbf{x}}}, \ldots \frac{x_n}{\sqrt{\mathbf{x} \cdot \mathbf{x}}} \right\rangle = \frac{x_i \mathbf{e}_i}{\sqrt{x_j x_j}}.$$

In three dimensions, we have

$$\nabla u(x, y, z) = \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle.$$

This is a unit vector because the sum of the squared components sums to unity.

$$\nabla u \cdot \nabla u = \frac{x_i \mathbf{e}_i}{\sqrt{x_j x_j}} \cdot \frac{x_k \mathbf{e}_k}{\sqrt{x_l x_l}} \frac{x_i x_i}{x_j x_j} = 1$$

Figure 7.2 shows a plot of the vector field of ∇u in two dimensions.

Example 7.2.4 Consider an ellipse. An implicit equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We can also express an ellipse as $u(x, y) + v(x, y) = c$ where u and v are the distance from the two foci. That is, an ellipse is the set of points such that the sum of the distances from the two foci is a constant. Let $\mathbf{n} = \nabla(u + v)$.

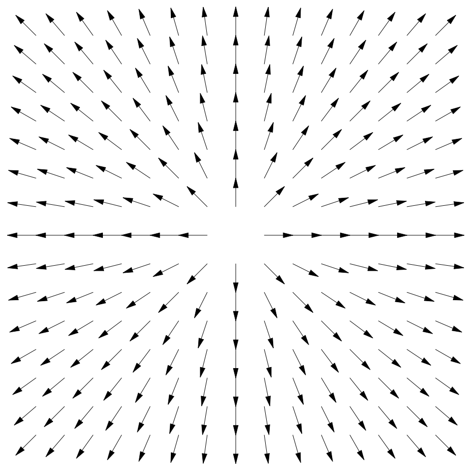


Figure 7.2: The gradient of the distance from the origin.

This is a vector which is orthogonal to the ellipse when evaluated on the surface. Let $\mathbf{t}$ be a unit tangent to the surface. Since $\mathbf{n}$ and $\mathbf{t}$ are orthogonal,

$$\begin{aligned} \mathbf{n} \cdot \mathbf{t} &= 0 \\ (\nabla u + \nabla v) \cdot \mathbf{t} &= 0 \\ \nabla u \cdot \mathbf{t} &= \nabla v \cdot (-\mathbf{t}). \end{aligned}$$

Since these are unit vectors, the angle between ∇u and $\mathbf{t}$ is equal to the angle between ∇v and $-\mathbf{t}$. In other words: If we draw rays from the foci to a point on the ellipse, the rays make equal angles with the ellipse. If the ellipse were a reflective surface, a wave starting at one focus would be reflected from the ellipse and travel to the other focus. See Figure 8.3. This result also holds for ellipsoids, $u(x, y, z) + v(x, y, z) = c$.

We see that an ellipsoidal dish could be used to collect spherical waves, (waves emanating from a point). If the dish is shaped so that the source of the waves is located at one foci and a collector is placed at the second, then any wave starting at the source and reflecting off the dish will travel to the collector. See Figure 7.4.

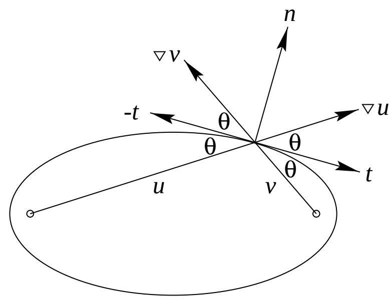


Figure 7.3: An ellipse and rays from the foci.

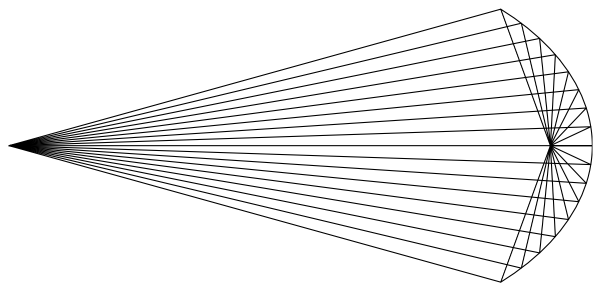


Figure 7.4: An elliptical dish.

7.3 Exercises

Vector Functions

Exercise 7.1
Consider the parametric curve

$$\mathbf{r} = \cos\left(\frac{t}{2}\right)\mathbf{i} + \sin\left(\frac{t}{2}\right)\mathbf{j}.$$

Calculate $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$. Plot the position and some velocity and acceleration vectors.

Exercise 7.2
Let $\mathbf{r}(t)$ be the position of an object moving with constant speed. Show that the acceleration of the object is orthogonal to the velocity of the object.

Vector Fields

Exercise 7.3
Consider the paraboloid $x^2+y^2-z=0$. What is the angle between the two tangent planes that touch the surface at $(1,1,2)$ and $(1,-1,2)$? What are the equations of the tangent planes at these points?

Exercise 7.4
Consider the paraboloid $x^2+y^2-z=0$. What is the point on the paraboloid that is closest to $(1,0,0)$?

7.4 Hints

Vector Functions

Hint 7.1
Plot the velocity and acceleration vectors at regular intervals along the path of motion.

Hint 7.2
If $\mathbf{r}(t)$ has constant speed, then $|\mathbf{r}'(t)| = c$. The condition that the acceleration is orthogonal to the velocity can be stated mathematically in terms of the dot product, $\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0$. Write the condition of constant speed in terms of a dot product and go from there.

Vector Fields

Hint 7.3
The angle between two planes is the angle between the vectors orthogonal to the planes. The angle between the two vectors is

$$\theta = \arccos\left(\frac{\langle 2, 2, -1\rangle \cdot \langle 2, -2, -1\rangle}{|\langle 2, 2, -1\rangle||\langle 2, -2, -1\rangle|}\right)$$

The equation of a line orthogonal to $\mathbf{a}$ and passing through the point $\mathbf{b}$ is $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{b}$.

Hint 7.4
Since the paraboloid is a differentiable surface, the normal to the surface at the closest point will be parallel to the vector from the closest point to $(1, 0, 0)$. We can express this using the gradient and the cross product. If (x, y, z) is the closest point on the paraboloid, then a vector orthogonal to the surface there is $\nabla f = \langle 2x, 2y, -1\rangle$. The vector from the surface to the point $(1, 0, 0)$ is $\langle 1 - x, -y, -z\rangle$. These two vectors are parallel if their cross product is zero.

7.5 Solutions

Vector Functions

Solution 7.1
The velocity is

$$\mathbf{r}' = -\frac{1}{2}\sin\left(\frac{t}{2}\right)\mathbf{i} + \frac{1}{2}\cos\left(\frac{t}{2}\right)\mathbf{j}.$$

The acceleration is

$$\mathbf{r}' = -\frac{1}{4}\cos\left(\frac{t}{2}\right)\mathbf{i} - \frac{1}{4}\sin\left(\frac{t}{2}\right)\mathbf{j}.$$

See Figure 7.5 for plots of position, velocity and acceleration.

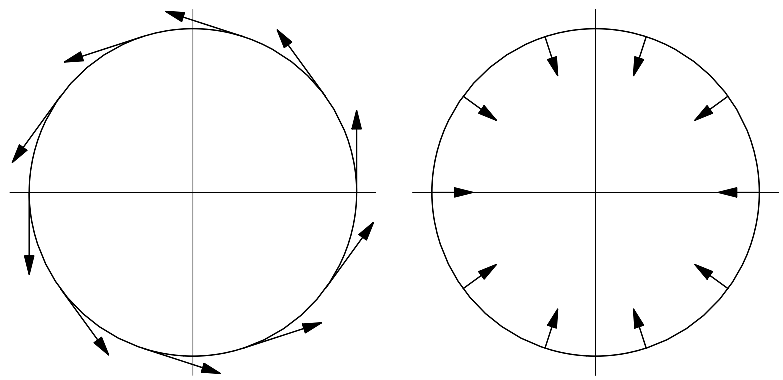


Figure 7.5: A Graph of Position and Velocity and of Position and Acceleration

Solution 7.2
If $\mathbf{r}(t)$ has constant speed, then $|\mathbf{r}'(t)| = c$. The condition that the acceleration is orthogonal to the velocity can be stated mathematically in terms of the dot product, $\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0$. Note that we can write the condition of constant speed in terms of a dot product,

$$\sqrt{\mathbf{r}'(t) \cdot \mathbf{r}'(t)} = c,$$

$$\mathbf{r}'(t) \cdot \mathbf{r}'(t) = c^2.$$

Differentiating this equation yields,

$$\mathbf{r}''(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$$

$$\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0.$$

This shows that the acceleration is orthogonal to the velocity.

Vector Fields

Solution 7.3
The gradient, which is orthogonal to the surface when evaluated there is $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$. $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ are orthogonal to the paraboloid, (and hence the tangent planes), at the points $(1, 1, 2)$ and $(1, -1, 2)$, respectively. The angle between the tangent planes is the angle between the vectors orthogonal to the planes. The angle between the two vectors is

$$\theta = \arccos\left(\frac{\langle 2, 2, -1\rangle \cdot \langle 2, -2, -1\rangle}{|\langle 2, 2, -1\rangle||\langle 2, -2, -1\rangle|}\right)$$

$$\boxed{\theta = \arccos\left(\frac{1}{9}\right) \approx 1.45946.}$$

Recall that the equation of a line orthogonal to $\mathbf{a}$ and passing through the point $\mathbf{b}$ is $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{b}$. The equations of the tangent planes are

$$\langle 2, \pm 2, -1 \rangle \cdot \langle x, y, z \rangle = \langle 2, \pm 2, -1 \rangle \cdot \langle 1, \pm 1, 2 \rangle,$$

$$\boxed{2x \pm 2y - z = 2.}$$

The paraboloid and the tangent planes are shown in Figure 7.6.

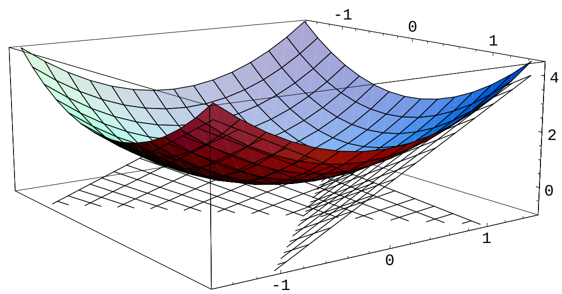


Figure 7.6: Paraboloid and Two Tangent Planes

Solution 7.4
Since the paraboloid is a differentiable surface, the normal to the surface at the closest point will be parallel to the vector from the closest point to $(1, 0, 0)$. We can express this using the gradient and the cross product. If (x, y, z) is the closest point on the paraboloid, then a vector orthogonal to the surface there is $\nabla f = \langle 2x, 2y, -1 \rangle$. The vector from the surface to the point $(1, 0, 0)$ is $\langle 1 - x, -y, -z \rangle$. These two vectors are parallel if their cross product is zero,

$$\langle 2x, 2y, -1 \rangle \times \langle 1 - x, -y, -z \rangle = \langle -y - 2yz, -1 + x + 2xz, -2y \rangle = \mathbf{0}.$$

This gives us the three equations,

$$\begin{aligned} -y - 2yz &= 0, \\ -1 + x + 2xz &= 0, \\ -2y &= 0. \end{aligned}$$

The third equation requires that $y = 0$. The first equation then becomes trivial and we are left with the second equation,

$$-1 + x + 2xz = 0.$$

Substituting $z = x^2 + y^2$ into this equation yields,

$$2x^3 + x - 1 = 0.$$

The only real valued solution of this polynomial is

$$x = \frac{6^{-2/3}\left(9+\sqrt{87}\right)^{2/3} - 6^{-1/3}}{\left(9+\sqrt{87}\right)^{1/3}} \approx 0.589755.$$

Thus the closest point to $(1, 0, 0)$ on the paraboloid is

$$\left(\frac{6^{-2/3}\left(9+\sqrt{87}\right)^{2/3} - 6^{-1/3}}{\left(9+\sqrt{87}\right)^{1/3}}, 0, \left(\frac{6^{-2/3}\left(9+\sqrt{87}\right)^{2/3} - 6^{-1/3}}{\left(9+\sqrt{87}\right)^{1/3}} \right)^2 \right) \approx (0.589755, 0, 0.34781).$$

The closest point is shown graphically in Figure 7.7.

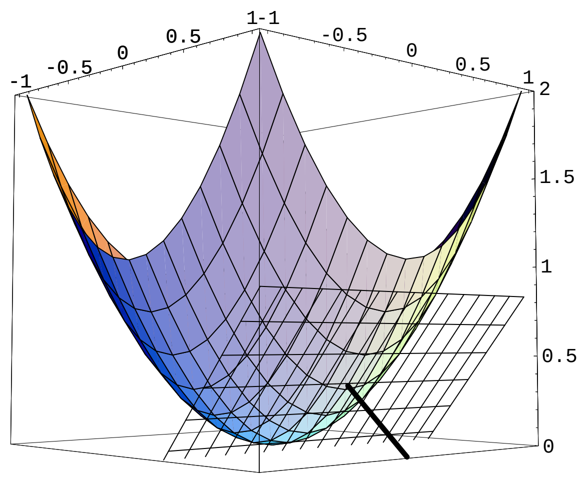


Figure 7.7: Paraboloid, Tangent Plane and Line Connecting $(1, 0, 0)$ to Closest Point

Part III

Functions of a Complex Variable

Chapter 8

Complex Numbers

For every complex problem, there is a solution that is simple, neat, and wrong.

- H. L. Mencken

8.1 Complex Numbers

When you started algebra, you learned that the quadratic equation: $x^2 + 2ax + b = 0$ has either two, one or no solutions. For example:

- $x^2 - 3x + 2 = 0$ has the two solutions $x = 1$ and $x = 2$.
- $x^2 - 2x + 1 = 0$ has the one solution $x = 1$.
- $x^2 + 1 = 0$ has no solutions.

This is a little unsatisfactory. We can formally solve the general quadratic equation.

$$x^2 + 2ax + b = 0$$
$$(x + a)^2 = a^2 - b$$
$$x = -a \pm \sqrt{a^2 - b}$$

However, the solutions are defined only when $a^2 \geq b$. The square root function, $\sqrt{x}$, is a bijection from $\mathbb{R}^{0+}$ to $\mathbb{R}^{0+}$. We cannot solve $x^2 = -1$ because $\sqrt{-1}$ is not defined. To overcome this apparent shortcoming of the real number system, we create a new symbolic constant, $i \equiv \sqrt{-1}$. Now we can express the solutions of $x^2 = -1$ as $x = i$ and $x = -i$. These satisfy the equation since $i^2 = \left(\sqrt{-1}\right)^2 = -1$ and $(-i)^2 = \left(-\sqrt{-1}\right)^2 = -1$. Note that we can express the square root of any negative real number in terms of i: $\sqrt{-r} = \sqrt{-1}\sqrt{r} = i\sqrt{r}$. We call any number of the form ib, $b \in \mathbb{R}$, a *pure imaginary number.* [1] We call numbers of the form $a + ib$, where $a, b \in \mathbb{R}$, *complex numbers* [2]

The quadratic with real coefficients, $x^2 + 2ax + b = 0$, has solutions $x = -a \pm \sqrt{a^2 - b}$. The solutions are real-valued only if $a^2 - b \geq 0$. If not, then we can define solutions as complex numbers. If the discriminant is negative, then we write $x = -a \pm i\sqrt{b - a^2}$. Thus every quadratic polynomial has exactly two solutions, counting multiplicities. The fundamental theorem of algebra states that an n^{th} degree polynomial with complex coefficients has n, not necessarily distinct, complex roots. We will prove this result later using the theory of functions of a complex variable.

Consider the complex number $z = x + iy$, $(x, y \in \mathbb{R})$. The *real part* of z is $\Re(z) = x$; the *imaginary part* of z is $\Im(z) = y$. Two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, are equal if and only if $x_1 = x_2$ and $y_1 = y_2$. The *complex conjugate* [3] of $z = x + iy$ is $\overline{z} = x - iy$. The notation $z^* = x - iy$ is also used.

[1] "Imaginary" is an unfortunate term. Real numbers are artificial; constructs of the mind. Real numbers are no more real than imaginary numbers.

[2] Here complex means "composed of two or more parts", not "hard to separate, analyze, or solve". Those who disagree have a complex number complex.

[3] Conjugate: having features in common but opposite or inverse in some particular.

The set of complex numbers, $\mathbb{C}$, form a field. That essentially means that we can do arithmetic with complex numbers. We treat i as a symbolic constant with the property that $i^2 = -1$. The field of complex numbers satisfy the following properties: (Let $z, z_1, z_2, z_3 \in \mathbb{C}$.)

1. Closure under addition and multiplication.

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \in \mathbb{C} \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \in \mathbb{C} \end{aligned}$$

2. Commutativity of addition and multiplication. $z_1 + z_2 = z_2 + z_1$. $z_1 z_2 = z_2 z_1$.

3. Associativity of addition and multiplication. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$. $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

4. Distributive law. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

5. Identity with respect to addition and multiplication. $z + 0 = z$. $z(1) = z$.

6. Inverse with respect to addition. $z + (-z) = (x + iy) + (-x - iy) = 0$.

7. Inverse with respect to multiplication for nonzero numbers. $zz^{-1} = 1$, where

$$z^{-1} = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Complex Conjugate. Using the field properties of complex numbers, we can derive the following properties of the complex conjugate, $\overline{z} = x - iy$.

1. $\overline{(\overline{z})} = z$,

2. $\overline{z + \zeta} = \overline{z} + \overline{\zeta}$,

3. $\overline{z\zeta} = \overline{z}\,\overline{\zeta}$,

4. $\overline{\left(\frac{z}{\zeta}\right)} = \frac{\overline{z}}{\overline{\zeta}}$.

8.2 The Complex Plane

We can denote a complex number $z = x + iy$ as an ordered pair of real numbers (x, y). Thus we can represent a complex number as a point in $\mathbb{R}^2$ where the x component is the real part and the y component is the imaginary part of z. This is called the *complex plane* or the *Argand diagram*. (See Figure 8.1.)

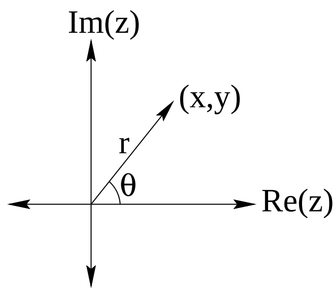


Figure 8.1: The Complex Plane

There are two ways of describing a point in the complex plane: an ordered pair of coordinates (x, y) that give the horizontal and vertical offset from the origin or the distance r from the origin and the angle θ from the positive horizontal axis. The angle θ is not unique. It is only determined up to an additive integer multiple of 2π.

Modulus. The *magnitude* or *modulus* of a complex number is the distance of the point from the origin. It is defined as $|z| = |x + iy| = \sqrt{x^2 + y^2}$. Note that $z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$. The modulus has the following properties.

1. $|z_1 z_2| = |z_1|\,|z_2|$
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ for $z_2 \neq 0$.
3. $|z_1 + z_2| \leq |z_1| + |z_2|$
4. $|z_1 + z_2| \geq ||z_1| - |z_2||$

We could prove the first two properties by expanding in $x + iy$ form, but it would be fairly messy. The proofs will become simple after polar form has been introduced. The second two properties follow from the triangle inequalities in geometry. This will become apparent after the relationship between complex numbers and vectors is introduced. One can show that

$$|z_1 z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|$$

and

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

with proof by induction.

Argument. The *argument* of a complex number is the angle that the vector with tail at the origin and head at $z = x + iy$ makes with the positive x-axis. The argument is denoted $\arg(z)$. Note that the argument is defined for all nonzero numbers and is only determined up to an additive integer multiple of 2π. That is, the argument of a complex number is the set of values: $\{\theta + 2\pi n \,|\, n \in \mathbb{Z}\}$. The *principal argument* of a complex number is that angle in the set $\arg(z)$ which lies in the range $(-\pi, \pi]$. The principal argument is denoted $\operatorname{Arg}(z)$. We prove the following identities in Exercise 8.7.

$$\arg(z\zeta) = \arg(z) + \arg(\zeta)$$
$$\operatorname{Arg}(z\zeta) \neq \operatorname{Arg}(z) + \operatorname{Arg}(\zeta)$$
$$\arg(z^2) = \arg(z) + \arg(z) \neq 2\arg(z)$$

Example 8.2.1 Consider the equation $|z - 1 - i| = 2$. The set of points satisfying this equation is a circle of radius 2 and center at $1 + i$ in the complex plane. You can see this by noting that $|z - 1 - i|$ is the distance from the point $(1, 1)$. (See Figure 8.2.)

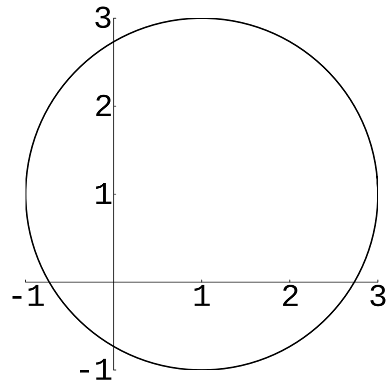


Figure 8.2: Solution of $|z - 1 - i| = 2$

Another way to derive this is to substitute $z = x + iy$ into the equation.

$$|x + iy - 1 - i| = 2$$
$$\sqrt{(x-1)^2 + (y-1)^2} = 2$$
$$(x-1)^2 + (y-1)^2 = 4$$

This is the analytic geometry equation for a circle of radius 2 centered about $(1, 1)$.

Example 8.2.2 Consider the curve described by

$$|z| + |z - 2| = 4.$$

Note that $|z|$ is the distance from the origin in the complex plane and $|z-2|$ is the distance from $z=2$. The equation is

$$(\text{distance from } (0,0)) + (\text{distance from } (2,0)) = 4.$$

From geometry, we know that this is an ellipse with foci at $(0,0)$ and $(2,0)$, major axis 2, and minor axis $\sqrt{3}$. (See Figure 8.3.)

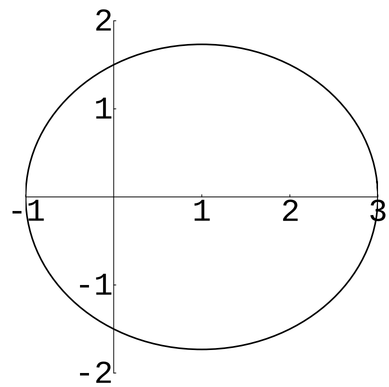


Figure 8.3: Solution of $|z| + |z-2| = 4$

We can use the substitution $z = x + iy$ to get the equation an algebraic form.

$$
\begin{aligned}
|z| + |z - 2| &= 4 \\
|x + iy| + |x + iy - 2| &= 4 \\
\sqrt{x^2 + y^2} + \sqrt{(x-2)^2 + y^2} &= 4 \\
x^2 + y^2 &= 16 - 8\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2 \\
x - 5 &= -2\sqrt{(x-2)^2 + y^2} \\
x^2 - 10x + 25 &= 4x^2 - 16x + 16 + 4y^2 \\
\frac{1}{4}(x-1)^2 + \frac{1}{3}y^2 &= 1
\end{aligned}
$$

Thus we have the standard form for an equation describing an ellipse.

8.3 Polar Form

Polar Form. A complex number written as $z = x + iy$ is said to be in *Cartesian form*, or *$a + ib$ form*. We can convert this representation to *polar form*, $z = r(\cos\theta + i\sin\theta)$, using trigonometry. Here $r = |z|$ is the modulus and $\theta = \arctan(x, y)$ is the argument of z. The argument is the angle between the x axis and the vector with its head at (x, y). (See Figure 8.4.) Note that θ is not unique. If $z = r(\cos\theta + i\sin\theta)$ then $z = r(\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi))$ for any $n \in \mathbb{Z}$.

The Arctangent. Note that $\arctan(x, y)$ is not the same thing as the old arctangent that you learned about in trigonometry, $\arctan\left(\frac{y}{x}\right)$. For example,

$$
\arctan(1, 1) = \frac{\pi}{4} + 2n\pi \quad \text{and} \quad \arctan(-1, -1) = \frac{-3\pi}{4} + 2n\pi,
$$

whereas

$$
\arctan\left(\frac{-1}{-1}\right) = \arctan\left(\frac{1}{1}\right) = \arctan(1).
$$

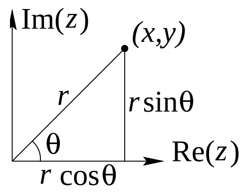


Figure 8.4: Polar Form

Euler's Formula. *Euler's formula*, $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$, allows us to write the polar form more compactly. Expressing the polar form in terms of the exponential function of imaginary argument makes arithmetic with complex numbers much more convenient. (See Exercise 8.14 for a proof of Euler's formula.)

$$z = r(\cos\theta + i\sin\theta) = r\,\mathrm{e}^{i\theta}$$

Arithmetic With Complex Numbers. Note that it is convenient to add complex numbers in Cartesian form.

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

However, it is difficult to multiply or divide them in Cartesian form.

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

On the other hand, it is difficult to add complex numbers in polar form.

$$
\begin{aligned}
r_1 \mathrm{e}^{i\theta_1} + r_2 \mathrm{e}^{i\theta_2} &= r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2) \\
&= r_1\cos\theta_1 + r_2\cos\theta_2 + i(r_1\sin\theta_1 + r_2\sin\theta_2) \\
&= \sqrt{(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2} \\
&\qquad \times \mathrm{e}^{i\arctan(r_1\cos\theta_1 + r_2\cos\theta_2, r_1\sin\theta_1 + r_2\sin\theta_2)} \\
&= \sqrt{r_1^2 + r_2^2 + 2\cos(\theta_1 - \theta_2)}\, \mathrm{e}^{i\arctan(r_1\cos\theta_1 + r_2\cos\theta_2, r_1\sin\theta_1 + r_2\sin\theta_2)}
\end{aligned}
$$

However, it is convenient to multiply and divide them in polar form.

$$r_1 \mathrm{e}^{i\theta_1} r_2 \mathrm{e}^{i\theta_2} = r_1 r_2 \mathrm{e}^{i(\theta_1+\theta_2)}$$

$$\frac{r_1 \mathrm{e}^{i\theta_1}}{r_2 \mathrm{e}^{i\theta_2}} = \frac{r_1}{r_2} \mathrm{e}^{i(\theta_1-\theta_2)}$$

Keeping this in mind will make working with complex numbers a shade or two less grungy.

Result 8.3.1 To change between Cartesian and polar form, use the identities

$$
\begin{aligned}
r\,\mathrm{e}^{i\theta} &= r\cos\theta + ir\sin\theta, \\
x + iy &= \sqrt{x^2 + y^2}\, \mathrm{e}^{i\arctan(x,y)}.
\end{aligned}
$$

Cartesian form is convenient for addition. Polar form is convenient for multiplication and division.

Example 8.3.1 The polar form of $5 + 7i$ is

$$5 + 7i = \sqrt{74}\, \mathrm{e}^{i\arctan(5,7)}.$$

$2\,e^{i\pi/6}$ in Cartesian form is

$$\begin{aligned} 2\,e^{i\pi/6} &= 2\cos\left(\frac{\pi}{6}\right) + 2i\sin\left(\frac{\pi}{6}\right) \\ &= \sqrt{3} + i. \end{aligned}$$

Example 8.3.2 We will show that

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}.$$

Recall that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

$$\begin{aligned} \cos^4\theta &= \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^4 \\ &= \frac{1}{16}(e^{4i\theta} + 4\,e^{2i\theta} + 6 + 4\,e^{-2i\theta} + e^{-4i\theta}) \\ &= \frac{1}{8}\left(\frac{e^{4i\theta} + e^{-4i\theta}}{2}\right) + \frac{1}{2}\left(\frac{e^{2i\theta} + e^{-2i\theta}}{2}\right) + \frac{3}{8} \\ &= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8} \end{aligned}$$

By the definition of exponentiation, we have $e^{in\theta} = (e^{i\theta})^n$ We apply Euler's formula to obtain a result which is useful in deriving trigonometric identities.

$$\cos(n\theta) + i\sin(n\theta) = (\cos(\theta) + i\sin(\theta))^n$$

Result 8.3.2 DeMoivre's Theorem. [a]

$$\cos(n\theta) + i\sin(n\theta) = (\cos(\theta) + i\sin(\theta))^n$$

[a] It's amazing what passes for a theorem these days. I would think that this would be a corollary at most.

Example 8.3.3 We will express $\cos 5\theta$ in terms of $\cos\theta$ and $\sin 5\theta$ in terms of $\sin\theta$.

We start with DeMoivre's theorem.

$$e^{i5\theta} = \left(e^{i\theta}\right)^5$$

$$\begin{aligned}
\cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5 \\
&= \binom{5}{0}\cos^5\theta + i\binom{5}{1}\cos^4\theta\sin\theta - \binom{5}{2}\cos^3\theta\sin^2\theta - i\binom{5}{3}\cos^2\theta\sin^3\theta \\
&\quad + \binom{5}{4}\cos\theta\sin^4\theta + i\binom{5}{5}\sin^5\theta \\
&= \left(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\right) + i\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right)
\end{aligned}$$

Equating the real and imaginary parts we obtain

$$\begin{aligned}
\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\
\sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta
\end{aligned}$$

Now we use the Pythagorean identity, $\cos^2\theta + \sin^2\theta = 1$.

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1 - \cos^2\theta) + 5\cos\theta(1 - \cos^2\theta)^2$$

$$\boxed{\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}$$

$$\sin 5\theta = 5(1 - \sin^2\theta)^2\sin\theta - 10(1 - \sin^2\theta)\sin^3\theta + \sin^5\theta$$

$$\boxed{\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta}$$

8.4 Arithmetic and Vectors

Addition. We can represent the complex number $z = x + iy = r\,e^{i\theta}$ as a vector in Cartesian space with tail at the origin and head at (x, y), or equivalently, the vector of length r and angle θ. With the vector representation, we can add complex numbers by connecting the tail of one vector to the head of the other. The vector $z + \zeta$ is the diagonal of the parallelogram defined by z and ζ. (See Figure 8.5.)

Negation. The negative of $z = x + iy$ is $-z = -x - iy$. In polar form we have $z = r\,e^{i\theta}$ and $-z = r\,e^{i(\theta+\pi)}$, (more generally, $z = r\,e^{i(\theta+(2n+1)\pi)}$, $n \in \mathbb{Z}$. In terms of vectors, $-z$ has the same magnitude but opposite direction as z. (See Figure 8.5.)

Multiplication. The product of $z = r\,e^{i\theta}$ and $\zeta = \rho\,e^{i\phi}$ is $z\zeta = r\rho\,e^{i(\theta+\phi)}$. The length of the vector $z\zeta$ is the product of the lengths of z and ζ. The angle of $z\zeta$ is the sum of the angles of z and ζ. (See Figure 8.5.)

Note that $\arg(z\zeta) = \arg(z) + \arg(\zeta)$. Each of these arguments has an infinite number of values. If we write out the multi-valuedness explicitly, we have

$$\{\theta + \phi + 2\pi n : n \in \mathbb{Z}\} = \{\theta + 2\pi n : n \in \mathbb{Z}\} + \{\phi + 2\pi n : n \in \mathbb{Z}\}$$

The same is not true of the principal argument. In general, $\operatorname{Arg}(z\zeta) \neq \operatorname{Arg}(z) + \operatorname{Arg}(\zeta)$. Consider the case $z = \zeta = e^{i3\pi/4}$. Then $\operatorname{Arg}(z) = \operatorname{Arg}(\zeta) = 3\pi/4$, however, $\operatorname{Arg}(z\zeta) = -\pi/2$.

Multiplicative Inverse. Assume that z is nonzero. The multiplicative inverse of $z = r\,e^{i\theta}$ is $\frac{1}{z} = \frac{1}{r}\,e^{-i\theta}$. The length of $\frac{1}{z}$ is the multiplicative inverse of the length of z. The angle of $\frac{1}{z}$ is the negative of the angle of z. (See Figure 8.6.)

Division. Assume that ζ is nonzero. The quotient of $z = r\,e^{i\theta}$ and $\zeta = \rho\,e^{i\phi}$ is $\frac{z}{\zeta} = \frac{r}{\rho}\,e^{i(\theta-\phi)}$. The length of the vector $\frac{z}{\zeta}$ is the quotient of the lengths of z and ζ. The angle of $\frac{z}{\zeta}$ is the difference of the angles of z and ζ. (See Figure 8.6.)

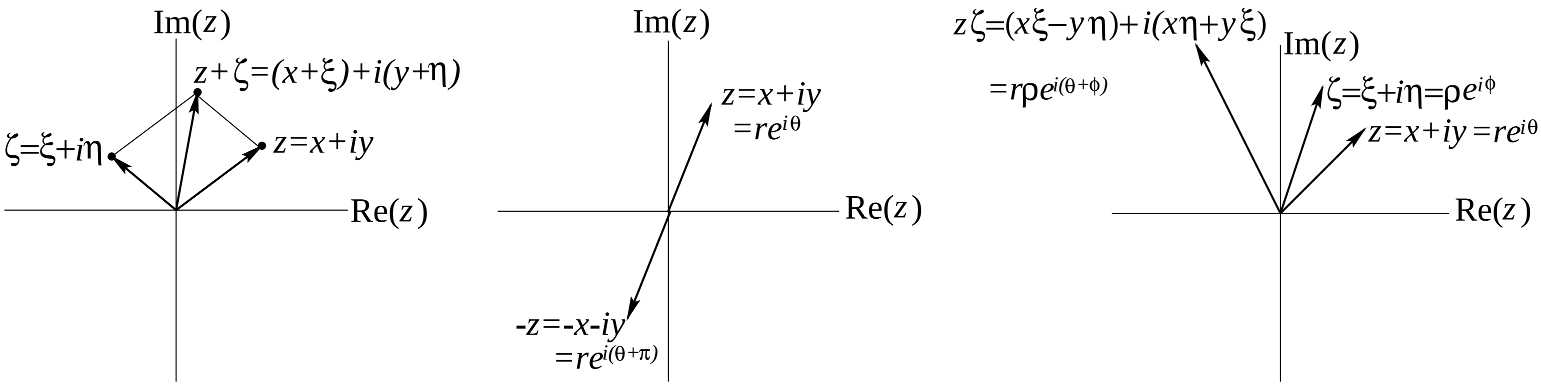


Figure 8.5: Addition, Negation and Multiplication

Complex Conjugate. The complex conjugate of $z = x + iy = r\,\mathrm{e}^{i\theta}$ is $\overline{z} = x - iy = r\,\mathrm{e}^{-i\theta}$. $\overline{z}$ is the mirror image of z, reflected across the x axis. In other words, $\overline{z}$ has the same magnitude as z and the angle of $\overline{z}$ is the negative of the angle of z. (See Figure 8.6.)

8.5 Integer Exponents

Consider the product $(a + b)^n$, $n \in \mathbb{Z}$. If we know $\arctan(a, b)$ then it will be most convenient to expand the product working in polar form. If not, we can write n in base 2 to efficiently do the multiplications.

Example 8.5.1 Suppose that we want to write $(\sqrt{3} + i)^{20}$ in Cartesian form. [4] We can do the multiplication directly. Note that 20 is 10100 in base 2. That is, $20 = 2^4 + 2^2$. We first calculate the powers of the form $(\sqrt{3}+i)^{2^n}$

[4] No, I have no idea why we would want to do that. Just humor me. If you pretend that you're interested, I'll do the same. Believe me, expressing your real feelings here isn't going to do anyone any good.

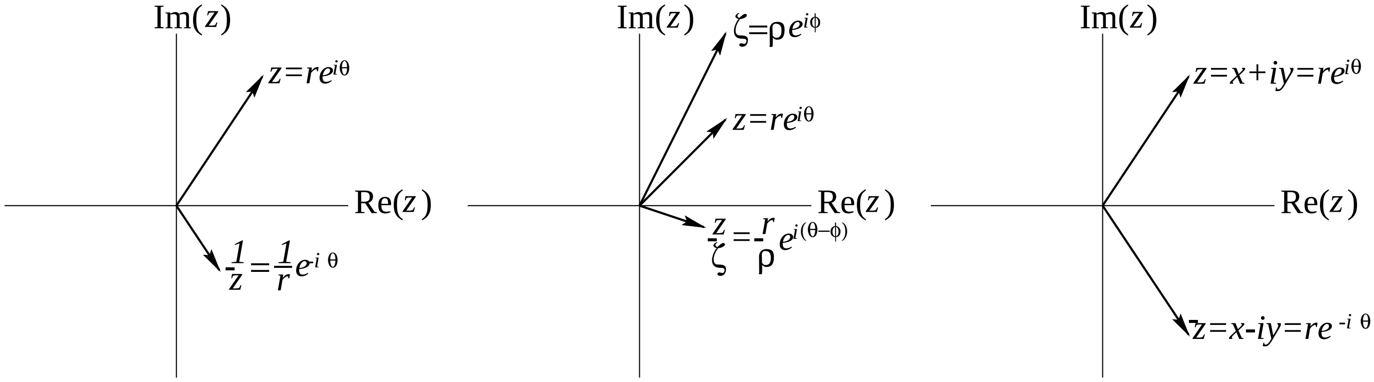


Figure 8.6: Multiplicative Inverse, Division and Complex Conjugate

by successive squaring.

$$
\begin{aligned}
(\sqrt{3}+i)^2 &= 2 + i2\sqrt{3} \\
(\sqrt{3}+i)^4 &= -8 + i8\sqrt{3} \\
(\sqrt{3}+i)^8 &= -128 - i128\sqrt{3} \\
(\sqrt{3}+i)^{16} &= -32768 + i32768\sqrt{3}
\end{aligned}
$$

Next we multiply $(\sqrt{3}+i)^4$ and $(\sqrt{3}+i)^{16}$ to obtain the answer.

$$(\sqrt{3}+i)^{20} = (-32768 + i32768\sqrt{3})(-8 + i8\sqrt{3}) = -524288 - i524288\sqrt{3}$$

Since we know that $\arctan(\sqrt{3}, 1) = \pi/6$, it is easiest to do this problem by first changing to modulus-argument form.

$$\begin{aligned}
\left(\sqrt{3}+i\right)^{20} &= \left(\sqrt{\left(\sqrt{3}\right)^2 + 1^2}\,\mathrm{e}^{i\arctan(\sqrt{3},1)}\right)^{20} \\
&= \left(2\,\mathrm{e}^{i\pi/6}\right)^{20} \\
&= 2^{20}\,\mathrm{e}^{i4\pi/3} \\
&= 1048576\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \\
&= -524288 - i524288\sqrt{3}
\end{aligned}$$

Example 8.5.2 Consider $(5+7i)^{11}$. We will do the exponentiation in polar form and write the result in Cartesian form.

$$\begin{aligned}
(5+7i)^{11} &= \left(\sqrt{74}\,\mathrm{e}^{i\arctan(5,7)}\right)^{11} \\
&= 74^5\sqrt{74}(\cos(11\arctan(5,7)) + i\sin(11\arctan(5,7))) \\
&= 2219006624\sqrt{74}\cos(11\arctan(5,7)) + i2219006624\sqrt{74}\sin(11\arctan(5,7))
\end{aligned}$$

The result is correct, but not very satisfying. This expression could be simplified. You could evaluate the trigonometric functions with some fairly messy trigonometric identities. This would take much more work than directly multiplying $(5+7i)^{11}$.

8.6 Rational Exponents

In this section we consider complex numbers with rational exponents, $z^{p/q}$, where p/q is a rational number. First we consider unity raised to the $1/n$ power. We define $1^{1/n}$ as the set of numbers $\{z\}$ such that $z^n = 1$.

$$1^{1/n} = \{z \mid z^n = 1\}$$

We can find these values by writing z in modulus-argument form.

$$
\begin{gathered}
z^n = 1 \\
r^n \mathrm{e}^{in\theta} = 1 \\
r^n = 1 \qquad n\theta = 0 \quad \bmod 2\pi \\
r = 1 \qquad \theta = 2\pi k \text{ for } k \in \mathbb{Z}
\end{gathered}
$$

There are only n distinct solutions as a result of the 2π periodicity of $\mathrm{e}^{i\theta}$. Thus

$$1^{1/n} = \{ \mathrm{e}^{i2\pi k/n} \mid k = 0, \dots, n-1 \}.$$

These values are equally spaced points on the unit circle in the complex plane.

Example 8.6.1 $1^{1/6}$ has the 6 values,

$$\left\{ \mathrm{e}^{i0}, \mathrm{e}^{i\pi/3}, \mathrm{e}^{i2\pi/3}, \mathrm{e}^{i\pi}, \mathrm{e}^{i4\pi/3}, \mathrm{e}^{i5\pi/3} \right\}.$$

In Cartesian form this is

$$\left\{ 1, \frac{1+i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}, -1, \frac{-1-i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2} \right\}.$$

The sixth roots of unity are plotted in Figure 8.7.

The n^{th} roots of the complex number $c = \alpha \mathrm{e}^{i\beta}$ are the set of numbers $z = r \mathrm{e}^{i\theta}$ such that

$$
\begin{gathered}
z^n = c = \alpha \mathrm{e}^{i\beta} \\
r^n \mathrm{e}^{in\theta} = \alpha \mathrm{e}^{i\beta} \\
r = \sqrt[n]{\alpha} \qquad n\theta = \beta \quad \bmod 2\pi \\
r = \sqrt[n]{\alpha} \qquad \theta = (\beta + 2\pi k)/n \text{ for } k = 0, \dots, n-1.
\end{gathered}
$$

Thus

$$c^{1/n} = \{ \sqrt[n]{\alpha} \, \mathrm{e}^{i(\beta+2\pi k)/n} \mid k = 0, \dots, n-1 \} = \{ \sqrt[n]{|c|} \, \mathrm{e}^{i(\operatorname{Arg}(c)+2\pi k)/n} \mid k = 0, \dots, n-1 \}$$

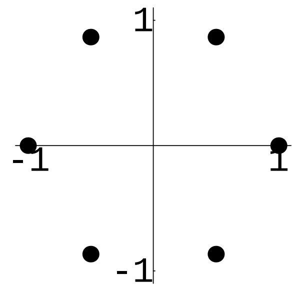


Figure 8.7: The Sixth Roots of Unity.

Principal Roots. The *principal n^{th} root* is denoted

$$\sqrt[n]{z} \equiv \sqrt[n]{z}\,\mathrm{e}^{i \operatorname{Arg}(z)/n}.$$

Thus the principal root has the property

$$-\pi/n < \operatorname{Arg}\left(\sqrt[n]{z}\right) \leq \pi/n.$$

This is consistent with the notation you learned back in algebra where $\sqrt[n]{x}$ denoted the positive n^{th} root of a positive real number. We adopt the convention that $z^{1/n}$ denotes the n^{th} roots of z, which is a set of n numbers and $\sqrt[n]{z}$ is the principal n^{th} root of z, which is a single number. With the principal root we can write,

$$\begin{aligned} z^{1/n} &= \sqrt[n]{r}\,\mathrm{e}^{i(\operatorname{Arg}(z)+2\pi k)/n} \mid k = 0, \dots, n-1\} \\ &= \sqrt[n]{z}\,\mathrm{e}^{i2\pi k/n} \mid k = 0, \dots, n-1\} \end{aligned}$$

$$z^{1/n} = \sqrt[n]{z} 1^{1/n}.$$

That is, the n^{th} roots of z are the principal n^{th} root of z times the n^{th} roots of unity.

Rational Exponents. We interpret $z^{p/q}$ to mean $z^{(p/q)}$. That is, we first simplify the exponent, i.e. reduce the fraction, before carrying out the exponentiation. Therefore $z^{2/4} = z^{1/2}$ and $z^{10/5} = z^2$. If p/q is a reduced fraction, (p and q are relatively prime, in other words, they have no common factors), then

$$z^{p/q} \equiv (z^p)^{1/q}.$$

Thus $z^{p/q}$ is a set of q values. Note that for an un-reduced fraction r/s,

$$\left(z^r\right)^{1/s} \neq \left(z^{1/s}\right)^r.$$

The former expression is a set of s values while the latter is a set of no more that s values. For instance, $(1^2)^{1/2} = 1^{1/2} = \pm 1$ and $(1^{1/2})^2 = (\pm 1)^2 = 1$.

Example 8.6.2 Consider $2^{1/5}$, $(1+i)^{1/3}$ and $(2+i)^{5/6}$.

$$2^{1/5} = \sqrt[5]{2}\, \mathrm{e}^{i2\pi k/5}, \quad \text{for } k = 0, 1, 2, 3, 4$$

$$\begin{aligned}(1+i)^{1/3} &= \left(\sqrt{2}\, \mathrm{e}^{i\pi/4}\right)^{1/3} \\ &= \sqrt[6]{2}\, \mathrm{e}^{i\pi/12}\, \mathrm{e}^{i2\pi k/3}, \quad \text{for } k = 0, 1, 2\end{aligned}$$

$$\begin{aligned}(2+i)^{5/6} &= \left(\sqrt{5}\, \mathrm{e}^{i \operatorname{Arctan}(2,1)}\right)^{5/6} \\ &= \left(\sqrt{5^5}\, \mathrm{e}^{i5 \operatorname{Arctan}(2,1)}\right)^{1/6} \\ &= \sqrt[12]{5^5}\, \mathrm{e}^{i\frac{5}{6} \operatorname{Arctan}(2,1)}\, \mathrm{e}^{i\pi k/3}, \quad \text{for } k = 0, 1, 2, 3, 4, 5\end{aligned}$$

Example 8.6.3 The roots of the polynomial $z^5 + 4$ are

$$\begin{aligned}(-4)^{1/5} &= \left(4\, \mathrm{e}^{i\pi}\right)^{1/5} \\ &= \sqrt[5]{4}\, \mathrm{e}^{i\pi(1+2k)/5}, \quad \text{for } k = 0, 1, 2, 3, 4.\end{aligned}$$

8.7 Exercises

Complex Numbers

Exercise 8.1
Verify that:

1. $\frac{1+2i}{3-4i}+\frac{2-i}{5i}=-\frac{2}{5}$

2. $(1-i)^4=-4$

Exercise 8.2
Write the following complex numbers in the form $a+ib$.

1. $\left(1+i\sqrt{3}\right)^{-10}$

2. $(11+4i)^2$

Exercise 8.3
Write the following complex numbers in the form $a+ib$

1. $\left(\frac{2+i}{i6-(1-i2)}\right)^2$

2. $(1-i)^7$

Exercise 8.4
If $z=x+iy$, write the following in the form $u(x,y)+iv(x,y)$.

1. $\overline{\left(\frac{\overline{z}}{z}\right)}$

2. $\frac{z + 2i}{2 - i\overline{z}}$

Exercise 8.5
Quaternions are sometimes used as a generalization of complex numbers. A quaternion u may be defined as

$$u = u_0 + iu_1 + ju_2 + ku_3$$

where u_0, u_1, u_2 and u_3 are real numbers and i, j and k are objects which satisfy

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad ji = -k$$

and the usual associative and distributive laws. Show that for any quaternions u, w there exists a quaternion v such that

$$uv = w$$

except for the case $u_0 = u_1 = u_2 = u_3$.

Exercise 8.6
Let $\alpha \neq 0$, $\beta \neq 0$ be two complex numbers. Show that $\alpha = t\beta$ for some real number t (i.e. the vectors defined by α and β are parallel) if and only if $\Im(\alpha\overline{\beta}) = 0$.

The Complex Plane

Exercise 8.7
Prove the following identities.

1. $\arg(z\zeta) = \arg(z) + \arg(\zeta)$

2. $\text{Arg}\,(z\zeta) \neq \text{Arg}\,(z) + \text{Arg}\,(\zeta)$

3. $\arg(z^2) = \arg(z) + \arg(z) \neq 2\arg(z)$

Exercise 8.8
Show, both by geometric and algebraic arguments, that for complex numbers z_1 and z_2 the inequalities

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

hold.

Exercise 8.9
Find all the values of

1. $(-1)^{-3/4}$

2. $8^{1/6}$

and show them graphically.

Exercise 8.10
Find all values of

1. $(-1)^{-1/4}$

2. $16^{1/8}$

and show them graphically.

Exercise 8.11
Sketch the regions or curves described by

1. $1 < |z - 2i| < 2$

2. $|\Re(z)| + 5|\Im(z)| = 1$

Exercise 8.12
Sketch the regions or curves described by

1. $|z - 1 + i| \leq 1$
2. $|z - i| = |z + i|$
3. $\Re(z) - \Im(z) = 5$
4. $|z - i| + |z + i| = 1$

Exercise 8.13
Solve the equation

$$|\,\mathrm{e}^{i\theta} - 1| = 2$$

for θ $(0 \leq \theta \leq \pi)$ and verify the solution geometrically.

Polar Form

Exercise 8.14
Prove Euler's formula, $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$. Consider the Taylor series of e^z to be the definition of the exponential function.

Exercise 8.15
Use de Moivre's formula to derive the trigonometric identity

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta).$$

Exercise 8.16
Establish the formula

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}, \qquad (z \neq 1),$$

for the sum of a finite geometric series; then derive the formulas

1. $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \dfrac{1}{2} + \dfrac{\sin((n + 1/2))}{2\sin(\theta/2)}$

2. $\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \dfrac{1}{2}\cot\dfrac{\theta}{2} - \dfrac{\cos((n + 1/2))}{2\sin(\theta/2)}$

where $0 < \theta < 2\pi$.

Arithmetic and Vectors

Exercise 8.17
Prove $|z_1 z_2| = |z_1||z_2|$ and $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ using polar form.

Exercise 8.18
Prove that

$$|z + \zeta|^2 + |z - \zeta|^2 = 2\left(|z|^2 + |\zeta|^2\right).$$

Interpret this geometrically.

Integer Exponents

Exercise 8.19
Write $(1+i)^{10}$ in Cartesian form with the following two methods:

1. Just do the multiplication. If it takes you more than four multiplications, you suck.

2. Do the multiplication in polar form.

Rational Exponents

Exercise 8.20
Show that each of the numbers $z = -a + (a^2 - b)^{1/2}$ satisfies the equation $z^2 + 2az + b = 0$.

8.8 Hints

Complex Numbers

Hint 8.1

Hint 8.2

Hint 8.3

Hint 8.4

Hint 8.5

Hint 8.6

The Complex Plane

Hint 8.7
Write the multivaluedness explicitly.

Hint 8.8
Consider a triangle with vertices at 0, z_1 and $z_1 + z_2$.

Hint 8.9

Hint 8.10

Hint 8.11

Hint 8.12

Hint 8.13

Polar Form

Hint 8.14
Find the Taylor series of $\mathrm{e}^{i\theta}$, $\cos\theta$ and $\sin\theta$. Note that $i^{2n} = (-1)^n$.

Hint 8.15

Hint 8.16

Arithmetic and Vectors

Hint 8.17
$|\,\mathrm{e}^{i\theta}| = 1$.

Hint 8.18
Consider the parallelogram defined by z and ζ.

Integer Exponents

Hint 8.19
For the first part,

$$(1+i)^{10} = \left(\left((1+i)^2\right)^2\right)^2 (1+i)^2.$$

Rational Exponents

Hint 8.20
Substitite the numbers into the equation.

8.9 Solutions

Complex Numbers

Solution 8.1

1.

$$
\begin{aligned}
\frac{1+2i}{3-4i} + \frac{2-i}{5i} &= \frac{1+2i}{3-4i}\frac{3+4i}{3+4i} + \frac{2-i}{5i}\frac{-i}{-i} \\
&= \frac{-5+10i}{25} + \frac{-1-2i}{5} \\
&= -\frac{2}{5}
\end{aligned}
$$

2.

$$
(1-i)^4 = (-2i)^2 = -4
$$

Solution 8.2

1. First we do the multiplication in Cartesian form.

$$
\begin{aligned}
\left(1+i\sqrt{3}\right)^{-10} &= \left(\left(1+i\sqrt{3}\right)^2\left(1+i\sqrt{3}\right)^8\right)^{-1} \\
&= \left(\left(-2+i2\sqrt{3}\right)\left(-2+i2\sqrt{3}\right)^4\right)^{-1} \\
&= \left(\left(-2+i2\sqrt{3}\right)\left(-8-i8\sqrt{3}\right)^2\right)^{-1} \\
&= \left(\left(-2+i2\sqrt{3}\right)\left(-128+i128\sqrt{3}\right)\right)^{-1} \\
&= \left(-512-i512\sqrt{3}\right)^{-1} \\
&= \frac{1}{512}\frac{-1}{1+i\sqrt{3}} \\
&= \frac{1}{512}\frac{-1}{1+i\sqrt{3}}\frac{1-i\sqrt{3}}{1-i\sqrt{3}} \\
&= -\frac{1}{2048}+i\frac{\sqrt{3}}{2048}
\end{aligned}
$$

Now we do the multiplication in modulus-argument, (polar), form.

$$\begin{aligned}
\left(1+i\sqrt{3}\right)^{-10} &= \left(2\,\mathrm{e}^{i\pi/3}\right)^{-10} \\
&= 2^{-10}\,\mathrm{e}^{-i10\pi/3} \\
&= \frac{1}{1024}\left(\cos\left(-\frac{10\pi}{3}\right) + i\sin\left(-\frac{10\pi}{3}\right)\right) \\
&= \frac{1}{1024}\left(\cos\left(\frac{4\pi}{3}\right) - i\sin\left(\frac{4\pi}{3}\right)\right) \\
&= \frac{1}{1024}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= -\frac{1}{2048} + i\frac{\sqrt{3}}{2048}
\end{aligned}$$

2.

$$(11+4i)^2 = 105 + i88$$

Solution 8.3

1.

$$\begin{aligned}
\left(\frac{2+i}{i6-(1-i2)}\right)^2 &= \left(\frac{2+i}{-1+i8}\right)^2 \\
&= \frac{3+i4}{-63-i16} \\
&= \frac{3+i4}{-63-i16}\,\frac{-63+i16}{-63+i16} \\
&= -\frac{253}{4225} - i\frac{204}{4225}
\end{aligned}$$

2.

$$
\begin{aligned}
(1-i)^7 &= ((1-i)^2)^2(1-i)^2(1-i) \\
&= (-i2)^2(-i2)(1-i) \\
&= (-4)(-2-i2) \\
&= 8+i8
\end{aligned}
$$

Solution 8.4

1.

$$
\begin{aligned}
\overline{\left(\frac{\overline{z}}{z}\right)} &= \overline{\left(\frac{\overline{x+iy}}{x+iy}\right)} \\
&= \overline{\left(\frac{x-iy}{x+iy}\right)} \\
&= \frac{x+iy}{x-iy} \\
&= \frac{x+iy}{x-iy}\,\frac{x+iy}{x+iy} \\
&= \frac{x^2-y^2}{x^2+y^2} + i\frac{2xy}{x^2+y^2}
\end{aligned}
$$

2.

$$
\begin{aligned}
\frac{z+2i}{2-i\overline{z}} &= \frac{x+iy+2i}{2-i(x-iy)} \\
&= \frac{x+i(y+2)}{2-y-ix} \\
&= \frac{x+i(y+2)}{2-y-ix}\,\frac{2-y+ix}{2-y+ix} \\
&= \frac{x(2-y)-(y+2)x}{(2-y)^2+x^2} + i\frac{x^2+(y+2)(2-y)}{(2-y)^2+x^2} \\
&= \frac{-2xy}{(2-y)^2+x^2} + i\frac{4+x^2-y^2}{(2-y)^2+x^2}
\end{aligned}
$$

Solution 8.5

Method 1. We expand the equation $uv = w$ in its components.

$$
\begin{gathered}
uv = w \\
(u_0 + iu_1 + ju_2 + ku_3)(v_0 + iv_1 + jv_2 + kv_3) = w_0 + iw_1 + jw_2 + kw_3
\end{gathered}
$$

$$
\begin{multlined}
(u_0v_0 - u_1v_1 - u_2v_2 - u_3v_3) + i(u_1v_0 + u_0v_1 - u_3v_2 + u_2v_3) + j(u_2v_0 + u_3v_1 + u_0v_2 - u_1v_3) \\
+ k(u_3v_0 - u_2v_1 + u_1v_2 + u_0v_3) = w_0 + iw_1 + jw_2 + kw_3
\end{multlined}
$$

We can write this as a matrix equation.

$$
\begin{pmatrix} u_0 & -u_1 & -u_2 & -u_3 \\ u_1 & u_0 & -u_3 & u_2 \\ u_2 & u_3 & u_0 & -u_1 \\ u_3 & -u_2 & u_1 & u_0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix}
$$

This linear system of equations has a unique solution for v if and only if the determinant of the matrix is nonzero. The determinant of the matrix is $(u_0^2 + u_1^2 + u_2^2 + u_3^2)^2$. This is zero if and only if $u_0 = u_1 = u_2 = u_3 = 0$. Thus there exists a unique v such that $uv = w$ if u is nonzero. This v is

$$v = \big((u_0w_0 + u_1w_1 + u_2w_2 + u_3w_3) + i(-u_1w_0 + u_0w_1 + u_3w_2 - u_2w_3) + j(-u_2w_0 - u_3w_1 + u_0w_2 + u_1w_3) \\ + k(-u_3w_0 + u_2w_1 - u_1w_2 + u_0w_3)\big)/(u_0^2 + u_1^2 + u_2^2 + u_3^2)$$

Method 2. Note that $\overline{u}u$ is a real number.

$$\begin{aligned}
\overline{u}u &= (u_0 - iu_1 - ju_2 - ku_3)(u_0 + iu_1 + ju_2 + ku_3) \\
&= (u_0^2 + u_1^2 + u_2^2 + u_3^2) + i(u_0u_1 - u_1u_0 - u_2u_3 + u_3u_2) \\
&\quad + j(u_0u_2 + u_1u_3 - u_2u_0 - u_3u_1) + k(u_0u_3 - u_1u_2 + u_2u_1 - u_3u_0) \\
&= (u_0^2 + u_1^2 + u_2^2 + u_3^2)
\end{aligned}$$

$\overline{u}u = 0$ only if $u = 0$. We solve for v by multiplying by the conjugate of u and divide by $\overline{u}u$.

$$\begin{aligned}
uv &= w \\
\overline{u}uv &= \overline{u}w \\
v &= \frac{\overline{u}w}{\overline{u}u} \\
v &= \frac{(u_0 - iu_1 - ju_2 - ku_3)(w_0 + iw_1 + jw_2 + kw_3)}{u_0^2 + u_1^2 + u_2^2 + u_3^2}
\end{aligned}$$

$$v = \big((u_0w_0 + u_1w_1 + u_2w_2 + u_3w_3) + i(-u_1w_0 + u_0w_1 + u_3w_2 - u_2w_3) + j(-u_2w_0 - u_3w_1 + u_0w_2 + u_1w_3) \\ + k(-u_3w_0 + u_2w_1 - u_1w_2 + u_0w_3)\big)/(u_0^2 + u_1^2 + u_2^2 + u_3^2)$$

Solution 8.6

If $\alpha = t\beta$, then $\alpha\overline{\beta} = t|\beta|^2$, which is a real number. Hence $\Im(\alpha\overline{\beta}) = 0$.

Now assume that $\Im(\alpha\overline{\beta}) = 0$. This implies that $\alpha\overline{\beta} = r$ for some $r \in \mathbb{R}$. We multiply by β and simplify.

$$\alpha|\beta|^2 = r\beta$$
$$\alpha = \frac{r}{|\beta|^2}\beta$$

By taking $t = \frac{r}{|\beta|^2}$ We see that $\alpha = t\beta$ for some real number t.

The Complex Plane

Solution 8.7
Let $z = r\,\mathrm{e}^{i\theta}$ and $\zeta = \rho\,\mathrm{e}^{i\vartheta}$.

1.

$$\arg(z\zeta) = \arg(z) + \arg(\zeta)$$
$$\arg(r\rho\,\mathrm{e}^{i(\theta+\vartheta)}) = \{\theta + 2\pi m\} + \{\vartheta + 2\pi n\}$$
$$\{\theta + \vartheta + 2\pi k\} = \{\theta + \vartheta + 2\pi m\}$$

2.

$$\operatorname{Arg}(z\zeta) \neq \operatorname{Arg}(z) + \operatorname{Arg}(\zeta)$$

Consider $z = \zeta = -1$. $\operatorname{Arg}(z) = \operatorname{Arg}(\zeta) = \pi$, however $\operatorname{Arg}(z\zeta) = \operatorname{Arg}(1) = 0$. The identity becomes $0 \neq 2\pi$.

3.

$$\arg(z^2) = \arg(z) + \arg(z) \neq 2\arg(z)$$
$$\arg(r^2\,\mathrm{e}^{i2\theta}) = \{\theta + 2\pi k\} + \{\theta + 2\pi m\} \neq 2\{\theta + 2\pi n\}$$
$$\{2\theta + 2\pi k\} = \{2\theta + 2\pi m\} \neq \{2\theta + 4\pi n\}$$

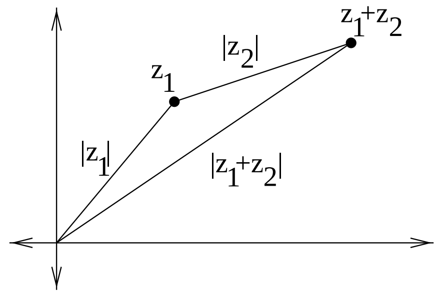


Figure 8.8: Triangle Inequality

Solution 8.8
Consider a triangle in the complex plane with vertices at 0, z_1 and $z_1 + z_2$. (See Figure 8.8.)

The lengths of the sides of the triangle are $|z_1|$, $|z_2|$ and $|z_1 + z_2|$ The second inequality shows that one side of the triangle must be less than or equal to the sum of the other two sides.

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

The first inequality shows that the length of one side of the triangle must be greater than or equal to the difference in the length of the other two sides.

$$|z_1 + z_2| \geq ||z_1| - |z_2||$$

Now we prove the inequalities algebraically. We will reduce the inequality to an identity. Let $z_1 = r_1 \mathrm{e}^{i\theta_1}$, $z_2 = r_2 \mathrm{e}^{i\theta_2}$.

$$
\begin{gathered}
||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2| \\
|r_1 - r_2| \leq |r_1 \mathrm{e}^{i\theta_1} + r_2 \mathrm{e}^{i\theta_2}| \leq r_1 + r_2 \\
(r_1 - r_2)^2 \leq (r_1 \mathrm{e}^{i\theta_1} + r_2 \mathrm{e}^{i\theta_2})(r_1 \mathrm{e}^{-i\theta_1} + r_2 \mathrm{e}^{-i\theta_2}) \leq (r_1 + r_2)^2 \\
r_1^2 + r_2^2 - 2r_1 r_2 \leq r_1^2 + r_2^2 + r_1 r_2 \mathrm{e}^{i(\theta_1 - \theta_2)} + r_1 r_2 \mathrm{e}^{i(-\theta_1 + \theta_2)} \leq r_1^2 + r_2^2 + 2r_1 r_2 \\
-2r_1 r_2 \leq 2r_1 r_2 \cos(\theta_1 - \theta_2) \leq 2r_1 r_2 \\
-1 \leq \cos(\theta_1 - \theta_2) \leq 1
\end{gathered}
$$

Solution 8.9

1.

$$
\begin{aligned}
(-1)^{-3/4} &= \left((-1)^{-3}\right)^{1/4} \\
&= (-1)^{1/4} \\
&= (\mathrm{e}^{i\pi})^{1/4} \\
&= \mathrm{e}^{i\pi/4} 1^{1/4} \\
&= \mathrm{e}^{i\pi/4} \mathrm{e}^{ik\pi/2}, \quad k = 0, 1, 2, 3 \\
&= \left\{ \mathrm{e}^{i\pi/4}, \mathrm{e}^{i3\pi/4}, \mathrm{e}^{i5\pi/4}, \mathrm{e}^{i7\pi/4} \right\} \\
&= \left\{ \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}} \right\}
\end{aligned}
$$

See Figure 8.9.

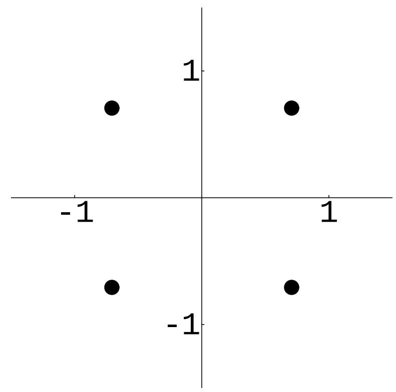


Figure 8.9: $(-1)^{-3/4}$

2.

$$
\begin{aligned}
8^{1/6} &= \sqrt[6]{8}1^{1/6} \\
&= \sqrt{2}\,\mathrm{e}^{ik\pi/3}, \quad k = 0, 1, 2, 3, 4, 5 \\
&= \left\{\sqrt{2}, \sqrt{2}\,\mathrm{e}^{i\pi/3}, \sqrt{2}\,\mathrm{e}^{i2\pi/3}, \sqrt{2}\,\mathrm{e}^{i\pi}, \sqrt{2}\,\mathrm{e}^{i4\pi/3}, \sqrt{2}\,\mathrm{e}^{i5\pi/3}\right\} \\
&= \left\{\sqrt{2}, \frac{1+i\sqrt{3}}{\sqrt{2}}, \frac{-1+i\sqrt{3}}{\sqrt{2}}, -\sqrt{2}\frac{-1-i\sqrt{3}}{\sqrt{2}}, \frac{1-i\sqrt{3}}{\sqrt{2}}\right\}
\end{aligned}
$$

See Figure 8.10.

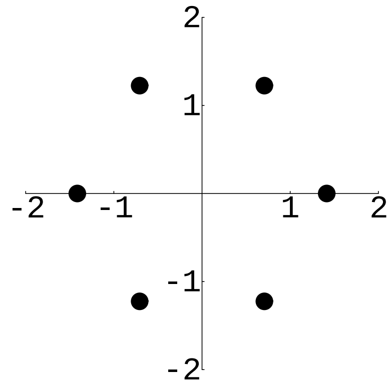


Figure 8.10: $8^{1/6}$

Solution 8.10

1.

$$
\begin{aligned}
(-1)^{-1/4} &= ((-1)^{-1})^{1/4} \\
&= (-1)^{1/4} \\
&= (\mathrm{e}^{i\pi})^{1/4} \\
&= \mathrm{e}^{i\pi/4} 1^{1/4} \\
&= \mathrm{e}^{i\pi/4} \mathrm{e}^{ik\pi/2}, \quad k = 0, 1, 2, 3 \\
&= \left\{ \mathrm{e}^{i\pi/4}, \mathrm{e}^{i3\pi/4}, \mathrm{e}^{i5\pi/4}, \mathrm{e}^{i7\pi/4} \right\} \\
&= \left\{ \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}} \right\}
\end{aligned}
$$

See Figure 8.11.

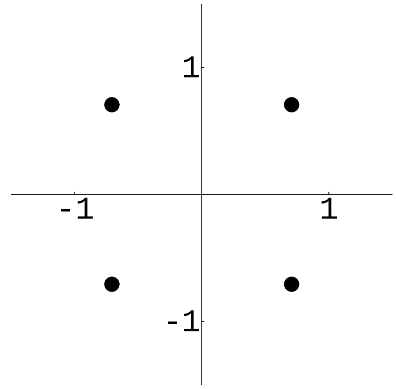


Figure 8.11: $(-1)^{-1/4}$

2.

$$
\begin{aligned}
16^{1/8} &= \sqrt[8]{16}\, 1^{1/8} \\
&= \sqrt{2}\, \mathrm{e}^{ik\pi/4}, \quad k = 0,1,2,3,4,5,6,7 \\
&= \left\{\sqrt{2}, \sqrt{2}\,\mathrm{e}^{i\pi/4}, \sqrt{2}\,\mathrm{e}^{i\pi/2}, \sqrt{2}\,\mathrm{e}^{i3\pi/4}, \sqrt{2}\,\mathrm{e}^{i\pi}, \sqrt{2}\,\mathrm{e}^{i5\pi/4}, \sqrt{2}\,\mathrm{e}^{i3\pi/2}, \sqrt{2}\,\mathrm{e}^{i7\pi/4}\right\} \\
&= \left\{\sqrt{2}, 1+i, \sqrt{2}i, -1+i, -\sqrt{2}, -1-i, -\sqrt{2}i, 1-i\right\}
\end{aligned}
$$

See Figure 8.12.

Solution 8.11

1. $|z - 2i|$ is the distance from the point $2i$ in the complex plane. Thus $1 < |z - 2i| < 2$ is an annulus. See Figure 8.13.

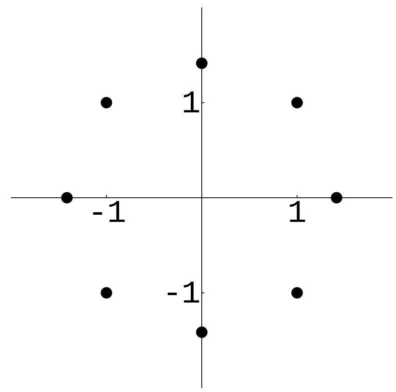


Figure 8.12: $16^{-1/8}$

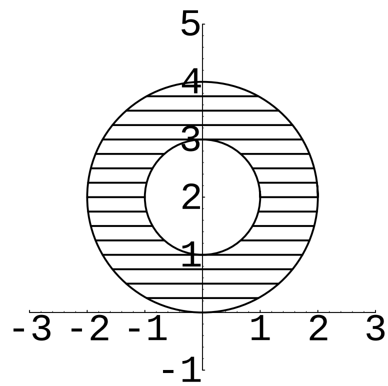


Figure 8.13: $1 < |z - 2i| < 2$

2.

$$|\Re(z)| + 5|\Im(z)| = 1$$
$$|x| + 5|y| = 1$$

In the first quadrant this is the line $y = (1-x)/5$. We reflect this line segment across the coordinate axes to obtain line segments in the other quadrants. Explicitly, we have the set of points: $\{z = x + iy : -1 < x < 1 \wedge y = \pm(1 - |x|)/5\}$. See Figure 8.14.

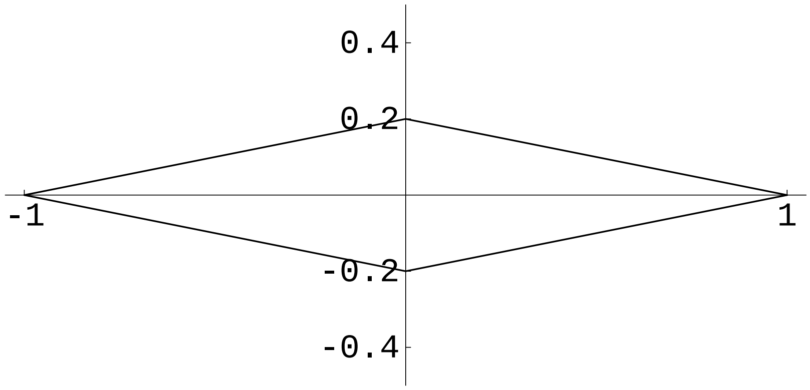


Figure 8.14: $|\Re(z)| + 5|\Im(z)| = 1$

Solution 8.12

1. $|z - 1 + i|$ is the distance from the point $(1 - i)$. Thus $|z - 1 + i| \leq 1$ is the disk of unit radius centered at $(1 - i)$. See Figure 8.15.

2. The set of points equidistant from i and $-i$ is the real axis. See Figure 8.16.

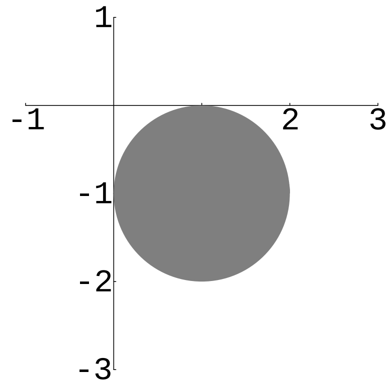


Figure 8.15: $|z - 1 + i| < 1$

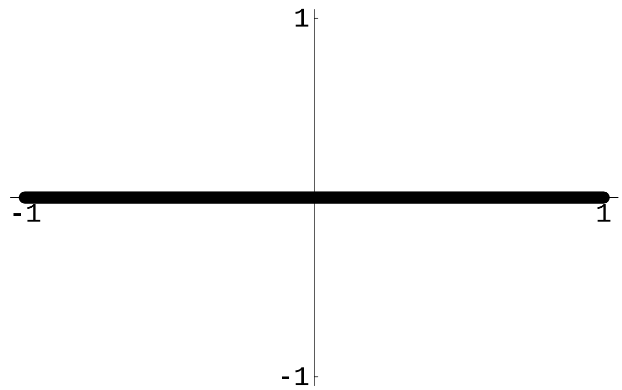


Figure 8.16: $|z - i| = |z + i|$

3.

$$\Re(z) - \Im(z) = 5$$
$$x - y = 5$$
$$y = x - 5$$

See Figure 8.17.

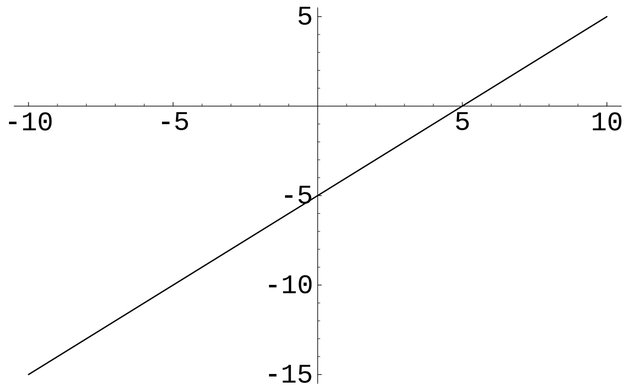


Figure 8.17: $\Re(z) - \Im(z) = 5$

4. Since $|z - i| + |z + i| \geq 2$, there are no solutions of $|z - i| + |z + i| = 1$.

Solution 8.13

$$
\begin{aligned}
|\, e^{i\theta} - 1| &= 2 \\
(\, e^{i\theta} - 1)(\, e^{-i\theta} - 1) &= 4 \\
1 - \, e^{i\theta} - \, e^{-i\theta} + 1 &= 4 \\
-2\cos(\theta) &= 2 \\
\theta &= \pi
\end{aligned}
$$

$\{\, e^{i\theta} \,|\, 0 \leq \theta \leq \pi\}$ is a unit semi-circle in the upper half of the complex plane from 1 to -1. The only point on this semi-circle that is a distance 2 from the point 1 is the point -1, which corresponds to $\theta = \pi$.

Polar Form

Solution 8.14

The Taylor series expansion of e^x is

$$
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.
$$

Taking this as the definition of the exponential function for complex argument,

$$
\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\
&= \sum_{n=0}^{\infty} i^n \frac{\theta^n}{n!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}.
\end{aligned}
$$

The sine and cosine have the Taylor series

$$\cos\theta = \sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n}}{(2n)!}, \qquad \sin\theta = \sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n+1}}{(2n+1)!},$$

Thus $e^{i\theta}$ and $\cos\theta + i\sin\theta$ have the same Taylor series expansions about $\theta = 0$. Since the radius of convergence of the series is infinite we conclude that,

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta.}$$

Solution 8.15

$$\cos(3\theta) + i\sin(3\theta) = (\cos(\theta) + i\sin(\theta))^3$$
$$\cos(3\theta) + i\sin(3\theta) = \cos^3(\theta) + i3\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)$$

We equate the real parts of the equation.

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$$

Solution 8.16
Define the partial sum,

$$S_n(z) = \sum_{k=0}^{n} z^k.$$

Now consider $(1-z)S_n(z)$.

$$
\begin{aligned}
(1-z)S_n(z) &= (1-z)\sum_{k=0}^{n} z^k \\
(1-z)S_n(z) &= \sum_{k=0}^{n} z^k - \sum_{k=1}^{n+1} z^k \\
(1-z)S_n(z) &= 1 - z^{n+1}
\end{aligned}
$$

We divide by $1-z$. Note that $1-z$ is nonzero.

$$
\begin{aligned}
S_n(z) &= \frac{1-z^{n+1}}{1-z} \\
1 + z + z^2 + \cdots + z^n &= \frac{1-z^{n+1}}{1-z}, \qquad (z \neq 1)
\end{aligned}
$$

Now consider $z = \mathrm{e}^{i\theta}$ where $0 < \theta < 2\pi$ so that z is not unity.

$$
\begin{aligned}
\sum_{k=0}^{n} \left(\mathrm{e}^{i\theta}\right)^k &= \frac{1-\left(\mathrm{e}^{i\theta}\right)^{n+1}}{1-\mathrm{e}^{i\theta}} \\
\sum_{k=0}^{n} \mathrm{e}^{ik\theta} &= \frac{1-\mathrm{e}^{i(n+1)\theta}}{1-\mathrm{e}^{i\theta}}
\end{aligned}
$$

In order to get $\sin(\theta/2)$ in the denominator, we multiply top and bottom by $e^{-i\theta/2}$.

$$\sum_{k=0}^{n}(\cos k\theta + i\sin k\theta) = \frac{e^{-i\theta/2} - e^{i(n+1/2)\theta}}{e^{-i\theta/2} - e^{i\theta/2}}$$

$$\sum_{k=0}^{n}\cos k\theta + i\sum_{k=0}^{n}\sin k\theta = \frac{\cos(\theta/2) - i\sin(\theta/2) - \cos((n+1/2)\theta) - i\sin((n+1/2)\theta)}{-2i\sin(\theta/2)}$$

$$\sum_{k=0}^{n}\cos k\theta + i\sum_{k=1}^{n}\sin k\theta = \frac{1}{2} + \frac{\sin((n+1/2)\theta)}{\sin(\theta/2)} + i\left(\frac{1}{2}\cot(\theta/2) - \frac{\cos((n+1/2)\theta)}{\sin(\theta/2)}\right)$$

1. We take the real and imaginary part of this to obtain the identities.

$$\boxed{\sum_{k=0}^{n}\cos k\theta = \frac{1}{2} + \frac{\sin((n+1/2)\theta)}{2\sin(\theta/2)}}$$

2.

$$\boxed{\sum_{k=1}^{n}\sin k\theta = \frac{1}{2}\cot(\theta/2) - \frac{\cos((n+1/2)\theta)}{2\sin(\theta/2)}}$$

Arithmetic and Vectors

Solution 8.17

$$\begin{aligned}
|z_1 z_2| &= |r_1 e^{i\theta_1} r_2 e^{i\theta_2}| \\
&= |r_1 r_2 e^{i(\theta_1+\theta_2)}| \\
&= |r_1 r_2| \\
&= |r_1||r_2| \\
&= |z_1||z_2|
\end{aligned}$$

$$\begin{aligned}
\left|\frac{z_1}{z_2}\right| &= \left|\frac{r_1 \, \mathrm{e}^{i\theta_1}}{r_2 \, \mathrm{e}^{i\theta_2}}\right| \\
&= \left|\frac{r_1}{r_2} \, \mathrm{e}^{i(\theta_1 - \theta_2)}\right| \\
&= \left|\frac{r_1}{r_2}\right| \\
&= \frac{|r_1|}{|r_2|} \\
&= \frac{|z_1|}{|z_2|}
\end{aligned}$$

Solution 8.18

$$\begin{aligned}
|z+\zeta|^2 + |z-\zeta|^2 &= (z+\zeta)\left(\overline{z}+\overline{\zeta}\right) + (z-\zeta)\left(\overline{z}-\overline{\zeta}\right) \\
&= z\overline{z} + z\overline{\zeta} + \zeta\overline{z} + \zeta\overline{\zeta} + z\overline{z} - z\overline{\zeta} - \zeta\overline{z} + \zeta\overline{\zeta} \\
&= 2\left(|z|^2 + |\zeta|^2\right)
\end{aligned}$$

Consider the parallelogram defined by the vectors z and ζ. The lengths of the sides are z and ζ and the lengths of the diagonals are $z+\zeta$ and $z-\zeta$. We know from geometry that the sum of the squared lengths of the diagonals of a parallelogram is equal to the sum of the squared lengths of the four sides. (See Figure 8.18.)

Integer Exponents

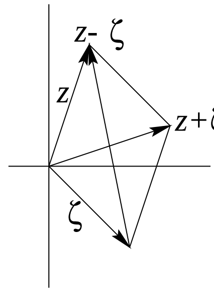


Figure 8.18: The parallelogram defined by z and ζ.

Solution 8.19

1.

$$
\begin{aligned}
(1+i)^{10} &= \left(\left((1+i)^2\right)^2\right)^2 (1+i)^2 \\
&= \left((i2)^2\right)^2 (i2) \\
&= (-4)^2 (i2) \\
&= 16(i2) \\
&= i32
\end{aligned}
$$

2.

$$
\begin{aligned}
(1+i)^{10} &= \left(\sqrt{2}\,\mathrm{e}^{i\pi/4}\right)^{10} \\
&= \left(\sqrt{2}\right)^{10} \mathrm{e}^{i10\pi/4} \\
&= 32\,\mathrm{e}^{i\pi/2} \\
&= i32
\end{aligned}
$$

Rational Exponents

Solution 8.20

We substititite the numbers into the equation to obtain an identity.

$$
\begin{gathered}
z^2 + 2az + b = 0 \\
(-a + (a^2 - b)^{1/2})^2 + 2a(-a + (a^2 - b)^{1/2}) + b = 0 \\
a^2 - 2a(a^2 - b)^{1/2} + a^2 - b - 2a^2 + 2a(a^2 - b)^{1/2} + b = 0 \\
0 = 0
\end{gathered}
$$

Chapter 9

Functions of a Complex Variable

If brute force isn't working, youŕe not using enough of it.

-Tim Mauch

In this chapter we introduce the algebra of functions of a complex variable. We will cover the trigonometric and inverse trigonometric functions. The properties of trigonometric function carry over directly from real-variable theory. However, because of multi-valuedness, the inverse trigonometric functions are significantly trickier than their real-variable counterparts.

9.1 Curves and Regions

In this section we introduce curves and regions in the complex plane. This material is necessary for the study of branch points in this chapter and later for contour integration.

Curves. Consider two continuous functions, $x(t)$ and $y(t)$, defined on the interval $t \in [t_0 \dots t_1]$. The set of points in the complex plane

$$\{z(t) = x(t) + iy(t) \mid t \in [t_0 \dots t_1]\}$$

defines a *continuous curve* or simply a *curve*. If the endpoints coincide, $z(t_0) = z(t_1)$, it is a *closed curve*. (We assume that $t_0 \neq t_1$.) If the curve does not intersect itself, then it is said to be a *simple curve*.

If $x(t)$ and $y(t)$ have continuous derivatives and the derivatives do not both vanish at any point [1] , then it is a *smooth curve*. This essentially means that the curve does not have any corners or other nastiness.

A continuous curve which is composed of a finite number of smooth curves is called a *piecewise smooth curve*. We will use the word *contour* as a synonym for a piecewise smooth curve.

See Figure 9.1 for a smooth curve, a piecewise smooth curve, a simple closed curve and a non-simple closed curve.

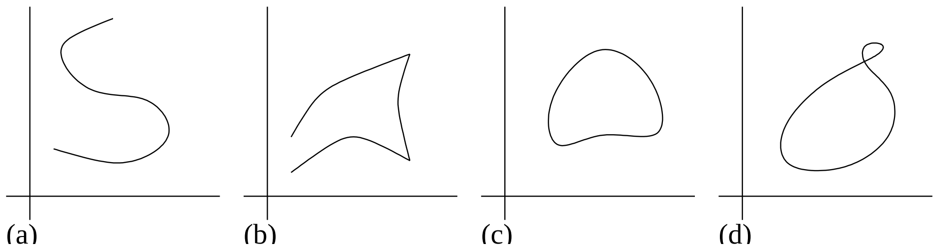


Figure 9.1: (a) Smooth Curve, (b) Piecewise Smooth Curve, (c) Simple Closed Curve, (d) Non-Simple Closed Curve

Regions. A region R is *connected* if any two points in R can be connected by a curve which lies entirely in R. A region is *simply-connected* if every closed curve in R can be continuously shrunk to a point without leaving R. A region which is not simply-connected is said to be *multiply-connected region*. Another way of defining

[1]Why is it necessary that the derivatives do not both vanish?

simply-connected is that a path connecting two points in R can be continuously deformed into any other path that connects those points. Figure 9.2 shows a simply-connected region with two paths which can be continuously deformed into one another and a multiply-connected region with paths which cannot be deformed into one another.

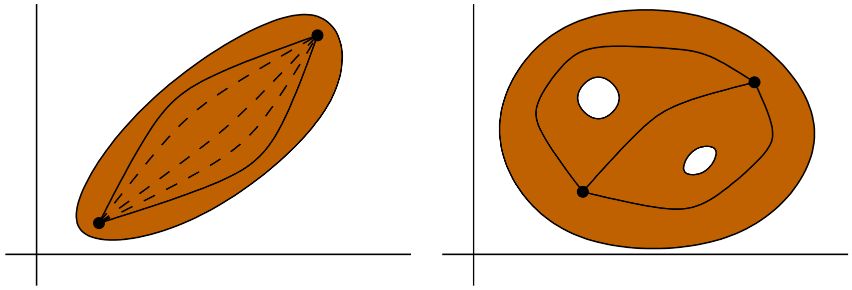


Figure 9.2: Simply-connected and multiply-connected regions.

Jordan Curve Theorem. A continuous, simple, closed curve is known as a *Jordan curve*. The Jordan Curve Theorem, which seems intuitively obvious but is difficult to prove, states that a Jordan curve divides the plane into a simply-connected, bounded region and an unbounded region. These two regions are called the interior and exterior regions, respectively. The two regions share the curve as a boundary. Points in the interior are said to be inside the curve; points in the exterior are said to be outside the curve.

Traversal of a Contour. Consider a Jordan curve. If you traverse the curve in the *positive* direction, then the inside is to your left. If you traverse the curve in the opposite direction, then the outside will be to your left and you will go around the curve in the negative direction. For circles, the positive direction is the *counter-clockwise* direction. The positive direction is consistent with the way angles are measured in a right-handed coordinate system, i.e. for a circle centered on the origin, the positive direction is the direction of increasing angle. For an oriented contour C, we denote the contour with opposite orientation as $-C$.

Boundary of a Region. Consider a simply-connected region. The boundary of the region is traversed in the positive direction if the region is to the left as you walk along the contour. For multiply-connected regions, the boundary may be a set of contours. In this case the boundary is traversed in the positive direction if each of the contours is traversed in the positive direction. When we refer to the boundary of a region we will assume it is given the positive orientation. In Figure 9.3 the boundaries of three regions are traversed in the positive direction.

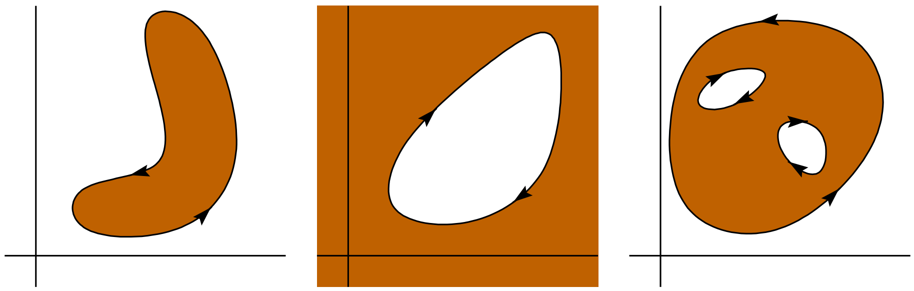


Figure 9.3: Traversing the boundary in the positive direction.

Two Interpretations of a Curve. Consider a simple closed curve as depicted in Figure 9.4a. By giving it an orientation, we can make a contour that either encloses the bounded domain Figure 9.4b or the unbounded domain Figure 9.4c. Thus a curve has two interpretations. It can be thought of as enclosing either the points which are "inside" or the points which are "outside". [2]

[2] A farmer wanted to know the most efficient way to build a pen to enclose his sheep, so he consulted an engineer, a physicist and a mathematician. The engineer suggested that he build a circular pen to get the maximum area for any given perimeter. The physicist suggested that he build a fence at infinity and then shrink it to fit the sheep. The mathematician constructed a little fence around himself and then defined himself to be outside.

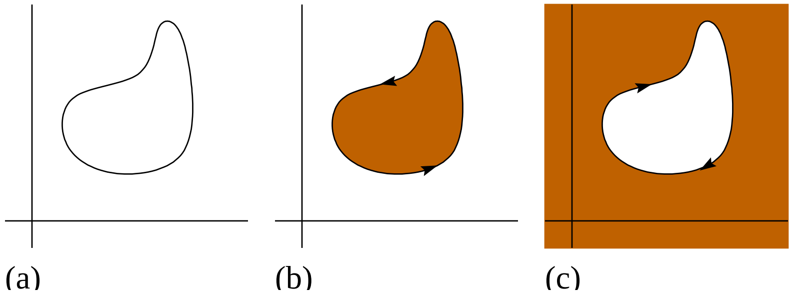


Figure 9.4: Two interpretations of a curve.

9.2 Cartesian and Modulus-Argument Form

We can write a function of a complex variable z as a function of x and y or as a function of r and θ with the substitutions $z = x + iy$ and $z = r\,\mathrm{e}^{i\theta}$, respectively. Then we can separate the real and imaginary components or write the function in modulus-argument form,

$$f(z) = u(x, y) + iv(x, y), \quad \text{or} \quad f(z) = u(r, \theta) + iv(r, \theta),$$

$$f(z) = \rho(x, y)\,\mathrm{e}^{i\phi(x,y)}, \quad \text{or} \quad f(z) = \rho(r, \theta)\,\mathrm{e}^{i\phi(r,\theta)}.$$

Example 9.2.1 Consider the functions $f(z) = z$, $f(z) = z^3$ and $f(z) = \frac{1}{1-z}$. We write the functions in terms of x and y and separate them into their real and imaginary components.

$$\begin{aligned} f(z) &= z \\ &= x + iy \end{aligned}$$

$$
\begin{aligned}
f(z) &= z^3 \\
&= (x + iy)^3 \\
&= x^3 + ix^2y - xy^2 - iy^3 \\
&= \left(x^3 - xy^2\right) + i\left(x^2y - y^3\right)
\end{aligned}
$$

$$
\begin{aligned}
f(z) &= \frac{1}{1-z} \\
&= \frac{1}{1-x-iy} \\
&= \frac{1}{1-x-iy}\frac{1-x+iy}{1-x+iy} \\
&= \frac{1-x}{(1-x)^2+y^2} + i\frac{y}{(1-x)^2+y^2}
\end{aligned}
$$

Example 9.2.2 Consider the functions $f(z) = z$, $f(z) = z^3$ and $f(z) = \frac{1}{1-z}$. We write the functions in terms of r and θ and write them in modulus-argument form.

$$
\begin{aligned}
f(z) &= z \\
&= r\,\mathrm{e}^{i\theta}
\end{aligned}
$$

$$
\begin{aligned}
f(z) &= z^3 \\
&= \left(r\,\mathrm{e}^{i\theta}\right)^3 \\
&= r^3\,\mathrm{e}^{i3\theta}
\end{aligned}
$$

$$
\begin{aligned}
f(z) &= \frac{1}{1-z} \\
&= \frac{1}{1 - r\,\mathrm{e}^{i\theta}} \\
&= \frac{1}{1 - r\,\mathrm{e}^{i\theta}} \frac{1}{1 - r\,\mathrm{e}^{-i\theta}} \\
&= \frac{1 - r\,\mathrm{e}^{-i\theta}}{1 - r\,\mathrm{e}^{i\theta} - r\,\mathrm{e}^{-i\theta} + r^2} \\
&= \frac{1 - r\cos\theta + ir\sin\theta}{1 - 2r\cos\theta + r^2}
\end{aligned}
$$

Note that the denominator is real and non-negative.

$$
\begin{aligned}
&= \frac{1}{1 - 2r\cos\theta + r^2} |1 - r\cos\theta + ir\sin\theta|\, \mathrm{e}^{i\arctan(1 - r\cos\theta, r\sin\theta)} \\
&= \frac{1}{1 - 2r\cos\theta + r^2} \sqrt{(1 - r\cos\theta)^2 + r^2\sin^2\theta}\, \mathrm{e}^{i\arctan(1 - r\cos\theta, r\sin\theta)} \\
&= \frac{1}{1 - 2r\cos\theta + r^2} \sqrt{1 - 2r\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta}\, \mathrm{e}^{i\arctan(1 - r\cos\theta, r\sin\theta)} \\
&= \frac{1}{\sqrt{1 - 2r\cos\theta + r^2}}\, \mathrm{e}^{i\arctan(1 - r\cos\theta, r\sin\theta)}
\end{aligned}
$$

9.3 Graphing Functions of a Complex Variable

We cannot directly graph a function of a complex variable as they are mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. To do so would require four dimensions. However, we can can use a surface plot to graph the real part, the imaginary part, the modulus or the argument of a function of a complex variable. Each of these are scalar fields, mappings from $\mathbb{R}^2$ to $\mathbb{R}$.

Example 9.3.1 Consider the identity function, $f(z) = z$. In Cartesian coordinates and Cartesian form, the function is $f(z) = x + iy$. The real and imaginary components are $u(x, y) = x$ and $v(x, y) = y$. (See Figure 9.5.) In modulus argument form the function is

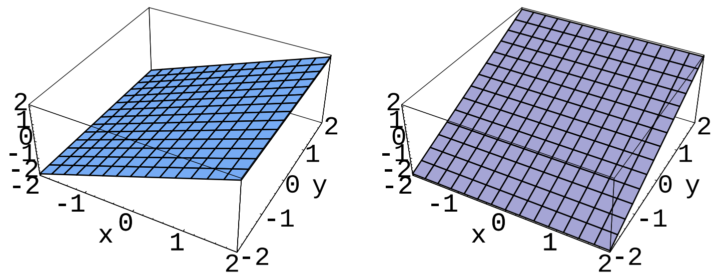


Figure 9.5: The real and imaginary parts of $f(z) = z = x + iy$

$$f(z) = z = r\,\mathrm{e}^{i\theta} = \sqrt{x^2 + y^2}\,\mathrm{e}^{i \arctan(x,y)}.$$

The modulus of $f(z)$ is a single-valued function which is the distance from the origin. The argument of $f(z)$ is a multi-valued function. Recall that $\arctan(x, y)$ has an infinite number of values each of which differ by an integer multiple of 2π. A few branches of $\arg(f(z))$ are plotted in Figure 9.6. The modulus and principal argument of $f(z) = z$ are plotted in Figure 9.7.

Example 9.3.2 Consider the function $f(z) = z^2$. In Cartesian coordinates and separated into its real and imaginary components the function is

$$f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + i2xy.$$

Figure 9.8 shows surface plots of the real and imaginary parts of z^2. The magnitude of z^2 is

$$|z^2| = \sqrt{z^2\overline{z^2}} = z\overline{z} = (x + iy)(x - iy) = x^2 + y^2.$$

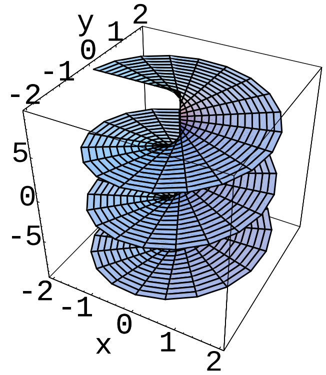


Figure 9.6: A Few Branches of $\arg(z)$

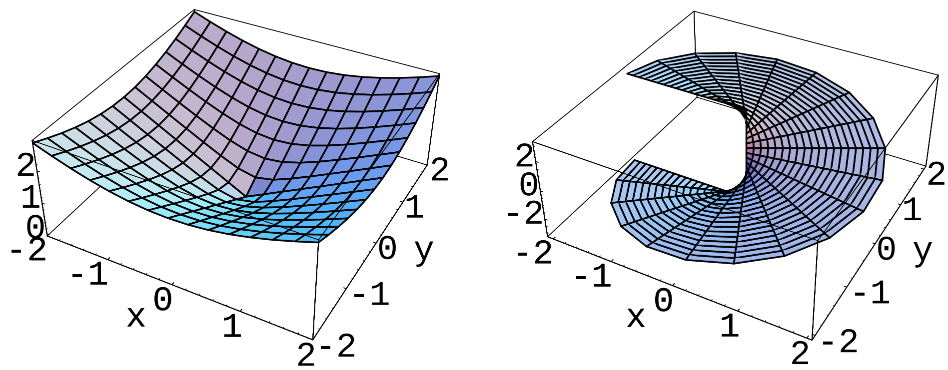


Figure 9.7: Plots of $|z|$ and $\operatorname{Arg}(z)$

Note that

$$z^2 = (r\,\mathrm{e}^{i\theta})^2 = r^2\,\mathrm{e}^{i2\theta}.$$

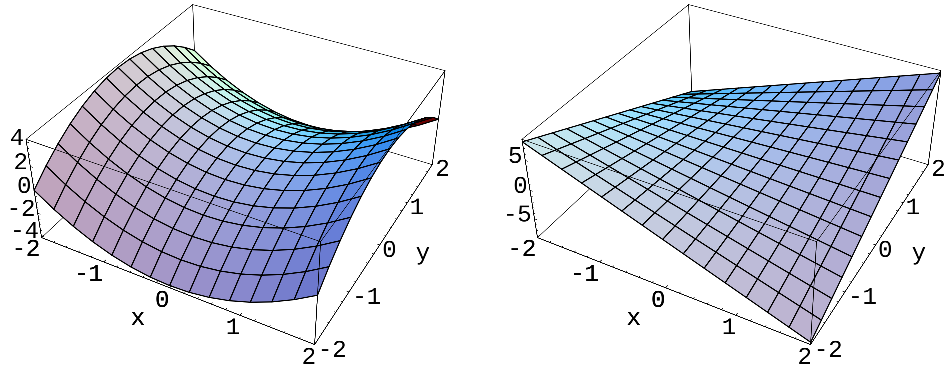


Figure 9.8: Plots of $\Re(z^2)$ and $\Im(z^2)$

In Figure 9.9 are plots of $|z^2|$ and a branch of $\arg(z^2)$.

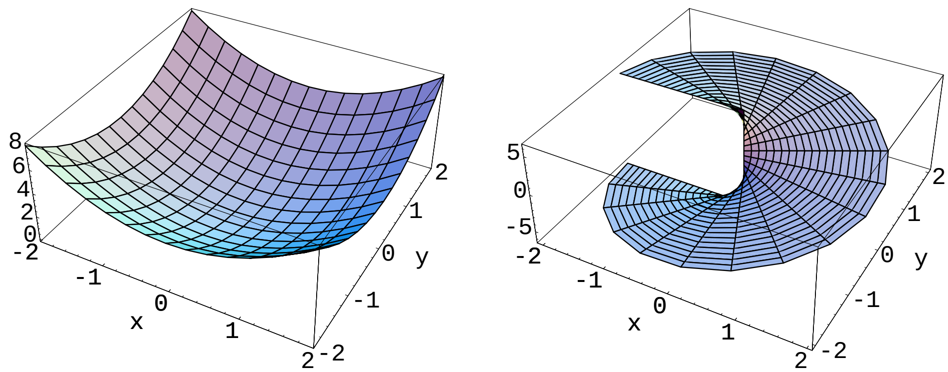


Figure 9.9: Plots of $|z^2|$ and a branch of $\arg(z^2)$

9.4 Trigonometric Functions

The Exponential Function. Consider the exponential function e^z. We can use Euler's formula to write $e^z = e^{x+iy}$ in terms of its real and imaginary parts.

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + i e^x \sin y$$

From this we see that the exponential function is $i2\pi$ periodic: $e^{z+i2\pi} = e^z$, and $i\pi$ odd periodic: $e^{z+i\pi} = -e^z$. Figure 9.10 has surface plots of the real and imaginary parts of e^z which show this periodicity.

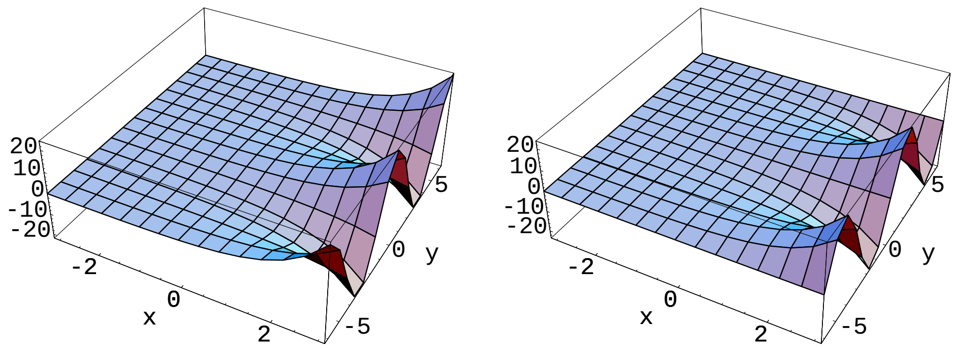


Figure 9.10: Plots of $\Re(e^z)$ and $\Im(e^z)$

The modulus of e^z is a function of x alone.

$$|e^z| = \left|e^{x+iy}\right| = e^x$$

The argument of e^z is a function of y alone.

$$\arg\left(e^z\right) = \arg\left(e^{x+iy}\right) = \{y + 2\pi n \mid n \in \mathbb{Z}\}$$

In Figure 9.11 are plots of $|e^z|$ and a branch of $\arg(e^z)$.

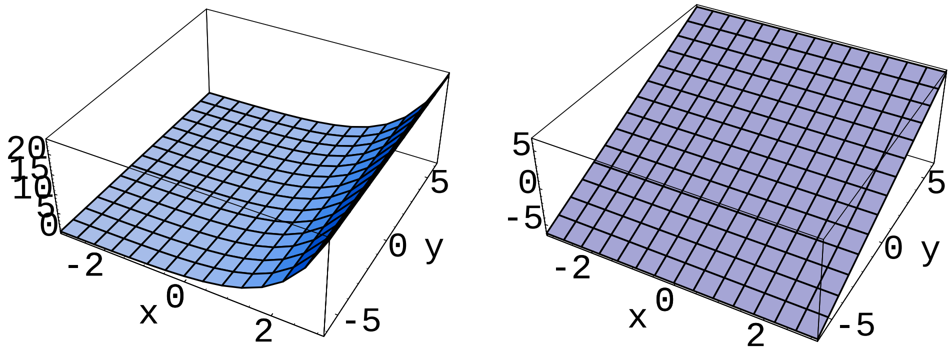


Figure 9.11: Plots of $|\mathrm{e}^z|$ and a branch of $\arg(\mathrm{e}^z)$

Example 9.4.1 Show that the transformation $w = \mathrm{e}^z$ maps the infinite strip, $-\infty < x < \infty$, $0 < y < \pi$, onto the upper half-plane.

Method 1. Consider the line $z = x + ic$, $-\infty < x < \infty$. Under the transformation, this is mapped to

$$w = \mathrm{e}^{x+ic} = \mathrm{e}^{ic}\,\mathrm{e}^{x}, \quad -\infty < x < \infty.$$

This is a ray from the origin to infinity in the direction of e^{ic}. Thus we see that $z = x$ is mapped to the positive, real w axis, $z = x + i\pi$ is mapped to the negative, real axis, and $z = x + ic$, $0 < c < \pi$ is mapped to a ray with angle c in the upper half-plane. Thus the strip is mapped to the upper half-plane. See Figure 9.12.

Method 2. Consider the line $z = c + iy$, $0 < y < \pi$. Under the transformation, this is mapped to

$$w = \mathrm{e}^{c+iy} + \mathrm{e}^{c}\,\mathrm{e}^{iy}, \quad 0 < y < \pi.$$

This is a semi-circle in the upper half-plane of radius e^c. As $c \to -\infty$, the radius goes to zero. As $c \to \infty$, the radius goes to infinity. Thus the strip is mapped to the upper half-plane. See Figure 9.13.

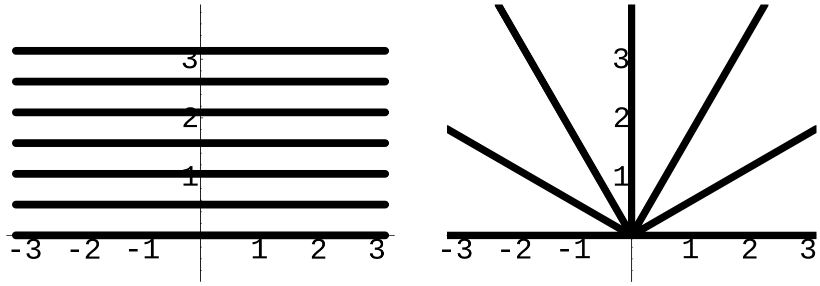


Figure 9.12: e^z maps horizontal lines to rays.

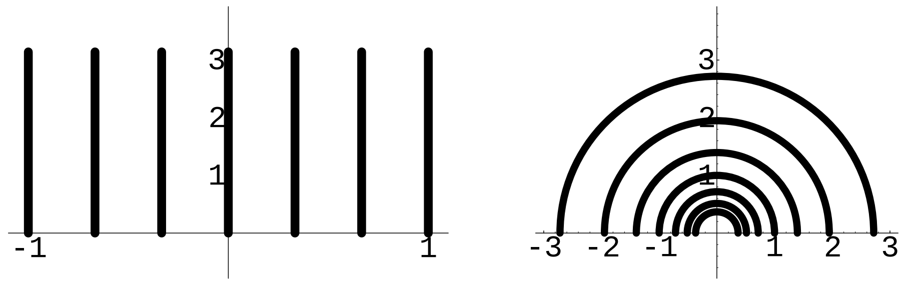


Figure 9.13: e^z maps vertical lines to circular arcs.

The Sine and Cosine. We can write the sine and cosine in terms of the exponential function.

$$\begin{aligned}
\frac{e^{iz} + e^{-iz}}{2} &= \frac{\cos(z) + i\sin(z) + \cos(-z) + i\sin(-z)}{2} \\
&= \frac{\cos(z) + i\sin(z) + \cos(z) - i\sin(z)}{2} \\
&= \cos z
\end{aligned}$$

$$
\begin{aligned}
\frac{e^{iz} - e^{-iz}}{i2} &= \frac{\cos(z) + i\sin(z) - \cos(-z) - i\sin(-z)}{2} \\
&= \frac{\cos(z) + i\sin(z) - \cos(z) + i\sin(z)}{2} \\
&= \sin z
\end{aligned}
$$

We separate the sine and cosine into their real and imaginary parts.

$$
\cos z = \cos x \cosh y - i \sin x \sinh y \qquad \sin z = \sin x \cosh y + i \cos x \sinh y
$$

For fixed y, the sine and cosine are oscillatory in x. The amplitude of the oscillations grows with increasing $|y|$. See Figure 9.14 and Figure 9.15 for plots of the real and imaginary parts of the cosine and sine, respectively. Figure 9.16 shows the modulus of the cosine and the sine.

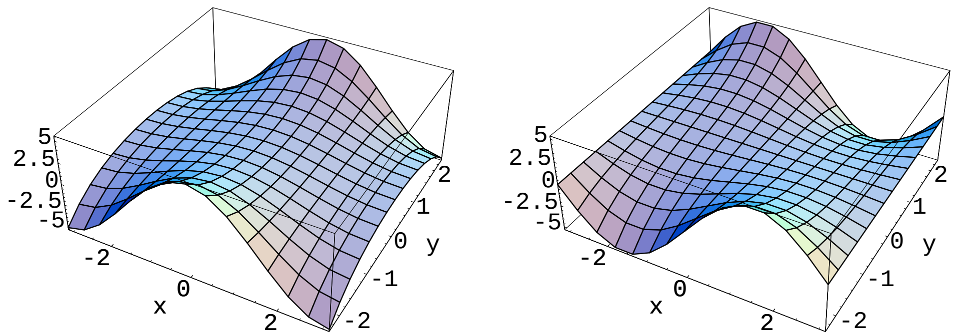


Figure 9.14: Plots of $\Re(\cos(z))$ and $\Im(\cos(z))$

The Hyperbolic Sine and Cosine. The hyperbolic sine and cosine have the familiar definitions in terms of the exponential function. Thus not surprisingly, we can write the sine in terms of the hyperbolic sine and write the cosine in terms of the hyperbolic cosine. Below is a collection of trigonometric identities.

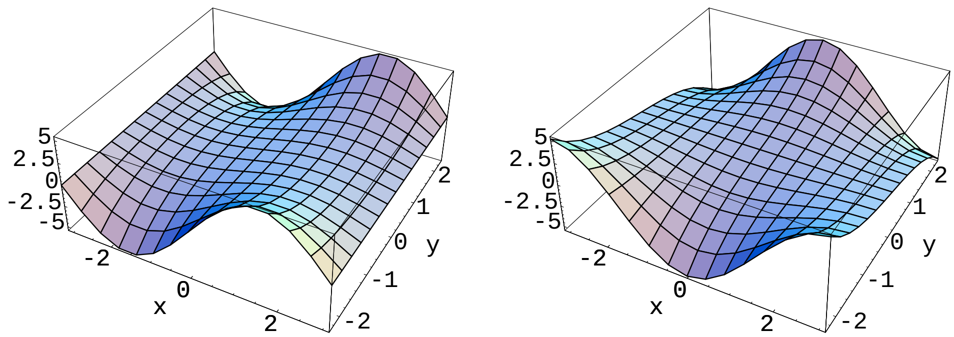


Figure 9.15: Plots of $\Re(\sin(z))$ and $\Im(\sin(z))$

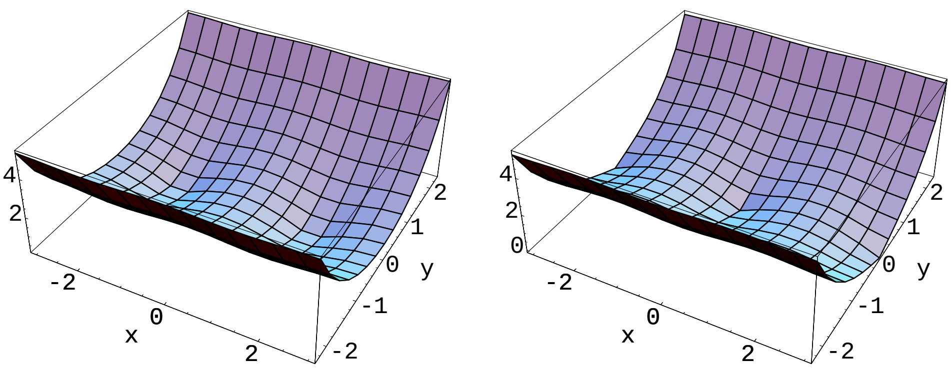


Figure 9.16: Plots of $|\cos(z)|$ and $|\sin(z)|$

Result 9.4.1

$$\mathrm{e}^z = \mathrm{e}^x(\cos y + i \sin y)$$

$$\cos z = \frac{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}{2} \qquad \sin z = \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{2i}$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y \qquad \sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\cosh z = \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2} \qquad \sinh z = \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2}$$

$$\cosh z = \cosh x \cos y + i \sinh x \sin y \qquad \sinh z = \sinh x \cos y + i \cosh x \sin y$$

$$\sin iz = i \sinh z \qquad \sinh iz = i \sin z$$

$$\cos iz = \cosh z \qquad \cosh iz = \cos z$$

$$\log z = \operatorname{Log}|z| + i \arg(z) = \operatorname{Log}|z| + i \operatorname{Arg}(z) + 2i\pi n$$

9.5 Inverse Trigonometric Functions

The Logarithm. The logarithm, $\log(z)$, is defined as the inverse of the exponential function e^z. The exponential function is many-to-one and thus has a multi-valued inverse. From what we know of many-to-one functions, we conclude that

$$\mathrm{e}^{\log z} = z, \quad \text{but} \quad \log(\mathrm{e}^z) \neq z.$$

This is because $\mathrm{e}^{\log z}$ is single-valued but $\log(\mathrm{e}^z)$ is not. Because e^z is $i2\pi$ periodic, the logarithm of a number is a set of numbers which differ by integer multiples of $i2\pi$. For instance, $\mathrm{e}^{i2\pi n} = 1$ so that $\log(1) = \{i2\pi n : n \in \mathbb{Z}\}$. The logarithmic function has an infinite number of branches. The value of the function on the branches differs by integer multiples of $i2\pi$. It has singularities at zero and infinity. $|\log(z)| \to \infty$ as either $z \to 0$ or $z \to \infty$.

We will derive the formula for the complex variable logarithm. For now, let $\operatorname{Log}(x)$ denote the real variable logarithm that is defined for positive real numbers. Consider $w = \log z$. This means that $\mathrm{e}^w = z$. We write $w = u + iv$ in Cartesian form and $z = r\,\mathrm{e}^{i\theta}$ in polar form.

$$\mathrm{e}^{u+iv} = r\,\mathrm{e}^{i\theta}$$

We equate the modulus and argument of this expression.

$$\begin{aligned} \mathrm{e}^u &= r & v &= \theta + 2\pi n \\ u &= \operatorname{Log} r & v &= \theta + 2\pi n \end{aligned}$$

With $\log z = u + iv$, we have a formula form the logarithm.

$$\boxed{\log z = \operatorname{Log}|z| + i\arg(z)}$$

If we write out the multi-valuedness of the argument function we note that this has the form that we expected.

$$\log z = \operatorname{Log}|z| + i(\operatorname{Arg}(z) + 2\pi n), \quad n \in \mathbb{Z}$$

We check that our formula is correct by showing that $\mathrm{e}^{\log z} = z$

$$\mathrm{e}^{\log z} = \mathrm{e}^{\operatorname{Log}|z| + i\arg(z)} = \mathrm{e}^{\operatorname{Log} r + i\theta + i2\pi n} = r\,\mathrm{e}^{i\theta} = z$$

Note again that $\log(\mathrm{e}^z) \neq z$.

$$\log(\mathrm{e}^z) = \operatorname{Log}|\mathrm{e}^z| + i\arg(\mathrm{e}^z) = \operatorname{Log}(\mathrm{e}^x) + i\arg(\mathrm{e}^{x+iy}) = x + i(y + 2\pi n) = z + i2n\pi \neq z$$

The real part of the logarithm is the single-valued $\operatorname{Log} r$; the imaginary part is the multi-valued $\arg(z)$. We define the principal branch of the logarithm $\operatorname{Log} z$ to be the branch that satisfies $-\pi < \Im(\operatorname{Log} z) \leq \pi$. For positive, real numbers the principal branch, $\operatorname{Log} x$ is real-valued. We can write $\operatorname{Log} z$ in terms of the principal argument, $\operatorname{Arg} z$.

$$\operatorname{Log} z = \operatorname{Log}|z| + i\operatorname{Arg}(z)$$

See Figure 9.17 for plots of the real and imaginary part of $\operatorname{Log} z$.

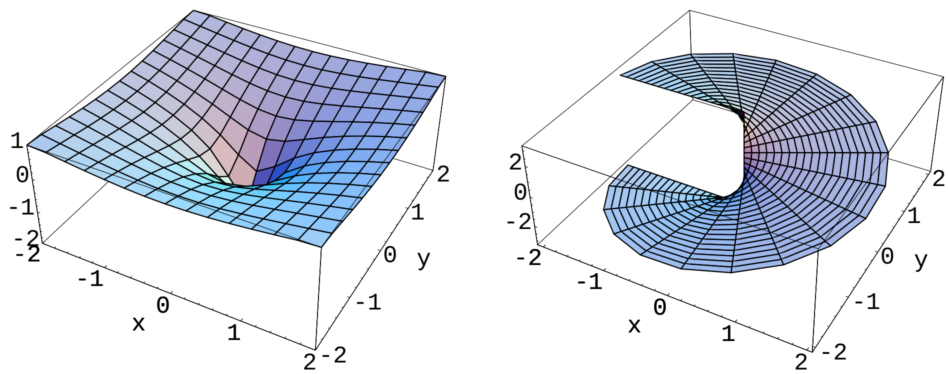


Figure 9.17: Plots of $\Re(\operatorname{Log} z)$ and $\Im(\operatorname{Log} z)$.

The Form: $\mathbf{a^b}$. Consider a^b where a and b are complex and a is nonzero. We define this expression in terms of the exponential and the logarithm as

$$a^b = \mathrm{e}^{b\log a}.$$

Note that the multi-valuedness of the logarithm may make a^b multi-valued. First consider the case that the exponent is an integer.

$$a^m = e^{m \log a} = e^{m(\operatorname{Log} a + i2n\pi)} = e^{m \operatorname{Log} a} e^{i2mn\pi} = e^{m \operatorname{Log} a}$$

Thus we see that a^m has a single value where m is an integer.

Now consider the case that the exponent is a rational number. Let p/q be a rational number in reduced form.

$$a^{p/q} = e^{\frac{p}{q} \log a} = e^{\frac{p}{q}(\operatorname{Log} a + i2n\pi)} = e^{\frac{p}{q} \operatorname{Log} a} e^{i2np\pi/q}.$$

This expression has q distinct values as

$$e^{i2np\pi/q} = e^{i2mp\pi/q} \quad \text{if and only if} \quad n = m \mod q.$$

Finally consider the case that the exponent b is an irrational number.

$$a^b = e^{b \log a} = e^{b(\operatorname{Log} a + i2n\pi)} = e^{b \operatorname{Log} a} e^{i2bn\pi}$$

Note that $e^{i2bn\pi}$ and $e^{i2bm\pi}$ are equal if and only if $i2bn\pi$ and $i2bm\pi$ differ by an integer multiple of $i2\pi$, which means that bn and bm differ by an integer. This occurs only when $n = m$. Thus $e^{i2bn\pi}$ has a distinct value for each different integer n. We conclude that a^b has an infinite number of values.

You may have noticed something a little fishy. If b is not an integer and a is any non-zero complex number, then a^b is multi-valued. Then why have we been treating e^b as single-valued, when it is merely the case $a = e$? The answer is that in the realm of functions of a complex variable, e^z is an abuse of notation. We write e^z when we mean $\exp(z)$, the single-valued exponential function. Thus when we write e^z we do not mean "the number e raised to the z power", we mean "the exponential function of z". We denote the former scenario as $(e)^z$, which is multi-valued.

Logarithmic Identities. Back in high school trigonometry when you thought that the logarithm was only defined for positive real numbers you learned the identity $\log x^a = a \log x$. This identity doesn't hold when the logarithm is defined for nonzero complex numbers. Consider the logarithm of z^a.

$$\log z^a = \operatorname{Log} z^a + i2\pi n$$

$$a \log z = a(\operatorname{Log} z + i2\pi n) = a \operatorname{Log} z + i2a\pi n$$

Note that

$$\log z^a \neq a \log z$$

Furthermore, since

$$\operatorname{Log} z^a = \operatorname{Log} |z^a| + i \operatorname{Arg}(z^a), \quad a \operatorname{Log} z = a \operatorname{Log} |z| + ia \operatorname{Arg}(z)$$

and $\operatorname{Arg}(z^a)$ is not necessarily the same as $a \operatorname{Arg}(z)$ we see that

$$\operatorname{Log} z^a \neq a \operatorname{Log} z.$$

Consider the logarithm of a product.

$$\begin{aligned} \log(ab) &= \operatorname{Log} |ab| + i \arg(ab) \\ &= \operatorname{Log} |a| + \operatorname{Log} |b| + i \arg(a) + i \arg(b) \\ &= \log a + \log b \end{aligned}$$

There is not an analogous identity for the principal branch of the logarithm since $\operatorname{Arg}(ab)$ is not in general the same as $\operatorname{Arg}(a) + \operatorname{Arg}(b)$.

Using $\log(ab) = \log(a) + \log(b)$ we can deduce that $\log(a^n) = \sum_{k=1}^n \log a = n \log a$, where n is a positive integer. This result is simple, straightforward and wrong. I have led you down the merry path to damnation. [3] In fact, $\log(a^2) \neq 2 \log a$. Just write the multi-valuedness explicitly,

$$\log(a^2) = \operatorname{Log}(a^2) + i2n\pi, \qquad 2 \log a = 2(\operatorname{Log} a + i2n\pi) = 2 \operatorname{Log} a + i4n\pi.$$

[3] Don't feel bad if you fell for it. The logarithm is a tricky bastard.

You can verify that

$$\log\left(\frac{1}{a}\right) = -\log a.$$

We can use this and the product identity to expand the logarithm of a quotient.

$$\log\left(\frac{a}{b}\right) = \log a - \log b$$

For general values of a, $\log z^a \neq a \log z$. However, for some values of a, equality holds. We already know that $a = 1$ and $a = -1$ work. To determine if equality holds for other values of a, we explicitly write the multi-valuedness.

$$\log z^a = \log\left(\mathrm{e}^{a \log z}\right) = a \log z + i2\pi k, \quad k \in \mathbb{Z}$$
$$a \log z = a \operatorname{Log}|z| + ia \operatorname{Arg} z + ia2\pi m, \quad m \in \mathbb{Z}$$

We see that $\log z^a = a \log z$ if and only if

$$\{am \mid m \in \mathbb{Z}\} = \{am + k \mid k, m \in \mathbb{Z}\}.$$

The sets are equal if and only if $a = 1/n$, $n \in \mathbb{Z}^{\pm}$. Thus we have the identity:

$$\log\left(z^{1/n}\right) = \frac{1}{n} \log z, \quad n \in \mathbb{Z}^{\pm}$$

Result 9.5.1 Logarithmic Identities.

$$
\begin{aligned}
a^b &= \mathrm{e}^{b \log a} \\
\mathrm{e}^{\log z} &= \mathrm{e}^{\operatorname{Log} z} = z \\
\log(ab) &= \log a + \log b \\
\log(1/a) &= -\log a \\
\log(a/b) &= \log a - \log b \\
\log\left(z^{1/n}\right) &= \frac{1}{n} \log z, \quad n \in \mathbb{Z}^{\pm}
\end{aligned}
$$

Logarithmic Inequalities.

$$
\begin{aligned}
&\operatorname{Log}(uv) \neq \operatorname{Log}(u) + \operatorname{Log}(v) \\
&\log z^a \neq a \log z \\
&\operatorname{Log} z^a \neq a \operatorname{Log} z \\
&\log \mathrm{e}^z \neq z
\end{aligned}
$$

Example 9.5.1 Consider 1^π. We apply the definition $a^b = \mathrm{e}^{b \log a}$.

$$
\begin{aligned}
1^\pi &= \mathrm{e}^{\pi \log(1)} \\
&= \mathrm{e}^{\pi(\operatorname{Log}(1) + i2n\pi)} \\
&= \mathrm{e}^{i2n\pi^2}
\end{aligned}
$$

Thus we see that 1^π has an infinite number of values, all of which lie on the unit circle $|z| = 1$ in the complex plane. However, the set 1^π is not equal to the set $|z| = 1$. There are points in the latter which are not in the former. This is analogous to the fact that the rational numbers are dense in the real numbers, but are a subset of the real numbers.

Example 9.5.2 We find the zeros of $\sin z$.

$$
\begin{aligned}
\sin z = \frac{e^{iz} - e^{-iz}}{2i} &= 0 \\
e^{iz} &= e^{-iz} \\
e^{i2z} &= 1 \\
2z \mod 2\pi &= 0 \\
\boxed{z = n\pi, \quad n \in \mathbb{Z}}
\end{aligned}
$$

Equivalently, we could use the identity

$$\sin z = \sin x \cosh y + i \cos x \sinh y = 0.$$

This becomes the two equations (for the real and imaginary parts)

$$\sin x \cosh y = 0 \qquad \text{and} \qquad \cos x \sinh y = 0.$$

Since cosh is real-valued and positive for real argument, the first equation dictates that $x = n\pi$, $n \in \mathbb{Z}$. Since $\cos(n\pi) = (-1)^n$ for $n \in \mathbb{Z}$, the second equation implies that $\sinh y = 0$. For real argument, $\sinh y$ is only zero at $y = 0$. Thus the zeros are

$$\boxed{z = n\pi, \quad n \in \mathbb{Z}}$$

Example 9.5.3 Since we can express $\sin z$ in terms of the exponential function, one would expect that we could express the $\sin^{-1} z$ in terms of the logarithm.

$$w = \sin^{-1} z$$
$$z = \sin w$$
$$z = \frac{e^{iw} - e^{-iw}}{2i}$$
$$e^{2iw} - 2iz\, e^{iw} - 1 = 0$$
$$e^{iw} = iz \pm \sqrt{1 - z^2}$$
$$w = -i \log\left(iz \pm \sqrt{1 - z^2}\right)$$

Thus we see how the multi-valued $\sin^{-1}$ is related to the logarithm.

$$\boxed{\sin^{-1} z = -i \log\left(iz \pm \sqrt{1 - z^2}\right)}$$

Example 9.5.4 Consider the equation $\sin^3 z = 1$.

$$\sin^3 z = 1$$
$$\sin z = 1^{1/3}$$
$$\frac{e^{iz} - e^{-iz}}{2i} = 1^{1/3}$$
$$e^{iz} - 2i(1)^{1/3} - e^{-iz} = 0$$
$$e^{2iz} - 2i(1)^{1/3} e^{iz} - 1 = 0$$
$$e^{iz} = \frac{2i(1)^{1/3} \pm \sqrt{-4(1)^{2/3} + 4}}{2}$$
$$e^{iz} = i(1)^{1/3} \pm \sqrt{1 - (1)^{2/3}}$$
$$\boxed{z = -i \log\left(i(1)^{1/3} \pm \sqrt{1 - 1^{2/3}}\right)}$$

Note that there are three sources of multi-valuedness in the expression for z. The two values of the square root are shown explicitly. There are three cube roots of unity. Finally, the logarithm has an infinite number of branches. To show this multi-valuedness explicitly, we could write

$$\boxed{z = -i \operatorname{Log}\left(i\,\mathrm{e}^{i2m\pi/3} \pm \sqrt{1 - \mathrm{e}^{i4m\pi/3}}\right) + 2\pi n, \qquad m = 0, 1, 2, \quad n = \ldots, -1, 0, 1, \ldots}$$

Example 9.5.5 Consider the harmless looking equation, $i^z = 1$.

Before we start with the algebra, note that the right side of the equation is a single number. i^z is single-valued only when z is an integer. Thus we know that if there are solutions for z, they are integers. We now proceed to solve the equation.

$$i^z = 1$$
$$\left(\mathrm{e}^{i\pi/2}\right)^z = 1$$

Use the fact that z is an integer.

$$\mathrm{e}^{i\pi z/2} = 1$$
$$i\pi z/2 = 2in\pi, \quad \text{for some } n \in \mathbb{Z}$$
$$\boxed{z = 4n, \quad n \in \mathbb{Z}}$$

Here is a different approach. We write down the multi-valued form of i^z. We solve the equation by requiring that all the values of i^z are 1.

$$i^z = 1$$
$$\mathrm{e}^{z \log i} = 1$$
$$z \log i = 2\pi i n, \quad \text{for some } n \in \mathbb{Z}$$
$$z\left(i\frac{\pi}{2} + 2\pi i m\right) = 2\pi i n, \quad \forall m \in \mathbb{Z}, \quad \text{for some } n \in \mathbb{Z}$$
$$i\frac{\pi}{2} z + 2\pi i m z = 2\pi i n, \quad \forall m \in \mathbb{Z}, \quad \text{for some } n \in \mathbb{Z}$$

The only solutions that satisfy the above equation are

$$\boxed{z = 4k, \quad k \in \mathbb{Z}.}$$

Now let's consider a slightly different problem: $1 \in i^z$. For what values of z does i^z have 1 as one of its values.

$$1 \in i^z$$
$$1 \in \mathrm{e}^{z \log i}$$
$$1 \in \{ \mathrm{e}^{z(i\pi/2 + i2\pi n)} \,|\, n \in \mathbb{Z}\}$$
$$z(i\pi/2 + i2\pi n) = i2\pi m, \quad m, n \in \mathbb{Z}$$
$$\boxed{z = \frac{4m}{1 + 4n}, \quad m, n \in \mathbb{Z}}$$

There are an infinite set of rational numbers for which i^z has 1 as one of its values. For example,

$$i^{4/5} = 1^{1/5} = \left\{1,\ \mathrm{e}^{i2\pi/5},\ \mathrm{e}^{i4\pi/5},\ \mathrm{e}^{i6\pi/5},\ \mathrm{e}^{i8\pi/5}\right\}$$

9.6 Branch Points

Example 9.6.1 Consider the function $z^{1/2}$. For each value of z, there are two values of $z^{1/2}$. We write $z^{1/2}$ in modulus-argument and Cartesian form.

$$z^{1/2} = \sqrt{|z|}\,\mathrm{e}^{i\arg(z)/2}$$
$$z^{1/2} = \sqrt{|z|}\cos(\arg(z)/2) + i\sqrt{|z|}\sin(\arg(z)/2)$$

Figures 9.18 and 9.19 show the real and imaginary parts of $z^{1/2}$ from three different viewpoints. The second and third views are looking down the x axis and y axis, respectively. Consider $\Re(z^{1/2})$. This is a double layered sheet which intersects itself on the negative real axis. ($\Im(z^{1/2})$ has a similar structure, but intersects itself on the positive real axis.) Let's start at a point on the positive real axis on the lower sheet. If we walk around the origin once and return to the positive real axis, we will be on the upper sheet. If we do this again, we will return to the lower sheet.

Suppose we are at a point in the complex plane. We pick one of the two values of $z^{1/2}$. If the function varies continuously as we walk around the origin and back to our starting point, the value of $z^{1/2}$ will have changed. We will be on the other branch. Because walking around the point $z = 0$ takes us to a different branch of the function, we refer to $z = 0$ as a *branch point*.

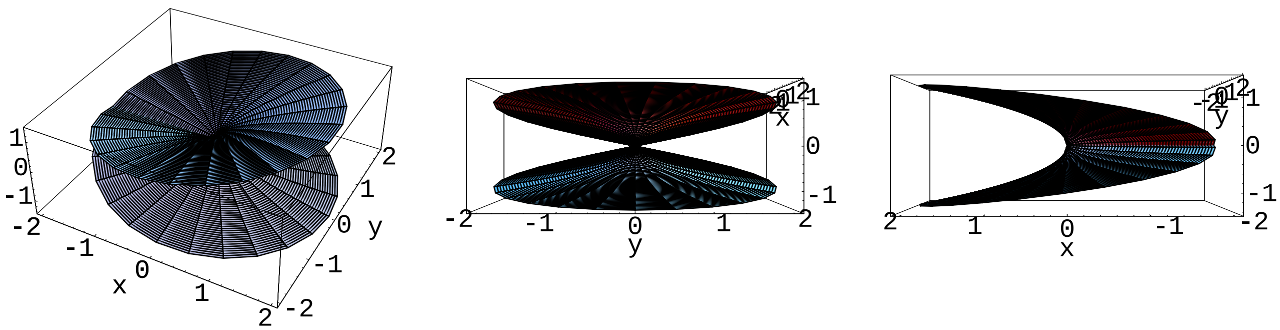


Figure 9.18: Plots of $\Re(z^{1/2})$ from three viewpoints.

Now consider the modulus-argument form of $z^{1/2}$:

$$z^{1/2} = \sqrt{|z|}\,\mathrm{e}^{i\arg(z)/2}.$$

Figure 9.20 shows the modulus and the principal argument of $z^{1/2}$. We see that each time we walk around the origin, the argument of $z^{1/2}$ changes by π. This means that the value of the function changes by the factor $\mathrm{e}^{i\pi} = -1$, i.e. the function changes sign. If we walk around the origin twice, the argument changes by 2π, so that the value of the function does not change, $\mathrm{e}^{i2\pi} = 1$.

$z^{1/2}$ is a continuous function except at $z = 0$. Suppose we start at $z = 1 = \mathrm{e}^{i0}$ and the function value $(\mathrm{e}^{i0})^{1/2} = 1$. If we follow the first path in Figure 9.21, the argument of z varies from up to about $\frac{\pi}{4}$, down to about $-\frac{\pi}{4}$ and back to 0. The value of the function is still $(\mathrm{e}^{i0})^{1/2}$.

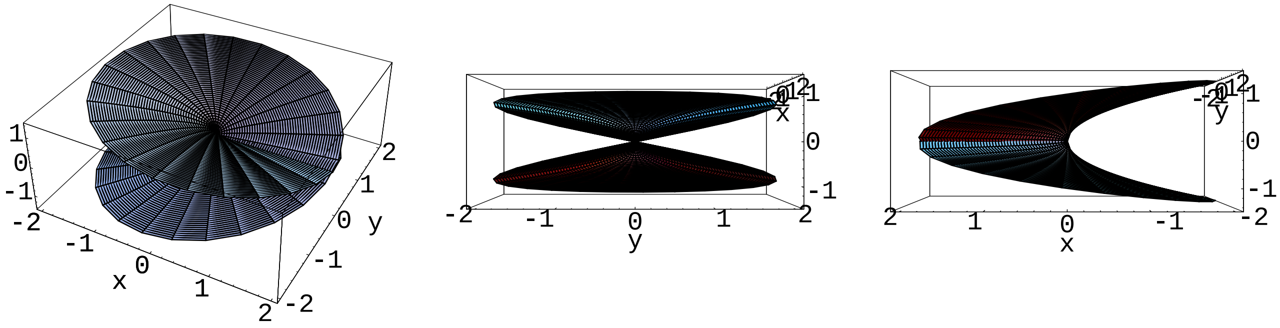


Figure 9.19: Plots of $\Im(z^{1/2})$ from three viewpoints.

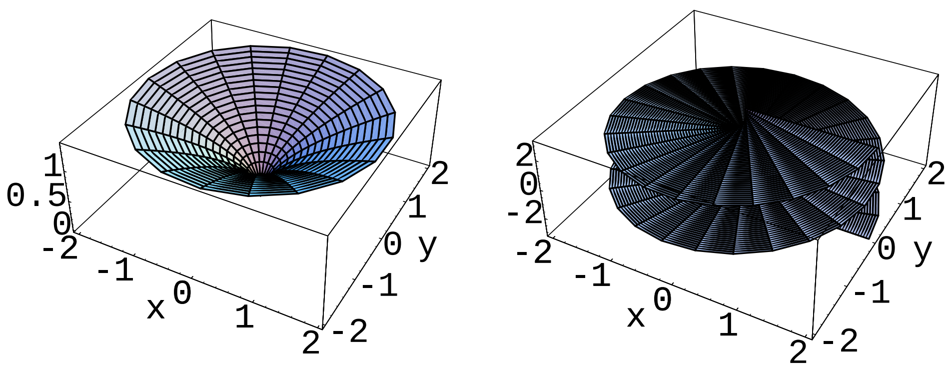


Figure 9.20: Plots of $|z^{1/2}|$ and $\operatorname{Arg}(z^{1/2})$.

Now suppose we follow a circular path around the origin in the positive, counter-clockwise, direction. (See the second path in Figure 9.21.) The argument of z increases by 2π. The value of the function at half turns on the

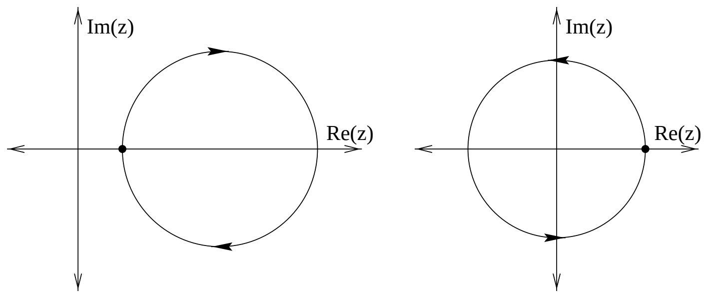


Figure 9.21: A path that does not encircle the origin and a path around the origin

path is

$$
\begin{aligned}
(\,\mathrm{e}^{i0})^{1/2} &= 1, \\
(\,\mathrm{e}^{i\pi})^{1/2} &= \,\mathrm{e}^{i\pi/2} = i, \\
(\,\mathrm{e}^{i2\pi})^{1/2} &= \,\mathrm{e}^{i\pi} = -1
\end{aligned}
$$

As we return to the point $z = 1$, the argument of the function has changed by π and the value of the function has changed from 1 to -1. If we were to walk along the circular path again, the argument of z would increase by another 2π. The argument of the function would increase by another π and the value of the function would return to 1.

$$(\,\mathrm{e}^{4\pi i})^{1/2} = \,\mathrm{e}^{2\pi i} = 1$$

In general, any time we walk around the origin, the value of $z^{1/2}$ changes by the factor -1. We call $z = 0$ a branch point. If we want a single-valued square root, we need something to prevent us from walking around the origin. We achieve this by introducing a branch cut. Suppose we have the complex plane drawn on an infinite sheet of paper. With a scissors we cut the paper from the origin to $-\infty$ along the real axis. Then if we start at $z = e^{i0}$, and draw a continuous line without leaving the paper, the argument of z will always be in the range $-\pi < \arg z < \pi$. This means that $-\frac{\pi}{2} < \arg(z^{1/2}) < \frac{\pi}{2}$. No matter what path we follow in this cut plane, $z = 1$ has argument zero and $(1)^{1/2} = 1$. By never crossing the negative real axis, we have constructed a single valued **branch** of the square root function. We call the cut along the negative real axis a **branch cut**.

Example 9.6.2 Consider the logarithmic function $\log z$. For each value of z, there are an infinite number of values of $\log z$. We write $\log z$ in Cartesian form.

$$\log z = \operatorname{Log}|z| + i \arg z$$

Figure 9.22 shows the real and imaginary parts of the logarithm. The real part is single-valued. The imaginary part is multi-valued and has an infinite number of branches. The values of the logarithm form an infinite-layered sheet. If we start on one of the sheets and walk around the origin once in the positive direction, then the value of the logarithm increases by $i2\pi$ and we move to the next branch. $z = 0$ is a branch point of the logarithm.

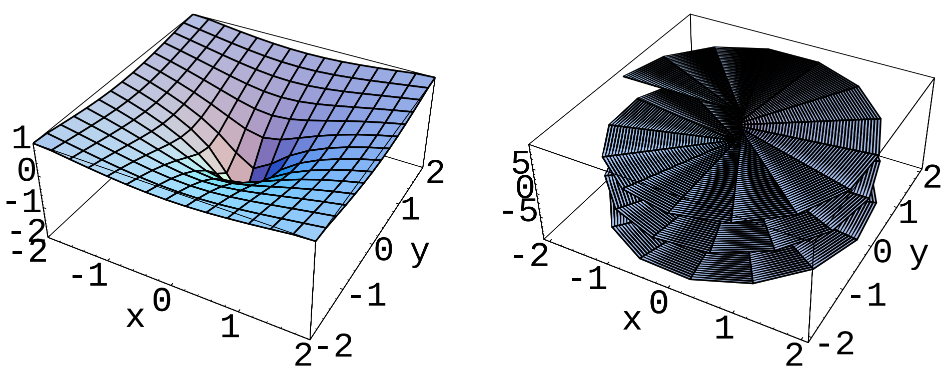


Figure 9.22: Plots of $\Re(\log z)$ and a portion of $\Im(\log z)$.

The logarithm is a continuous function except at $z = 0$. Suppose we start at $z = 1 = e^{i0}$ and the function value $\log(e^{i0}) = \operatorname{Log}(1) + i0 = 0$. If we follow the first path in Figure 9.21, the argument of z and thus the imaginary part of the logarithm varies from up to about $\frac{\pi}{4}$, down to about $-\frac{\pi}{4}$ and back to 0. The value of the logarithm is still 0.

Now suppose we follow a circular path around the origin in the positive direction. (See the second path in Figure 9.21.) The argument of z increases by 2π. The value of the logarithm at half turns on the path is

$$\begin{aligned}\log(\mathrm{e}^{i0}) &= 0,\\ \log(\mathrm{e}^{i\pi}) &= i\pi,\\ \log(\mathrm{e}^{i2\pi}) &= i2\pi\end{aligned}$$

As we return to the point $z = 1$, the value of the logarithm has changed by $i2\pi$. If we were to walk along the circular path again, the argument of z would increase by another 2π and the value of the logarithm would increase by another $i2\pi$.

Result 9.6.1 A point z_0 is a **branch point** of a function $f(z)$ if the function changes value when you walk around the point on any path that encloses no singularities other than the one at $z = z_0$.

Branch Points at Infinity : Mapping Infinity to the Origin. Up to this point we have considered only branch points in the finite plane. Now we consider the possibility of a branch point at infinity. As a first method of approaching this problem we map the point at infinity to the origin with the transformation $\zeta = 1/z$ and examine the point $\zeta = 0$.

Example 9.6.3 Again consider the function $z^{1/2}$. Mapping the point at infinity to the origin, we have $f(\zeta) = (1/\zeta)^{1/2} = \zeta^{-1/2}$. For each value of ζ, there are two values of $\zeta^{-1/2}$. We write $\zeta^{-1/2}$ in modulus-argument form.

$$\zeta^{-1/2} = \frac{1}{\sqrt{|\zeta|}}\,\mathrm{e}^{-i\arg(\zeta)/2}$$

Like $z^{1/2}$, $\zeta^{-1/2}$ has a double-layered sheet of values. Figure 9.23 shows the modulus and the principal argument of $\zeta^{-1/2}$. We see that each time we walk around the origin, the argument of $\zeta^{-1/2}$ changes by $-\pi$. This means that the value of the function changes by the factor $e^{-i\pi} = -1$, i.e. the function changes sign. If we walk around the origin twice, the argument changes by -2π, so that the value of the function does not change, $e^{-i2\pi} = 1$.

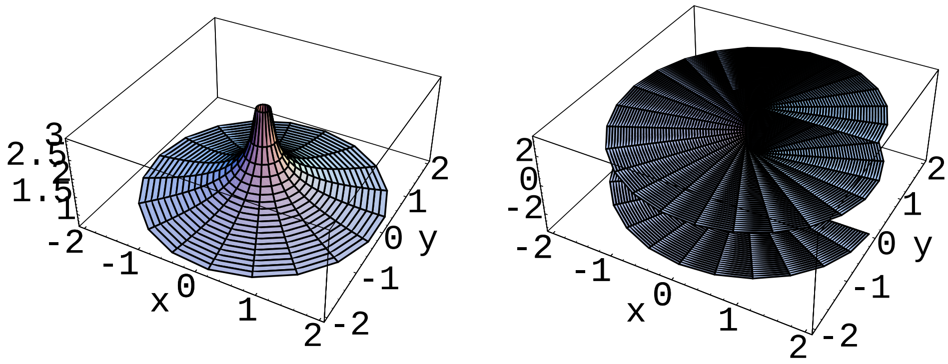


Figure 9.23: Plots of $|\zeta^{-1/2}|$ and $\operatorname{Arg}(\zeta^{-1/2})$.

Since $\zeta^{-1/2}$ has a branch point at zero, we conclude that $z^{1/2}$ has a branch point at infinity.

Example 9.6.4 Again consider the logarithmic function $\log z$. Mapping the point at infinity to the origin, we have $f(\zeta) = \log(1/\zeta) = -\log(\zeta)$. From Example 9.6.2 we known that $-\log(\zeta)$ has a branch point at $\zeta = 0$. Thus $\log z$ has a branch point at infinity.

Branch Points at Infinity : Paths Around Infinity. We can also check for a branch point at infinity by following a path that encloses the point at infinity and no other singularities. Just draw a simple closed curve that separates the complex plane into a bounded component that contains all the singularities of the function in the finite plane. Then, depending on orientation, the curve is a contour enclosing all the finite singularities, or the point at infinity and no other singularities.

Example 9.6.5 Once again consider the function $z^{1/2}$. We know that the function changes value on a curve that goes once around the origin. Such a curve can be considered to be either a path around the origin or a path around infinity. In either case the path encloses one singularity. There are branch points at the origin and at infinity. Now consider a curve that does not go around the origin. Such a curve can be considered to be either a path around neither of the branch points or both of them. Thus we see that $z^{1/2}$ does not change value when we follow a path that encloses neither or both of its branch points.

Example 9.6.6 Consider $f(z) = (z^2 - 1)^{1/2}$. We factor the function.

$$f(z) = (z-1)^{1/2}(z+1)^{1/2}$$

There are branch points at $z = \pm 1$. Now consider the point at infinity.

$$f(\zeta^{-1}) = (\zeta^{-2} - 1)^{1/2} = \pm\zeta^{-1}(1 - \zeta^2)^{1/2}$$

Since $f(\zeta^{-1})$ does not have a branch point at $\zeta = 0$, $f(z)$ does not have a branch point at infinity. We could reach the same conclusion by considering a path around infinity. Consider a path that circles the branch points at $z = \pm 1$ once in the positive direction. Such a path circles the point at infinity once in the negative direction. In traversing this path, the value of $f(z)$ is multiplied by the factor $(\mathrm{e}^{i2\pi})^{1/2}(\mathrm{e}^{i2\pi})^{1/2} = \mathrm{e}^{i2\pi} = 1$. Thus the value of the function does not change. There is no branch point at infinity.

Diagnosing Branch Points. We have the definition of a branch point, but we do not have a convenient criterion for determining if a particular function has a branch point. We have seen that $\log z$ and z^α for non-integer α have branch points at zero and infinity. The inverse trigonometric functions like the arcsine also have branch points, but they can be written in terms of the logarithm and the square root. In fact all the elementary functions with branch points can be written in terms of the functions $\log z$ and z^α. Furthermore, note that the multi-valuedness of z^α comes from the logarithm, $z^\alpha = \mathrm{e}^{\alpha \log z}$. This gives us a way of quickly determining if and where a function may have branch points.

Result 9.6.2 Let $f(z)$ be a single-valued function. Then $\log(f(z))$ and $(f(z))^\alpha$ may have branch points only where $f(z)$ is zero or singular.

Example 9.6.7 Consider the functions,

1. $(z^2)^{1/2}$
2. $(z^{1/2})^2$
3. $(z^{1/2})^3$

Are they multi-valued? Do they have branch points?

1.
$$(z^2)^{1/2} = \pm\sqrt{z^2} = \pm z$$
Because of the $(\cdot)^{1/2}$, the function is multi-valued. The only possible branch points are at zero and infinity. If $((\,\mathrm{e}^{i0})^2)^{1/2} = 1$, then $((\,\mathrm{e}^{2\pi i})^2)^{1/2} = (\,\mathrm{e}^{4\pi i})^{1/2} = \mathrm{e}^{2\pi i} = 1$. Thus we see that the function does not change value when we walk around the origin. We can also consider this to be a path around infinity. This function is multi-valued, but has no branch points.

2.
$$(z^{1/2})^2 = (\pm\sqrt{z})^2 = z$$
This function is single-valued.

3.
$$(z^{1/2})^3 = (\pm\sqrt{z})^3 = \pm(\sqrt{z})^3$$
This function is multi-valued. We consider the possible branch point at $z = 0$. If $((\,\mathrm{e}^0)^{1/2})^3 = 1$, then $((\,\mathrm{e}^{2\pi i})^{1/2})^3 = (\,\mathrm{e}^{\pi i})^3 = \mathrm{e}^{3\pi i} = -1$. Since the function changes value when we walk around the origin, it has a branch point at $z = 0$. Since this is also a path around infinity, there is a branch point there.

Example 9.6.8 Consider the function $f(z) = \log\left(\frac{1}{z-1}\right)$. Since $\frac{1}{z-1}$ is only zero at infinity and its only singularity is at $z = 1$, the only possibilities for branch points are at $z = 1$ and $z = \infty$. Since

$$\log\left(\frac{1}{z-1}\right) = -\log(z-1)$$

and $\log w$ has branch points at zero and infinity, we see that $f(z)$ has branch points at $z = 1$ and $z = \infty$.

Example 9.6.9 Consider the functions,

1. $\mathrm{e}^{\log z}$
2. $\log \mathrm{e}^z$.

Are they multi-valued? Do they have branch points?

1. $$\mathrm{e}^{\log z} = \exp(\operatorname{Log} z + 2\pi i n) = \mathrm{e}^{\operatorname{Log} z}\,\mathrm{e}^{2\pi i n} = z$$

   This function is single-valued.

2. $$\log \mathrm{e}^z = \operatorname{Log} \mathrm{e}^z + 2\pi i n = z + 2\pi i m$$

   This function is multi-valued. It may have branch points only where e^z is zero or infinite. This only occurs at $z = \infty$. Thus there are no branch points in the finite plane. The function does not change when traversing a simple closed path. Since this path can be considered to enclose infinity, there is no branch point at infinity.

Consider $(f(z))^\alpha$ where $f(z)$ is single-valued and $f(z)$ has either a zero or a singularity at $z = z_0$. $(f(z))^\alpha$ may have a branch point at $z = z_0$. If $f(z)$ is not a power of z, then it may be difficult to tell if $(f(z))^\alpha$ changes value when we walk around z_0. Factor $f(z)$ into $f(z) = g(z)h(z)$ where $h(z)$ is nonzero and finite at z_0. Then $g(z)$ captures the important behavior of $f(z)$ at the z_0. $g(z)$ tells us how fast $f(z)$ vanishes or blows up. Since $(f(z))^\alpha = (g(z))^\alpha(h(z))^\alpha$ and $(h(z))^\alpha$ does not have a branch point at z_0, $(f(z))^\alpha$ has a branch point at z_0 if and only if $(g(z))^\alpha$ has a branch point there.

Similarly, we can decompose

$$\log(f(z)) = \log(g(z)h(z)) = \log(g(z)) + \log(h(z))$$

to see that $\log(f(z))$ has a branch point at z_0 if and only if $\log(g(z))$ has a branch point there.

Result 9.6.3 Consider a single-valued function $f(z)$ that has either a zero or a singularity at $z = z_0$. Let $f(z) = g(z)h(z)$ where $h(z)$ is nonzero and finite. $(f(z))^\alpha$ has a branch point at $z = z_0$ if and only if $(g(z))^\alpha$ has a branch point there. $\log(f(z))$ has a branch point at $z = z_0$ if and only if $\log(g(z))$ has a branch point there.

Example 9.6.10 Consider the functions,

1. $\sin z^{1/2}$
2. $(\sin z)^{1/2}$
3. $z^{1/2} \sin z^{1/2}$
4. $(\sin z^2)^{1/2}$

Find the branch points and the number of branches.

1.

$$\sin z^{1/2} = \sin(\pm\sqrt{z}) = \pm \sin \sqrt{z}$$

$\sin z^{1/2}$ is multi-valued. It has two branches. There may be branch points at zero and infinity. Consider the unit circle which is a path around the origin or infinity. If $\sin((\mathrm{e}^{i0})^{1/2}) = \sin(1)$, then $\sin((\mathrm{e}^{i2\pi})^{1/2}) = \sin(\mathrm{e}^{i\pi}) = \sin(-1) = -\sin(1)$. There are branch points at the origin and infinity.

2.

$$(\sin z)^{1/2} = \pm\sqrt{\sin z}$$

The function is multi-valued with two branches. The sine vanishes at $z = n\pi$ and is singular at infinity. There could be branch points at these locations. Consider the point $z = n\pi$. We can write

$$\sin z = (z - n\pi)\frac{\sin z}{z - n\pi}$$

Note that $\frac{\sin z}{z-n\pi}$ is nonzero and has a removable singularity at $z = n\pi$.

$$\lim_{z \to n\pi} \frac{\sin z}{z - n\pi} = \lim_{z \to n\pi} \frac{\cos z}{1} = (-1)^n$$

Since $(z - n\pi)^{1/2}$ has a branch point at $z = n\pi$, $(\sin z)^{1/2}$ has branch points at $z = n\pi$.

Since the branch points at $z = n\pi$ go all the way out to infinity. It is not possible to make a path that encloses infinity and no other singularities. The point at infinity is a non-isolated singularity. A point can be a branch point only if it is an isolated singularity.

3.

$$\begin{aligned} z^{1/2} \sin z^{1/2} &= \pm\sqrt{z}\sin(\pm\sqrt{z}) \\ &= \pm\sqrt{z}(\pm\sin\sqrt{z}) \\ &= \sqrt{z}\sin\sqrt{z} \end{aligned}$$

The function is single-valued. Thus there could be no branch points.

4.

$$(\sin z^2)^{1/2} = \pm\sqrt{\sin z^2}$$

This function is multi-valued. Since $\sin z^2 = 0$ at $z = (n\pi)^{1/2}$, there may be branch points there. First consider the point $z = 0$. We can write

$$\sin z^2 = z^2 \frac{\sin z^2}{z^2}$$

where $\sin(z^2)/z^2$ is nonzero and has a removable singularity at $z = 0$.

$$\lim_{z\to 0} \frac{\sin z^2}{z^2} = \lim_{z\to 0} \frac{2z\cos z^2}{2z} = 1.$$

Since $(z^2)^{1/2}$ does not have a branch point at $z = 0$, $(\sin z^2)^{1/2}$ does not have a branch point there either.

Now consider the point $z = \sqrt{n\pi}$.

$$\sin z^2 = (z - \sqrt{n\pi})\frac{\sin z^2}{z - \sqrt{n\pi}}$$

$\sin(z^2)/(z - \sqrt{n\pi})$ in nonzero and has a removable singularity at $z = \sqrt{n\pi}$.

$$\lim_{z\to\sqrt{n\pi}} \frac{\sin z^2}{z - \sqrt{n\pi}} = \lim_{z\to\sqrt{n\pi}} \frac{2z\cos z^2}{1} = 2\sqrt{n\pi}(-1)^n$$

Since $(z - \sqrt{n\pi})^{1/2}$ has a branch point at $z = \sqrt{n\pi}$, $(\sin z^2)^{1/2}$ also has a branch point there.

Thus we see that $(\sin z^2)^{1/2}$ has branch points at $z = (n\pi)^{1/2}$ for $n \in \mathbb{Z} \setminus \{0\}$. This is the set of numbers: $\{\pm\sqrt{\pi}, \pm\sqrt{2\pi}, \ldots, \pm i\sqrt{\pi}, \pm i\sqrt{2\pi}, \ldots\}$. The point at infinity is a non-isolated singularity.

Example 9.6.11 Find the branch points of

$$f(z) = (z^3 - z)^{1/3}.$$

Introduce branch cuts. If $f(2) = \sqrt[3]{6}$ then what is $f(-2)$?

We expand $f(z)$.

$$f(z) = z^{1/3}(z-1)^{1/3}(z+1)^{1/3}.$$

There are branch points at $z = -1, 0, 1$. We consider the point at infinity.

$$\begin{aligned} f\left(\frac{1}{\zeta}\right) &= \left(\frac{1}{\zeta}\right)^{1/3}\left(\frac{1}{\zeta}-1\right)^{1/3}\left(\frac{1}{\zeta}+1\right)^{1/3} \\ &= \frac{1}{\zeta}\left(1-\zeta\right)^{1/3}\left(1+\zeta\right)^{1/3} \end{aligned}$$

Since $f(1/\zeta)$ does not have a branch point at $\zeta = 0$, $f(z)$ does not have a branch point at infinity. Consider the three possible branch cuts in Figure 9.24.

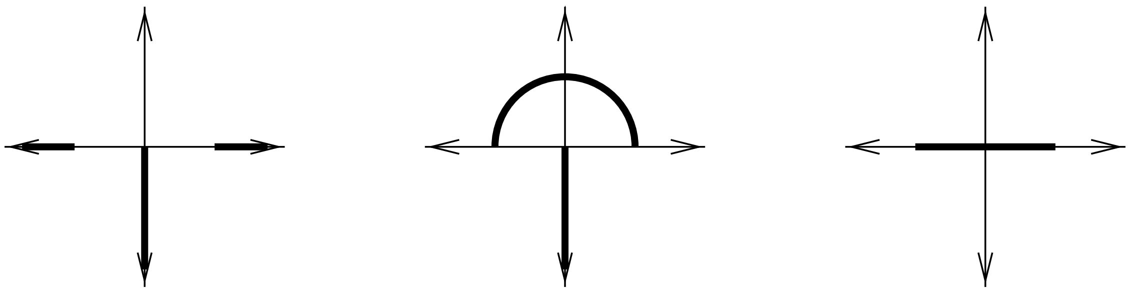


Figure 9.24: Three Possible Branch Cuts for $f(z) = (z^3 - z)^{1/3}$

The first and the third branch cuts will make the function single valued, the second will not. It is clear that the first set makes the function single valued since it is not possible to walk around any of the branch points.

The second set of branch cuts would allow you to walk around the branch points at $z = \pm 1$. If you walked around these two once in the positive direction, the value of the function would change by the factor $\mathrm{e}^{4\pi i/3}$.

The third set of branch cuts would allow you to walk around all three branch points together. You can verify that if you walk around the three branch points, the value of the function will not change ($\mathrm{e}^{6\pi i/3} = \mathrm{e}^{2\pi i} = 1$).

Suppose we introduce the third set of branch cuts and are on the branch with $f(2) = \sqrt[3]{6}$.

$$f(2) = (2\,\mathrm{e}^{i0})^{1/3}(1\,\mathrm{e}^{i0})^{1/3}(3\,\mathrm{e}^{i0})^{1/3} = \sqrt[3]{6}.$$

The value of $f(-2)$ is

$$\begin{aligned} f(-2) &= (2\,\mathrm{e}^{i\pi})^{1/3}(3\,\mathrm{e}^{i\pi})^{1/3}(1\,\mathrm{e}^{i\pi})^{1/3} \\ &= \sqrt[3]{2}\,\mathrm{e}^{i\pi/3}\sqrt[3]{3}\,\mathrm{e}^{i\pi/3}\sqrt[3]{1}\,\mathrm{e}^{i\pi/3} \\ &= \sqrt[3]{6}\,\mathrm{e}^{i\pi} \\ &= -\sqrt[3]{6}. \end{aligned}$$

Example 9.6.12 Find the branch points and number of branches for

$$f(z) = z^{z^2}.$$

$$z^{z^2} = \exp(z^2 \log z)$$

There may be branch points at the origin and infinity due to the logarithm. Consider walking around a circle of radius r centered at the origin in the positive direction. Since the logarithm changes by $i2\pi$, the value of $f(z)$ changes by the factor $\mathrm{e}^{i2\pi r^2}$. There are branch points at the origin and infinity. The function has an infinite number of branches.

Example 9.6.13 Construct a branch of

$$f(z) = (z^2 + 1)^{1/3}$$

such that

$$f(0) = \frac{1}{2}(-1 + \sqrt{3}i).$$

First we factor $f(z)$.

$$f(z) = (z - i)^{1/3}(z + i)^{1/3}$$

There are branch points at $z = \pm i$. Figure 9.25 shows one way to introduce branch cuts.

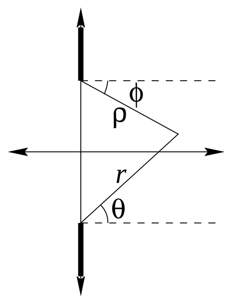


Figure 9.25: Branch Cuts for $f(z) = (z^2 + 1)^{1/3}$

Since it is not possible to walk around any branch point, these cuts make the function single valued. We introduce the coordinates:

$$z - i = \rho\, e^{i\phi}, \quad z + i = r\, e^{i\theta}.$$

$$\begin{aligned} f(z) &= (\rho\, e^{i\phi})^{1/3}(r\, e^{i\theta})^{1/3} \\ &= \sqrt[3]{\rho r}\, e^{i(\phi+\theta)/3} \end{aligned}$$

The condition

$$f(0) = \frac{1}{2}(-1 + \sqrt{3}i) = \mathrm{e}^{i(2\pi/3 + 2\pi n)}$$

can be stated

$$\sqrt[3]{1}\,\mathrm{e}^{i(\phi+\theta)/3} = \mathrm{e}^{i(2\pi/3+2\pi n)}$$
$$\phi + \theta = 2\pi + 6\pi n$$

The angles must be defined to satisfy this relation. One choice is

$$\boxed{\frac{\pi}{2} < \phi < \frac{5\pi}{2}, \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2}.}$$

Principal Branches. We construct the principal branch of the logarithm by putting a branch cut on the negative real axis choose $z = r\,\mathrm{e}^{i\theta}$, $\theta \in (-\pi, \pi)$. Thus the principal branch of the logarithm is

$$\operatorname{Log} z = \operatorname{Log} r + i\theta, \qquad -\pi < \theta < \pi.$$

Note that the if x is a negative real number, (and thus lies on the branch cut), then $\operatorname{Log} x$ is undefined.

The principal branch of z^α is

$$z^\alpha = \mathrm{e}^{\alpha \operatorname{Log} z}.$$

Note that there is a branch cut on the negative real axis.

$$-\alpha\pi < \arg(\mathrm{e}^{\alpha \operatorname{Log} z}) < \alpha\pi$$

The principal branch of the $z^{1/2}$ is denoted $\sqrt{z}$. The principal branch of $z^{1/n}$ is denoted $\sqrt[n]{z}$.

Example 9.6.14 Construct $\sqrt{1-z^2}$, the principal branch of $(1-z^2)^{1/2}$.

First note that since $(1-z^2)^{1/2} = (1-z)^{1/2}(1+z)^{1/2}$ there are branch points at $z = 1$ and $z = -1$. The principal branch of the square root has a branch cut on the negative real axis. $1 - z^2$ is a negative real number for $z \in (-\infty, -1) \cup (1, \infty)$. Thus we put branch cuts on $(-\infty, -1]$ and $[1, \infty)$.

9.7 Exercises

Cartesian and Modulus-Argument Form

Exercise 9.1
For a given real number ϕ, $0 \leq \phi < 2\pi$, find the image of the sector $0 \leq \arg(z) < \phi$ under the transformation $w = z^4$. How large should ϕ be so that the w plane is covered exactly once?
Hint, Solution

Trigonometric Functions

Exercise 9.2
In Cartesian coordinates, $z = x + iy$, write $\sin(z)$ in Cartesian and modulus-argument form.
Hint, Solution

Exercise 9.3
Show that e^z in nonzero for all finite z.
Hint, Solution

Exercise 9.4
Show that

$$\left| \mathrm{e}^{z^2} \right| \leq \mathrm{e}^{|z|^2}.$$

When does equality hold?
Hint, Solution

Exercise 9.5
Solve $\coth(z) = 1$.
Hint, Solution

Exercise 9.6
Solve $2 \in 2^z$. That is, for what values of z is 2 one of the values of 2^z? Derive this result then verify your answer by evaluating 2^z for the solutions that your find.
Hint, Solution

Exercise 9.7
Solve $1 \in 1^z$. That is, for what values of z is 1 one of the values of 1^z? Derive this result then verify your answer by evaluating 1^z for the solutions that your find.
Hint, Solution

Logarithmic Identities

Exercise 9.8
Find the fallacy in the following arguments:

1. $\log(-1) = \log\left(\frac{1}{-1}\right) = \log(1) - \log(-1) = -\log(-1)$, therefore, $\log(-1) = 0$.
2. $1 = 1^{1/2} = ((-1)(-1))^{1/2} = (-1)^{1/2}(-1)^{1/2} = ii = -1$, therefore, $1 = -1$.

Hint, Solution

Exercise 9.9
Write the following expressions in modulus-argument or Cartesian form. Denote any multi-valuedness explicitly.

$$2^{2/5}, \quad 3^{1+i}, \quad (\sqrt{3} - i)^{1/4}, \quad 1^{i/4}.$$

Hint, Solution

Exercise 9.10
Solve $\cos z = 69$.
Hint, Solution

Exercise 9.11
Solve $\cot z = i47$.
Hint, Solution

Exercise 9.12
Determine all values of

1. $\log(-i)$
2. $(-i)^{-i}$
3. 3^{π}
4. $\log(\log(i))$

and plot them in the complex plane.
Hint, Solution

Exercise 9.13
Determine all values of i^i and $\log((1+i)^{i\pi})$ and plot them in the complex plane.
Hint, Solution

Exercise 9.14
Find all z for which

1. $\mathrm{e}^z = i$
2. $\cos z = \sin z$
3. $\tan^2 z = -1$

Hint, Solution

Exercise 9.15
Show that

$$\tan^{-1}(z) = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right)$$

and

$$\tanh^{-1}(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right).$$

Hint, Solution

Branch Points and Branch Cuts

Exercise 9.16
Determine the branch points of the function

$$f(z) = (z^3 - 1)^{1/2}.$$

Construct cuts and define a branch so that $z = 0$ and $z = -1$ do not lie on a cut, and such that $f(0) = -i$. What is $f(-1)$ for this branch?
Hint, Solution

Exercise 9.17
Determine the branch points of the function

$$w(z) = ((z-1)(z-6)(z+2))^{1/2}$$

Construct cuts and define a branch so that $z = 4$ does not lie on a cut, and such that $w = 6i$ when $z = 4$.
Hint, Solution

Exercise 9.18
Give the number of branches and locations of the branch points for the functions

1. $\cos z^{1/2}$
2. $(z + i)^{-z}$

Hint, Solution

Exercise 9.19
Find the branch points of the following functions in the extended complex plane, (the complex plane including the point at infinity).

1. $(z^2 + 1)^{1/2}$
2. $(z^3 - z)^{1/2}$
3. $\log(z^2 - 1)$
4. $\log\left(\frac{z+1}{z-1}\right)$

Introduce branch cuts to make the functions single valued.
Hint, Solution

Exercise 9.20
Find all branch points and introduce cuts to make the following functions single-valued: For the first function, choose cuts so that there is no cut within the disk $|z| < 2$.

1. $f(z) = \left(z^3 + 8\right)^{1/2}$
2. $f(z) = \log\left(5 + \left(\frac{z+1}{z-1}\right)^{1/2}\right)$

3. $f(z) = (z + i3)^{1/2}$

Hint, Solution

Exercise 9.21
Let $f(z)$ have branch points at $z = 0$ and $z = \pm i$, but nowhere else in the extended complex plane. How does the value and argument of $f(z)$ change while traversing the contour in Figure 9.26? Does the branch cut in Figure 9.26 make the function single-valued?

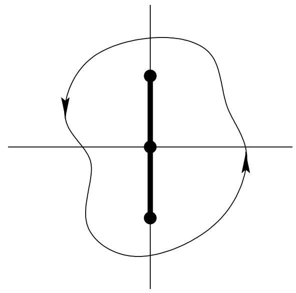


Figure 9.26: Contour Around the Branch Points and Branch Cut.

Hint, Solution

Exercise 9.22
Let $f(z)$ be analytic except for no more than a countably infinite number of singularities. Suppose that $f(z)$ has only one branch point in the finite complex plane. Does $f(z)$ have a branch point at infinity? Now suppose that $f(z)$ has two or more branch points in the finite complex plane. Does $f(z)$ have a branch point at infinity?
Hint, Solution

Exercise 9.23
Find all branch points of $(z^4 + 1)^{1/4}$ in the extended complex plane. Which of the branch cuts in Figure 9.27 make the function single-valued.

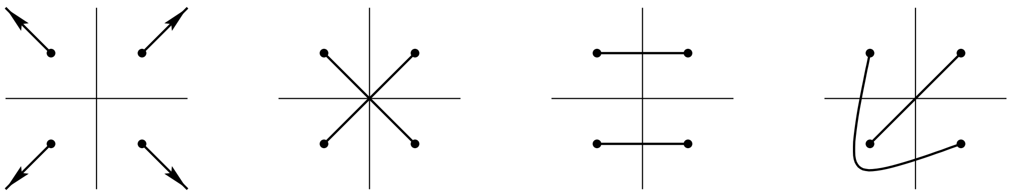


Figure 9.27: Four Candidate Sets of Branch Cuts for $(z^4 + 1)^{1/4}$

Hint, Solution

Exercise 9.24
Find the branch points of

$$f(z) = \left(\frac{z}{z^2 + 1}\right)^{1/3}$$

in the extended complex plane. Introduce branch cuts that make the function single-valued and such that the function is defined on the positive real axis. Define a branch such that $f(1) = 1/\sqrt[3]{2}$. Write down an explicit formula for the value of the branch. What is $f(1 + i)$? What is the value of $f(z)$ on either side of the branch cuts?
Hint, Solution

Exercise 9.25
Find all branch points of

$$f(z) = ((z - 1)(z - 2)(z - 3))^{1/2}$$

in the extended complex plane. Which of the branch cuts in Figure 9.28 will make the function single-valued. Using the first set of branch cuts in this figure define a branch on which $f(0) = i\sqrt{6}$. Write out an explicit formula for the value of the function on this branch.

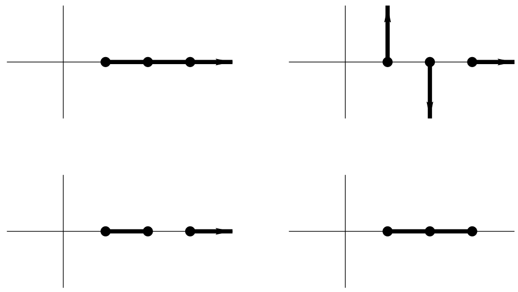


Figure 9.28: Four Candidate Sets of Branch Cuts for $((z-1)(z-2)(z-3))^{1/2}$

Hint, Solution

Exercise 9.26
Determine the branch points of the function

$$w = \left((z^2 - 2)(z + 2)\right)^{1/3}.$$

Construct and define a branch so that the resulting cut is one line of finite extent and $w(2) = 2$. What is $w(-3)$ for this branch? What are the limiting values of w on either side of the branch cut?
Hint, Solution

Exercise 9.27
Construct the principal branch of $\arccos(z)$. ($\operatorname{Arccos}(z)$ has the property that if $x \in [-1, 1]$ then $\operatorname{Arccos}(x) \in [0, \pi]$. In particular, $\operatorname{Arccos}(0) = \frac{\pi}{2}$).
Hint, Solution

Exercise 9.28
Find the branch points of $(z^{1/2} - 1)^{1/2}$ in the finite complex plane. Introduce branch cuts to make the function single-valued.
Hint, Solution

Exercise 9.29
For the linkage illustrated in Figure 9.29, use complex variables to outline a scheme for expressing the angular position, velocity and acceleration of arm c in terms of those of arm a. (You needn't work out the equations.)

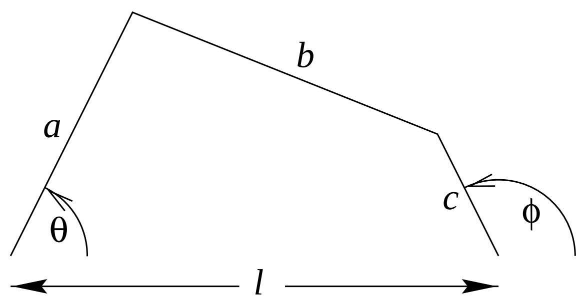


Figure 9.29: A linkage

Hint, Solution

Exercise 9.30
Find the image of the strip $|\Re(z)| < 1$ and of the strip $1 < \Im(z) < 2$ under the transformations:

1. $w = 2z^2$

2. $w = \frac{z+1}{z-1}$

Hint, Solution

Exercise 9.31
Locate and classify all the singularities of the following functions:

1. $\dfrac{(z+1)^{1/2}}{z+2}$

2. $\cos\left(\dfrac{1}{1+z}\right)$

3. $\dfrac{1}{(1-\mathrm{e}^z)^2}$

In each case discuss the possibility of a singularity at the point ∞.
Hint, Solution

Exercise 9.32
Describe how the mapping $w = \sinh(z)$ transforms the infinite strip $-\infty < x < \infty$, $0 < y < \pi$ into the w-plane. Find cuts in the w-plane which make the mapping continuous both ways. What are the images of the lines (a) $y = \pi/4$; (b) $x = 1$?
Hint, Solution

9.8 Hints

Cartesian and Modulus-Argument Form

Hint 9.1

Trigonometric Functions

Hint 9.2
Recall that $\sin(z) = \frac{1}{2i}(\mathrm{e}^{iz} - \mathrm{e}^{-iz})$. Use Result 8.3.1 to convert between Cartesian and modulus-argument form.

Hint 9.3
Write e^z in polar form.

Hint 9.4
The exponential is an increasing function for real variables.

Hint 9.5
Write the hyperbolic cotangent in terms of exponentials.

Hint 9.6
Write out the multi-valuedness of 2^z. There is a doubly-infinite set of solutions to this problem.

Hint 9.7
Write out the multi-valuedness of 1^z.

Logarithmic Identities

Hint 9.8
Write out the multi-valuedness of the expressions.

Hint 9.9
Do the exponentiations in polar form.

Hint 9.10
Write the cosine in terms of exponentials. Multiply by e^{iz} to get a quadratic equation for e^{iz}.

Hint 9.11
Write the cotangent in terms of exponentials. Get a quadratic equation for e^{iz}.

Hint 9.12

Hint 9.13
i^i has an infinite number of real, positive values. $i^i = \mathrm{e}^{i \log i}$. $\log((1+i)^{i\pi})$ has a doubly infinite set of values. $\log((1+i)^{i\pi}) = \log(\exp(i\pi \log(1+i)))$.

Hint 9.14

Hint 9.15

Branch Points and Branch Cuts

Hint 9.16

Hint 9.17

Hint 9.18

Hint 9.19

1. $(z^2 + 1)^{1/2} = (z - i)^{1/2}(z + i)^{1/2}$
2. $(z^3 - z)^{1/2} = z^{1/2}(z - 1)^{1/2}(z + 1)^{1/2}$
3. $\log(z^2 - 1) = \log(z - 1) + \log(z + 1)$
4. $\log\left(\frac{z+1}{z-1}\right) = \log(z + 1) - \log(z - 1)$

Hint 9.20

Hint 9.21

Reverse the orientation of the contour so that it encircles infinity and does not contain any branch points.

Hint 9.22

Consider a contour that encircles all the branch points in the finite complex plane. Reverse the orientation of the contour so that it contains the point at infinity and does not contain any branch points in the finite complex plane.

Hint 9.23
Factor the polynomial. The argument of $z^{1/4}$ changes by $\pi/2$ on a contour that goes around the origin once in the positive direction.

Hint 9.24

Hint 9.25
To define the branch, define angles from each of the branch points in the finite complex plane.

Hint 9.26

Hint 9.27

Hint 9.28

Hint 9.29

Hint 9.30

Hint 9.31

Hint 9.32

9.9 Solutions

Cartesian and Modulus-Argument Form

Solution 9.1
We write the mapping $w = z^4$ in polar coordinates.

$$w = z^4 = (r\,\mathrm{e}^{i\theta})^4 = r^4\,\mathrm{e}^{i4\theta}$$

Thus we see that

$$w : \{r\,\mathrm{e}^{i\theta} \,|\, r \geq 0, 0 \leq \theta < \phi\} \to \{r^4\,\mathrm{e}^{i4\theta} \,|\, r \geq 0, 0 \leq \theta < \phi\} = \{r\,\mathrm{e}^{i\theta} \,|\, r \geq 0, 0 \leq \theta < 4\phi\}.$$

We can state this in terms of the argument.

$$\boxed{w : \{z \,|\, 0 \leq \arg(z) < \phi\} \to \{z \,|\, 0 \leq \arg(z) < 4\phi\}}$$

If $\phi = \pi/2$, the sector will be mapped exactly to the whole complex plane.

Trigonometric Functions

Solution 9.2

$$\begin{aligned}
\sin z &= \frac{1}{2i}\left(\mathrm{e}^{iz} - \mathrm{e}^{-iz}\right) \\
&= \frac{1}{2i}\left(\mathrm{e}^{-y+ix} - \mathrm{e}^{y-ix}\right) \\
&= \frac{1}{2i}\left(\mathrm{e}^{-y}(\cos x + i\sin x) - \mathrm{e}^{y}(\cos x - i\sin x)\right) \\
&= \frac{1}{2}\left(\mathrm{e}^{-y}(\sin x - i\cos x) + \mathrm{e}^{y}(\sin x + i\cos x)\right) \\
&= \sin x \cosh y + i \cos x \sinh y
\end{aligned}$$

$$
\begin{aligned}
\sin z &= \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}\, \exp(i \arctan(\sin x \cosh y, \cos x \sinh y)) \\
&= \sqrt{\cosh^2 y - \cos^2 x}\, \exp(i \arctan(\sin x \cosh y, \cos x \sinh y)) \\
&= \sqrt{\frac{1}{2} (\cosh(2y) - \cos(2x))}\, \exp(i \arctan(\sin x \cosh y, \cos x \sinh y))
\end{aligned}
$$

Solution 9.3
In order that e^z be zero, the modulus, e^x must be zero. Since e^x has no finite solutions, $\mathrm{e}^z = 0$ has no finite solutions.

Solution 9.4

$$
\begin{aligned}
\left| \mathrm{e}^{z^2} \right| &= \left| \mathrm{e}^{(x+iy)^2} \right| \\
&= \left| \mathrm{e}^{x^2 - y^2 + i2xy} \right| \\
&= \mathrm{e}^{x^2 - y^2}
\end{aligned}
$$

$$
\mathrm{e}^{|z|^2} = \mathrm{e}^{|x+iy|^2} = \mathrm{e}^{x^2+y^2}
$$

The exponential function is an increasing function for real variables. Since $x^2 - y^2 \leq x^2 + y^2$,

$$
\left| \mathrm{e}^{z^2} \right| \leq \mathrm{e}^{|z|^2}.
$$

Equality holds when $y = 0$.

Solution 9.5

$$\coth(z) = 1$$

$$\frac{(\mathrm{e}^z + \mathrm{e}^{-z})/2}{(\mathrm{e}^z - \mathrm{e}^{-z})/2} = 1$$

$$\mathrm{e}^z + \mathrm{e}^{-z} = \mathrm{e}^z - \mathrm{e}^{-z}$$

$$\mathrm{e}^{-z} = 0$$

There are no solutions.

Solution 9.6

We write out the multi-valuedness of 2^z.

$$\begin{gathered}
2 \in 2^z \\
\mathrm{e}^{\operatorname{Log} 2} \in \mathrm{e}^{z \log(2)} \\
\mathrm{e}^{\operatorname{Log} 2} \in \{ \mathrm{e}^{z(\operatorname{Log}(2) + i2\pi n)} \,|\, n \in \mathbb{Z} \} \\
\operatorname{Log} 2 \in z\{ \operatorname{Log} 2 + i2\pi n + i2\pi m \,|\, m, n \in \mathbb{Z} \} \\
\boxed{z = \left\{ \frac{\operatorname{Log}(2) + i2\pi m}{\operatorname{Log}(2) + i2\pi n} \,|\, m, n \in \mathbb{Z} \right\}}
\end{gathered}$$

We verify this solution. Consider m and n to be fixed integers. We express the multi-valuedness in terms of k.

$$\begin{aligned}
2^{(\operatorname{Log}(2)+i2\pi m)/(\operatorname{Log}(2)+i2\pi n)} &= \mathrm{e}^{(\operatorname{Log}(2)+i2\pi m)/(\operatorname{Log}(2)+i2\pi n) \log(2)} \\
&= \mathrm{e}^{(\operatorname{Log}(2)+i2\pi m)/(\operatorname{Log}(2)+i2\pi n)(\operatorname{Log}(2)+i2\pi k)}
\end{aligned}$$

For $k = n$, this has the value, $\mathrm{e}^{\operatorname{Log}(2)+i2\pi m} = \mathrm{e}^{\operatorname{Log}(2)} = 2$.

Solution 9.7
We write out the multi-valuedness of 1^z.

$$1 \in 1^z$$
$$1 \in \mathrm{e}^{z \log(1)}$$
$$1 \in \{ \mathrm{e}^{zi2\pi n} \,|\, n \in \mathbb{Z}\}$$

The element corresponding to $n = 0$ is $\mathrm{e}^0 = 1$. Thus $1 \in 1^z$ has the solutions,

$$\boxed{z \in \mathbb{C}.}$$

That is, z may be any complex number. We verify this solution.

$$1^z = \mathrm{e}^{z \log(1)} = \mathrm{e}^{zi2\pi n}$$

For $n = 0$, this has the value 1.

Logarithmic Identities

Solution 9.8

1. The algebraic manipulations are fine. We write out the multi-valuedness of the logarithms.

$$\log(-1) = \log\left(\frac{1}{-1}\right) = \log(1) - \log(-1) = -\log(-1)$$
$$\{i\pi + i2\pi n : n \in \mathbb{Z}\} = \{i\pi + i2\pi n : n \in \mathbb{Z}\} = \{i2\pi n : n \in \mathbb{Z}\} - \{i\pi + i2\pi n : n \in \mathbb{Z}\} = \{-i\pi - i2\pi n : n \in \mathbb{Z}\}$$

Thus $\log(-1) = -\log(-1)$. However this does not imply that $\log(-1) = 0$. This is because the logarithm is a set-valued function $\log(-1) = -\log(-1)$ is really saying:

$$\{i\pi + i2\pi n : n \in \mathbb{Z}\} = \{-i\pi - i2\pi n : n \in \mathbb{Z}\}$$

2. We consider

$$1 = 1^{1/2} = ((-1)(-1))^{1/2} = (-1)^{1/2}(-1)^{1/2} = ii = -1.$$

There are three multi-valued expressions above.

$$1^{1/2} = \pm 1$$
$$((-1)(-1))^{1/2} = \pm 1$$
$$(-1)^{1/2}(-1)^{1/2} = (\pm i)(\pm i) = \pm 1$$

Thus we see that the first and fourth equalities are incorrect.

$$1 \neq 1^{1/2}, \quad (-1)^{1/2}(-1)^{1/2} \neq ii$$

Solution 9.9

$$\begin{aligned} 2^{2/5} &= 4^{1/5} \\ &= \sqrt[5]{4}\, 1^{1/5} \\ &= \sqrt[5]{4}\, \mathrm{e}^{i2n\pi/5}, \quad n = 0, 1, 2, 3, 4 \end{aligned}$$

$$\begin{aligned} 3^{1+i} &= \mathrm{e}^{(1+i)\log 3} \\ &= \mathrm{e}^{(1+i)(\operatorname{Log} 3 + i2\pi n)} \\ &= \mathrm{e}^{\operatorname{Log} 3 - 2\pi n}\, \mathrm{e}^{i(\operatorname{Log} 3 + 2\pi n)}, \quad n \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} (\sqrt{3} - i)^{1/4} &= (2\,\mathrm{e}^{-i\pi/6})^{1/4} \\ &= \sqrt[4]{2}\, \mathrm{e}^{-i\pi/24} 1^{1/4} \\ &= \sqrt[4]{2}\, \mathrm{e}^{i(\pi n/2 - \pi/24)}, \quad n = 0, 1, 2, 3 \end{aligned}$$

$$
\begin{aligned}
1^{i/4} &= e^{(i/4)\log 1} \\
&= e^{(i/4)(i2\pi n)} \\
&= e^{-\pi n/2}, \quad n \in \mathbb{Z}
\end{aligned}
$$

Solution 9.10

$$
\begin{gathered}
\cos z = 69 \\
\frac{e^{iz} + e^{-iz}}{2} = 69 \\
e^{i2z} - 138\, e^{iz} + 1 = 0 \\
e^{iz} = \frac{1}{2}\left(138 \pm \sqrt{138^2 - 4}\right) \\
z = -i \log\left(69 \pm 2\sqrt{1190}\right) \\
z = -i\left(\operatorname{Log}\left(69 \pm 2\sqrt{1190}\right) + i2\pi n\right) \\
\boxed{z = 2\pi n - i \operatorname{Log}\left(69 \pm 2\sqrt{1190}\right), \quad n \in \mathbb{Z}}
\end{gathered}
$$

Solution 9.11

$$
\begin{gathered}
\cot z = i47 \\
\frac{(\mathrm{e}^{iz} + \mathrm{e}^{-iz})/2}{(\mathrm{e}^{iz} - \mathrm{e}^{-iz})/(2i)} = i47 \\
\mathrm{e}^{iz} + \mathrm{e}^{-iz} = 47(\mathrm{e}^{iz} - \mathrm{e}^{-iz}) \\
46\,\mathrm{e}^{i2z} - 48 = 0 \\
i2z = \log \frac{24}{23} \\
z = -\frac{i}{2} \log \frac{24}{23} \\
z = -\frac{i}{2}\left(\operatorname{Log}\frac{24}{23} + i2\pi n\right), \quad n \in \mathbb{Z} \\
\boxed{z = \pi n - \frac{i}{2}\operatorname{Log}\frac{24}{23}, \quad n \in \mathbb{Z}}
\end{gathered}
$$

Solution 9.12

1.

$$
\begin{aligned}
\log(-i) &= \operatorname{Log}(|-i|) + i\arg(-i) \\
&= \operatorname{Log}(1) + i\left(-\frac{\pi}{2} + 2\pi n\right), \quad n \in \mathbb{Z}
\end{aligned}
$$

$$
\boxed{\log(-i) = -i\frac{\pi}{2} + i2\pi n, \quad n \in \mathbb{Z}}
$$

These are equally spaced points in the imaginary axis. See Figure 9.30.

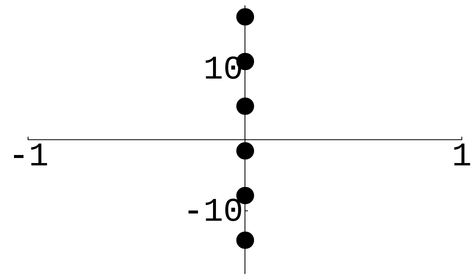


Figure 9.30: $\log(-i)$

2.

$$\begin{aligned}(-i)^{-i} &= \mathrm{e}^{-i\log(-i)} \\ &= \mathrm{e}^{-i(-i\pi/2+i2\pi n)}, \quad n \in \mathbb{Z}\end{aligned}$$

$$\boxed{(-i)^{-i} = \mathrm{e}^{-\pi/2+2\pi n}, \quad n \in \mathbb{Z}}$$

These are points on the positive real axis with an accumulation point at the origin. See Figure 9.31.

3.

$$\begin{aligned}3^{\pi} &= \mathrm{e}^{\pi\log(3)} \\ &= \mathrm{e}^{\pi(\operatorname{Log}(3)+i\arg(3))}\end{aligned}$$

$$\boxed{3^{\pi} = \mathrm{e}^{\pi(\operatorname{Log}(3)+i2\pi n)}, \quad n \in \mathbb{Z}}$$

These points all lie on the circle of radius $|e^{\pi}|$ centered about the origin in the complex plane. See Figure 9.32.

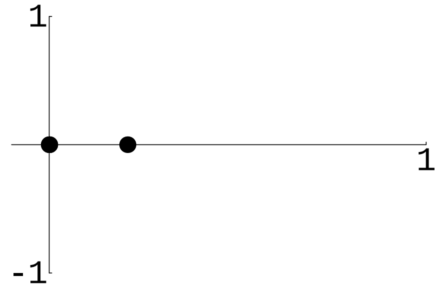


Figure 9.31: $(-i)^{-i}$

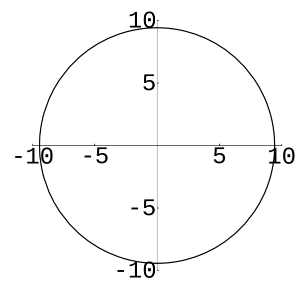


Figure 9.32: 3^π

4.

$$\begin{aligned}
\log(\log(i)) &= \log\left(i\left(\frac{\pi}{2}+2\pi m\right)\right), \quad m\in\mathbb{Z}\\
&= \operatorname{Log}\left|\frac{\pi}{2}+2\pi m\right| + i\operatorname{Arg}\left(i\left(\frac{\pi}{2}+2\pi m\right)\right) + i2\pi n, \quad m,n\in\mathbb{Z}\\
&= \operatorname{Log}\left|\frac{\pi}{2}+2\pi m\right| + i\operatorname{sign}(1+4m)\frac{\pi}{2} + i2\pi n, \quad m,n\in\mathbb{Z}
\end{aligned}$$

These points all lie in the right half-plane. See Figure 9.33.

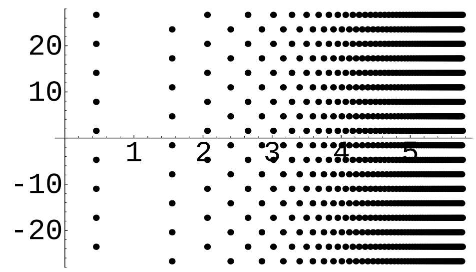


Figure 9.33: $\log(\log(i))$

Solution 9.13

$$
\begin{aligned}
i^i &= \mathrm{e}^{i \log(i)} \\
&= \mathrm{e}^{i(\operatorname{Log}|i| + i \operatorname{Arg}(i) + i2\pi n)}, \quad n \in \mathbb{Z} \\
&= \mathrm{e}^{i(i\pi/2 + i2\pi n)}, \quad n \in \mathbb{Z} \\
&= \mathrm{e}^{-\pi(1/2+2n)}, \quad n \in \mathbb{Z}
\end{aligned}
$$

These are points on the positive real axis. There is an accumulation point at $z = 0$. See Figure 9.34.

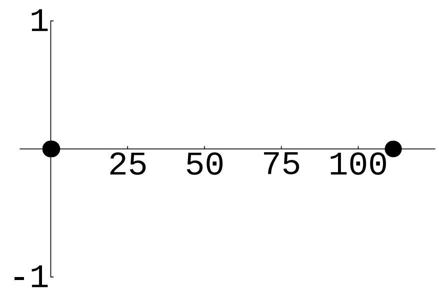


Figure 9.34: i^i

$$
\begin{aligned}
\log\left((1+i)^{i\pi}\right) &= \log\left(\mathrm{e}^{i\pi\log(1+i)}\right) \\
&= i\pi\log(1+i) + i2\pi n, \quad n \in \mathbb{Z} \\
&= i\pi(\operatorname{Log}|1+i| + i\operatorname{Arg}(1+i) + i2\pi m) + i2\pi n, \quad m,n \in \mathbb{Z} \\
&= i\pi\left(\frac{1}{2}\operatorname{Log}2 + i\frac{\pi}{4} + i2\pi m\right) + i2\pi n, \quad m,n \in \mathbb{Z} \\
&= -\pi^2\left(\frac{1}{4} + 2m\right) + i\pi\left(\frac{1}{2}\operatorname{Log}2 + 2n\right), \quad m,n \in \mathbb{Z}
\end{aligned}
$$

See Figure 9.35 for a plot.

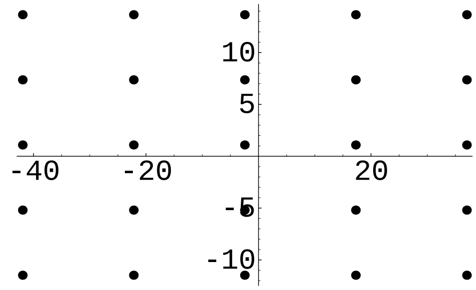


Figure 9.35: $\log((1+i)^{i\pi})$

Solution 9.14

1.

$$e^z = i$$
$$z = \log i$$
$$z = \operatorname{Log}(|i|) + i \arg(i)$$
$$z = \operatorname{Log}(1) + i\left(\frac{\pi}{2} + 2\pi n\right), \quad n \in \mathbb{Z}$$
$$\boxed{z = i\frac{\pi}{2} + i2\pi n, \quad n \in \mathbb{Z}}$$

2. We can solve the equation by writing the cosine and sine in terms of the exponential.

$$\begin{aligned}
\cos z &= \sin z \\
\frac{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}{2} &= \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{i2} \\
(1+i)\,\mathrm{e}^{iz} &= (-1+i)\,\mathrm{e}^{-iz} \\
\mathrm{e}^{i2z} &= \frac{-1+i}{1+i} \\
\mathrm{e}^{i2z} &= i \\
i2z &= \log(i) \\
i2z &= i\frac{\pi}{2} + i2\pi n, \quad n \in \mathbb{Z} \\
&\boxed{z = \frac{\pi}{4} + \pi n, \quad n \in \mathbb{Z}}
\end{aligned}$$

3.

$$\begin{aligned}
\tan^2 z &= -1 \\
\sin^2 z &= -\cos^2 z \\
\cos z &= \pm i \sin z \\
\frac{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}{2} &= \pm i \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{2i} \\
\mathrm{e}^{-iz} = -\mathrm{e}^{-iz} \quad &\text{or} \quad \mathrm{e}^{iz} = -\mathrm{e}^{iz} \\
\mathrm{e}^{-iz} = 0 \quad &\text{or} \quad \mathrm{e}^{iz} = 0 \\
\mathrm{e}^{y-ix} = 0 \quad &\text{or} \quad \mathrm{e}^{-y+ix} = 0 \\
\mathrm{e}^{y} = 0 \quad &\text{or} \quad \mathrm{e}^{-y} = 0 \\
&\boxed{z = \emptyset}
\end{aligned}$$

There are no solutions for finite z.

Solution 9.15
First we consider $\tan^{-1}(z)$.

$$
\begin{gathered}
w = \tan^{-1}(z) \\
z = \tan(w) \\
z = \frac{\sin(w)}{\cos(w)} \\
z = \frac{(\,\mathrm{e}^{iw} - \,\mathrm{e}^{-iw})/(2i)}{(\,\mathrm{e}^{iw} + \,\mathrm{e}^{-iw})/2} \\
z\,\mathrm{e}^{iw} + z\,\mathrm{e}^{-iw} = -i\,\mathrm{e}^{iw} + i\,\mathrm{e}^{-iw} \\
(i+z)\,\mathrm{e}^{i2w} = (i-z) \\
\mathrm{e}^{iw} = \left(\frac{i-z}{i+z}\right)^{1/2} \\
w = -i\log\left(\frac{i-z}{i+z}\right)^{1/2} \\
\boxed{\tan^{-1}(z) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)}
\end{gathered}
$$

Now we consider $\tanh^{-1}(z)$.

$$w = \tanh^{-1}(z)$$
$$z = \tanh(w)$$
$$z = \frac{\sinh(w)}{\cosh(w)}$$
$$z = \frac{(\mathrm{e}^w - \mathrm{e}^{-w})/2}{(\mathrm{e}^w + \mathrm{e}^{-w})/2}$$
$$z\,\mathrm{e}^w + z\,\mathrm{e}^{-w} = \mathrm{e}^w - \mathrm{e}^{-w}$$
$$(z-1)\,\mathrm{e}^{2w} = -z-1$$
$$\mathrm{e}^w = \left(\frac{-z-1}{z-1}\right)^{1/2}$$
$$w = \log\left(\frac{z+1}{1-z}\right)^{1/2}$$
$$\boxed{\tanh^{-1}(z) = \frac{1}{2}\log\left(\frac{z+1}{1-z}\right)}$$

Branch Points and Branch Cuts

Solution 9.16

The cube roots of 1 are

$$\left\{1,\, \mathrm{e}^{i2\pi/3},\, \mathrm{e}^{i4\pi/3}\right\} = \left\{1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\right\}.$$

Thus we can write

$$\left(z^3-1\right)^{1/2} = (z-1)^{1/2}\left(z+\frac{1-i\sqrt{3}}{2}\right)^{1/2}\left(z+\frac{1+i\sqrt{3}}{2}\right)^{1/2}.$$

There are three branch points on the circle of radius 1.

$$z = \left\{ 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2} \right\}$$

We examine the point at infinity.

$$f(1/\zeta) = (1/\zeta^3 - 1)^{1/2} = \zeta^{-3/2}(1 - \zeta^3)^{1/2}$$

Since $f(1/\zeta)$ has a branch point at $\zeta = 0$, $f(z)$ has a branch point at infinity.

There are several ways of introducing branch cuts to separate the branches of the function. The easiest approach is to put a branch cut from each of the three branch points in the finite complex plane out to the branch point at infinity. See Figure 9.36a. Clearly this makes the function single valued as it is impossible to walk around any of the branch points. Another approach is to have a branch cut from one of the branch points in the finite plane to the branch point at infinity and a branch cut connecting the remaining two branch points. See Figure 9.36bcd. Note that in walking around any one of the finite branch points, (in the positive direction), the argument of the function changes by π. This means that the value of the function changes by $\mathrm{e}^{i\pi}$, which is to say the value of the function changes sign. In walking around any two of the finite branch points, (again in the positive direction), the argument of the function changes by 2π. This means that the value of the function changes by $\mathrm{e}^{i2\pi}$, which is to say that the value of the function does not change. This demonstrates that the latter branch cut approach makes the function single-valued.

Now we construct a branch. We will use the branch cuts in Figure 9.36a. We introduce variables to measure radii and angles from the three finite branch points.

$$z - 1 = r_1 \,\mathrm{e}^{i\theta_1}, \quad 0 < \theta_1 < 2\pi$$
$$z + \frac{1 - i\sqrt{3}}{2} = r_2 \,\mathrm{e}^{i\theta_2}, \quad -\frac{2\pi}{3} < \theta_2 < \frac{\pi}{3}$$
$$z + \frac{1 + i\sqrt{3}}{2} = r_3 \,\mathrm{e}^{i\theta_3}, \quad -\frac{\pi}{3} < \theta_3 < \frac{2\pi}{3}$$

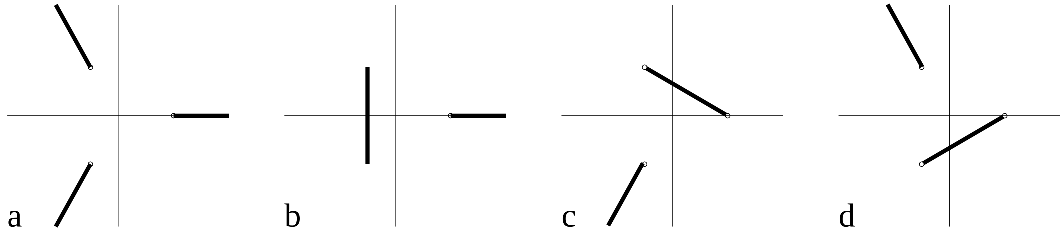


Figure 9.36: $(z^3 - 1)^{1/2}$

We compute $f(0)$ to see if it has the desired value.

$$f(z) = \sqrt{r_1 r_2 r_3}\,\mathrm{e}^{i(\theta_1+\theta_2+\theta_3)/2}$$
$$f(0) = \mathrm{e}^{i(\pi-\pi/3+\pi/3)/2} = i$$

Since it does not have the desired value, we change the range of θ_1.

$$z - 1 = r_1\,\mathrm{e}^{i\theta_1}, \quad 2\pi < \theta_1 < 4\pi$$

$f(0)$ now has the desired value.

$$f(0) = \mathrm{e}^{i(3\pi-\pi/3+\pi/3)/2} = -i$$

We compute $f(-1)$.

$$f(-1) = \sqrt{2}\,\mathrm{e}^{i(3\pi-2\pi/3+2\pi/3)/2} = -i\sqrt{2}$$

Solution 9.17

First we factor the function.

$$w(z) = ((z+2)(z-1)(z-6))^{1/2} = (z+2)^{1/2}(z-1)^{1/2}(z-6)^{1/2}$$

There are branch points at $z = -2, 1, 6$. Now we examine the point at infinity.

$$w\left(\frac{1}{\zeta}\right) = \left(\left(\frac{1}{\zeta} + 2\right)\left(\frac{1}{\zeta} - 1\right)\left(\frac{1}{\zeta} - 6\right)\right)^{1/2} = \zeta^{-3/2}\left(\left(1 + \frac{2}{\zeta}\right)\left(1 - \frac{1}{\zeta}\right)\left(1 - \frac{6}{\zeta}\right)\right)^{1/2}$$

Since $\zeta^{-3/2}$ has a branch point at $\zeta = 0$ and the rest of the terms are analytic there, $w(z)$ has a branch point at infinity.

Consider the set of branch cuts in Figure 9.37. These cuts let us walk around the branch points at $z = -2$ and $z = 1$ together or if we change our perspective, we would be walking around the branch points at $z = 6$ and $z = \infty$ together. Consider a contour in this cut plane that encircles the branch points at $z = -2$ and $z = 1$. Since the argument of $(z - z_0)^{1/2}$ changes by π when we walk around z_0, the argument of $w(z)$ changes by 2π when we traverse the contour. Thus the value of the function does not change and it is a valid set of branch cuts.

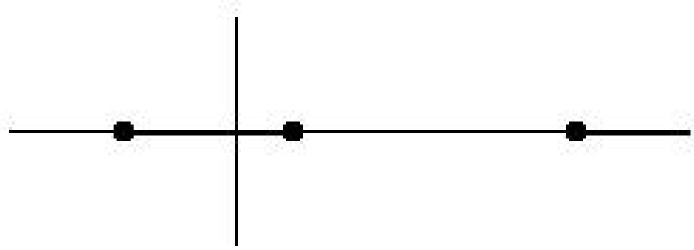


Figure 9.37: Branch Cuts for $((z + 2)(z - 1)(z - 6))^{1/2}$

Now to define the branch. We make a choice of angles.

$$\begin{aligned}
z + 2 &= r_1 \mathrm{e}^{i\theta_1}, &\quad \theta_1 &= \theta_2 \text{ for } z \in (1..6),\\
z - 1 &= r_2 \mathrm{e}^{i\theta_2}, &\quad \theta_2 &= \theta_1 \text{ for } z \in (1..6),\\
z - 6 &= r_3 \mathrm{e}^{i\theta_3}, &\quad 0 &< \theta_3 < 2\pi
\end{aligned}$$

The function is

$$w(z) = \left(r_1 \mathrm{e}^{i\theta_1} r_2 \mathrm{e}^{i\theta_2} r_3 \mathrm{e}^{i\theta_3}\right)^{1/2} = \sqrt{r_1 r_2 r_3}\, \mathrm{e}^{i(\theta_1+\theta_2+\theta_3)/2}.$$

We evaluate the function at $z = 4$.

$$w(4) = \sqrt{(6)(3)(2)}\, \mathrm{e}^{i(2\pi n + 2\pi n + \pi)/2} = i6$$

We see that our choice of angles gives us the desired branch.

Solution 9.18

1.

$$\cos z^{1/2} = \cos(\pm\sqrt{z}) = \cos(\sqrt{z})$$

This is a single-valued function. There are no branch points.

2.

$$\begin{aligned}(z+i)^{-z} &= \mathrm{e}^{-z \log(z+i)} \\ &= \mathrm{e}^{-z(\operatorname{Log}|z+i| + i \operatorname{Arg}(z+i) + i2\pi n)}, \quad n \in \mathbb{Z}\end{aligned}$$

There is a branch point at $z = -i$. There are an infinite number of branches.

Solution 9.19

1.

$$f(z) = (z^2+1)^{1/2} = (z+i)^{1/2}(z-i)^{1/2}$$

We see that there are branch points at $z = \pm i$. To examine the point at infinity, we substitute $z = 1/\zeta$ and examine the point $\zeta = 0$.

$$\left(\left(\frac{1}{\zeta}\right)^2 + 1\right)^{1/2} = \frac{1}{(\zeta^2)^{1/2}}(1+\zeta^2)^{1/2}$$

Since there is no branch point at $\zeta = 0$, $f(z)$ has no branch point at infinity.

A branch cut connecting $z = \pm i$ would make the function single-valued. We could also accomplish this with two branch cuts starting $z = \pm i$ and going to infinity.

2.

$$f(z) = (z^3 - z)^{1/2} = z^{1/2}(z-1)^{1/2}(z+1)^{1/2}$$

There are branch points at $z = -1, 0, 1$. Now we consider the point at infinity.

$$f\left(\frac{1}{\zeta}\right) = \left(\left(\frac{1}{\zeta}\right)^3 - \frac{1}{\zeta}\right)^{1/2} = \zeta^{-3/2}(1-\zeta^2)^{1/2}$$

There is a branch point at infinity.

One can make the function single-valued with three branch cuts that start at $z = -1, 0, 1$ and each go to infinity. We can also make the function single-valued with a branch cut that connects two of the points $z = -1, 0, 1$ and another branch cut that starts at the remaining point and goes to infinity.

3.

$$f(z) = \log(z^2 - 1) = \log(z-1) + \log(z+1)$$

There are branch points at $z = \pm 1$.

$$f\left(\frac{1}{\zeta}\right) = \log\left(\frac{1}{\zeta^2} - 1\right) = \log(\zeta^{-2}) + \log(1 - \zeta^2)$$

$\log(\zeta^{-2})$ has a branch point at $\zeta = 0$.

$$\log(\zeta^{-2}) = \operatorname{Log}|\zeta^{-2}| + i \arg(\zeta^{-2}) = \operatorname{Log}|\zeta^{-2}| - i2\arg(\zeta)$$

Every time we walk around the point $\zeta = 0$ in the positive direction, the value of the function changes by $-i4\pi$. $f(z)$ has a branch point at infinity.

We can make the function single-valued by introducing two branch cuts that start at $z = \pm 1$ and each go to infinity.

4.
$$f(z) = \log\left(\frac{z+1}{z-1}\right) = \log(z+1) - \log(z-1)$$
There are branch points at $z = \pm 1$.
$$f\left(\frac{1}{\zeta}\right) = \log\left(\frac{1/\zeta+1}{1/\zeta-1}\right) = \log\left(\frac{1+\zeta}{1-\zeta}\right)$$
There is no branch point at $\zeta = 0$. $f(z)$ has no branch point at infinity.

We can make the function single-valued by introducing two branch cuts that start at $z = \pm 1$ and each go to infinity. We can also make the function single-valued with a branch cut that connects the points $z = \pm 1$. This is because $\log(z + 1)$ and $-\log(z - 1)$ change by $i2\pi$ and $-i2\pi$, respectively, when you walk around their branch points once in the positive direction.

Solution 9.20

1. The cube roots of -8 are
$$\left\{-2, -2\,\mathrm{e}^{i2\pi/3}, -2\,\mathrm{e}^{i4\pi/3}\right\} = \left\{-2, 1+i\sqrt{3}, 1-i\sqrt{3}\right\}.$$
Thus we can write
$$\left(z^3+8\right)^{1/2} = (z+2)^{1/2}(z-1-i\sqrt{3})^{1/2}(z-1+i\sqrt{3})^{1/2}.$$
There are three branch points on the circle of radius 2.
$$z = \left\{-2, 1+i\sqrt{3}, 1-i\sqrt{3}\right\}.$$
We examine the point at infinity.
$$f(1/\zeta) = (1/\zeta^3+8)^{1/2} = \zeta^{-3/2}(1+8\zeta^3)^{1/2}$$

Since $f(1/\zeta)$ has a branch point at $\zeta = 0$, $f(z)$ has a branch point at infinity.

There are several ways of introducing branch cuts outside of the disk $|z| < 2$ to separate the branches of the function. The easiest approach is to put a branch cut from each of the three branch points in the finite complex plane out to the branch point at infinity. See Figure 9.38a. Clearly this makes the function single valued as it is impossible to walk around any of the branch points. Another approach is to have a branch cut from one of the branch points in the finite plane to the branch point at infinity and a branch cut connecting the remaining two branch points. See Figure 9.38bcd. Note that in walking around any one of the finite branch points, (in the positive direction), the argument of the function changes by π. This means that the value of the function changes by $e^{i\pi}$, which is to say the value of the function changes sign. In walking around any two of the finite branch points, (again in the positive direction), the argument of the function changes by 2π. This means that the value of the function changes by $e^{i2\pi}$, which is to say that the value of the function does not change. This demonstrates that the latter branch cut approach makes the function single-valued.

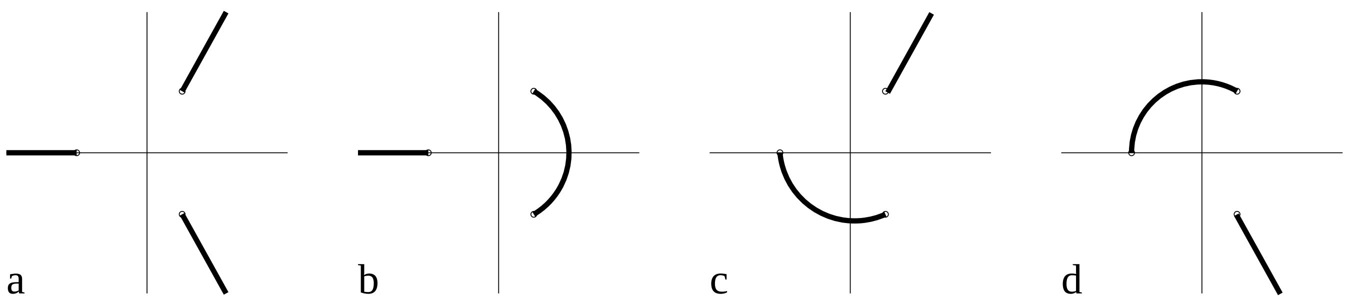


Figure 9.38: $(z^3 + 8)^{1/2}$

2.

$$f(z) = \log\left(5 + \left(\frac{z+1}{z-1}\right)^{1/2}\right)$$

First we deal with the function

$$g(z) = \left(\frac{z+1}{z-1}\right)^{1/2}$$

Note that it has branch points at $z = \pm 1$. Consider the point at infinity.

$$g(1/\zeta) = \left(\frac{1/\zeta + 1}{1/\zeta - 1}\right)^{1/2} = \left(\frac{1+\zeta}{1-\zeta}\right)^{1/2}$$

Since $g(1/\zeta)$ has no branch point at $\zeta = 0$, $g(z)$ has no branch point at infinity. This means that if we walk around both of the branch points at $z = \pm 1$, the function does not change value. We can verify this with another method: When we walk around the point $z = -1$ once in the positive direction, the argument of $z + 1$ changes by 2π, the argument of $(z+1)^{1/2}$ changes by π and thus the value of $(z+1)^{1/2}$ changes by $\mathrm{e}^{i\pi} = -1$. When we walk around the point $z = 1$ once in the positive direction, the argument of $z - 1$ changes by 2π, the argument of $(z-1)^{-1/2}$ changes by $-\pi$ and thus the value of $(z-1)^{-1/2}$ changes by $\mathrm{e}^{-i\pi} = -1$. $f(z)$ has branch points at $z = \pm 1$. When we walk around both points $z = \pm 1$ once in the positive direction, the value of $\left(\frac{z+1}{z-1}\right)^{1/2}$ does not change. Thus we can make the function single-valued with a branch cut which enables us to walk around either none or both of these branch points. We put a branch cut from -1 to 1 on the real axis.

$f(z)$ has branch points where

$$5 + \left(\frac{z+1}{z-1}\right)^{1/2}$$

is either zero or infinite. The only place in the extended complex plane where the expression becomes infinite is at $z = 1$. Now we look for the zeros.

$$
\begin{aligned}
5 + \left(\frac{z+1}{z-1}\right)^{1/2} &= 0. \\
\left(\frac{z+1}{z-1}\right)^{1/2} &= -5. \\
\frac{z+1}{z-1} &= 25. \\
z + 1 &= 25z - 25 \\
z &= \frac{13}{12}
\end{aligned}
$$

Note that

$$
\left(\frac{13/12+1}{13/12-1}\right)^{1/2} = 25^{1/2} = \pm 5.
$$

On one branch, (which we call the positive branch), of the function $g(z)$ the quantity

$$
5 + \left(\frac{z+1}{z-1}\right)^{1/2}
$$

is always nonzero. On the other (negative) branch of the function, this quantity has a zero at $z = 13/12$.

The logarithm introduces branch points at $z = 1$ on both the positive and negative branch of $g(z)$. It introduces a branch point at $z = 13/12$ on the negative branch of $g(z)$. To determine if additional branch cuts are needed to separate the branches, we consider

$$
w = 5 + \left(\frac{z+1}{z-1}\right)^{1/2}
$$

and see where the branch cut between ± 1 gets mapped to in the w plane. We rewrite the mapping.

$$w = 5 + \left(1 + \frac{2}{z-1}\right)^{1/2}$$

The mapping is the following sequence of simple transformations:

(a) $z \mapsto z - 1$

(b) $z \mapsto \frac{1}{z}$

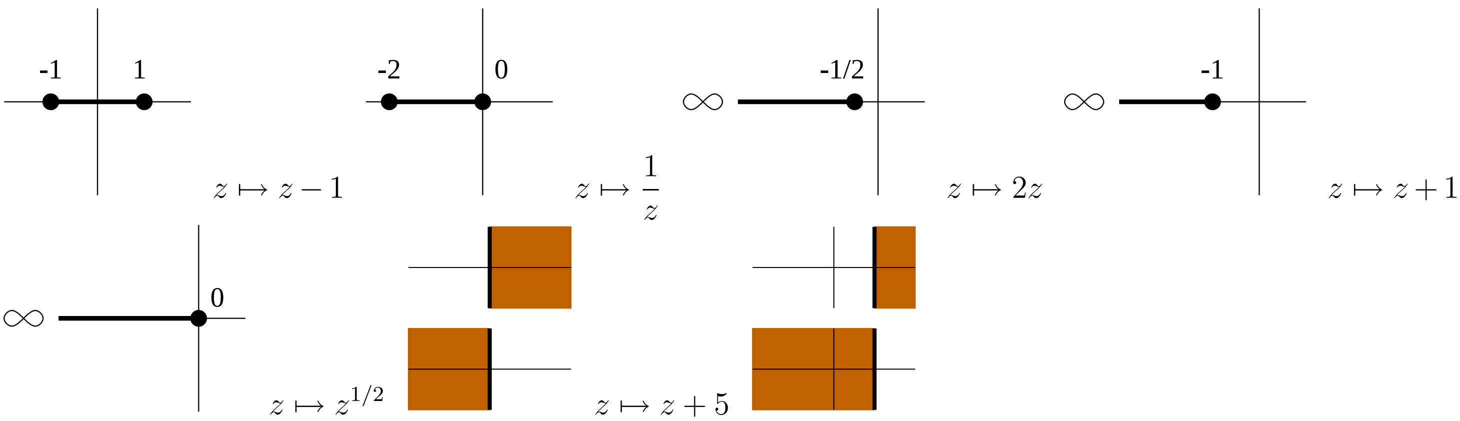
(c) $z \mapsto 2z$

(d) $z \mapsto z + 1$

(e) $z \mapsto z^{1/2}$

(f) $z \mapsto z + 5$

We show these transformations graphically below.



For the positive branch of $g(z)$, the branch cut is mapped to the line $x = 5$ and the z plane is mapped to the half-plane $x > 5$. $\log(w)$ has branch points at $w = 0$ and $w = \infty$. It is possible to walk around only one of these points in the half-plane $x > 5$. Thus no additional branch cuts are needed in the positive sheet of $g(z)$.

For the negative branch of $g(z)$, the branch cut is mapped to the line $x = 5$ and the z plane is mapped to the half-plane $x < 5$. It is possible to walk around either $w = 0$ or $w = \infty$ alone in this half-plane. Thus we need an additional branch cut. On the negative sheet of $g(z)$, we put a branch cut beteen $z = 1$ and $z = 13/12$. This puts a branch cut between $w = \infty$ and $w = 0$ and thus separates the branches of the logarithm.

Figure 9.39 shows the branch cuts in the positive and negative sheets of $g(z)$.

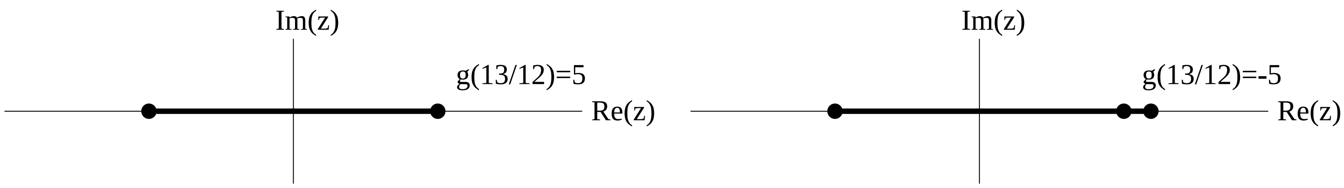


Figure 9.39: The branch cuts for $f(z) = \log\left(5 + \left(\frac{z+1}{z-1}\right)^{1/2}\right)$.

3. The function $f(z) = (z + i3)^{1/2}$ has a branch point at $z = -i3$. The function is made single-valued by connecting this point and the point at infinity with a branch cut.

Solution 9.21

Note that the curve with opposite orientation goes around infinity in the positive direction and does not enclose any branch points. Thus the value of the function does not change when traversing the curve, (with either orientation, of course). This means that the argument of the function must change my an integer multiple of 2π. Since the branch cut only allows us to encircle all three or none of the branch points, it makes the function single valued.

Solution 9.22

We suppose that $f(z)$ has only one branch point in the finite complex plane. Consider any contour that encircles this branch point in the positive direction. $f(z)$ changes value if we traverse the contour. If we reverse the orientation of the contour, then it encircles infinity in the positive direction, but contains no branch points in the finite complex plane. Since the function changes value when we traverse the contour, we conclude that the point at infinity must be a branch point. If $f(z)$ has only a single branch point in the finite complex plane then it must have a branch point at infinity.

If $f(z)$ has two or more branch points in the finite complex plane then it may or may not have a branch point at infinity. This is because the value of the function may or may not change on a contour that encircles all the branch points in the finite complex plane.

Solution 9.23

First we factor the function,

$$f(z) = \left(z^4+1\right)^{1/4} = \left(z - \frac{1+i}{\sqrt{2}}\right)^{1/4} \left(z - \frac{-1+i}{\sqrt{2}}\right)^{1/4} \left(z - \frac{-1-i}{\sqrt{2}}\right)^{1/4} \left(z - \frac{1-i}{\sqrt{2}}\right)^{1/4}.$$

There are branch points at $z = \frac{\pm 1 \pm i}{\sqrt{2}}$. We make the substitution $z = 1/\zeta$ to examine the point at infinity.

$$\begin{aligned} f\left(\frac{1}{\zeta}\right) &= \left(\frac{1}{\zeta^4}+1\right)^{1/4} \\ &= \frac{1}{\left(\zeta^4\right)^{1/4}} \left(1+\zeta^4\right)^{1/4} \end{aligned}$$

$(\zeta^{1/4})^4$ has a removable singularity at the point $\zeta = 0$, but no branch point there. Thus $(z^4+1)^{1/4}$ has no branch point at infinity.

Note that the argument of $(z^4 - z_0)^{1/4}$ changes by $\pi/2$ on a contour that goes around the point z_0 once in the positive direction. The argument of $(z^4+1)^{1/4}$ changes by $n\pi/2$ on a contour that goes around n of its branch points. Thus any set of branch cuts that permit you to walk around only one, two or three of the branch points will not make the function single valued. A set of branch cuts that permit us to walk around only zero or all four of the branch points will make the function single-valued. Thus we see that the first two sets of branch cuts in Figure 9.27 will make the function single-valued, while the remaining two will not.

Consider the contour in Figure ??. There are two ways to see that the function does not change value while traversing the contour. The first is to note that each of the branch points makes the argument of the function increase by $\pi/2$. Thus the argument of $(z^4+1)^{1/4}$ changes by $4(\pi/2) = 2\pi$ on the contour. This means that the value of the function changes by the factor $e^{i2\pi} = 1$. If we change the orientation of the contour, then it is a contour that encircles infinity once in the positive direction. There are no branch points inside the this contour with opposite orientation. (Recall that the inside of a contour lies to your left as you walk around it.) Since there are no branch points inside this contour, the function cannot change value as we traverse it.

Solution 9.24

$$f(z) = \left(\frac{z}{z^2+1}\right)^{1/3} = z^{1/3}(z-i)^{-1/3}(z+i)^{-1/3}$$

There are branch points at $z = 0, \pm i$.

$$f\left(\frac{1}{\zeta}\right) = \left(\frac{1/\zeta}{(1/\zeta)^2+1}\right)^{1/3} = \frac{\zeta^{1/3}}{(1+\zeta^2)^{1/3}}$$

There is a branch point at $\zeta = 0$. $f(z)$ has a branch point at infinity.

We introduce branch cuts from $z = 0$ to infinity on the negative real axis, from $z = i$ to infinity on the positive imaginary axis and from $z = -i$ to infinity on the negative imaginary axis. As we cannot walk around any of the branch points, this makes the function single-valued.

We define a branch by defining angles from the branch points. Let

$$\begin{aligned}
z &= r\,e^{i\theta} && -\pi < \theta < \pi, \\
(z-i) &= s\,e^{i\phi} && -3\pi/2 < \phi < \pi/2, \\
(z+i) &= t\,e^{i\psi} && -\pi/2 < \psi < 3\pi/2.
\end{aligned}$$

With

$$
\begin{aligned}
f(z) &= z^{1/3}(z-i)^{-1/3}(z+i)^{-1/3} \\
&= \sqrt[3]{r}\, \mathrm{e}^{i\theta/3} \frac{1}{\sqrt[3]{s}}\, \mathrm{e}^{-i\phi/3} \frac{1}{\sqrt[3]{t}}\, \mathrm{e}^{-i\psi/3} \\
&= \sqrt[3]{\frac{r}{st}}\, \mathrm{e}^{i(\theta-\phi-\psi)/3}
\end{aligned}
$$

we have an explicit formula for computing the value of the function for this branch. Now we compute $f(1)$ to see if we chose the correct ranges for the angles. (If not, we'll just change one of them.)

$$
f(1) = \sqrt[3]{\frac{1}{\sqrt{2}\sqrt{2}}}\, \mathrm{e}^{i(0-\pi/4-(-\pi/4))/3} = \frac{1}{\sqrt[3]{2}}
$$

We made the right choice for the angles. Now to compute $f(1+i)$.

$$
f(1+i) = \sqrt[3]{\frac{\sqrt{2}}{1\sqrt{5}}}\, \mathrm{e}^{i(\pi/4-0-\operatorname{Arctan}(2))/3} = \sqrt[6]{\frac{2}{5}}\, \mathrm{e}^{i(\pi/4-\operatorname{Arctan}(2))/3}
$$

Consider the value of the function above and below the branch cut on the negative real axis. Above the branch cut the function is

$$
f(-x+i0) = \sqrt[3]{\frac{x}{\sqrt{x^2+1}\sqrt{x^2+1}}}\, \mathrm{e}^{i(\pi-\phi-\psi)/3}
$$

Note that $\phi = -\psi$ so that

$$
f(-x+i0) = \sqrt[3]{\frac{x}{x^2+1}}\, \mathrm{e}^{i(\pi)/3} = \sqrt[3]{\frac{x}{x^2+1}} \frac{1+i\sqrt{3}}{2}.
$$

Below the branch cut $\theta = -\pi$ and

$$
f(-x-i0) = \sqrt[3]{\frac{x}{x^2+1}}\, \mathrm{e}^{i(-\pi)/3} = \sqrt[3]{\frac{x}{x^2+1}} \frac{1-i\sqrt{3}}{2}.
$$

For the branch cut along the positive imaginary axis,

$$
\begin{aligned}
f(iy+0) &= \sqrt[3]{\frac{y}{(y-1)(y+1)}}\,\mathrm{e}^{i(\pi/2-\pi/2-\pi/2)/3} \\
&= \sqrt[3]{\frac{y}{(y-1)(y+1)}}\,\mathrm{e}^{-i\pi/6} \\
&= \sqrt[3]{\frac{y}{(y-1)(y+1)}}\,\frac{\sqrt{3}-i}{2},
\end{aligned}
$$

$$
\begin{aligned}
f(iy-0) &= \sqrt[3]{\frac{y}{(y-1)(y+1)}}\,\mathrm{e}^{i(\pi/2-(-3\pi/2)-\pi/2)/3} \\
&= \sqrt[3]{\frac{y}{(y-1)(y+1)}}\,\mathrm{e}^{i\pi/2} \\
&= i\sqrt[3]{\frac{y}{(y-1)(y+1)}}.
\end{aligned}
$$

For the branch cut along the negative imaginary axis,

$$
\begin{aligned}
f(-iy+0) &= \sqrt[3]{\frac{y}{(y+1)(y-1)}}\,\mathrm{e}^{i(-\pi/2-(-\pi/2)-(-\pi/2))/3} \\
&= \sqrt[3]{\frac{y}{(y+1)(y-1)}}\,\mathrm{e}^{i\pi/6} \\
&= \sqrt[3]{\frac{y}{(y+1)(y-1)}}\,\frac{\sqrt{3}+i}{2},
\end{aligned}
$$

$$
\begin{aligned}
f(-iy-0) &= \sqrt[3]{\frac{y}{(y+1)(y-1)}}\,\mathrm{e}^{i(-\pi/2-(-\pi/2)-(3\pi/2))/3} \\
&= \sqrt[3]{\frac{y}{(y+1)(y-1)}}\,\mathrm{e}^{-i\pi/2} \\
&= -i\sqrt[3]{\frac{y}{(y+1)(y-1)}}.
\end{aligned}
$$

Solution 9.25

First we factor the function.

$$
f(z) = ((z-1)(z-2)(z-3))^{1/2} = (z-1)^{1/2}(z-2)^{1/2}(z-3)^{1/2}
$$

There are branch points at $z = 1, 2, 3$. Now we examine the point at infinity.

$$
f\left(\frac{1}{\zeta}\right) = \left(\left(\frac{1}{\zeta}-1\right)\left(\frac{1}{\zeta}-2\right)\left(\frac{1}{\zeta}-3\right)\right)^{1/2} = \zeta^{-3/2}\left(\left(1-\frac{1}{\zeta}\right)\left(1-\frac{2}{\zeta}\right)\left(1-\frac{3}{\zeta}\right)\right)^{1/2}
$$

Since $\zeta^{-3/2}$ has a branch point at $\zeta = 0$ and the rest of the terms are analytic there, $f(z)$ has a branch point at infinity.

The first two sets of branch cuts in Figure 9.28 do not permit us to walk around any of the branch points, including the point at infinity, and thus make the function single-valued. The third set of branch cuts lets us walk around the branch points at $z = 1$ and $z = 2$ together or if we change our perspective, we would be walking around the branch points at $z = 3$ and $z = \infty$ together. Consider a contour in this cut plane that encircles the branch points at $z = 1$ and $z = 2$. Since the argument of $(z - z_0)^{1/2}$ changes by π when we walk around z_0, the argument of $f(z)$ changes by 2π when we traverse the contour. Thus the value of the function does not change and it is a valid set of branch cuts. Clearly the fourth set of branch cuts does not make the function single-valued as there are contours that encircle the branch point at infinity and no other branch points. The other way to see this is to note that the argument of $f(z)$ changes by 3π as we traverse a contour that goes around the branch points at $z = 1, 2, 3$ once in the positive direction.

Now to define the branch. We make the preliminary choice of angles,

$$\begin{aligned} z-1 &= r_1 \,\mathrm{e}^{i\theta_1}, \quad 0 < \theta_1 < 2\pi, \\ z-2 &= r_2 \,\mathrm{e}^{i\theta_2}, \quad 0 < \theta_2 < 2\pi, \\ z-3 &= r_3 \,\mathrm{e}^{i\theta_3}, \quad 0 < \theta_3 < 2\pi. \end{aligned}$$

The function is

$$f(z) = \left(r_1 \,\mathrm{e}^{i\theta_1} r_2 \,\mathrm{e}^{i\theta_2} r_3 \,\mathrm{e}^{i\theta_3}\right)^{1/2} = \sqrt{r_1 r_2 r_3}\,\mathrm{e}^{i(\theta_1+\theta_2+\theta_3)/2}.$$

The value of the function at the origin is

$$f(0) = \sqrt{6}\,\mathrm{e}^{i(3\pi)/2} = -i\sqrt{6},$$

which is not what we wanted. We will change range of one of the angles to get the desired result.

$$\begin{aligned} z-1 &= r_1 \,\mathrm{e}^{i\theta_1}, \quad 0 < \theta_1 < 2\pi, \\ z-2 &= r_2 \,\mathrm{e}^{i\theta_2}, \quad 0 < \theta_2 < 2\pi, \\ z-3 &= r_3 \,\mathrm{e}^{i\theta_3}, \quad 2\pi < \theta_3 < 4\pi. \end{aligned}$$

$$f(0) = \sqrt{6}\,\mathrm{e}^{i(5\pi)/2} = i\sqrt{6},$$

Solution 9.26

$$w = \left((z^2-2)(z+2)\right)^{1/3} (z+\sqrt{2})^{1/3}(z-\sqrt{2})^{1/3}(z+2)^{1/3}$$

There are branch points at $z = \pm\sqrt{2}$ and $z = -2$. If we walk around any one of the branch points once in the positive direction, the argument of w changes by $2\pi/3$ and thus the value of the function changes by $\mathrm{e}^{i2\pi/3}$. If we walk around all three branch points then the argument of w changes by $3 \times 2\pi/3 = 2\pi$. The value of the function is unchanged as $e^{i2\pi} = 1$. Thus the branch cut on the real axis from -2 to $\sqrt{2}$ makes the function single-valued.

Now we define a branch. Let

$$z - \sqrt{2} = a\,e^{i\alpha}, \quad z + \sqrt{2} = b\,e^{i\beta}, \quad z + 2 = c\,e^{i\gamma}.$$

We constrain the angles as follows: On the positive real axis, $\alpha = \beta = \gamma$. See Figure 9.40.

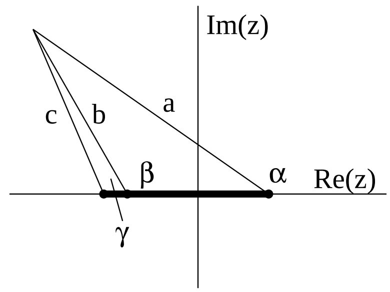


Figure 9.40: A branch of $((z^2 - 2)(z + 2))^{1/3}$.

Now we determine $w(2)$.

$$\begin{aligned} w(2) &= (2 - \sqrt{2})^{1/3}(2 + \sqrt{2})^{1/3}(2 + 2)^{1/3} \\ &= \sqrt[3]{2 - \sqrt{2}}\,e^{i0}\,\sqrt[3]{2 + \sqrt{2}}\,e^{i0}\,\sqrt[3]{4}\,e^{i0} \\ &= \sqrt[3]{2}\sqrt[3]{4} \\ &= 2. \end{aligned}$$

Note that we didn't have to choose the angle from each of the branch points as zero. Choosing any integer multiple of 2π would give us the same result.

$$
\begin{aligned}
w(-3) &= (-3-\sqrt{2})^{1/3}(-3+\sqrt{2})^{1/3}(-3+2)^{1/3} \\
&= \sqrt[3]{3+\sqrt{2}}\,\mathrm{e}^{i\pi/3}\sqrt[3]{3-\sqrt{2}}\,\mathrm{e}^{i\pi/3}\sqrt[3]{1}\,\mathrm{e}^{i\pi/3} \\
&= \sqrt[3]{7}\,\mathrm{e}^{i\pi} \\
&= -\sqrt[3]{7}
\end{aligned}
$$

The value of the function is

$$
w = \sqrt[3]{abc}\,\mathrm{e}^{i(\alpha+\beta+\gamma)/3}.
$$

Consider the interval $(-\sqrt{2} \ldots \sqrt{2})$. As we approach the branch cut from above, the function has the value,

$$
w = \sqrt[3]{abc}\,\mathrm{e}^{i\pi/3} = \sqrt[3]{(\sqrt{2}-x)(x+\sqrt{2})(x+2)}\,\mathrm{e}^{i\pi/3}.
$$

As we approach the branch cut from below, the function has the value,

$$
w = \sqrt[3]{abc}\,\mathrm{e}^{-i\pi/3} = \sqrt[3]{(\sqrt{2}-x)(x+\sqrt{2})(x+2)}\,\mathrm{e}^{-i\pi/3}.
$$

Consider the interval $(-2 \ldots -\sqrt{2})$. As we approach the branch cut from above, the function has the value,

$$
w = \sqrt[3]{abc}\,\mathrm{e}^{i2\pi/3} = \sqrt[3]{(\sqrt{2}-x)(-x-\sqrt{2})(x+2)}\,\mathrm{e}^{i2\pi/3}.
$$

As we approach the branch cut from below, the function has the value,

$$
w = \sqrt[3]{abc}\,\mathrm{e}^{-i2\pi/3} = \sqrt[3]{(\sqrt{2}-x)(-x-\sqrt{2})(x+2)}\,\mathrm{e}^{-i2\pi/3}.
$$

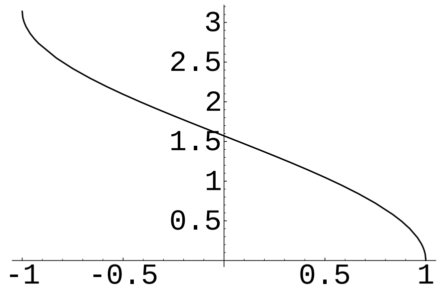


Figure 9.41: The Principal Branch of the arc cosine, *Arccos*(x).

Solution 9.27

Arccos (x) is shown in Figure 9.41 for real variables in the range $[-1, 1]$.

First we write $\arccos(z)$ in terms of $\log(z)$. If $\cos(w) = z$, then $w = \arccos(z)$.

$$\cos(w) = z$$
$$\frac{\mathrm{e}^{iw} + \mathrm{e}^{-iw}}{2} = z$$
$$(\mathrm{e}^{iw})^2 - 2z\,\mathrm{e}^{iw} + 1 = 0$$
$$\mathrm{e}^{iw} = z + (z^2 - 1)^{1/2}$$
$$w = -i\log(z + (z^2 - 1)^{1/2})$$

Thus we have

$$\boxed{\arccos(z) = -i\log(z + (z^2 - 1)^{1/2}).}$$

Since Arccos $(0) = \frac{\pi}{2}$, we must find the branch such that

$$-i\log(0 + (0^2 - 1)^{1/2}) = 0$$
$$-i\log((-1)^{1/2}) = 0.$$

Since

$$-i\log(i) = -i\left(i\frac{\pi}{2} + i2\pi n\right) = \frac{\pi}{2} + 2\pi n$$

and

$$-i\log(-i) = -i\left(-i\frac{\pi}{2} + i2\pi n\right) = -\frac{\pi}{2} + 2\pi n$$

we must choose the branch of the square root such that $(-1)^{1/2} = i$ and the branch of the logarithm such that $\log(i) = i\frac{\pi}{2}$.

First we construct the branch of the square root.

$$(z^2 - 1)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}$$

We see that there are branch points at $z = -1$ and $z = 1$. In particular we want the Arccos to be defined for $z = x$, $x \in [-1, 1]$. Hence we introduce branch cuts on the lines $-\infty < x \leq -1$ and $1 \leq x < \infty$. Define the local coordinates

$$z + 1 = r\,\mathrm{e}^{i\theta}, \qquad z - 1 = \rho\,\mathrm{e}^{i\phi}.$$

With the given branch cuts, the angles have the possible ranges

$$\{\theta\} = \{\dots, (-\pi..\pi), (\pi..3\pi), \dots\}, \qquad \{\phi\} = \{\dots, (0..2\pi), (2\pi..4\pi), \dots\}.$$

Now we choose ranges for θ and ϕ and see if we get the desired branch. If not, we choose a different range for one of the angles. First we choose the ranges

$$\theta \in (-\pi..\pi), \qquad \phi \in (0..2\pi).$$

If we substitute in $z = 0$ we get

$$(0^2 - 1)^{1/2} = (1\,\mathrm{e}^{i0})^{1/2}(1\,\mathrm{e}^{i\pi})^{1/2} = \mathrm{e}^{i0}\,\mathrm{e}^{i\pi/2} = i$$

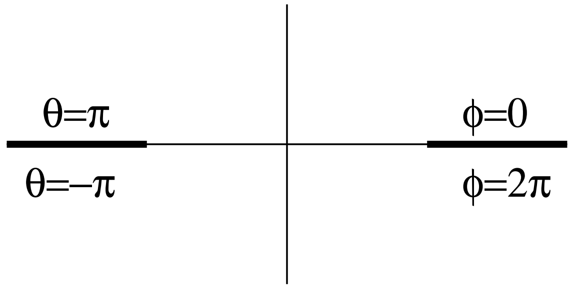


Figure 9.42: Branch Cuts and Angles for $(z^2 - 1)^{1/2}$

Thus we see that this choice of angles gives us the desired branch.

Now we go back to the expression

$$\arccos(z) = -i \log(z + (z^2 - 1)^{1/2}).$$

We have already seen that there are branch points at $z = -1$ and $z = 1$ because of $(z^2 - 1)^{1/2}$. Now we must determine if the logarithm introduces additional branch points. The only possibilities for branch points are where the argument of the logarithm is zero.

$$\begin{aligned} z + (z^2 - 1)^{1/2} &= 0 \\ z^2 &= z^2 - 1 \\ 0 &= -1 \end{aligned}$$

We see that the argument of the logarithm is nonzero and thus there are no additional branch points. Introduce the variable, $w = z + (z^2 - 1)^{1/2}$. What is the image of the branch cuts in the w plane? We parameterize the branch cut connecting $z = 1$ and $z = +\infty$ with $z = r + 1$, $r \in [0, \infty)$.

$$\begin{aligned} w &= r + 1 + ((r + 1)^2 - 1)^{1/2} \\ &= r + 1 \pm \sqrt{r(r + 2)} \\ &= r(1 \pm r\sqrt{1 + 2/r}) + 1 \end{aligned}$$

$r(1+\sqrt{1+2/r})+1$ is the interval $[1,\infty)$; $r(1-\sqrt{1+2/r})+1$ is the interval $(0,1]$. Thus we see that this branch cut is mapped to the interval $(0,\infty)$ in the w plane. Similarly, we could show that the branch cut $(-\infty,-1]$ in the z plane is mapped to $(-\infty,0)$ in the w plane. In the w plane there is a branch cut along the real w axis from $-\infty$ to ∞. Thus cut makes the logarithm single-valued. For the branch of the square root that we chose, all the points in the z plane get mapped to the upper half of the w plane.

With the branch cuts we have introduced so far and the chosen branch of the square root we have

$$\begin{aligned}\arccos(0) &= -i\log(0+(0^2-1)^{1/2})\\ &= -i\log i\\ &= -i\left(i\frac{\pi}{2}+i2\pi n\right)\\ &= \frac{\pi}{2}+2\pi n\end{aligned}$$

Choosing the $n=0$ branch of the logarithm will give us $\operatorname{Arccos}(z)$. We see that we can write

$$\operatorname{Arccos}(z) = -i\operatorname{Log}(z+(z^2-1)^{1/2}).$$

Solution 9.28

We consider the function $f(z)=(z^{1/2}-1)^{1/2}$. First note that $z^{1/2}$ has a branch point at $z=0$. We place a branch cut on the negative real axis to make it single valued. $f(z)$ will have a branch point where $z^{1/2}-1=0$. This occurs at $z=1$ on the branch of $z^{1/2}$ on which $1^{1/2}=1$. ($1^{1/2}$ has the value 1 on one branch of $z^{1/2}$ and -1 on the other branch.) For this branch we introduce a branch cut connecting $z=1$ with the point at infinity. (See Figure 9.43.)

Solution 9.29

The distance between the end of rod a and the end of rod c is b. In the complex plane, these points are $a\,\mathrm{e}^{i\theta}$ and

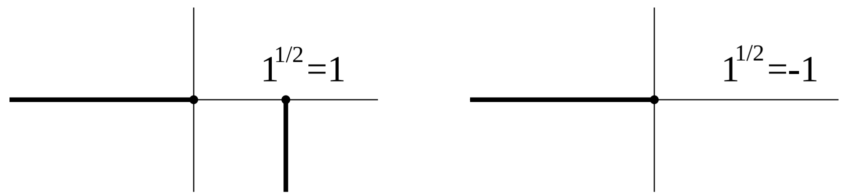


Figure 9.43: Branch Cuts for $(z^{1/2} - 1)^{1/2}$

$l + c\,\mathrm{e}^{i\phi}$, respectively. We write this out mathematically.

$$\left| l + c\,\mathrm{e}^{i\phi} - a\,\mathrm{e}^{i\theta} \right| = b$$
$$\left(l + c\,\mathrm{e}^{i\phi} - a\,\mathrm{e}^{i\theta} \right) \left(l + c\,\mathrm{e}^{-i\phi} - a\,\mathrm{e}^{-i\theta} \right) = b^2$$
$$l^2 + cl\,\mathrm{e}^{-i\phi} - al\,\mathrm{e}^{-i\theta} + cl\,\mathrm{e}^{i\phi} + c^2 - ac\,\mathrm{e}^{i(\phi-\theta)} - al\,\mathrm{e}^{i\theta} - ac\,\mathrm{e}^{i(\theta-\phi)} + a^2 = b^2$$
$$\boxed{cl\cos\phi - ac\cos(\phi - \theta) - al\cos\theta = \frac{1}{2}\left(b^2 - a^2 - c^2 - l^2\right)}$$

This equation relates the two angular positions. One could differentiate the equation to relate the velocities and accelerations.

Solution 9.30

1. Let $w = u + iv$. First we do the strip: $|\Re(z)| < 1$. Consider the vertical line: $z = c + iy$, $y \in \mathbb{R}$. This line is mapped to

$$w = 2(c + iy)^2$$
$$w = 2c^2 - 2y^2 + i4cy$$
$$u = 2c^2 - 2y^2, \quad v = 4cy$$

This is a parabola that opens to the left. For the case $c = 0$ it is the negative u axis. We can parametrize the curve in terms of v.

$$u = 2c^2 - \frac{1}{8c^2}v^2, \quad v \in \mathbb{R}$$

The boundaries of the region are both mapped to the parabolas:

$$u = 2 - \frac{1}{8}v^2, \quad v \in \mathbb{R}.$$

The image of the mapping is

$$\boxed{\left\{w = u + iv : v \in \mathbb{R} \text{ and } u < 2 - \frac{1}{8}v^2\right\}.}$$

Note that the mapping is two-to-one.

Now we do the strip $1 < \Im(z) < 2$. Consider the horizontal line: $z = x + ic$, $x \in \mathbb{R}$. This line is mapped to

$$w = 2(x + ic)^2$$
$$w = 2x^2 - 2c^2 + i4cx$$
$$u = 2x^2 - 2c^2, \quad v = 4cx$$

This is a parabola that opens upward. We can parametrize the curve in terms of v.

$$u = \frac{1}{8c^2}v^2 - 2c^2, \quad v \in \mathbb{R}$$

The boundary $\Im(z) = 1$ is mapped to

$$u = \frac{1}{8}v^2 - 2, \quad v \in \mathbb{R}.$$

The boundary $\Im(z) = 2$ is mapped to

$$u = \frac{1}{32}v^2 - 8, \quad v \in \mathbb{R}$$

The image of the mapping is

$$\boxed{\left\{w = u + iv : v \in \mathbb{R} \text{ and } \frac{1}{32}v^2 - 8 < u < \frac{1}{8}v^2 - 2\right\}.}$$

2. We write the transformation as

$$\frac{z+1}{z-1} = 1 + \frac{2}{z-1}.$$

Thus we see that the transformation is the sequence:

(a) translation by -1
(b) inversion
(c) magnification by 2
(d) translation by 1

Consider the strip $|\Re(z)| < 1$. The translation by -1 maps this to $-2 < \Re(z) < 0$. Now we do the inversion. The left edge, $\Re(z) = 0$, is mapped to itself. The right edge, $\Re(z) = -2$, is mapped to the circle $|z + 1/4| = 1/4$. Thus the current image is the left half plane minus a circle:

$$\Re(z) < 0 \quad \text{and} \quad \left|z + \frac{1}{4}\right| > \frac{1}{4}.$$

The magnification by 2 yields

$$\Re(z) < 0 \quad \text{and} \quad \left|z + \frac{1}{2}\right| > \frac{1}{2}.$$

The final step is a translation by 1.

$$\boxed{\Re(z) < 1 \quad \text{and} \quad \left|z - \frac{1}{2}\right| > \frac{1}{2}.}$$

Now consider the strip $1 < \Im(z) < 2$. The translation by -1 does not change the domain. Now we do the inversion. The bottom edge, $\Im(z) = 1$, is mapped to the circle $|z + i/2| = 1/2$. The top edge, $\Im(z) = 2$, is mapped to the circle $|z + i/4| = 1/4$. Thus the current image is the region between two circles:

$$\left| z + \frac{i}{2} \right| < \frac{1}{2} \quad \text{and} \quad \left| z + \frac{i}{4} \right| > \frac{1}{4}.$$

The magnification by 2 yields

$$|z + i| < 1 \quad \text{and} \quad \left| z + \frac{i}{2} \right| > \frac{1}{2}.$$

The final step is a translation by 1.

$$\boxed{|z - 1 + i| < 1 \quad \text{and} \quad \left| z - 1 + \frac{i}{2} \right| > \frac{1}{2}.}$$

Solution 9.31

1. There is a simple pole at $z = -2$. The function has a branch point at $z = -1$. Since this is the only branch point in the finite complex plane there is also a branch point at infinity. We can verify this with the substitution $z = 1/\zeta$.

$$\begin{aligned} f\left(\frac{1}{\zeta}\right) &= \frac{(1/\zeta + 1)^{1/2}}{1/\zeta + 2} \\ &= \frac{\zeta^{1/2}(1 + \zeta)^{1/2}}{1 + 2\zeta} \end{aligned}$$

   Since $f(1/\zeta)$ has a branch point at $\zeta = 0$, $f(z)$ has a branch point at infinity.

2. $\cos z$ is an entire function with an essential singularity at infinity. Thus $f(z)$ has singularities only where $1/(1 + z)$ has singularities. $1/(1 + z)$ has a first order pole at $z = -1$. It is analytic everywhere else,

including the point at infinity. Thus we conclude that $f(z)$ has an essential singularity at $z = -1$ and is analytic elsewhere. To explicitly show that $z = -1$ is an essential singularity, we can find the Laurent series expansion of $f(z)$ about $z = -1$.

$$\cos\left(\frac{1}{1+z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z+1)^{-2n}$$

3. $1 - \mathrm{e}^z$ has simple zeros at $z = i2n\pi$, $n \in \mathbb{Z}$. Thus $f(z)$ has second order poles at those points.

   The point at infinity is a non-isolated singularity. To justify this: Note that

$$f(z) = \frac{1}{(1 - \mathrm{e}^z)^2}$$

   has second order poles at $z = i2n\pi$, $n \in \mathbb{Z}$. This means that $f(1/\zeta)$ has second order poles at $\zeta = \frac{1}{i2n\pi}$, $n \in \mathbb{Z}$. These second order poles get arbitrarily close to $\zeta = 0$. There is no deleted neighborhood around $\zeta = 0$ in which $f(1/\zeta)$ is analytic. Thus the point $\zeta = 0$, $(z = \infty)$, is a non-isolated singularity. There is no Laurent series expansion about the point $\zeta = 0$, $(z = \infty)$.

   The point at infinity is neither a branch point nor a removable singularity. It is not a pole either. If it were, there would be an n such that $\lim_{z\to\infty} z^{-n} f(z) = \text{const} \neq 0$. Since $z^{-n} f(z)$ has second order poles in every deleted neighborhood of infinity, the above limit does not exist. Thus we conclude that the point at infinity is an essential singularity.

Solution 9.32
We write $\sinh z$ in Cartesian form.

$$w = \sinh z = \sinh x \cos y + i \cosh x \sin y = u + iv$$

Consider the line segment $x = c$, $y \in (0 \ldots \pi)$. Its image is

$$\{\sinh c \cos y + i \cosh c \sin y \mid y \in (0 \ldots \pi)\}.$$

This is the parametric equation for the upper half of an ellipse. Also note that u and v satisfy the equation for an ellipse.

$$\frac{u^2}{\sinh^2 c} + \frac{v^2}{\cosh^2 c} = 1$$

The ellipse starts at the point $(\sinh(c), 0)$, passes through the point $(0, \cosh(c))$ and ends at $(-sinh(c), 0)$. As c varies from zero to ∞ or from zero to $-\infty$, the semi-ellipses cover the upper half w plane. Thus the mapping is 2-to-1.

Consider the infinite line $y = c$, $x \in (-\infty \ldots \infty)$.Its image is

$$\{\sinh x \cos c + i \cosh x \sin c \,|\, x \in (-\infty \ldots \infty)\}.$$

This is the parametric equation for the upper half of a hyperbola. Also note that u and v satisfy the equation for a hyperbola.

$$-\frac{u^2}{\cos^2 c} + \frac{v^2}{\sin^2 c} = 1$$

As c varies from 0 to $\pi/2$ or from $\pi/2$ to π, the semi-hyperbola cover the upper half w plane. Thus the mapping is 2-to-1.

We look for branch points of $\sinh^{-1} w$.

$$\begin{aligned} w &= \sinh z \\ w &= \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{2} \\ \mathrm{e}^{2z} - 2w\,\mathrm{e}^z - 1 &= 0 \\ \mathrm{e}^z &= w + (w^2 + 1)^{1/2} \\ z &= \log\left(w + (w - i)^{1/2}(w + i)^{1/2}\right) \end{aligned}$$

There are branch points at $w = \pm i$. Since $w + (w^2 + 1)^{1/2}$ is nonzero and finite in the finite complex plane, the logarithm does not introduce any branch points in the finite plane. Thus the only branch point in the upper half w plane is at $w = i$. Any branch cut that connects $w = i$ with the boundary of $\Im(w) > 0$ will separate the branches under the inverse mapping.

Consider the line $y = \pi/4$. The image under the mapping is the upper half of the hyperbola

$$2u^2 + 2v^2 = 1.$$

Consider the segment $x = 1$.The image under the mapping is the upper half of the ellipse

$$\frac{u^2}{\sinh^2 1} + \frac{v^2}{\cosh^2 1} = 1.$$

Chapter 10

Analytic Functions

Students need encouragement. So if a student gets an answer right, tell them it was a lucky guess. That way, they develop a good, lucky feeling. [1]

-Jack Handey

10.1 Complex Derivatives

Functions of a Real Variable. The derivative of a function of a real variable is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

If the limit exists then the function is differentiable at the point x. Note that Δx can approach zero from above or below. The limit cannot depend on the direction in which Δx vanishes.

Consider $f(x) = |x|$. The function is not differentiable at $x = 0$ since

$$\lim_{\Delta x \to 0^+} \frac{|0 + \Delta x| - |0|}{\Delta x} = 1$$

[1]Quote slightly modified.

and

$$\lim_{\Delta x \to 0^-} \frac{|0 + \Delta x| - |0|}{\Delta x} = -1.$$

Analyticity. The *complex derivative*, (or simply *derivative* if the context is clear), is defined,

$$\frac{\mathrm{d}}{\mathrm{d}z} f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

The complex derivative exists if this limit exists. This means that the value of the limit is independent of the manner in which $\Delta z \to 0$. If the complex derivative exists at a point, then we say that the function is *complex differentiable* there.

A function of a complex variable is *analytic* at a point z_0 if the complex derivative exists in a neighborhood about that point. The function is analytic in an open set if it has a complex derivative at each point in that set. Note that complex differentiable has a different meaning than analytic. Analyticity refers to the behavior of a function on an open set. A function can be complex differentiable at isolated points, but the function would not be analytic at those points. Analytic functions are also called *regular* or *holomorphic*. If a function is analytic everywhere in the finite complex plane, it is called *entire*.

Example 10.1.1 Consider z^n, $n \in \mathbb{Z}^+$, Is the function differentiable? Is it analytic? What is the value of the derivative?

We determine differentiability by trying to differentiate the function. We use the limit definition of differentiation. We will use Newton's binomial formula to expand $(z + \Delta z)^n$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z} z^n &= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\left(z^n + n z^{n-1} \Delta z + \frac{n(n-1)}{2} z^{n-2} \Delta z^2 + \cdots + \Delta z^n \right) - z^n}{\Delta z} \\
&= \lim_{\Delta z \to 0} \left(n z^{n-1} + \frac{n(n-1)}{2} z^{n-2} \Delta z + \cdots + \Delta z^{n-1} \right) \\
&= n z^{n-1}
\end{aligned}$$

The derivative exists everywhere. The function is analytic in the whole complex plane so it is entire. The value of the derivative is $\frac{\mathrm{d}}{\mathrm{d}z} = nz^{n-1}$.

Example 10.1.2 We will show that $f(z) = \overline{z}$ is not differentiable. The derivative is,

$$\frac{\mathrm{d}}{\mathrm{d}z} f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z} \overline{z} &= \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} \end{aligned}$$

If we take $\Delta z = \Delta x$, the limit is

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

If we take $\Delta z = i \Delta y$, the limit is

$$\lim_{\Delta y \to 0} \frac{-i \Delta y}{i \Delta y} = -1.$$

Since the limit depends on the way that $\Delta z \to 0$, the function is nowhere differentiable. Thus the function is not analytic.

Complex Derivatives in Terms of Plane Coordinates. Let $z = \zeta(\xi, \eta)$ be a system of coordinates in the complex plane. (For example, we could have Cartesian coordinates $z = \zeta(x, y) = x + iy$ or polar coordinates $z = \zeta(r, \theta) = r\,\mathrm{e}^{i\theta}$). Let $f(z) = \psi(\xi, \eta)$ be a complex-valued function. (For example we might have a function in the form $\psi(x, y) = u(x, y) + iv(x, y)$ or $\psi(r, \theta) = R(r, \theta)\,\mathrm{e}^{i\Theta(r,\theta)}$.) If $f(z) = \psi(\xi, \eta)$ is analytic, its complex derivative is equal to the derivative in any direction. In particular, it is equal to the derivatives in the coordinate directions.

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{\Delta\xi\to 0, \Delta\eta=0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta\xi\to 0} \frac{\psi(\xi+\Delta\xi, \eta) - \psi(\xi, \eta)}{\frac{\partial\zeta}{\partial\xi}\Delta\xi} = \left(\frac{\partial\zeta}{\partial\xi}\right)^{-1} \frac{\partial\psi}{\partial\xi},$$
$$\frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{\Delta\xi=0, \Delta\eta\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta\eta\to 0} \frac{\psi(\xi, \eta+\Delta\eta) - \psi(\xi, \eta)}{\frac{\partial\zeta}{\partial\eta}\Delta\eta} = \left(\frac{\partial\zeta}{\partial\eta}\right)^{-1} \frac{\partial\psi}{\partial\eta}.$$

Example 10.1.3 Consider the Cartesian coordinates $z = x + iy$. We write the complex derivative as derivatives in the coordinate directions for $f(z) = \psi(x, y)$.

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \left(\frac{\partial(x+iy)}{\partial x}\right)^{-1} \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial x},$$
$$\frac{\mathrm{d}f}{\mathrm{d}z} = \left(\frac{\partial(x+iy)}{\partial y}\right)^{-1} \frac{\partial\psi}{\partial y} = -i\frac{\partial\psi}{\partial y}.$$

We write this in operator notation.

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{\partial}{\partial x} = -i\frac{\partial}{\partial y}.$$

Example 10.1.4 In Example 10.1.1 we showed that z^n, $n \in \mathbb{Z}^+$, is an entire function and that $\frac{\mathrm{d}}{\mathrm{d}z} z^n = nz^{n-1}$. Now we corroborate this by calculating the complex derivative in the Cartesian coordinate directions.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z} z^n &= \frac{\partial}{\partial x}(x+iy)^n \\ &= n(x+iy)^{n-1} \\ &= nz^{n-1}\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z} z^n &= -i \frac{\partial}{\partial y}(x + iy)^n \\
&= -i(i)n(x + iy)^{n-1} \\
&= n z^{n-1}
\end{aligned}
$$

Complex Derivatives are Not the Same as Partial Derivatives Recall from calculus that

$$
f(x, y) = g(s, t) \quad \Rightarrow \quad \frac{\partial f}{\partial x} = \frac{\partial g}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial g}{\partial t}\frac{\partial t}{\partial x}
$$

Do not make the mistake of using a similar formula for functions of a complex variable. If $f(z) = \psi(x, y)$ then

$$
\frac{\mathrm{d}f}{\mathrm{d}z} \neq \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial z}.
$$

This is because the $\frac{\mathrm{d}}{\mathrm{d}z}$ operator means "The derivative in any direction in the complex plane." Since $f(z)$ is analytic, $f'(z)$ is the same no matter in which direction we take the derivative.

Rules of Differentiation. For an analytic function defined in terms of z we can calculate the complex derivative using all the usual rules of differentiation that we know from calculus like the product rule,

$$
\frac{\mathrm{d}}{\mathrm{d}z} f(z)g(z) = f'(z)g(z) + f(z)g'(z),
$$

or the chain rule,

$$
\frac{\mathrm{d}}{\mathrm{d}z} f(g(z)) = f'(g(z))g'(z).
$$

This is because the complex derivative derives its properties from properties of limits, just like its real variable counterpart.

Result 10.1.1 The complex derivative is,

$$\frac{\mathrm{d}}{\mathrm{d}z} f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

The complex derivative is defined if the limit exists and is independent of the manner in which $\Delta z \to 0$. A function is analytic at a point if the complex derivative exists in a neighborhood of that point.

Let $z = \zeta(\xi, \eta)$ be coordinates in the complex plane. The complex derivative in the coordinate directions is

$$\frac{\mathrm{d}}{\mathrm{d}z} = \left(\frac{\partial \zeta}{\partial \xi}\right)^{-1} \frac{\partial}{\partial \xi} = \left(\frac{\partial \zeta}{\partial \eta}\right)^{-1} \frac{\partial}{\partial \eta}.$$

In Cartesian coordinates, this is

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{\partial}{\partial x} = -i \frac{\partial}{\partial y}.$$

In polar coordinates, this is

$$\frac{\mathrm{d}}{\mathrm{d}z} = \mathrm{e}^{-i\theta} \frac{\partial}{\partial r} = -\frac{i}{r} \mathrm{e}^{-i\theta} \frac{\partial}{\partial \theta}$$

Since the complex derivative is defined with the same limit formula as real derivatives, all the rules from the calculus of functions of a real variable may be used to differentiate functions of a complex variable.

Example 10.1.5 We have shown that z^n, $n \in \mathbb{Z}^+$, is an entire function. Now we corroborate that $\frac{\mathrm{d}}{\mathrm{d}z} z^n = n z^{n-1}$ by calculating the complex derivative in the polar coordinate directions.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z} z^n &= \mathrm{e}^{-i\theta} \frac{\partial}{\partial r} r^n \, \mathrm{e}^{in\theta} \\
&= \mathrm{e}^{-i\theta} n r^{n-1} \, \mathrm{e}^{in\theta} \\
&= n r^{n-1} \, \mathrm{e}^{i(n-1)\theta} \\
&= n z^{n-1}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z} z^n &= -\frac{i}{r} \, \mathrm{e}^{-i\theta} \frac{\partial}{\partial \theta} r^n \, \mathrm{e}^{in\theta} \\
&= -\frac{i}{r} \, \mathrm{e}^{-i\theta} r^n i n \, \mathrm{e}^{in\theta} \\
&= n r^{n-1} \, \mathrm{e}^{i(n-1)\theta} \\
&= n z^{n-1}
\end{aligned}$$

Analytic Functions can be Written in Terms of z. Consider an analytic function expressed in terms of x and y, $\psi(x, y)$. We can write ψ as a function of $z = x + iy$ and $\overline{z} = x - iy$.

$$f(z, \overline{z}) = \psi \left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{i2} \right)$$

We treat z and $\overline{z}$ as independent variables. We find the partial derivatives with respect to these variables.

$$\begin{aligned}
\frac{\partial}{\partial z} &= \frac{\partial x}{\partial z} \frac{\partial}{\partial x} + \frac{\partial y}{\partial z} \frac{\partial}{\partial y} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \\
\frac{\partial}{\partial \overline{z}} &= \frac{\partial x}{\partial \overline{z}} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \overline{z}} \frac{\partial}{\partial y} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\end{aligned}$$

Since ψ is analytic, the complex derivatives in the x and y directions are equal.

$$\frac{\partial \psi}{\partial x} = -i\frac{\partial \psi}{\partial y}$$

The partial derivative of $f(z, \overline{z})$ with respect to $\overline{z}$ is zero.

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial \psi}{\partial x} + i\frac{\partial \psi}{\partial y}\right) = 0$$

Thus $f(z, \overline{z})$ has no functional dependence on $\overline{z}$, it can be written as a function of z alone.

If we were considering an analytic function expressed in polar coordinates $\phi(r, \theta)$, then we could write it in Cartesian coordinates with the substitutions:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(x, y).$$

Thus we could write $\phi(r, \theta)$ as a function of z alone.

Result 10.1.2 Any analytic function $\psi(x, y)$ or $\phi(r, \theta)$ can be written as a function of z alone.

10.2 Cauchy-Riemann Equations

If we know that a function is analytic, then we have a convenient way of determining its complex derivative. We just express the complex derivative in terms of the derivative in a coordinate direction. However, we don't have a nice way of determining if a function is analytic. The definition of complex derivative in terms of a limit is cumbersome to work with. In this section we remedy this problem.

Consider a function $f(z) = \psi(x, y)$. If $f(z)$ is analytic, the complex derivative is equal to the derivatives in the coordinate directions. We equate the derivatives in the x and y directions to obtain the *Cauchy-Riemann equations* in Cartesian coordinates.

$$\psi_x = -i\psi_y \tag{10.1}$$

This equation is a necessary condition for the analyticity of $f(z)$.

Let $\psi(x, y) = u(x, y) + iv(x, y)$ where u and v are real-valued functions. We equate the real and imaginary parts of Equation 10.1 to obtain another form for the Cauchy-Riemann equations in Cartesian coordinates.

$$u_x = v_y, \qquad u_y = -v_x.$$

Note that this is a necessary and not a sufficient condition for analyticity of $f(z)$. That is, u and v may satisfy the Cauchy-Riemann equations but $f(z)$ may not be analytic. The Cauchy-Riemann equations give us an easy test for determining if a function is not analytic.

Example 10.2.1 In Example 10.1.2 we showed that $\overline{z}$ is not analytic using the definition of complex differentiation. Now we obtain the same result using the Cauchy-Riemann equations.

$$\overline{z} = x - iy$$
$$u_x = 1, \quad v_y = -1$$

We see that the first Cauchy-Riemann equation is not satisfied; the function is not analytic at any point.

A sufficient condition for $f(z) = \psi(x, y)$ to be analytic at a point $z_0 = (x_0, y_0)$ is that the partial derivatives of $\psi(x, y)$ exist and are continuous in some neighborhood of z_0 and satisfy the Cauchy-Riemann equations there. If the partial derivatives of ψ exist and are continuous then

$$\psi(x + \Delta x, y + \Delta y) = \psi(x, y) + \Delta x \psi_x(x, y) + \Delta y \psi_y(x, y) + o(\Delta x) + o(\Delta y).$$

Here the notation $o(\Delta x)$ means "terms smaller than Δx". We calculate the derivative of $f(z)$.

$$\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta x, \Delta y \to 0} \frac{\psi(x + \Delta x, y + \Delta y) - \psi(x, y)}{\Delta x + i\Delta y} \\
&= \lim_{\Delta x, \Delta y \to 0} \frac{\psi(x, y) + \Delta x \psi_x(x, y) + \Delta y \psi_y(x, y) + o(\Delta x) + o(\Delta y) - \psi(x, y)}{\Delta x + i\Delta y} \\
&= \lim_{\Delta x, \Delta y \to 0} \frac{\Delta x \psi_x(x, y) + \Delta y \psi_y(x, y) + o(\Delta x) + o(\Delta y)}{\Delta x + i\Delta y}.
\end{aligned}$$

Here we use the Cauchy-Riemann equations.

$$
\begin{aligned}
&= \lim_{\Delta x, \Delta y \to 0} \frac{(\Delta x + i\Delta y)\psi_x(x,y)}{\Delta x + i\Delta y} + \lim_{\Delta x, \Delta y \to 0} \frac{o(\Delta x) + o(\Delta y)}{\Delta x + i\Delta y}. \\
&= \psi_x(x,y)
\end{aligned}
$$

Thus we see that the derivative is well defined.

Cauchy-Riemann Equations in General Coordinates Let $z = \zeta(\xi, \eta)$ be a system of coordinates in the complex plane. Let $\psi(\xi, \eta)$ be a function which we write in terms of these coordinates, A necessary condition for analyticity of $\psi(\xi, \eta)$ is that the complex derivatives in the coordinate directions exist and are equal. Equating the derivatives in the ξ and η directions gives us the *Cauchy-Riemann equations*.

$$
\left(\frac{\partial \zeta}{\partial \xi}\right)^{-1} \frac{\partial \psi}{\partial \xi} = \left(\frac{\partial \zeta}{\partial \eta}\right)^{-1} \frac{\partial \psi}{\partial \eta}
$$

We could separate this into two equations by equating the real and imaginary parts or the modulus and argument.

Result 10.2.1 A necessary condition for analyticity of $\psi(\xi, \eta)$, where $z = \zeta(\xi, \eta)$, at $z = z_0$ is that the Cauchy-Riemann equations are satisfied in a neighborhood of $z = z_0$.

$$\left(\frac{\partial \zeta}{\partial \xi}\right)^{-1} \frac{\partial \psi}{\partial \xi} = \left(\frac{\partial \zeta}{\partial \eta}\right)^{-1} \frac{\partial \psi}{\partial \eta}.$$

(We could equate the real and imaginary parts or the modulus and argument of this to obtain two equations.) A sufficient condition for analyticity of $f(z)$ is that the Cauchy-Riemann equations hold and the first partial derivatives of ψ exist and are continuous in a neighborhood of $z = z_0$.
Below are the Cauchy-Riemann equations for various forms of $f(z)$.

$$\begin{aligned}
&f(z) = \psi(x, y), && \psi_x = -i\psi_y \\
&f(z) = u(x, y) + iv(x, y), && u_x = v_y, \quad u_y = -v_x \\
&f(z) = \psi(r, \theta), && \psi_r = -\frac{i}{r}\psi_\theta \\
&f(z) = u(r, \theta) + iv(r, \theta), && u_r = \frac{1}{r}v_\theta, \quad u_\theta = -rv_r \\
&f(z) = R(r, \theta)\,\mathrm{e}^{i\Theta(r,\theta)}, && R_r = \frac{R}{r}\Theta_\theta, \quad \frac{1}{r}R_\theta = -R\Theta_r
\end{aligned}$$

Example 10.2.2 Consider the Cauchy-Riemann equations for $f(z) = u(r, \theta) + iv(r, \theta)$. From Exercise 10.2 we know that the complex derivative in the polar coordinate directions is

$$\frac{\mathrm{d}}{\mathrm{d}z} = \mathrm{e}^{-i\theta}\frac{\partial}{\partial r} = -\frac{i}{r}\,\mathrm{e}^{-i\theta}\frac{\partial}{\partial \theta}.$$

From Result 10.2.1 we have the equation,

$$\mathrm{e}^{-i\theta} \frac{\partial}{\partial r}[u + iv] = -\frac{i}{r} \mathrm{e}^{-i\theta} \frac{\partial}{\partial \theta}[u + iv].$$

We multiply by $\mathrm{e}^{i\theta}$ and equate the real and imaginary components to obtain the Cauchy-Riemann equations.

$$u_r = \frac{1}{r} v_\theta, \qquad u_\theta = -r v_r$$

Example 10.2.3 Consider the exponential function.

$$\mathrm{e}^z = \psi(x, y) = \mathrm{e}^x (\cos y + i \sin(y))$$

We use the Cauchy-Riemann equations to show that the function is entire.

$$\begin{aligned} \psi_x &= -i \psi_y \\ \mathrm{e}^x (\cos y + i \sin(y)) &= -i \, \mathrm{e}^x (-\sin y + i \cos(y)) \\ \mathrm{e}^x (\cos y + i \sin(y)) &= \mathrm{e}^x (\cos y + i \sin(y)) \end{aligned}$$

Since the function satisfies the Cauchy-Riemann equations and the first partial derivatives are continuous everywhere in the finite complex plane, the exponential function is entire.

Now we find the value of the complex derivative.

$$\frac{\mathrm{d}}{\mathrm{d}z} \mathrm{e}^z = \frac{\partial \psi}{\partial x} = \mathrm{e}^x (\cos y + i \sin(y)) = \mathrm{e}^z$$

The differentiability of the exponential function implies the differentiability of the trigonometric functions, as they can be written in terms of the exponential.

In Exercise 10.11 you can show that the logarithm $\log z$ is differentiable for $z \neq 0$. This implies the differentiability of z^α and the inverse trigonometric functions as they can be written in terms of the logarithm.

Example 10.2.4 We compute the derivative of z^z.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\left(z^z\right) &= \frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{z\log z} \\ &= (1+\log z)\,\mathrm{e}^{z\log z} \\ &= (1+\log z)z^z \\ &= z^z + z^z\log z\end{aligned}$$

10.3 Harmonic Functions

A function u is harmonic if its second partial derivatives exist, are continuous and satisfy Laplace's equation $\Delta u = 0$. [2] (In Cartesian coordinates the Laplacian is $\Delta u \equiv u_{xx} + u_{yy}$.) If $f(z) = u + iv$ is an analytic function then u and v are harmonic functions. To see why this is so, we start with the Cauchy-Riemann equations.

$$u_x = v_y, \quad u_y = -v_x$$

We differentiate the first equation with respect to x and the second with respect to y. (We assume that u and v are twice continuously differentiable. We will see later that they are infinitely differentiable.)

$$u_{xx} = v_{xy}, \quad u_{yy} = -v_{yx}$$

Thus we see that u is harmonic.

$$\Delta u \equiv u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0$$

One can use the same method to show that $\Delta v = 0$.

[2] The capital Greek letter Δ is used to denote the Laplacian, like $\Delta u(x, y)$, and differentials, like Δx.

If u is harmonic on some simply-connected domain, then there exists a harmonic function v such that $f(z) = u + iv$ is analytic in the domain. v is called the *harmonic conjugate* of u. The harmonic conjugate is unique up to an additive constant. To demonstrate this, let w be another harmonic conjugate of u. Both the pair u and v and the pair u and w satisfy the Cauchy-Riemann equations.

$$u_x = v_y, \quad u_y = -v_x, \qquad u_x = w_y, \quad u_y = -w_x$$

We take the difference of these equations.

$$v_x - w_x = 0, \quad v_y - w_y = 0$$

On a simply connected domain, the difference between v and w is thus a constant.

To prove the existence of the harmonic conjugate, we first write v as an integral.

$$v(x,y) = v(x_0,y_0) + \int_{(x_0,y_0)}^{(x,y)} v_x \, \mathrm{d}x + v_y \, \mathrm{d}y$$

On a simply connected domain, the integral is path independent and defines a unique v in terms of v_x and v_y. We use the Cauchy-Riemann equations to write v in terms of u_x and u_y.

$$v(x,y) = v(x_0,y_0) + \int_{(x_0,y_0)}^{(x,y)} -u_y \, \mathrm{d}x + u_x \, \mathrm{d}y$$

Changing the starting point (x_0, y_0) changes v by an additive constant. The harmonic conjugate of u to within an additive constant is

$$v(x,y) = \int -u_y \, \mathrm{d}x + u_x \, \mathrm{d}y.$$

This proves the existence [3] of the harmonic conjugate. This is not the formula one would use to construct the harmonic conjugate of a u. One accomplishes this by solving the Cauchy-Riemann equations.

[3] A mathematician returns to his office to find that a cigarette tossed in the trash has started a small fire. Being calm and a quick thinker he notes that there is a fire extinguisher by the window. He then closes the door and walks away because “the solution exists.”

Result 10.3.1 If $f(z) = u + iv$ is an analytic function then u and v are harmonic functions. That is, the Laplacians of u and v vanish $\Delta u = \Delta v = 0$. The Laplacian in Cartesian and polar coordinates is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Given a harmonic function u in a simply connected domain, there exists a harmonic function v, (unique up to an additive constant), such that $f(z) = u + iv$ is analytic in the domain. One can construct v by solving the Cauchy-Riemann equations.

Example 10.3.1 Is x^2 the real part of an analytic function?

The Laplacian of x^2 is

$$\Delta[x^2] = 2 + 0$$

x^2 is not harmonic and thus is not the real part of an analytic function.

Example 10.3.2 Show that $u = \mathrm{e}^{-x}(x \sin y - y \cos y)$ is harmonic.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \mathrm{e}^{-x} \sin y - \mathrm{e}^{x}(x \sin y - y \cos y) \\ &= \mathrm{e}^{-x} \sin y - x\,\mathrm{e}^{-x} \sin y + y\,\mathrm{e}^{-x} \cos y \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= -\mathrm{e}^{-x} \sin y - \mathrm{e}^{-x} \sin y + x\,\mathrm{e}^{-x} \sin y - y\,\mathrm{e}^{-x} \cos y \\ &= -2\,\mathrm{e}^{-x} \sin y + x\,\mathrm{e}^{-x} \sin y - y\,\mathrm{e}^{-x} \cos y \end{aligned}$$

$$\frac{\partial u}{\partial y} = \mathrm{e}^{-x}(x\cos y - \cos y + y\sin y)$$

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \mathrm{e}^{-x}(-x\sin y + \sin y + y\cos y + \sin y)\\ &= -x\,\mathrm{e}^{-x}\sin y + 2\,\mathrm{e}^{-x}\sin y + y\,\mathrm{e}^{-x}\cos y\end{aligned}$$

Thus we see that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and u is harmonic.

Example 10.3.3 Consider $u = \cos x \cosh y$. This function is harmonic.

$$u_{xx} + u_{yy} = -\cos x \cosh y + \cos x \cosh y = 0$$

Thus it is the real part of an analytic function, $f(z)$. We find the harmonic conjugate, v, with the Cauchy-Riemann equations. We integrate the first Cauchy-Riemann equation.

$$\begin{gathered} v_y = u_x = -\sin x \cosh y\\ v = -\sin x \sinh y + a(x)\end{gathered}$$

Here $a(x)$ is a constant of integration. We substitute this into the second Cauchy-Riemann equation to determine $a(x)$.

$$\begin{gathered} v_x = -u_y\\ -\cos x \sinh y + a'(x) = -\cos x \sinh y\\ a'(x) = 0\\ a(x) = c\end{gathered}$$

Here c is a real constant. Thus the harmonic conjugate is

$$v = -\sin x \sinh y + c.$$

The analytic function is

$$f(z) = \cos x \cosh y - i \sin x \sinh y + ic$$

We recognize this as

$$f(z) = \cos z + ic.$$

Example 10.3.4 Here we consider an example that demonstrates the need for a simply connected domain. Consider $u = \operatorname{Log} r$ in the multiply connected domain, $r > 0$. u is harmonic.

$$\Delta \operatorname{Log} r = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \operatorname{Log} r \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \operatorname{Log} r = 0$$

We solve the Cauchy-Riemann equations to try to find the harmonic conjugate.

$$\begin{gathered} u_r = \frac{1}{r} v_\theta, \quad u_\theta = -r v_r \\ v_r = 0, \quad v_\theta = 1 \\ v = \theta + c \end{gathered}$$

We are able to solve for v, but it is multi-valued. Any single-valued branch of θ that we choose will not be continuous on the domain. Thus there is no harmonic conjugate of $u = \operatorname{Log} r$ for the domain $r > 0$.

If we had instead considered the simply-connected domain $r > 0$, $|\arg(z)| < \pi$ then the harmonic conjugate would be $v = \operatorname{Arg}(z) + c$. The corresponding analytic function is $f(z) = \operatorname{Log} z + ic$.

Example 10.3.5 Consider $u = x^3 - 3xy^2 + x$. This function is harmonic.

$$u_{xx} + u_{yy} = 6x - 6x = 0$$

Thus it is the real part of an analytic function, $f(z)$. We find the harmonic conjugate, v, with the Cauchy-Riemann equations. We integrate the first Cauchy-Riemann equation.

$$v_y = u_x = 3x^2 - 3y^2 + 1$$
$$v = 3x^2 y - y^3 + y + a(x)$$

Here $a(x)$ is a constant of integration. We substitute this into the second Cauchy-Riemann equation to determine $a(x)$.

$$v_x = -u_y$$
$$6xy + a'(x) = 6xy$$
$$a'(x) = 0$$
$$a(x) = c$$

Here c is a real constant. The harmonic conjugate is

$$v = 3x^2 y - y^3 + y + c.$$

The analytic function is

$$f(z) = x^3 - 3xy^2 + x + i(3x^2 y - y^3 + y) + ic$$
$$f(z) = x^3 + i3x^2 y - 3xy^2 - iy^2 + x + iy + ic$$
$$f(z) = z^3 + z + ic$$

10.4 Singularities

Any point at which a function is not analytic is called a *singularity*. In this section we will classify the different flavors of singularities.

Result 10.4.1 Singularities. If a function is not analytic at a point, then that point is a *singular point* or a *singularity* of the function.

10.4.1 Categorization of Singularities

Branch Points. If $f(z)$ has a branch point at z_0, then we cannot define a branch of $f(z)$ that is continuous in a neighborhood of z_0. Continuity is necessary for analyticity. Thus all branch points are singularities. Since function are discontinuous across branch cuts, all points on a branch cut are singularities.

Example 10.4.1 Consider $f(z) = z^{3/2}$. The origin and infinity are branch points and are thus singularities of $f(z)$. We choose the branch $g(z) = \sqrt{z^3}$. All the points on the negative real axis, including the origin, are singularities of $g(z)$.

Removable Singularities.

Example 10.4.2 Consider

$$f(z) = \frac{\sin z}{z}.$$

This function is undefined at $z = 0$ because $f(0)$ is the indeterminate form $0/0$. $f(z)$ is analytic everywhere in the finite complex plane except $z = 0$. Note that the limit as $z \to 0$ of $f(z)$ exists.

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \frac{\cos z}{1} = 1$$

If we were to fill in the hole in the definition of $f(z)$, we could make it differentiable at $z = 0$. Consider the function

$$g(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0, \\ 1 & z = 0. \end{cases}$$

We calculate the derivative at $z = 0$ to verify that $g(z)$ is analytic there.

$$\begin{aligned} f'(0) &= \lim_{z \to 0} \frac{f(0) - f(z)}{z} \\ &= \lim_{z \to 0} \frac{1 - \sin(z)/z}{z} \\ &= \lim_{z \to 0} \frac{z - \sin(z)}{z^2} \\ &= \lim_{z \to 0} \frac{1 - \cos(z)}{2z} \\ &= \lim_{z \to 0} \frac{\sin(z)}{2} \\ &= 0 \end{aligned}$$

We call the point at $z = 0$ a *removable singularity* of $\sin(z)/z$ because we can remove the singularity by defining the value of the function to be its limiting value there.

Consider a function $f(z)$ that is analytic in a deleted neighborhood of $z = z_0$. If $f(z)$ is not analytic at z_0, but $\lim_{z \to z_0} f(z)$ exists, then the function has a removable singularity at z_0. The function

$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ \lim_{z \to z_0} f(z) & z = z_0 \end{cases}$$

is analytic in a neighborhood of $z = z_0$. We show this by calculating $g'(z_0)$.

$$\begin{aligned} g'(z_0) &= \lim_{z \to z_0} \frac{g(z_0) - g(z)}{z_0 - z} \\ &= \lim_{z \to z_0} \frac{-g'(z)}{-1} \\ &= \lim_{z \to z_0} f'(z) \end{aligned}$$

This limit exists because $f(z)$ is analytic in a deleted neighborhood of $z = z_0$.

Poles. If a function $f(z)$ behaves like $c/(z-z_0)^n$ near $z=z_0$ then the function has an n^{th} *order pole* at that point. More mathematically we say

$$\lim_{z \to z_0} (z-z_0)^n f(z) = c \neq 0.$$

We require the constant c to be nonzero so we know that it is not a pole of lower order. We can denote a removable singularity as a pole of order zero.

Another way to say that a function has an n^{th} order pole is that $f(z)$ is not analytic at $z=z_0$, but $(z-z_0)^n f(z)$ is either analytic or has a removable singularity at that point.

Example 10.4.3 $1/\sin(z^2)$ has a second order pole at $z=0$ and first order poles at $z=(n\pi)^{1/2}$, $n \in \mathbb{Z}^{\pm}$.

$$\begin{aligned}
\lim_{z \to 0} \frac{z^2}{\sin(z^2)} &= \lim_{z \to 0} \frac{2z}{2z\cos(z^2)} \\
&= \lim_{z \to 0} \frac{2}{2\cos(z^2) - 4z^2\sin(z^2)} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\lim_{z \to (n\pi)^{1/2}} \frac{z-(n\pi)^{1/2}}{\sin(z^2)} &= \lim_{z \to (n\pi)^{1/2}} \frac{1}{2z\cos(z^2)} \\
&= \frac{1}{2(n\pi)^{1/2}(-1)^n}
\end{aligned}$$

Example 10.4.4 $\mathrm{e}^{1/z}$ is singular at $z=0$. The function is not analytic as $\lim_{z \to 0} \mathrm{e}^{1/z}$ does not exist. We check if the function has a pole of order n at $z=0$.

$$\begin{aligned}
\lim_{z \to 0} z^n \, \mathrm{e}^{1/z} &= \lim_{\zeta \to \infty} \frac{\mathrm{e}^{\zeta}}{\zeta^n} \\
&= \lim_{\zeta \to \infty} \frac{\mathrm{e}^{\zeta}}{n!}
\end{aligned}$$

Since the limit does not exist for any value of n, the singularity is not a pole. We could say that $\mathrm{e}^{1/z}$ is more singular than any power of $1/z$.

Essential Singularities. If a function $f(z)$ is singular at $z = z_0$, but the singularity is not a branch point, or a pole, the the point is an *essential singularity* of the function.

The point at infinity. We can consider the point at infinity $z \to \infty$ by making the change of variables $z = 1/\zeta$ and considering $\zeta \to 0$. If $f(1/\zeta)$ is analytic at $\zeta = 0$ then $f(z)$ is analytic at infinity. We have encountered branch points at infinity before (Section 9.6). Assume that $f(z)$ is not analytic at infinity. If $\lim_{z\to\infty} f(z)$ exists then $f(z)$ has a removable singularity at infinity. If $\lim_{z\to\infty} f(z)/z^n = c \neq 0$ then $f(z)$ has an n^{th} order pole at infinity.

Result 10.4.2 Categorization of Singularities. Consider a function $f(z)$ that has a singularity at the point $z = z_0$. Singularities come in four flavors:

Branch Points. Branch points of multi-valued functions are singularities.

Removable Singularities. If $\lim_{z\to z_0} f(z)$ exists, then z_0 is a removable singularity. It is thus named because the singularity could be removed and thus the function made analytic at z_0 by redefining the value of $f(z_0)$.

Poles. If $\lim_{z\to z_0} (z - z_0)^n f(z) = \text{const} \neq 0$ then $f(z)$ has an n^{th} order pole at z_0.

Essential Singularities. Instead of defining what an essential singularity is, we say what it is not. If z_0 neither a branch point, a removable singularity nor a pole, it is an essential singularity.

A pole may be called a non-essential singularity. This is because multiplying the function by an integral power of $z - z_0$ will make the function analytic. Then an essential singularity is a point z_0 such that there does not exist an n such that $(z - z_0)^n f(z)$ is analytic there.

10.4.2 Isolated and Non-Isolated Singularities

Result 10.4.3 Isolated and Non-Isolated Singularities. Suppose $f(z)$ has a singularity at z_0. If there exists a deleted neighborhood of z_0 containing no singularities then the point is an **isolated singularity**. Otherwise it is a **non-isolated singularity**.

If you don't like the abstract notion of a deleted neighborhood, you can work with a deleted circular neighborhood. However, this will require the introduction of more math symbols and a Greek letter. $z = z_0$ is an isolated singularity if there exists a $\delta > 0$ such that there are no singularities in $0 < |z - z_0| < \delta$.

Example 10.4.5 We classify the singularities of $f(z) = z/\sin z$.

z has a simple zero at $z = 0$. $\sin z$ has simple zeros at $z = n\pi$. Thus $f(z)$ has a removable singularity at $z = 0$ and has first order poles at $z = n\pi$ for $n \in \mathbb{Z}^{\pm}$. We can corroborate this by taking limits.

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z}{\sin z} = \lim_{z \to 0} \frac{1}{\cos z} = 1$$

$$\begin{aligned}
\lim_{z \to n\pi} (z - n\pi) f(z) &= \lim_{z \to n\pi} \frac{(z - n\pi)z}{\sin z} \\
&= \lim_{z \to n\pi} \frac{2z - n\pi}{\cos z} \\
&= \frac{n\pi}{(-1)^n} \\
&\neq 0
\end{aligned}$$

Now to examine the behavior at infinity. There is no neighborhood of infinity that does not contain first order poles of $f(z)$. (Another way of saying this is that there does not exist an R such that there are no singularities in $R < |z| < \infty$.) Thus $z = \infty$ is a non-isolated singularity.

We could also determine this by setting $\zeta = 1/z$ and examining the point $\zeta = 0$. $f(1/\zeta)$ has first order poles at $\zeta = 1/(n\pi)$ for $n \in \mathbb{Z} \setminus \{0\}$. These first order poles come arbitrarily close to the point $\zeta = 0$ There is no deleted neighborhood of $\zeta = 0$ which does not contain singularities. Thus $\zeta = 0$, and hence $z = \infty$ is a non-isolated singularity.

The point at infinity is an essential singularity. It is certainly not a branch point or a removable singularity. It is not a pole, because there is no n such that $\lim_{z \to \infty} z^{-n} f(z) = \text{const} \neq 0$. $z^{-n} f(z)$ has first order poles in any neighborhood of infinity, so this limit does not exist.

10.5 Exercises

Complex Derivatives

Exercise 10.1
Show that if $f(z)$ is analytic and $\psi(x, y) = f(z)$ is twice continuously differentiable then $f'(z)$ is analytic.

Exercise 10.2
Find the complex derivative in the coordinate directions for $f(z) = \psi(r, \theta)$.
Hint, Solution

Exercise 10.3
Show that the following functions are nowhere analytic by checking where the derivative with respect to z exists.

1. $\sin x \cosh y - i \cos x \sinh y$
2. $x^2 - y^2 + x + i(2xy - y)$

Hint, Solution

Exercise 10.4
$f(z)$ is analytic for all z, ($|z| < \infty$). $f(z_1 + z_2) = f(z_1) f(z_2)$ for all z_1 and z_2. (This is known as a *functional equation*). Prove that $f(z) = \exp(f'(0) z)$.
Hint, Solution

Cauchy-Riemann Equations

Exercise 10.5
Find the Cauchy-Riemann equations for

$$f(z) = R(r, \theta) \, \mathrm{e}^{i\Theta(r,\theta)}.$$

Hint, Solution

Exercise 10.6
Let

$$f(z) = \begin{cases} \frac{x^{4/3}y^{5/3} + i x^{5/3} y^{4/3}}{x^2+y^2} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations hold at $z = 0$, but that f is not differentiable at this point.
Hint, Solution

Exercise 10.7
Consider the complex function

$$f(z) = u + iv = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}$$

Show that the partial derivatives of u and v with respect to x and y exist at $z = 0$ and that $u_x = v_y$ and $u_y = -v_x$ there: the Cauchy-Riemann equations are satisfied at $z = 0$. On the other hand, show that

$$\lim_{z \to 0} \frac{f(z)}{z}$$

does not exist, that is, f is not complex-differentiable at $z = 0$.
Hint, Solution

Exercise 10.8
Show that the function

$$f(z) = \begin{cases} e^{-z^{-4}} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}$$

satisfies the Cauchy-Riemann equations everywhere, including at $z = 0$, but $f(z)$ is not analytic at the origin.
Hint, Solution

Exercise 10.9

1. Show that $e^{\overline{z}}$ is not analytic.

2. $f(z)$ is an analytic function of z. Show that $\overline{f}(z) = \overline{f(\overline{z})}$ is also an analytic function of z.

Hint, Solution

Exercise 10.10

1. Determine all points $z = x + iy$ where the following functions are differentiable with respect to z:

$$\text{(i)} \quad x^3 + y^3 \qquad \text{(ii)} \quad \frac{x-1}{(x-1)^2 + y^2} - i\frac{y}{(x-1)^2 + y^2}$$

2. Determine all points z where the functions in part (a) are analytic.

3. Determine which of the following functions $v(x, y)$ are the imaginary part of an analytic function $u(x, y) + iv(x, y)$. For those that are, compute the real part $u(x, y)$ and re-express the answer as an explicit function of $z = x + iy$:

$$\text{(i)} \quad x^2 - y^2 \qquad \text{(ii)} \quad 3x^2y$$

Hint, Solution

Exercise 10.11

Show that the logarithm $\log z$ is differentiable for $z \neq 0$. Find the derivative of the logarithm.
Hint, Solution

Exercise 10.12
Show that the Cauchy-Riemann equations for the analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ are

$$u_r = v_\theta / r, \quad u_\theta = -r v_r.$$

Hint, Solution

Exercise 10.13
$w = u + iv$ is an analytic function of z. $\phi(x, y)$ is an arbitrary smooth function of x and y. When expressed in terms of u and v, $\phi(x, y) = \Phi(u, v)$. Show that $(w' \neq 0)$

$$\frac{\partial \Phi}{\partial u} - i \frac{\partial \Phi}{\partial v} = \left(\frac{\mathrm{d}w}{\mathrm{d}z} \right)^{-1} \left(\frac{\partial \phi}{\partial x} - i \frac{\partial \phi}{\partial y} \right).$$

Deduce

$$\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right|^{-2} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right).$$

Hint, Solution

Exercise 10.14
Show that the functions defined by $f(z) = \log |z| + i \arg(z)$ and $f(z) = \sqrt{|z|}\, \mathrm{e}^{i \arg(z)/2}$ are analytic in the sector $|z| > 0$, $|\arg(z)| < \pi$. What are the corresponding derivatives $\mathrm{d}f / \mathrm{d}z$?
Hint, Solution

Exercise 10.15
Show that the following functions are harmonic. For each one of them find its harmonic conjugate and form the corresponding holomorphic function.

1. $u(x, y) = x \operatorname{Log}(r) - y \arctan(x, y) \ (r \neq 0)$

2. $u(x, y) = \arg(z)$ ($|\arg(z)| < \pi$, $r \neq 0$)

3. $u(x, y) = r^n \cos(n\theta)$

4. $u(x, y) = y/r^2$ ($r \neq 0$)

Hint, Solution

10.6 Hints

Complex Derivatives

Hint 10.1
Start with the Cauchy-Riemann equation and then differentiate with respect to x.

Hint 10.2
Read Example 10.1.3 and use Result 10.1.1.

Hint 10.3
Use Result 10.1.1.

Hint 10.4
Take the logarithm of the equation to get a linear equation.

Cauchy-Riemann Equations

Hint 10.5
Use the result of Exercise 10.2.

Hint 10.6
To evaluate $u_x(0,0)$, etc. use the definition of differentiation. Try to find $f'(z)$ with the definition of complex differentiation. Consider $\Delta z = \Delta r \, \mathrm{e}^{i\theta}$.

Hint 10.7
To evaluate $u_x(0,0)$, etc. use the definition of differentiation. Try to find $f'(z)$ with the definition of complex differentiation. Consider $\Delta z = \Delta r \, \mathrm{e}^{i\theta}$.

Hint 10.8

Hint 10.9
Use the Cauchy-Riemann equations.

Hint 10.10

Hint 10.11

Hint 10.12

Hint 10.13

Hint 10.14

Hint 10.15

10.7 Solutions

Complex Derivatives

Solution 10.1
We start with the Cauchy-Riemann equation and then differentiate with respect to x.

$$\psi_x = -i\psi_y$$
$$\psi_{xx} = -i\psi_{yx}$$

We interchange the order of differentiation.

$$(\psi_x)_x = -i(\psi_x)_y$$
$$(f')_x = -i(f')_y$$

Since $f'(z)$ satisfies the Cauchy-Riemann equation and its partial derivatives exist and are continuous, it is analytic.

Solution 10.2
The complex derivative in the coordinate directions is

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \left(\frac{\partial r\,\mathrm{e}^{i\theta}}{\partial r}\right)^{-1}\frac{\partial\psi}{\partial r} = \mathrm{e}^{-i\theta}\frac{\partial\psi}{\partial r},$$
$$\frac{\mathrm{d}f}{\mathrm{d}z} = \left(\frac{\partial r\,\mathrm{e}^{i\theta}}{\partial\theta}\right)^{-1}\frac{\partial\psi}{\partial\theta} = -\frac{i}{r}\,\mathrm{e}^{-i\theta}\frac{\partial\psi}{\partial\theta}.$$

We write this in operator notation.

$$\frac{\mathrm{d}}{\mathrm{d}z} = \mathrm{e}^{-i\theta}\frac{\partial}{\partial r} = -\frac{i}{r}\,\mathrm{e}^{-i\theta}\frac{\partial}{\partial\theta}$$

Solution 10.3

1. Consider $f(x,y) = \sin x \cosh y - i \cos x \sinh y$. The derivatives in the x and y directions are
$$\frac{\partial f}{\partial x} = \cos x \cosh y + i \sin x \sinh y$$
$$-i\frac{\partial f}{\partial y} = -\cos x \cosh y - i \sin x \sinh y$$
These derivatives exist and are everywhere continuous. We equate the expressions to get a set of two equations.
$$\cos x \cosh y = -\cos x \cosh y, \qquad \sin x \sinh y = -\sin x \sinh y$$
$$\cos x \cosh y = 0, \qquad \sin x \sinh y = 0$$
$$\left(x = \frac{\pi}{2} + n\pi\right) \text{ and } (x = m\pi \text{ or } y = 0)$$
The function may be differentiable only at the points
$$\boxed{x = \frac{\pi}{2} + n\pi, \quad y = 0.}$$
Thus the function is nowhere analytic.

2. Consider $f(x,y) = x^2 - y^2 + x + i(2xy - y)$. The derivatives in the x and y directions are
$$\frac{\partial f}{\partial x} = 2x + 1 + i2y$$
$$-i\frac{\partial f}{\partial y} = i2y + 2x - 1$$
These derivatives exist and are everywhere continuous. We equate the expressions to get a set of two equations.
$$2x + 1 = 2x - 1, \qquad 2y = 2y.$$
Since this set of equations has no solutions, there are no points at which the function is differentiable. The function is nowhere analytic.

Solution 10.4

$$f(z_1 + z_2) = f(z_1)f(z_2)$$
$$\log(f(z_1 + z_2)) = \log(f(z_1)) + \log(f(z_2))$$

We define $g(z) = \log(f(z))$.

$$g(z_1 + z_2) = g(z_1) + g(z_2)$$

This is a linear equation which has exactly the solutions:

$$g(z) = cz.$$

Thus $f(z)$ has the solutions:

$$f(z) = \mathrm{e}^{cz},$$

where c is any complex constant. We can write this constant in terms of $f'(0)$. We differentiate the original equation with respect to z_1 and then substitute $z_1 = 0$.

$$f'(z_1 + z_2) = f'(z_1)f(z_2)$$
$$f'(z_2) = f'(0)f(z_2)$$
$$f'(z) = f'(0)f(z)$$

We substitute in the form of the solution.

$$c\,\mathrm{e}^{cz} = f'(0)\,\mathrm{e}^{cz}$$
$$c = f'(0)$$

Thus we see that

$$f(z) = \mathrm{e}^{f'(0)z}.$$

Cauchy-Riemann Equations

Solution 10.5

We find the Cauchy-Riemann equations for

$$f(z) = R(r,\theta)\,\mathrm{e}^{i\Theta(r,\theta)}.$$

From Exercise 10.2 we know that the complex derivative in the polar coordinate directions is

$$\frac{\mathrm{d}}{\mathrm{d}z} = \mathrm{e}^{-i\theta}\frac{\partial}{\partial r} = -\frac{i}{r}\,\mathrm{e}^{-i\theta}\frac{\partial}{\partial \theta}.$$

We equate the derivatives in the two directions.

$$\mathrm{e}^{-i\theta}\frac{\partial}{\partial r}\left[R\,\mathrm{e}^{i\Theta}\right] = -\frac{i}{r}\,\mathrm{e}^{-i\theta}\frac{\partial}{\partial \theta}\left[R\,\mathrm{e}^{i\Theta}\right]$$
$$(R_r + iR\Theta_r)\,\mathrm{e}^{i\Theta} = -\frac{i}{r}(R_\theta + iR\Theta_\theta)\,\mathrm{e}^{i\Theta}$$

We divide by $\mathrm{e}^{i\Theta}$ and equate the real and imaginary components to obtain the Cauchy-Riemann equations.

$$R_r = \frac{R}{r}\Theta_\theta, \qquad \frac{1}{r}R_\theta = -R\Theta_r$$

Solution 10.6

$$u = \begin{cases} \frac{x^{4/3}y^{5/3}}{x^2+y^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}, \qquad v = \begin{cases} \frac{x^{5/3}y^{4/3}}{x^2+y^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

The Cauchy-Riemann equations are

$$u_x = v_y, \qquad u_y = -v_x.$$

The partial derivatives of u and v at the point $x = y = 0$ are,

$$\begin{aligned}
u_x(0,0) &= \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
v_x(0,0) &= \lim_{\Delta x \to 0} \frac{v(\Delta x, 0) - v(0,0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
u_y(0,0) &= \lim_{\Delta y \to 0} \frac{u(0, \Delta y) - u(0,0)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
v_y(0,0) &= \lim_{\Delta y \to 0} \frac{v(0, \Delta y) - v(0,0)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} \\
&= 0.
\end{aligned}$$

Since $u_x(0,0) = u_y(0,0) = v_x(0,0) = v_y(0,0) = 0$ the Cauchy-Riemann equations are satisfied.

$f(z)$ is not analytic at the point $z = 0$. We show this by calculating the derivative.

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z}$$

Let $\Delta z = \Delta r \, e^{i\theta}$, that is, we approach the origin at an angle of θ. Then $x = \Delta r \cos\theta$ and $y = \Delta r \sin\theta$.

$$\begin{aligned}
f'(0) &= \lim_{\Delta r \to 0} \frac{f(\Delta r \, e^{i\theta})}{\Delta r \, e^{i\theta}} \\
&= \lim_{\Delta r \to 0} \frac{\frac{\Delta r^{4/3} \cos^{4/3}\theta \Delta r^{5/3} \sin^{5/3}\theta + i \Delta r^{5/3} \cos^{5/3}\theta \Delta r^{4/3} \sin^{4/3}\theta}{\Delta r^2}}{\Delta r \, e^{i\theta}} \\
&= \lim_{\Delta r \to 0} \frac{\cos^{4/3}\theta \sin^{5/3}\theta + i \cos^{5/3}\theta \sin^{4/3}\theta}{e^{i\theta}}
\end{aligned}$$

The value of the limit depends on θ and is not a constant. Thus this limit does not exist. The function is not differentiable at $z = 0$.

Solution 10.7

$$u = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}, \qquad v = \begin{cases} \frac{x^3 + y^3}{x^2 + y^2} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}$$

The Cauchy-Riemann equations are

$$u_x = v_y, \qquad u_y = -v_x.$$

The partial derivatives of u and v at the point $x = y = 0$ are,

$$\begin{aligned}
u_x(0,0) &= \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} \\
&= 1,
\end{aligned}$$

$$
\begin{aligned}
v_x(0,0) &= \lim_{\Delta x \to 0} \frac{v(\Delta x, 0) - v(0,0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} \\
&= 1,
\end{aligned}
$$

$$
\begin{aligned}
u_y(0,0) &= \lim_{\Delta y \to 0} \frac{u(0, \Delta y) - u(0,0)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{-\Delta y - 0}{\Delta y} \\
&= -1,
\end{aligned}
$$

$$
\begin{aligned}
v_y(0,0) &= \lim_{\Delta y \to 0} \frac{v(0, \Delta y) - v(0,0)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{\Delta y - 0}{\Delta y} \\
&= 1.
\end{aligned}
$$

We see that the Cauchy-Riemann equations are satisfied at $x = y = 0$

$f(z)$ is not analytic at the point $z = 0$. We show this by calculating the derivative.

$$
f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z}
$$

Let $\Delta z = \Delta r \, \mathrm{e}^{i\theta}$, that is, we approach the origin at an angle of θ. Then $x = \Delta r \cos\theta$ and $y = \Delta r \sin\theta$.

$$
\begin{aligned}
f'(0) &= \lim_{\Delta r \to 0} \frac{f(\Delta r \, \mathrm{e}^{i\theta})}{\Delta r \, \mathrm{e}^{i\theta}} \\
&= \lim_{\Delta r \to 0} \frac{\frac{\Delta r^3 \cos^3\theta(1+i) - \Delta r^3 \sin^3\theta(1-i)}{\Delta r^2}}{\Delta r \, \mathrm{e}^{i\theta}} \\
&= \lim_{\Delta r \to 0} \frac{\cos^3\theta(1+i) - \sin^3\theta(1-i)}{\mathrm{e}^{i\theta}}
\end{aligned}
$$

The value of the limit depends on θ and is not a constant. Thus this limit does not exist. The function is not differentiable at $z = 0$. Recall that satisfying the Cauchy-Riemann equations is a necessary, but not a sufficient condition for differentiability.

Solution 10.8

First we verify that the Cauchy-Riemann equations are satisfied for $z \neq 0$. Note that the form

$$f_x = -i f_y$$

will be far more convenient than the form

$$u_x = v_y, \quad u_y = -v_x$$

for this problem.

$$\begin{aligned} f_x &= 4(x+iy)^{-5} \, \mathrm{e}^{-(x+iy)^{-4}} \\ -i f_y &= -i4(x+iy)^{-5} i \, \mathrm{e}^{-(x+iy)^{-4}} = 4(x+iy)^{-5} \, \mathrm{e}^{-(x+iy)^{-4}} \end{aligned}$$

The Cauchy-Riemann equations are satisfied for $z \neq 0$.

Now we consider the point $z = 0$.

$$\begin{aligned} f_x(0,0) &= \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\mathrm{e}^{-\Delta x^{-4}}}{\Delta x} \\ &= 0 \end{aligned}$$

$$\begin{aligned} -i f_y(0,0) &= -i \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} \\ &= -i \lim_{\Delta y \to 0} \frac{\mathrm{e}^{-\Delta y^{-4}}}{\Delta y} \\ &= 0 \end{aligned}$$

The Cauchy-Riemann equations are satisfied for $z = 0$.

$f(z)$ is not analytic at the point $z = 0$. We show this by calculating the derivative.

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z}$$

Let $\Delta z = \Delta r \, \mathrm{e}^{i\theta}$, that is, we approach the origin at an angle of θ.

$$\begin{aligned} f'(0) &= \lim_{\Delta r \to 0} \frac{f(\Delta r \, \mathrm{e}^{i\theta})}{\Delta r \, \mathrm{e}^{i\theta}} \\ &= \lim_{\Delta r \to 0} \frac{\mathrm{e}^{-r^{-4} \mathrm{e}^{-i4\theta}}}{\Delta r \, \mathrm{e}^{i\theta}} \end{aligned}$$

For most values of θ the limit does not exist. Consider $\theta = \pi/4$.

$$f'(0) = \lim_{\Delta r \to 0} \frac{\mathrm{e}^{r^{-4}}}{\Delta r \, \mathrm{e}^{i\pi/4}} = \infty$$

Because the limit does not exist, the function is not differentiable at $z = 0$. Recall that satisfying the Cauchy-Riemann equations is a necessary, but not a sufficient condition for differentiability.

Solution 10.9

1. A necessary condition for analyticity in an open set is that the Cauchy-Riemann equations are satisfied in that set. We write $\mathrm{e}^{\overline{z}}$ in Cartesian form.

$$\mathrm{e}^{\overline{z}} = \mathrm{e}^{x - iy} = \mathrm{e}^x \cos y - i \, \mathrm{e}^x \sin y.$$

Now we determine where $u = \mathrm{e}^x \cos y$ and $v = -\mathrm{e}^x \sin y$ satisfy the Cauchy-Riemann equations.

$$\begin{aligned} u_x &= v_y, & u_y &= -v_x \\ \mathrm{e}^x \cos y &= -\mathrm{e}^x \cos y, & -\mathrm{e}^x \sin y &= \mathrm{e}^x \sin y \\ \cos y &= 0, & \sin y &= 0 \\ y &= \frac{\pi}{2} + \pi m, & y &= \pi n \end{aligned}$$

Thus we see that the Cauchy-Riemann equations are not satisfied anywhere. $\mathrm{e}^{\overline{z}}$ is nowhere analytic.

2. Since $f(z) = u + iv$ is analytic, u and v satisfy the Cauchy-Riemann equations and their first partial derivatives are continuous.

$$\overline{f}(z) = \overline{f(\overline{z})} = \overline{u(x,-y) + iv(x,-y)} = u(x,-y) - iv(x,-y)$$

We define $\overline{f}(z) \equiv \mu(x,y) + i\nu(x,y) = u(x,-y) - iv(x,y)$. Now we see if μ and ν satisfy the Cauchy-Riemann equations.

$$\begin{aligned} \mu_x &= \nu_y, & \mu_y &= -\nu_x \\ (u(x,-y))_x &= (-v(x,-y))_y, & (u(x,-y))_y &= -(-v(x,-y))_x \\ u_x(x,-y) &= v_y(x,-y), & -u_y(x,-y) &= v_x(x,-y) \\ u_x &= v_y, & u_y &= -v_x \end{aligned}$$

Thus we see that the Cauchy-Riemann equations for μ and ν are satisfied if and only if the Cauchy-Riemann equations for u and v are satisfied. The continuity of the first partial derivatives of u and v implies the same of μ and ν. Thus $\overline{f}(z)$ is analytic.

Solution 10.10

1. The necessary condition for a function $f(z) = u + iv$ to be differentiable at a point is that the Cauchy-Riemann equations hold and the first partial derivatives of u and v are continuous at that point.

   (a)

$$f(z) = x^3 + y^3 + i0$$

   The Cauchy-Riemann equations are

$$\begin{aligned} u_x &= v_y \quad \text{and} \quad u_y = -v_x \\ 3x^2 &= 0 \quad \text{and} \quad 3y^2 = 0 \\ x &= 0 \quad \text{and} \quad y = 0 \end{aligned}$$

   The first partial derivatives are continuous. Thus we see that the function is differentiable only at the point $z = 0$.

(b)

$$f(z) = \frac{x-1}{(x-1)^2 + y^2} - i\frac{y}{(x-1)^2 + y^2}$$

The Cauchy-Riemann equations are

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$
$$\frac{-(x-1)^2 + y^2}{((x-1)^2 + y^2)^2} = \frac{-(x-1)^2 + y^2}{((x-1)^2 + y^2)^2} \quad \text{and} \quad \frac{2(x-1)y}{((x-1)^2 + y^2)^2} = \frac{2(x-1)y}{((x-1)^2 + y^2)^2}$$

The Cauchy-Riemann equations are each identities. The first partial derivatives are continuous everywhere except the point $x = 1$, $y = 0$. Thus the function is differentiable everywhere except $z = 1$.

2. (a) The function is not differentiable in any open set. Thus the function is nowhere analytic.
   (b) The function is differentiable everywhere except $z = 1$. Thus the function is analytic everywhere except $z = 1$.

3. (a) First we determine if the function is harmonic.

$$v = x^2 - y^2$$
$$v_{xx} + v_{yy} = 0$$
$$2 - 2 = 0$$

The function is harmonic in the complex plane and this is the imaginary part of some analytic function. By inspection, we see that this function is

$$iz^2 + c = -2xy + c + i(x^2 - y^2),$$

where c is a real constant. We can also find the function by solving the Cauchy-Riemann equations.

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$
$$u_x = -2y \quad \text{and} \quad u_y = -2x$$

We integrate the first equation.

$$u = -2xy + g(y)$$

Here $g(y)$ is a function of integration. We substitute this into the second Cauchy-Riemann equation to determine $g(y)$.

$$\begin{aligned}
u_y &= -2x \\
-2x + g'(y) &= -2x \\
g'(y) &= 0 \\
g(y) &= c \\
u &= -2xy + c \\
f(z) &= -2xy + c + i(x^2 - y^2) \\
f(z) &= iz^2 + c
\end{aligned}$$

(b) First we determine if the function is harmonic.

$$\begin{aligned}
v &= 3x^2 y \\
v_{xx} + v_{yy} &= 6y
\end{aligned}$$

The function is not harmonic. It is not the imaginary part of some analytic function.

Solution 10.11

We show that the logarithm $\log z = \psi(r, \theta) = \operatorname{Log} r + i\theta$ satisfies the Cauchy-Riemann equations.

$$\begin{aligned}
\psi_r &= -\frac{i}{r}\psi_\theta \\
\frac{1}{r} &= -\frac{i}{r} i \\
\frac{1}{r} &= \frac{1}{r}
\end{aligned}$$

Since the logarithm satisfies the Cauchy-Riemann equations and the first partial derivatives are continuous for $z \neq 0$, the logarithm is analytic for $z \neq 0$.

Now we compute the derivative.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z} \log z &= \mathrm{e}^{-i\theta} \frac{\partial}{\partial r} (\operatorname{Log} r + i\theta) \\
&= \mathrm{e}^{-i\theta} \frac{1}{r} \\
&= \frac{1}{z}
\end{aligned}$$

Solution 10.12

The complex derivative in the coordinate directions is

$$\frac{\mathrm{d}}{\mathrm{d}z} = \mathrm{e}^{-i\theta} \frac{\partial}{\partial r} = -\frac{i}{r} \mathrm{e}^{-i\theta} \frac{\partial}{\partial \theta}.$$

We substitute $f = u + iv$ into this identity to obtain the Cauchy-Riemann equation in polar coordinates.

$$\begin{aligned}
\mathrm{e}^{-i\theta} \frac{\partial f}{\partial r} &= -\frac{i}{r} \mathrm{e}^{-i\theta} \frac{\partial f}{\partial \theta} \\
\frac{\partial f}{\partial r} &= -\frac{i}{r} \frac{\partial f}{\partial \theta} \\
u_r + i v_r &= -\frac{i}{r} (u_\theta + i v_\theta)
\end{aligned}$$

We equate the real and imaginary parts.

$$\begin{gathered}
u_r = \frac{1}{r} v_\theta, \quad v_r = -\frac{1}{r} u_\theta \\
u_r = \frac{1}{r} v_\theta, \quad u_\theta = -r v_r
\end{gathered}$$

Solution 10.13

Since w is analytic, u and v satisfy the Cauchy-Riemann equations,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Using the chain rule we can write the derivatives with respect to x and y in terms of u and v.

$$\frac{\partial}{\partial x} = u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v}$$
$$\frac{\partial}{\partial y} = u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v}$$

Now we examine $\phi_x - i\phi_y$.

$$\phi_x - i\phi_y = u_x \Phi_u + v_x \Phi_v - i(u_y \Phi_u + v_y \Phi_v)$$
$$\phi_x - i\phi_y = (u_x - i u_y)\Phi_u + (v_x - i v_y)\Phi_v$$
$$\phi_x - i\phi_y = (u_x - i u_y)\Phi_u - i(v_y + i v_x)\Phi_v$$

We use the Cauchy-Riemann equations to write u_y and v_y in terms of u_x and v_x.

$$\phi_x - i\phi_y = (u_x + i v_x)\Phi_u - i(u_x + i v_x)\Phi_v$$

Recall that $w' = u_x + i v_x = v_y - i u_y$.

$$\phi_x - i\phi_y = \frac{\mathrm{d}w}{\mathrm{d}z}(\Phi_u - i\Phi_v)$$

Thus we see that,

$$\frac{\partial \Phi}{\partial u} - i\frac{\partial \Phi}{\partial v} = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-1}\left(\frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}\right).$$

We write this in operator notation.

$$\frac{\partial}{\partial u} - i\frac{\partial}{\partial v} = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-1}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

The complex conjugate of this relation is

$$\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} = \left(\overline{\frac{\mathrm{d}w}{\mathrm{d}z}}\right)^{-1}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Now we apply both these operators to $\Phi = \phi$.

$$\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)\Phi = \left(\overline{\frac{\mathrm{d}w}{\mathrm{d}z}}\right)^{-1}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-1}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\phi$$

$$\begin{aligned}&\left(\frac{\partial^2}{\partial u^2} + i\frac{\partial^2}{\partial u \partial v} - i\frac{\partial^2}{\partial v \partial u} + \frac{\partial^2}{\partial v^2}\right)\Phi \\ &\quad = \left(\overline{\frac{\mathrm{d}w}{\mathrm{d}z}}\right)^{-1}\left[\left(\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-1}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-1}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right]\phi\end{aligned}$$

$(w')^{-1}$ is an analytic function. Recall that for analytic functions f, $f' = f_x = -i f_y$. So that $f_x + i f_y = 0$.

$$\begin{aligned}\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} &= \left(\overline{\frac{\mathrm{d}w}{\mathrm{d}z}}\right)^{-1}\left[\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-1}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\phi \\ \frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} &= \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^{-2}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)\end{aligned}$$

Solution 10.14

1. We consider

$$f(z) = \log|z| + i \arg(z) = \log r + i\theta.$$

The Cauchy-Riemann equations in polar coordinates are

$$u_r = \frac{1}{r} v_\theta, \quad u_\theta = -r v_r.$$

We calculate the derivatives.

$$u_r = \frac{1}{r}, \quad \frac{1}{r} v_\theta = \frac{1}{r}$$
$$u_\theta = 0, \quad -r v_r = 0$$

Since the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous, $f(z)$ is analytic in $|z| > 0$, $|\arg(z)| < \pi$. The complex derivative in terms of polar coordinates is

$$\frac{\mathrm{d}}{\mathrm{d}z} = \mathrm{e}^{-i\theta} \frac{\partial}{\partial r} = -\frac{i}{r} \mathrm{e}^{-i\theta} \frac{\partial}{\partial \theta}.$$

We use this to differentiate $f(z)$.

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \mathrm{e}^{-i\theta} \frac{\partial}{\partial r}[\log r + i\theta] = \mathrm{e}^{-i\theta} \frac{1}{r} = \frac{1}{z}$$

2. Next we consider

$$f(z) = \sqrt{|z|}\, \mathrm{e}^{i \arg(z)/2} = \sqrt{r}\, \mathrm{e}^{i\theta/2}.$$

The Cauchy-Riemann equations for polar coordinates and the polar form $f(z) = R(r,\theta)\, \mathrm{e}^{i\Theta(r,\theta)}$ are

$$R_r = \frac{R}{r} \Theta_\theta, \quad \frac{1}{r} R_\theta = -R \Theta_r.$$

We calculate the derivatives for $R = \sqrt{r}$, $\Theta = \theta/2$.

$$R_r = \frac{1}{2\sqrt{r}}, \quad \frac{R}{r}\Theta_\theta = \frac{1}{2\sqrt{r}}$$
$$\frac{1}{r}R_\theta = 0, \quad -R\Theta_r = 0$$

Since the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous, $f(z)$ is analytic in $|z| > 0$, $|\arg(z)| < \pi$. The complex derivative in terms of polar coordinates is

$$\frac{\mathrm{d}}{\mathrm{d}z} = \mathrm{e}^{-i\theta}\frac{\partial}{\partial r} = -\frac{i}{r}\,\mathrm{e}^{-i\theta}\frac{\partial}{\partial \theta}.$$

We use this to differentiate $f(z)$.

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \mathrm{e}^{-i\theta}\frac{\partial}{\partial r}[\sqrt{r}\,\mathrm{e}^{i\theta/2}] = \frac{1}{2\,\mathrm{e}^{i\theta/2}\sqrt{r}} = \frac{1}{2\sqrt{z}}$$

Solution 10.15

1. We consider the function

$$u = x \operatorname{Log} r - y \arctan(x, y) = r\cos\theta \operatorname{Log} r - r\theta\sin\theta$$

We compute the Laplacian.

$$\begin{aligned}
\Delta u &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(\cos\theta(r + r\operatorname{Log} r) - \theta\sin\theta\right) + \frac{1}{r^2}(r(\theta\sin\theta - 2\cos\theta) - r\cos\theta\operatorname{Log} r) \\
&= \frac{1}{r}(2\cos\theta + \cos\theta\operatorname{Log} r - \theta\sin\theta) + \frac{1}{r}(\theta\sin\theta - 2\cos\theta - \cos\theta\operatorname{Log} r) \\
&= 0
\end{aligned}$$

The function u is harmonic. We find the harmonic conjugate v by solving the Cauchy-Riemann equations.

$$v_r = -\frac{1}{r} u_\theta, \quad v_\theta = r u_r$$
$$v_r = \sin\theta(1 + \operatorname{Log} r) + \theta\cos\theta, \quad v_\theta = r(\cos\theta(1 + \operatorname{Log} r) - \theta\sin\theta)$$

We integrate the first equation with respect to r to determine v to within the constant of integration $g(\theta)$.

$$v = r(\sin\theta \operatorname{Log} r + \theta\cos\theta) + g(\theta)$$

We differentiate this expression with respect to θ.

$$v_\theta = r(\cos\theta(1 + \operatorname{Log} r) - \theta\sin\theta) + g'(\theta)$$

We compare this to the second Cauchy-Riemann equation to see that $g'(\theta) = 0$. Thus $g(\theta) = c$. We have determined the harmonic conjugate.

$$\boxed{v = r(\sin\theta \operatorname{Log} r + \theta\cos\theta) + c}$$

The corresponding analytic function is

$$f(z) = r\cos\theta \operatorname{Log} r - r\theta\sin\theta + i(r\sin\theta \operatorname{Log} r + r\theta\cos\theta + c).$$

On the positive real axis, $(\theta = 0)$, the function has the value

$$f(z = r) = r \operatorname{Log} r + ic.$$

We use analytic continuation to determine the function in the complex plane.

$$\boxed{f(z) = z \log z + ic}$$

2. We consider the function

$$u = \operatorname{Arg}(z) = \theta.$$

We compute the Laplacian.

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

The function u is harmonic. We find the harmonic conjugate v by solving the Cauchy-Riemann equations.

$$v_r = -\frac{1}{r}u_\theta, \quad v_\theta = r u_r$$
$$v_r = -\frac{1}{r}, \quad v_\theta = 0$$

We integrate the first equation with respect to r to determine v to within the constant of integration $g(\theta)$.

$$v = -\operatorname{Log} r + g(\theta)$$

We differentiate this expression with respect to θ.

$$v_\theta = g'(\theta)$$

We compare this to the second Cauchy-Riemann equation to see that $g'(\theta) = 0$. Thus $g(\theta) = c$. We have determined the harmonic conjugate.

$$\boxed{v = -\operatorname{Log} r + c}$$

The corresponding analytic function is

$$f(z) = \theta - i\operatorname{Log} r + ic$$

On the positive real axis, ($\theta = 0$), the function has the value

$$f(z = r) = -i \operatorname{Log} r + ic$$

We use analytic continuation to determine the function in the complex plane.

$$\boxed{f(z) = -i \log z + ic}$$

3. We consider the function

$$u = r^n \cos(n\theta)$$

We compute the Laplacian.

$$\begin{aligned}
\Delta u &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(n r^n \cos(n\theta)\right) - n^2 r^{n-2}\cos(n\theta) \\
&= n^2 r^{n-2}\cos(n\theta) - n^2 r^{n-2}\cos(n\theta) \\
&= 0
\end{aligned}$$

The function u is harmonic. We find the harmonic conjugate v by solving the Cauchy-Riemann equations.

$$v_r = -\frac{1}{r} u_\theta, \quad v_\theta = r u_r$$

$$v_r = n r^{n-1} \sin(n\theta), \quad v_\theta = n r^n \cos(n\theta)$$

We integrate the first equation with respect to r to determine v to within the constant of integration $g(\theta)$.

$$v = r^n \sin(n\theta) + g(\theta)$$

We differentiate this expression with respect to θ.

$$v_\theta = n r^n \cos(n\theta) + g'(\theta)$$

We compare this to the second Cauchy-Riemann equation to see that $g'(\theta) = 0$. Thus $g(\theta) = c$. We have determined the harmonic conjugate.

$$\boxed{v = r^n \sin(n\theta) + c}$$

The corresponding analytic function is

$$f(z) = r^n \cos(n\theta) + i r^n \sin(n\theta) + ic$$

On the positive real axis, ($\theta = 0$), the function has the value

$$f(z = r) = r^n + ic$$

We use analytic continuation to determine the function in the complex plane.

$$\boxed{f(z) = z^n}$$

4. We consider the function

$$u = \frac{y}{r^2} = \frac{\sin\theta}{r}$$

We compute the Laplacian.

$$\begin{aligned}
\Delta u &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{\sin\theta}{r}\right) - \frac{\sin\theta}{r^3} \\
&= \frac{\sin\theta}{r^3} - \frac{\sin\theta}{r^3} \\
&= 0
\end{aligned}$$

The function u is harmonic. We find the harmonic conjugate v by solving the Cauchy-Riemann equations.

$$v_r = -\frac{1}{r} u_\theta, \quad v_\theta = r u_r$$
$$v_r = -\frac{\cos\theta}{r^2}, \quad v_\theta = -\frac{\sin\theta}{r}$$

We integrate the first equation with respect to r to determine v to within the constant of integration $g(\theta)$.

$$v = \frac{\cos\theta}{r} + g(\theta)$$

We differentiate this expression with respect to θ.

$$v_\theta = -\frac{\sin\theta}{r} + g'(\theta)$$

We compare this to the second Cauchy-Riemann equation to see that $g'(\theta) = 0$. Thus $g(\theta) = c$. We have determined the harmonic conjugate.

$$\boxed{v = \frac{\cos\theta}{r} + c}$$

The corresponding analytic function is

$$f(z) = \frac{\sin\theta}{r} + i\frac{\cos\theta}{r} + ic$$

On the positive real axis, $(\theta = 0)$, the function has the value

$$f(z = r) = \frac{i}{r} + ic.$$

We use analytic continuation to determine the function in the complex plane.

$$\boxed{f(z) = \frac{i}{z} + ic}$$

Chapter 11

Analytic Continuation

I'm about two beers away from fine.

11.1 Analytic Continuation

Suppose there is a function, $f_1(z)$ that is analytic in the domain D_1 and another analytic function, $f_2(z)$ that is analytic in the domain D_2. (See Figure 11.1.)

If the two domains overlap and $f_1(z) = f_2(z)$ in the overlap region $D_1 \cap D_2$, then $f_2(z)$ is called an *analytic continuation* of $f_1(z)$. This is an appropriate name since $f_2(z)$ continues the definition of $f_1(z)$ outside of its original domain of definition D_1. We can define a function $f(z)$ that is analytic in the union of the domains $D_1 \cup D_2$. On the domain D_1 we have $f(z) = f_1(z)$ and $f(z) = f_2(z)$ on D_2. $f_1(z)$ and $f_2(z)$ are called *function elements*. There is an analytic continuation even if the two domains only share an arc and not a two dimensional region.

With more overlapping domains $D_3, D_4, \ldots$ we could perhaps extend $f_1(z)$ to more of the complex plane. Sometimes it is impossible to extend a function beyond the boundary of a domain. This is known as a *natural boundary*. If a function $f_1(z)$ is analytically continued to a domain D_n along two different paths, (See Figure 11.2.),

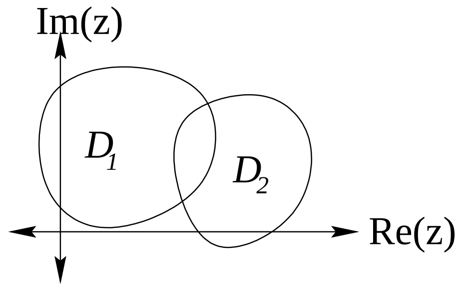


Figure 11.1: Overlapping Domains

then the two analytic continuations are identical as long as the paths do not enclose a branch point of the function. This is the *uniqueness theorem of analytic continuation*.

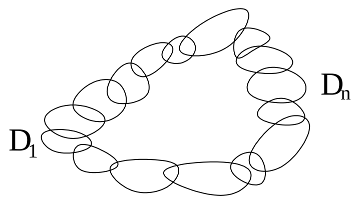


Figure 11.2: Two Paths of Analytic Continuation

Consider an analytic function $f(z)$ defined in the domain D. Suppose that $f(z) = 0$ on the arc AB, (see Figure 11.3.) Then $f(z) = 0$ in all of D.

Consider a point ζ on AB. The Taylor series expansion of $f(z)$ about the point $z = \zeta$ converges in a circle C at least up to the boundary of D. The derivative of $f(z)$ at the point $z = \zeta$ is

$$f'(\zeta) = \lim_{\Delta z \to 0} \frac{f(\zeta + \Delta z) - f(\zeta)}{\Delta z}$$

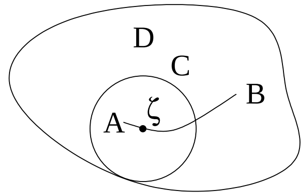


Figure 11.3: Domain Containing Arc Along Which $f(z)$ Vanishes

If Δz is in the direction of the arc, then $f'(\zeta)$ vanishes as well as all higher derivatives, $f'(\zeta) = f''(\zeta) = f'''(\zeta) = \cdots = 0$. Thus we see that $f(z) = 0$ inside C. By taking Taylor series expansions about points on AB or inside of C we see that $f(z) = 0$ in D.

Result 11.1.1 Let $f_1(z)$ and $f_2(z)$ be analytic functions defined in D. If $f_1(z) = f_2(z)$ for the points in a region or on an arc in D, then $f_1(z) = f_2(z)$ for all points in D.

To prove Result 11.1.1, we define the analytic function $g(z) = f_1(z) - f_2(z)$. Since $g(z)$ vanishes in the region or on the arc, then $g(z) = 0$ and hence $f_1(z) = f_2(z)$ for all points in D.

Result 11.1.2 Consider analytic functions $f_1(z)$ and $f_2(z)$ defined on the domains D_1 and D_2, respectively. Suppose that $D_1 \cap D_2$ is a region or an arc and that $f_1(z) = f_2(z)$ for all $z \in D_1 \cap D_2$. (See Figure 11.4.) Then the function

$$f(z) = \begin{cases} f_1(z) & \text{for } z \in D_1, \\ f_2(z) & \text{for } z \in D_2, \end{cases}$$

is analytic in $D_1 \cup D_2$.

Result 11.1.2 follows directly from Result 11.1.1.

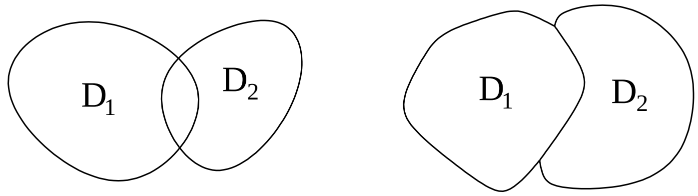


Figure 11.4: Domains that Intersect in a Region or an Arc

11.2 Analytic Continuation of Sums

Example 11.2.1 Consider the function

$$f_1(z) = \sum_{n=0}^{\infty} z^n.$$

The sum converges uniformly for $D_1 = |z| \leq r < 1$. Since the derivative also converges in this domain, the function is analytic there.

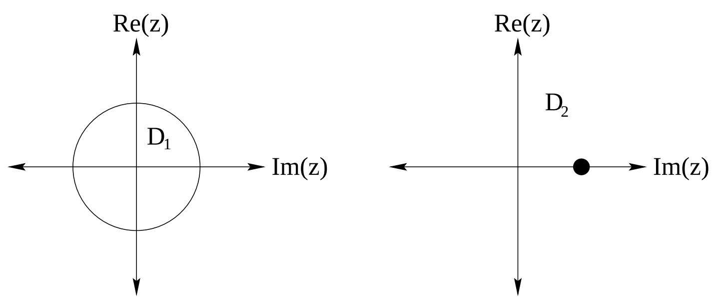


Figure 11.5: Domain of Convergence for $\sum_{n=0}^{\infty} z^n$.

Now consider the function

$$f_2(z) = \frac{1}{1-z}.$$

This function is analytic everywhere except the point $z = 1$. On the domain D_1,

$$f_2(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = f_1(z)$$

Analytic continuation tells us that there is a function that is analytic on the union of the two domains. Here, the domain is the entire z plane except the point $z = 1$ and the function is

$$f(z) = \frac{1}{1-z}.$$

$\frac{1}{1-z}$ is said to be an analytic continuation of $\sum_{n=0}^{\infty} z^n$.

11.3 Analytic Functions Defined in Terms of Real Variables

Result 11.3.1 An analytic function, $u(x, y) + iv(x, y)$ can be written in terms of a function of a complex variable, $f(z) = u(x, y) + iv(x, y)$.

Result 11.3.1 is proved in Exercise 11.1.

Example 11.3.1

$$\begin{aligned} f(z) = &\cosh y \sin x (x \mathrm{e}^x \cos y - y \mathrm{e}^x \sin y) - \cos x \sinh y (y \mathrm{e}^x \cos y + x \mathrm{e}^x \sin y) \\ &+ i \left[\cosh y \sin x (y \mathrm{e}^x \cos y + x \mathrm{e}^x \sin y) + \cos x \sinh y (x \mathrm{e}^x \cos y - y \mathrm{e}^x \sin y) \right] \end{aligned}$$

is an analytic function. Express $f(z)$ in terms of z.

On the real line, $y = 0$, $f(z)$ is

$$f(z = x) = x\,\mathrm{e}^x \sin x$$

(Recall that $\cos(0) = \cosh(0) = 1$ and $\sin(0) = \sinh(0) = 0$.)

The analytic continuation of $f(z)$ into the complex plane is

$$\boxed{f(z) = z\,\mathrm{e}^z \sin z.}$$

Alternatively, for $x = 0$ we have

$$f(z = iy) = y \sinh y(\cos y - i \sin y).$$

The analytic continuation from the imaginary axis to the complex plane is

$$\begin{aligned} f(z) &= -iz \sinh(-iz)(\cos(-iz) - i \sin(-iz)) \\ &= iz \sinh(iz)(\cos(iz) + i \sin(iz)) \\ &= z \sin z\,\mathrm{e}^z. \end{aligned}$$

Example 11.3.2 Consider $u = \mathrm{e}^{-x}(x \sin y - y \cos y)$. Find v such that $f(z) = u + iv$ is analytic.

From the Cauchy-Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \mathrm{e}^{-x} \sin y - x\,\mathrm{e}^{-x} \sin y + y\,\mathrm{e}^{-x} \cos y$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \mathrm{e}^{-x} \cos y - x\,\mathrm{e}^{-x} \cos y - y\,\mathrm{e}^{-x} \sin y$$

Integrate the first equation with respect to y.

$$
\begin{aligned}
v &= -\mathrm{e}^{-x}\cos y + x\,\mathrm{e}^{-x}\cos y + \mathrm{e}^{-x}(y\sin y + \cos y) + F(x) \\
&= y\,\mathrm{e}^{-x}\sin y + x\,\mathrm{e}^{-x}\cos y + F(x)
\end{aligned}
$$

$F(x)$ is an arbitrary function of x. Substitute this expression for v into the equation for $\partial v/\partial x$.

$$
-y\,\mathrm{e}^{-x}\sin y - x\,\mathrm{e}^{-x}\cos y + \mathrm{e}^{-x}\cos y + F'(x) = -y\,\mathrm{e}^{-x}\sin y - x\,\mathrm{e}^{-x}\cos y + \mathrm{e}^{-x}\cos y
$$

Thus $F'(x) = 0$ and $F(x) = c$.

$$
v = \mathrm{e}^{-x}(y\sin y + x\cos y) + c
$$

Example 11.3.3 Find $f(z)$ in the previous example. (Up to the additive constant.)

Method 1

$$
\begin{aligned}
f(z) &= u + iv \\
&= \mathrm{e}^{-x}(x\sin y - y\cos y) + i\,\mathrm{e}^{-x}(y\sin y + x\cos y) \\
&= \mathrm{e}^{-x}\left\{x\left(\frac{\mathrm{e}^{iy} - \mathrm{e}^{-iy}}{2i}\right) - y\left(\frac{\mathrm{e}^{iy} + \mathrm{e}^{-iy}}{2}\right)\right\} + i\,\mathrm{e}^{-x}\left\{y\left(\frac{\mathrm{e}^{iy} - \mathrm{e}^{-iy}}{2i}\right) + x\left(\frac{\mathrm{e}^{iy} + \mathrm{e}^{-iy}}{2}\right)\right\} \\
&= i(x + iy)\,\mathrm{e}^{-(x+iy)} \\
&= iz\,\mathrm{e}^{-z}
\end{aligned}
$$

Method 2 $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is an analytic function.
On the real axis, $y = 0$, $f(z)$ is

$$
\begin{aligned}
f(z = x) &= u(x, 0) + iv(x, 0) \\
&= \mathrm{e}^{-x}(x\sin 0 - 0\cos 0) + i\,\mathrm{e}^{-x}(0\sin 0 + x\cos 0) \\
&= ix\,\mathrm{e}^{-x}
\end{aligned}
$$

Suppose there is an analytic continuation of $f(z)$ into the complex plane. If such a continuation, $f(z)$, exists, then it must be equal to $f(z = x)$ on the real axis An obvious choice for the analytic continuation is

$$f(z) = u(z, 0) + iv(z, 0)$$

since this is clearly equal to $u(x, 0) + iv(x, 0)$ when z is real. Thus we obtain

$$f(z) = iz\,\mathrm{e}^{-z}$$

Example 11.3.4 Consider $f(z) = u(x, y) + iv(x, y)$. Show that $f'(z) = u_x(z, 0) - iu_y(z, 0)$.

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= u_x - iu_y \end{aligned}$$

$f'(z)$ is an analytic function. On the real axis, $z = x$, $f'(z)$ is

$$f'(z = x) = u_x(x, 0) - iu_y(x, 0)$$

Now $f'(z = x)$ is defined on the real line. An analytic continuation of $f'(z = x)$ into the complex plane is

$$\boxed{f'(z) = u_x(z, 0) - iu_y(z, 0).}$$

Example 11.3.5 Again consider the problem of finding $f(z)$ given that $u(x, y) = \mathrm{e}^{-x}(x \sin y - y \cos y)$. Now we can use the result of the previous example to do this problem.

$$\begin{aligned} u_x(x, y) &= \frac{\partial u}{\partial x} = \mathrm{e}^{-x} \sin y - x\,\mathrm{e}^{-x} \sin y + y\,\mathrm{e}^{-x} \cos y \\ u_y(x, y) &= \frac{\partial u}{\partial y} = x\,\mathrm{e}^{-x} \cos y + y\,\mathrm{e}^{-x} \sin y - \mathrm{e}^{-x} \cos y \end{aligned}$$

$$
\begin{aligned}
f'(z) &= u_x(z,0) - i u_y(z,0) \\
&= 0 - i(z\,\mathrm{e}^{-z} - \mathrm{e}^{-z}) \\
&= i(-z\,\mathrm{e}^{-z} + \mathrm{e}^{-z})
\end{aligned}
$$

Integration yields the result

$$
\boxed{f(z) = iz\,\mathrm{e}^{-z} + c}
$$

Example 11.3.6 Find $f(z)$ given that

$$
\begin{aligned}
u(x,y) &= \cos x \cosh^2 y \sin x + \cos x \sin x \sinh^2 y \\
v(x,y) &= \cos^2 x \cosh y \sinh y - \cosh y \sin^2 x \sinh y
\end{aligned}
$$

$f(z) = u(x,y) + v(x,y)$ is an analytic function. On the real line, $f(z)$ is

$$
\begin{aligned}
f(z = x) &= u(x,0) + iv(x,0) \\
&= \cos x \cosh^2 0 \sin x + \cos x \sin x \sinh^2 0 + i(\cos^2 x \cosh 0 \sinh 0 - \cosh 0 \sin^2 x \sinh 0) \\
&= \cos x \sin x
\end{aligned}
$$

Now we know the definition of $f(z)$ on the real line. We would like to find an analytic continuation of $f(z)$ into the complex plane. An obvious choice for $f(z)$ is

$$
f(z) = \cos z \sin z
$$

Using trig identities we can write this as

$$
\boxed{f(z) = \frac{\sin 2z}{2}.}
$$

Example 11.3.7 Find $f(z)$ given only that

$$u(x,y) = \cos x \cosh^2 y \sin x + \cos x \sin x \sinh^2 y.$$

Recall that

$$\begin{aligned} f'(z) &= u_x + i v_x \\ &= u_x - i u_y \end{aligned}$$

Differentiating $u(x,y)$,

$$\begin{aligned} u_x &= \cos^2 x \cosh^2 y - \cosh^2 y \sin^2 x + \cos^2 x \sinh^2 y - \sin^2 x \sinh^2 y \\ u_y &= 4 \cos x \cosh y \sin x \sinh y \end{aligned}$$

$f'(z)$ is an analytic function. On the real axis, $f'(z)$ is

$$f'(z = x) = \cos^2 x - \sin^2 x$$

Using trig identities we can write this as

$$f'(z = x) = \cos(2x)$$

Now we find an analytic continuation of $f'(z = x)$ into the complex plane.

$$f'(z) = \cos(2z)$$

Integration yields the result

$$\boxed{f(z) = \frac{\sin(2z)}{2} + c}$$

11.3.1 Polar Coordinates

Example 11.3.8 Is

$$u(r, \theta) = r\big(\log r \cos\theta - \theta \sin\theta\big)$$

the real part of an analytic function?

The Laplacian in polar coordinates is

$$\Delta\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}.$$

Calculating the partial derivatives of u,

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \cos\theta + \log r \cos\theta - \theta \sin\theta \\
r\frac{\partial u}{\partial r} &= r\cos\theta + r\log r \cos\theta - r\theta \sin\theta \\
\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) &= 2\cos\theta + \log r \cos\theta - \theta \sin\theta \\
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) &= \frac{1}{r}\big(2\cos\theta + \log r \cos\theta - \theta \sin\theta\big) \\
\frac{\partial u}{\partial \theta} &= -r\big(\theta\cos\theta + \sin\theta + \log r \sin\theta\big) \\
\frac{\partial^2 u}{\partial \theta^2} &= r\big(-2\cos\theta - \log r \cos\theta + \theta \sin\theta\big) \\
\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= \frac{1}{r}\big(-2\cos\theta - \log r \cos\theta + \theta \sin\theta\big)
\end{aligned}$$

From the above we see that

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Therefore u is harmonic and is the real part of some analytic function.

Example 11.3.9 Find an analytic function $f(z)$ whose real part is

$$u(r, \theta) = r\big(\log r \cos\theta - \theta \sin\theta\big).$$

Let $f(z) = u(r, \theta) + i v(r, \theta)$. The Cauchy-Riemann equations are

$$u_r = \frac{v_\theta}{r}, \qquad u_\theta = -r v_r.$$

Using the partial derivatives in the above example, we obtain two partial differential equations for $v(r, \theta)$.

$$\begin{aligned} v_r &= -\frac{u_\theta}{r} = \theta\cos\theta + \sin\theta + \log r \sin\theta \\ v_\theta &= r u_r = r\big(\cos\theta + \log r \cos\theta - \theta\sin\theta\big) \end{aligned}$$

Integrating the equation for v_θ yields

$$v = r\big(\theta\cos\theta + \log r \sin\theta\big) + F(r)$$

where $F(r)$ is a constant of integration.

Substituting our expression for v into the equation for v_r yields

$$\begin{aligned} \theta\cos\theta + \log r\sin\theta + \sin\theta + F'(r) &= \theta\cos\theta + \sin\theta + \log r\sin\theta \\ F'(r) &= 0 \\ F(r) &= \text{const} \end{aligned}$$

Thus we see that

$$\begin{aligned} f(z) &= u + iv \\ &= r\big(\log r\cos\theta - \theta\sin\theta\big) + ir\big(\theta\cos\theta + \log r\sin\theta\big) + \text{const} \end{aligned}$$

$f(z)$ is an analytic function. On the line $\theta = 0$, $f(z)$ is

$$\begin{aligned} f(z = r) &= r\big(\log r\big) + ir\big(0\big) + \text{const} \\ &= r\log r + \text{const} \end{aligned}$$

The analytic continuation into the complex plane is

$$\boxed{f(z) = z \log z + \text{const}}$$

Example 11.3.10 Find the formula in polar coordinates that is analogous to

$$f'(z) = u_x(z, 0) - i u_y(z, 0).$$

We know that

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \mathrm{e}^{-i\theta} \frac{\partial f}{\partial r}.$$

If $f(z) = u(r, \theta) + iv(r, \theta)$ then

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \mathrm{e}^{-i\theta} (u_r + i v_r)$$

From the Cauchy-Riemann equations, we have $v_r = -u_\theta / r$.

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \mathrm{e}^{-i\theta} \left(u_r - i \frac{u_\theta}{r} \right)$$

$f'(z)$ is an analytic function. On the line $\theta = 0$, $f(z)$ is

$$f'(z = r) = u_r(r, 0) - i \frac{u_\theta(r, 0)}{r}$$

The analytic continuation of $f'(z)$ into the complex plane is

$$\boxed{f'(z) = u_r(z, 0) - \frac{i}{r} u_\theta(z, 0).}$$

Example 11.3.11 Find an analytic function $f(z)$ whose real part is

$$u(r, \theta) = r\big(\log r \cos\theta - \theta \sin\theta\big).$$

$$u_r(r, \theta) = (\log r \cos\theta - \theta \sin\theta) + \cos\theta$$
$$u_\theta(r, \theta) = r(-\log r \sin\theta - \sin\theta - \theta \cos\theta$$

$$f'(z) = u_r(z, 0) - \frac{i}{r} u_\theta(z, 0)$$
$$= \log z + 1$$

Integrating $f'(z)$ yields

$$\boxed{f(z) = z \log z + ic.}$$

11.3.2 Analytic Functions Defined in Terms of Their Real or Imaginary Parts

Consider an analytic function: $f(z) = u(x, y) + iv(x, y)$. We differentiate this expression.

$$f'(z) = u_x(x, y) + iv_x(x, y)$$

We apply the Cauchy-Riemann equation $v_x = -u_y$.

$$f'(z) = u_x(x, y) - iu_y(x, y). \tag{11.1}$$

Now consider the function of a complex variable, $g(\zeta)$:

$$g(\zeta) = u_x(x, \zeta) - iu_y(x, \zeta) = u_x(x, \xi + i\eta) - iu_y(x, \xi + i\eta).$$

This function is analytic where $f(\zeta)$ is analytic. To show this we first verify that the derivatives in the ξ and η directions are equal.

$$\frac{\partial}{\partial \xi} g(\zeta) = u_{xy}(x, \xi + i\eta) - i u_{yy}(x, \xi + i\eta)$$
$$-i \frac{\partial}{\partial \eta} g(\zeta) = -i(i u_{xy}(x, \xi + i\eta) + u_{yy}(x, \xi + i\eta)) = u_{xy}(x, \xi + i\eta) - i u_{yy}(x, \xi + i\eta)$$

Since these partial derivatives are equal and continuous, $g(\zeta)$ is analytic. We evaluate the function $g(\zeta)$ at $\zeta = -ix$. (Substitute $y = -ix$ into Equation 11.1.)

$$f'(2x) = u_x(x, -ix) - i u_y(x, -ix)$$

We make a change of variables to solve for $f'(x)$.

$$f'(x) = u_x\left(\frac{x}{2}, -i\frac{x}{2}\right) - i u_y\left(\frac{x}{2}, -i\frac{x}{2}\right).$$

If the expression is nonsingular, then this defines the analytic function, $f'(z)$, on the real axis. The analytic continuation to the complex plane is

$$f'(z) = u_x\left(\frac{z}{2}, -i\frac{z}{2}\right) - i u_y\left(\frac{z}{2}, -i\frac{z}{2}\right).$$

Note that $\frac{\mathrm{d}}{\mathrm{d}z} 2u(z/2, -iz/2) = u_x(z/2, -iz/2) - i u_y(z/2, -iz/2)$. We integrate the equation to obtain:

$$f(z) = 2u\left(\frac{z}{2}, -i\frac{z}{2}\right) + c.$$

We know that the real part of an analytic function determines that function to within an additive constant. Assuming that the above expression is non-singular, we have found a formula for writing an analytic function in terms of its real part. With the same method, we can find how to write an analytic function in terms of its imaginary part, v.

We can also derive formulas if u and v are expressed in polar coordinates:

$$f(z) = u(r, \theta) + iv(r, \theta).$$

Result 11.3.2 If $f(z) = u(x, y) + iv(x, y)$ is analytic and the expressions are non-singular, then

$$f(z) = 2u\left(\frac{z}{2}, -i\frac{z}{2}\right) + \text{const} \tag{11.2}$$

$$f(z) = i2v\left(\frac{z}{2}, -i\frac{z}{2}\right) + \text{const.} \tag{11.3}$$

If $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic and the expressions are non-singular, then

$$f(z) = 2u\left(z^{1/2}, -\frac{i}{2}\log z\right) + \text{const} \tag{11.4}$$

$$f(z) = i2v\left(z^{1/2}, -\frac{i}{2}\log z\right) + \text{const.} \tag{11.5}$$

Example 11.3.12 Consider the problem of finding $f(z)$ given that $u(x, y) = \mathrm{e}^{-x}(x \sin y - y \cos y)$.

$$\begin{aligned}
f(z) &= 2u\left(\frac{z}{2}, -i\frac{z}{2}\right) \\
&= 2\,\mathrm{e}^{-z/2}\left(\frac{z}{2}\sin\left(-i\frac{z}{2}\right) + i\frac{z}{2}\cos\left(-i\frac{z}{2}\right)\right) + c \\
&= iz\,\mathrm{e}^{-z/2}\left(i\sin\left(i\frac{z}{2}\right) + \cos\left(-i\frac{z}{2}\right)\right) + c \\
&= iz\,\mathrm{e}^{-z/2}(\mathrm{e}^{-z/2}) + c \\
&= iz\,\mathrm{e}^{-z} + c
\end{aligned}$$

Example 11.3.13 Consider

$$\operatorname{Log} z = \frac{1}{2} \operatorname{Log}\left(x^2 + y^2\right) + i \operatorname{Arctan}(x, y).$$

We try to construct the analytic function from it's real part using Equation 11.2.

$$\begin{aligned}
f(z) &= 2u\left(\frac{z}{2}, -i\frac{z}{2}\right) + c \\
&= 2\frac{1}{2} \operatorname{Log}\left(\left(\frac{z}{2}\right)^2 + \left(-i\frac{z}{2}\right)^2\right) + c \\
&= \operatorname{Log}(0) + c
\end{aligned}$$

We obtain a singular expression, so the method fails.

Example 11.3.14 Again consider the logarithm, this time written in terms of polar coordinates,

$$\operatorname{Log} z = \operatorname{Log} r + i\theta.$$

We try to construct the analytic function from it's real part using Equation 11.4.

$$\begin{aligned}
f(z) &= 2u\left(z^{1/2}, -i\frac{i}{2} \log z\right) + c \\
&= 2 \operatorname{Log}\left(z^{1/2}\right) + c \\
&= \operatorname{Log} z + c
\end{aligned}$$

With this method we recover the analytic function.

11.4 Exercises

Exercise 11.1

Consider two functions, $f(x, y)$ and $g(x, y)$. They are said to be functionally dependent if there is a an $h(g)$ such that

$$f(x, y) = h(g(x, y)).$$

f and g will be functionally dependent if and only if their Jacobian vanishes.

If f and g are functionally dependent, then the derivatives of f are

$$f_x = h'(g) g_x$$
$$f_y = h'(g) g_y.$$

Thus we have

$$\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = f_x g_y - f_y g_x = h'(g) g_x g_y - h'(g) g_y g_x = 0.$$

If the Jacobian of f and g vanishes, then

$$f_x g_y - f_y g_x = 0.$$

This is a first order partial differential equation for f that has the general solution

$$f(x, y) = h(g(x, y)).$$

Prove that an analytic function $u(x, y) + iv(x, y)$ can be written in terms of a function of a complex variable, $f(z) = u(x, y) + iv(x, y)$.

Exercise 11.2

Which of the following functions are the real part of an analytic function? For those that are, find the harmonic conjugate, $v(x, y)$, and find the analytic function $f(z) = u(x, y) + iv(x, y)$ as a function of z.

1. $x^3 - 3xy^2 - 2xy + y$
2. $\mathrm{e}^x \sinh y$
3. $\mathrm{e}^x(\sin x \cos y \cosh y - \cos x \sin y \sinh y)$

Exercise 11.3
For an analytic function, $f(z) = u(r, \theta) + iv(r, \theta)$ prove that under suitable restrictions:

$$f(z) = 2u\left(z^{1/2}, -\frac{i}{2}\log z\right) + \text{const.}$$

11.5 Hints

Hint 11.1
Show that $u(x, y) + iv(x, y)$ is functionally dependent on $x + iy$ so that you can write $f(z) = f(x + iy) = u(x, y) + iv(x, y)$.

Hint 11.2

Hint 11.3
Check out the derivation of Equation 11.2.

11.6 Solutions

Solution 11.1

$u(x, y) + iv(x, y)$ is functionally dependent on $z = x + iy$ if and only if

$$\frac{\partial(u + iv, x + iy)}{\partial(x, y)} = 0.$$

$$\begin{aligned}\frac{\partial(u + iv, x + iy)}{\partial(x, y)} &= \begin{vmatrix} u_x + iv_x & u_y + iv_y \\ 1 & i \end{vmatrix} \\ &= -v_x - u_y + i(u_x - v_y)\end{aligned}$$

Since u and v satisfy the Cauchy-Riemann equations, this vanishes

$$= 0$$

Thus we see that $u(x, y) + iv(x, y)$ is functionally dependent on $x + iy$ so we can write

$$f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

Solution 11.2

1. Consider $u(x, y) = x^3 - 3xy^2 - 2xy + y$. The Laplacian of this function is

$$\begin{aligned}\Delta u &\equiv u_{xx} + u_{yy} \\ &= 6x - 6x \\ &= 0\end{aligned}$$

   Since the function is harmonic, it is the real part of an analytic function. Clearly the analytic function is of the form,

$$az^3 + bz^2 + cz + id,$$

with a, b and c complex-valued constants and d a real constant. Substituting $z = x + iy$ and expanding products yields,

$$a(x^3 + i3x^2y - 3xy^2 - iy^3) + b(x^2 + i2xy - y^2) + c(x + iy) + id.$$

By inspection, we see that the analytic function is

$$\boxed{f(z) = z^3 + iz^2 - iz + id.}$$

The harmonic conjugate of u is the imaginary part of $f(z)$,

$$\boxed{v(x, y) = 3x^2y - y^3 + x^2 - y^2 - x + d.}$$

We can also do this problem with analytic continuation. The derivatives of u are

$$u_x = 3x^2 - 3y^2 - 2y,$$
$$u_y = -6xy - 2x + 1.$$

The derivative of $f(z)$ is

$$f'(z) = u_x - iu_y = 3x^2 - 2y^2 - 2y + i(6xy - 2x + 1).$$

On the real axis we have

$$f'(z = x) = 3x^2 - i2x + i.$$

Using analytic continuation, we see that

$$f'(z) = 3z^2 - i2z + i.$$

Integration yields

$$f(z) = z^3 - iz^2 + iz + \text{const}$$

2. Consider $u(x, y) = e^x \sinh y$. The Laplacian of this function is

$$\begin{aligned} \Delta u &= e^x \sinh y + e^x \sinh y \\ &= 2 e^x \sinh y. \end{aligned}$$

Since the function is not harmonic, it is not the real part of an analytic function.

3. Consider $u(x, y) = e^x(\sin x \cos y \cosh y - \cos x \sin y \sinh y)$. The Laplacian of the function is

$$\begin{aligned} \Delta u &= \frac{\partial}{\partial x}\left(e^x(\sin x \cos y \cosh y - \cos x \sin y \sinh y + \cos x \cos y \cosh y + \sin x \sin y \sinh y)\right) \\ &\quad + \frac{\partial}{\partial y}\left(e^x(-\sin x \sin y \cosh y - \cos x \cos y \sinh y + \sin x \cos y \sinh y - \cos x \sin y \cosh y)\right) \\ &= 2 e^x(\cos x \cos y \cosh y + \sin x \sin y \sinh y) - 2 e^x(\cos x \cos y \cosh y + \sin x \sin y \sinh y) \\ &= 0. \end{aligned}$$

Thus u is the real part of an analytic function. The derivative of the analytic function is

$$f'(z) = u_x + i v_x = u_x - i u_y$$

From the derivatives of u we computed before, we have

$$\begin{aligned} f(z) &= \left(e^x(\sin x \cos y \cosh y - \cos x \sin y \sinh y + \cos x \cos y \cosh y + \sin x \sin y \sinh y)\right) \\ &\quad - i\left(e^x(-\sin x \sin y \cosh y - \cos x \cos y \sinh y + \sin x \cos y \sinh y - \cos x \sin y \cosh y)\right) \end{aligned}$$

Along the real axis, $f'(z)$ has the value,

$$f'(z = x) = e^x(\sin x + \cos x).$$

By analytic continuation, $f'(z)$ is

$$f'(z) = e^z(\sin z + \cos z)$$

We obtain $f(z)$ by integrating.

$$f(z) = \mathrm{e}^z \sin z + \text{const.}$$

u is the real part of the analytic function

$$\boxed{f(z) = \mathrm{e}^z \sin z + ic,}$$

where c is a real constant. We find the harmonic conjugate of u by taking the imaginary part of f.

$$f(z) = \mathrm{e}^x (cosy + i \sin y)(\sin x \cosh y + i \cos x \sinh y) + ic$$

$$\boxed{v(x, y) = \mathrm{e}^x \sin x \sin y \cosh y + \cos x \cos y \sinh y + c}$$

Solution 11.3

We consider the analytic function: $f(z) = u(r, \theta) + iv(r, \theta)$. Recall that the complex derivative in terms of polar coordinates is

$$\frac{}{z} = \mathrm{e}^{-i\theta} \frac{\partial}{\partial r} = -\frac{i}{r} \mathrm{e}^{-i\theta} \frac{\partial}{\partial \theta}.$$

The Cauchy-Riemann equations are

$$u_r = \frac{1}{r} v_\theta, \qquad v_r = -\frac{1}{r} u_\theta.$$

We differentiate $f(z)$ and use the partial derivative in r for the right side.

$$f'(z) = \mathrm{e}^{-i\theta} (u_r + iv_r)$$

We use the Cauchy-Riemann equations to right $f'(z)$ in terms of the derivatives of u.

$$f'(z) = \mathrm{e}^{-i\theta} \left(u_r - i\frac{1}{r} u_\theta \right) \tag{11.6}$$

Now consider the function of a complex variable, $g(\zeta)$:

$$g(\zeta) = \mathrm{e}^{-i\zeta}\left(u_r(r,\zeta) - i\frac{1}{r}u_\theta(r,\zeta)\right) = \mathrm{e}^{\eta - i\xi}\left(u_r(r,\xi + i\eta) - i\frac{1}{r}u_\theta(r,\xi + i\eta)\right)$$

This function is analytic where $f(\zeta)$ is analytic. It is a simple calculus exercise to show that the complex derivative in the ξ direction, $\frac{\partial}{\partial \xi}$, and the complex derivative in the η direction, $-i\frac{\partial}{\partial \eta}$, are equal. Since these partial derivatives are equal and continuous, $g(\zeta)$ is analytic. We evaluate the function $g(\zeta)$ at $\zeta = -i\log r$. (Substitute $\theta = -i\log r$ into Equation 11.6.)

$$f'\left(r\,\mathrm{e}^{i(-i\log r)}\right) = \mathrm{e}^{-i(-i\log r)}\left(u_r(r,-i\log r) - i\frac{1}{r}u_\theta(r,-i\log r)\right)$$

$$rf'\left(r^2\right) = u_r(r,-i\log r) - i\frac{1}{r}u_\theta(r,-i\log r)$$

If the expression is nonsingular, then it defines the analytic function, $f'(z)$, on a curve. The analytic continuation to the complex plane is

$$zf'\left(z^2\right) = u_r(z,-i\log z) - i\frac{1}{z}u_\theta(z,-i\log z).$$

We integrate to obtain an expression for $f(z^2)$.

$$\frac{1}{2}f\left(z^2\right) = u(z,-i\log z) + \text{const}$$

We make a change of variables and solve for $f(z)$.

$$\boxed{f(z) = 2u\left(z^{1/2}, -\frac{i}{2}\log z\right) + \text{const.}}$$

Assuming that the above expression is non-singular, we have found a formula for writing the analytic function in terms of its real part, $u(r,\theta)$. With the same method, we can find how to write an analytic function in terms of its imaginary part, $v(r,\theta)$.

Chapter 12

Contour Integration and Cauchy's Theorem

Between two evils, I always pick the one I never tried before.

- Mae West

12.1 Line Integrals

In this section we will recall the definition of a line integral of real-valued functions in the Cartesian plane. We will use this to define the contour integral of complex-valued functions in the complex plane.

Definition. Consider a curve C in the Cartesian plane joining the points (a_0, b_0) and (a_1, b_1). Partition the curve into $n+1$ segments with the points $(x_0, y_0), \ldots, (x_n, y_n)$ where the first and last points are at the endpoints of the curve. Define $\Delta x_k = x_{k+1} - x_k$ and $\Delta y_k = y_{k+1} - y_k$. Let (ξ_k, η_k) be points on the curve between (x_k, y_k) and (x_{k+1}, y_{k+1}). (See Figure 12.1.)

Consider the sum

$$\sum_{k=0}^{n-1} \left(P(\xi_k, \eta_k) \Delta x_k + Q(\xi_k, \eta_k) \Delta y_k \right),$$

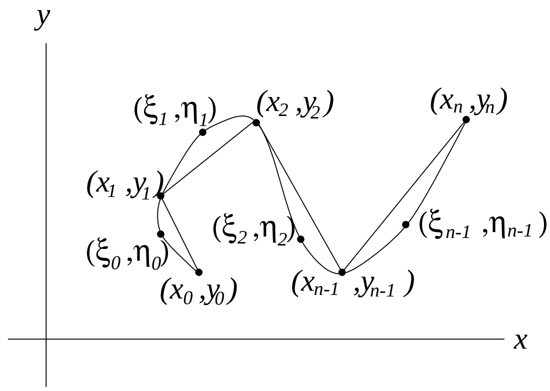


Figure 12.1: A curve in the Cartesian plane.

where P and Q are continuous functions on the curve. In the limit as each of the Δx_k and Δy_k approach zero the value of the sum, (if the limit exists), is denoted by

$$\int_C P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y.$$

This is a *line integral* along the curve C. The value of the line integral depends on the functions $P(x, y)$ and $Q(x, y)$, the endpoints of the curve and the curve C. One can also write a line integral in vector notation,

$$\int_C \mathbf{f}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x},$$

where $\mathbf{x} = (x, y)$ and $\mathbf{f}(\mathbf{x}) = (P(x, y), Q(x, y))$.

Evaluation. Let the curve C be parametrized by $x = x(t)$, $y = y(t)$ for $t_0 \leq t \leq t_1$. The differentials on the curve are $\mathrm{d}x = x'(t)\,\mathrm{d}t$ and $\mathrm{d}y = y'(t)\,dt$. Thus the line integral is

$$\int_{t_0}^{t_1} \big(P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\big)\,\mathrm{d}t,$$

which is a definite integral.

Example 12.1.1 Consider the line integral

$$\int_C x^2 \, \mathrm{d}x + (x+y) \, \mathrm{d}y,$$

where C is the semi-circle from $(1,0)$ to $(-1,0)$ in the upper half plane. We parameterize the curve with $x = \cos t$, $y = \sin t$ for $0 \leq t \leq \pi$.

$$\begin{aligned}\int_C x^2 \, \mathrm{d}x + (x+y) \, \mathrm{d}y &= \int_0^\pi \left(\cos^2 t(-\sin t) + (\cos t + \sin t) \cos t\right) \, \mathrm{d}t \\ &= \frac{\pi}{2} - \frac{2}{3}\end{aligned}$$

Complex Line Integrals. Consider a curve C in the complex plane joining the points c_0 and c_1. Partition the curve into $n+1$ segments with the points $z_0, \ldots, z_n$ where the first and last points are at the endpoints of the curve. Define $\Delta z_k = z_{k+1} - z_k$. Let ζ_k be points on the curve between z_k and z_{k+1}. Consider the sum

$$\sum_{k=0}^{n-1} f(\zeta_k) \Delta z_k,$$

where f is a continuous, complex-valued function on the curve. In the limit as each of the Δz_k approach zero the value of the sum, (if the limit exists), is denoted by

$$\int_C f(z) \, \mathrm{d}z.$$

This is a *complex line integral* along the curve C.

We can write a complex line integral in terms of real line integrals. Let $f(z) = u(x,y) + iv(x,y)$.

$$\int_C f(z) \, \mathrm{d}z = \int_C (u(x,y) + iv(x,y))(\,\mathrm{d}x + i \, \mathrm{d}y)$$

$$\int_C f(z) \, \mathrm{d}z = \int_C (u(x,y) \, \mathrm{d}x - v(x,y) \, \mathrm{d}y) + i \int_C (v(x,y) \, \mathrm{d}x + u(x,y) \, \mathrm{d}y). \tag{12.1}$$

Evaluation. Let the curve C be parametrized by $z = z(t)$ for $t_0 \leq t \leq t_1$. Then the complex line integral is

$$\int_{t_0}^{t_1} f(z(t)) z'(t) \, \mathrm{d}t,$$

which is a definite integral of a complex-valued function.

Example 12.1.2 Let C be the positively oriented unit circle about the origin in the complex plane. Evaluate:

1. $\int_C z \, \mathrm{d}z$
2. $\int_C \frac{1}{z} \, \mathrm{d}z$
3. $\int_C \frac{1}{z} |\mathrm{d}z|$

1. We parameterize the curve and then do the integral.

$$z = \mathrm{e}^{i\theta}, \quad \mathrm{d}z = i \, \mathrm{e}^{i\theta} \, d\theta$$

$$\begin{aligned}\int_C z \, \mathrm{d}z &= \int_0^{2\pi} \mathrm{e}^{i\theta} i \, \mathrm{e}^{i\theta} \, d\theta \\ &= \left[\frac{1}{2} \mathrm{e}^{i2\theta} \right]_0^{2\pi} \\ &= \left(\frac{1}{2} \mathrm{e}^{i4\pi} - \frac{1}{2} \mathrm{e}^{i0} \right) \\ &= 0\end{aligned}$$

2.

$$\int_C \frac{1}{z} \, \mathrm{d}z = \int_0^{2\pi} \frac{1}{\mathrm{e}^{i\theta}} i \, \mathrm{e}^{i\theta} \, d\theta = i \int_0^{2\pi} \mathrm{d}\theta = i2\pi$$

3.

$$|\mathrm{d}z| = \left|i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta\right| = \left|i\,\mathrm{e}^{i\theta}\right| |d\theta| = |d\theta|$$

Since $d\theta$ is positive in this case, $|d\theta| = d\theta$.

$$\int_C 1/z\,|\mathrm{d}z| = \int_0^{2\pi} \frac{1}{\mathrm{e}^{i\theta}}\,\mathrm{d}\theta = \left[i\,\mathrm{e}^{-i\theta}\right]_0^{2\pi} = 0$$

Maximum Modulus Integral Bound. The absolute value of a real integral obeys the inequality

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,|\mathrm{d}x| \leq (b-a) \max_{a\leq x\leq b} |f(x)|.$$

Now we prove the analogous result for the modulus of a complex line integral.

$$\begin{aligned}
\left|\int_C f(z)\,\mathrm{d}z\right| &= \left|\lim_{\Delta z\to 0} \sum_{k=0}^{n-1} f(\zeta_k)\Delta z_k\right| \\
&\leq \lim_{\Delta z\to 0} \sum_{k=0}^{n-1} |f(\zeta_k)|\;|\Delta z_k| \\
&= \int_C |f(z)|\,|\mathrm{d}z| \\
&\leq \int_C \left(\max_{z\in C} |f(z)|\right)\,|\mathrm{d}z| \\
&= \left(\max_{z\in C} |f(z)|\right) \int_C |\mathrm{d}z| \\
&= \left(\max_{z\in C} |f(z)|\right) \times (\text{length of } C)
\end{aligned}$$

Result 12.1.1 Maximum Modulus Integral Bound.

$$\left| \int_C f(z)\,\mathrm{d}z \right| \leq \int_C |f(z)|\,|\mathrm{d}z| \leq \left(\max_{z \in C} |f(z)| \right) (\text{length of } C)$$

12.2 Under Construction

Cauchy's Theorem. Let $f(z)$ be analytic in a compact, closed, connected domain D. Consider the integral of $f(z)$ on the boundary of the domain.

$$\int_{\partial D} f(z)\,\mathrm{d}z = \int_{\partial D} \psi(x,y)\,(\mathrm{d}x + i\mathrm{d}y) = \int_{\partial D} \psi\,\mathrm{d}x + i\psi\,\mathrm{d}y$$

Recall Green's Theorem.

$$\int_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_D (Q_x - P_y)\,\mathrm{d}xy$$

We apply Green's Theorem to the integral of $f(z)$ on ∂D.

$$\int_{\partial D} f(z)\,\mathrm{d}z = \int_{\partial D} \psi\,\mathrm{d}x + i\psi\,\mathrm{d}y = \int_D (i\psi_x - \psi_y)\,\mathrm{d}xy$$

Since $f(z)$ is analytic, $\psi_x = -i\psi_y$. The integrand $i\psi_x - \psi_y$ is zero. Thus we have

$$\int_{\partial D} f(z)\,\mathrm{d}z = 0.$$

This is known as *Cauchy's Theorem*.

Fundamental Theorem of Calculus. First note that $\Re(\cdot)$ and $\Im(\cdot)$ commute with derivatives and integrals. Let $P(x, y)$ and $Q(x, y)$ be defined on a simply connected domain. A necessary and sufficient condition for the existence of a primitive ϕ is that $P_y = Q_x$. The primitive satisfies

$$\mathrm{d}\phi = P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

Definite integral can be evaluated in terms of the primitive.

$$\int_{(a,b)}^{(c,d)} P\,\mathrm{d}x + Q\,\mathrm{d}y = \phi(c, d) - \phi(a, b)$$

Now consider integral along the contour C of the complex-valued function $\psi(x, y)$.

$$\int_C \psi\,\mathrm{d}z = \int_C \psi\,\mathrm{d}x + i\psi\,\mathrm{d}y$$

If $\psi(x, y)$ is analytic then there exists a function Ψ such that

$$\mathrm{d}\Psi = \psi\,\mathrm{d}x + i\psi\,\mathrm{d}y.$$

Then ψ satisfies the Cauchy-Riemann equations. How do we find the primitive Ψ that satisfies $\Psi_x = \psi$ and $\Psi_y = i\psi$? Note that choosing $\Psi(x, y) = F(z)$ where $F(z)$ is an anti-derivative of $f(z)$, $F'(z) = f(z)$, does the trick.

$$F'(z) = \Psi_x = -i\Psi_y = f = \psi$$

The differential of Ψ is

$$\mathrm{d}\Psi = \Psi_x\,\mathrm{d}x + \Psi_y\,\mathrm{d}y = \psi\,\mathrm{d}x + \psi\,\mathrm{d}y.$$

We can evaluate a definite integral of f in terms of F.

$$\int_a^b f(z)\,\mathrm{d}z = F(b) - F(a).$$

This is the *Fundamental Theorem of Calculus for functions of a complex variable.*

12.3 Cauchy's Theorem

Result 12.3.1 Cauchy's Theorem. If $f(z)$ is analytic in a compact, closed, connected domain D then the integral of $f(z)$ on the boundary of the domain vanishes.

$$\oint_{\partial D} f(z)\,\mathrm{d}z = \sum_k \oint_{C_k} f(z)\,\mathrm{d}z = 0$$

Here the set of contours $\{C_k\}$ make up the positively oriented boundary ∂D of the domain D.

This result follows from Green's Theorem. Since Green's theorem holds for both simply and multiply connected domains, so does Cauchy's theorem.

Proof of Cauchy's Theorem. We will assume that $f'(z)$ is continuous. This assumption is not necessary, but it allows us to use Green's Theorem, which makes for a simpler proof. We consider the integral of $f(z) = u(x,y) + iv(x,y)$ along the boundary of the domain. From Equation 12.1 we have,

$$\int_{\partial D} f(z)\,\mathrm{d}z = \int_{\partial D} (u\,\mathrm{d}x - v\,\mathrm{d}y) + i \int_{\partial D} (v\,\mathrm{d}x + u\,\mathrm{d}y)$$

We use Green's theorem to write this as an area integral.

$$\int_{\partial D} f(z)\,\mathrm{d}z = \int_D (-v_x - u_y)\,\mathrm{d}x\,\mathrm{d}y + i \int_D (u_x - v_y)\,\mathrm{d}x\,\mathrm{d}y$$

Since u and v satisfy the Cauchy-Riemann Equations, $u_x = v_y$ and $u_y = -v_x$, the two integrands on the right side are identically zero. Thus the two area integrals vanish and Cauchy's theorem is proved.

As a special case of Cauchy's theorem we can consider a simply-connected region. For this the boundary is a Jordan curve. We can state the theorem in terms of this curve instead of referring to the boundary.

Result 12.3.2 Cauchy's Theorem for Jordan Curves. If $f(z)$ is analytic inside and on a simple, closed contour C, then

$$\oint_C f(z)\,\mathrm{d}z = 0$$

Example 12.3.1 In Example 12.1.2 we calculated that

$$\int_C z\,\mathrm{d}z = 0$$

where C is the unit circle about the origin. Now we can evaluate the integral without parameterizing the curve. We simply note that the integrand is analytic inside and on the circle, which is simple and closed. By Cauchy's Theorem, the integral vanishes.

We cannot apply Cauchy's theorem to evaluate

$$\int_C \frac{1}{z}\,\mathrm{d}z = i2\pi$$

as the integrand is not analytic at $z = 0$.

Morera's Theorem. The converse of Cauchy's theorem, is Morera's Theorem. If the integrals of a continuous function $f(z)$ vanish along all possible simple, closed contours in a domain, then $f(z)$ is analytic on that domain. To prove Morera's Theorem we will assume that first partial derivatives of $f(z) = u(x, y) + iv(x, y)$ are continuous, although the result can be derived without this restriction. Let the simple, closed contour C be the boundary of

D which is contained in the domain Ω.

$$\begin{aligned}
\oint_C f(z)\,\mathrm{d}z &= \oint_C (u+iv)(\,\mathrm{d}x + i\,\mathrm{d}y) \\
&= \oint_C u\,\mathrm{d}x - v\,\mathrm{d}y + i\oint_C v\,\mathrm{d}x + u\,\mathrm{d}y \\
&= \int_D (-v_x - u_y)\,\mathrm{d}x\,\mathrm{d}y + i\int_D (u_x - v_y)\,\mathrm{d}x\,\mathrm{d}y \\
&= 0
\end{aligned}$$

Since the two integrands are continuous and vanish for all C in Ω, we conclude that the integrands are identically zero. This implies that the Cauchy-Riemann equations,

$$u_x = v_y, \qquad u_y = -v_x,$$

are satisfied. $f(z)$ is analytic in Ω.

Result 12.3.3 Morera's Theorem. If $f(z)$ is continuous in a simply connected domain Ω and

$$\oint_C f(z)\,\mathrm{d}z = 0$$

for all possible simple, closed contours C in the domain, the $f(z)$ is analytic in Ω.

12.4 Indefinite Integrals

Consider a function $f(z)$ which is analytic in a domain D. An *anti-derivative* or *indefinite integral* (or simply *integral*) is a function $F(z)$ which satisfies $F'(z) = f(z)$. This integral exists and is unique up to an additive constant. Note that if the domain is not connected, then the additive constants in each connected component are independent. The indefinite integrals are denoted:

$$\int f(z)\,\mathrm{d}z = F(z) + c.$$

We will prove existence in the next section by writing an indefinite integral as a contour integral. We consider uniqueness here. Let $F(z)$ and $G(z)$ be integrals of $f(z)$. Then $F'(z) - G'(z) = f(z) - f(z) = 0$. One can use this to show that $F(z) - G(z)$ is a constant on each connected component of the domain. This demonstrates uniqueness.

Integrals of analytic functions have all the nice properties of integrals of functions of a real variables. All the formulas from integral tables, including things like integration by parts, carry over directly.

12.5 Contour Integrals

Result 12.5.1 Path Independence. Let $f(z)$ be analytic on a simply connected domain. For points a and b in the domain, the contour integral,

$$\int_a^b f(z)\,\mathrm{d}z$$

is independent of the path connecting the points.

(Here we assume that the paths lie entirely in the domain.) This result is a direct consequence of Cauchy's Theorem. Let C_1 and C_2 be two different paths connecting the points. Let $-C_2$ denote the second curve with the opposite orientation. Let C be the contour which is the union of C_1 and $-C_2$. By Cauchy's theorem, the integral along this contour vanishes.

$$\int_{C_1} f(z)\,\mathrm{d}z + \int_{-C_2} f(z)\,\mathrm{d}z = 0$$

This implies that

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_{C_2} f(z)\,\mathrm{d}z.$$

Thus contour integrals on simply connected domains are independent of path. This result does not hold for multiply connected domains.

Result 12.5.2 Constructing an Indefinite Integral. If $f(z)$ is analytic in a simply connected domain D and a is a point in the domain, then

$$F(z) = \int_a^z f(\zeta)\,\mathrm{d}\zeta$$

is analytic in D and is an indefinite integral of $f(z)$, $(F'(z) = f(z))$.

To prove this, we use the limit definition of differentiation.

$$\begin{aligned} F'(z) &= \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left(\int_a^{z+\Delta z} f(\zeta)\,\mathrm{d}\zeta - \int_a^z f(\zeta)\,\mathrm{d}\zeta\right) \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\int_z^{z+\Delta z} f(\zeta)\,\mathrm{d}\zeta \end{aligned}$$

The integral is independent of path. We choose a straight line connecting z and $z+\Delta z$. We add and subtract $\Delta z f(z) = \int_z^{z+\Delta z} f(z)\,\mathrm{d}\zeta$ from the expression for $F'(z)$.

$$\begin{aligned} F'(z) &= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left(\Delta z f(z) + \int_z^{z+\Delta z} (f(\zeta) - f(z))\,\mathrm{d}\zeta\right) \\ &= f(z) + \lim_{\Delta z \to 0} \frac{1}{\Delta z}\int_z^{z+\Delta z} (f(\zeta) - f(z))\,\mathrm{d}\zeta \end{aligned}$$

Since $f(z)$ is analytic, it is certainly continuous. This means that

$$\lim_{\zeta \to z} f(\zeta) = 0.$$

The limit term vanishes as a result of this continuity.

$$\begin{aligned}
\lim_{\Delta z \to 0} \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} (f(\zeta) - f(z)) \, \mathrm{d}\zeta \right| &\leq \lim_{\Delta z \to 0} \frac{1}{|\Delta z|} |\Delta z| \max_{\zeta \in [z \ldots z+\Delta z]} |f(\zeta) - f(z)| \\
&= \lim_{\Delta z \to 0} \max_{\zeta \in [z \ldots z+\Delta z]} |f(\zeta) - f(z)| \\
&= 0
\end{aligned}$$

Thus $F'(z) = f(z)$.

This results demonstrates the existence of the indefinite integral. We will use this to prove the Fundamental Theorem of Calculus for functions of a complex variable.

Result 12.5.3 Fundamental Theorem of Calculus. If $f(z)$ is analytic in a simply connected domain D then

$$\int_a^b f(z) \, \mathrm{d}z = F(b) - F(a)$$

where $F(z)$ is any indefinite integral of $f(z)$.

From Result 12.5.2 we know that

$$\int_a^b f(z) \, \mathrm{d}z = F(b) + c.$$

(Here we are considering b to be a variable.) The case $b = a$ determines the constant.

$$\begin{aligned}
\int_a^a f(z) \, \mathrm{d}z = F(a) + c = 0 \\
c = -F(a)
\end{aligned}$$

This proves the Fundamental Theorem of Calculus for functions of a complex variable.

Example 12.5.1 Consider the integral

$$\int_C \frac{1}{z-a} \, \mathrm{d}z$$

where C is any closed contour that goes around the point $z = a$ once in the positive direction. We use the Fundamental Theorem of Calculus to evaluate the integral. We start at a point on the contour $z - a = r\,\mathrm{e}^{i\theta}$. When we traverse the contour once in the positive direction we end at the point $z - a = r\,\mathrm{e}^{i(\theta+2\pi)}$.

$$\begin{aligned}
\int_C \frac{1}{z-a} \, \mathrm{d}z &= [\log(z-a)]_{z-a=r\,\mathrm{e}^{i\theta}}^{z-a=r\,\mathrm{e}^{i(\theta+2\pi)}} \\
&= \operatorname{Log} r + i(\theta + 2\pi) - (\operatorname{Log} r + i\theta) \\
&= i2\pi
\end{aligned}$$

12.6 Exercises

Exercise 12.1

C is the arc corresponding to the unit semi-circle, $|z| = 1$, $\Im(z) \geq 0$, directed from $z = -1$ to $z = 1$. Evaluate

1. $\displaystyle\int_C z^2 \, dz$
2. $\displaystyle\int_C \left|z^2\right| \, dz$
3. $\displaystyle\int_C z^2 \, |dz|$
4. $\displaystyle\int_C \left|z^2\right| \, |dz|$

Exercise 12.2

Evaluate

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)} \, dx,$$

where $a, b \in \mathbb{C}$ and $\Re(a) > 0$. Use the fact that

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.$$

Exercise 12.3

Evaluate

$$2\int_0^{\infty} e^{-ax^2} \cos(\omega x) \, dx, \quad \text{and} \quad 2\int_0^{\infty} x \, e^{-ax^2} \sin(\omega x) \, dx,$$

where $\Re(a) > 0$ and $\omega \in \mathbb{R}$.

12.7 Hints

Hint 12.1

Hint 12.2
Let C be the parallelogram in the complex plane with corners at $\pm R$ and $\pm R + b/(2a)$. Consider the integral of e^{-az^2} on this contour. Take the limit as $R \to \infty$.

Hint 12.3
Extend the range of integration to $(-\infty \ldots \infty)$. Use $e^{i\omega x} = \cos(\omega x) + i \sin(\omega x)$ and the result of Exercise 12.2.

12.8 Solutions

Solution 12.1
We parameterize the path with $z = \mathrm{e}^{i\theta}$, with θ ranging from π to 0.

$$\mathrm{d}z = i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta$$

$$|\mathrm{d}z| = |i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta| = |d\theta| = -d\theta$$

1.

$$\begin{aligned}
\int_C z^2\,\mathrm{d}z &= \int_\pi^0 \mathrm{e}^{i2\theta} i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\
&= \int_\pi^0 i\,\mathrm{e}^{i3\theta}\,\mathrm{d}\theta \\
&= \left[\frac{1}{3}\,\mathrm{e}^{i3\theta}\right]_\pi^0 \\
&= \frac{1}{3}\left(\mathrm{e}^{i0} - \mathrm{e}^{i3\pi}\right) \\
&= \frac{1}{3}(1 - (-1)) \\
&= \frac{2}{3}
\end{aligned}$$

2.

$$\begin{aligned}\int_C |z^2| \,\mathrm{d}z &= \int_\pi^0 |\,\mathrm{e}^{i2\theta}| i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\ &= \int_\pi^0 i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\ &= \left[\,\mathrm{e}^{i\theta}\right]_\pi^0 \\ &= 1-(-1) \\ &= 2\end{aligned}$$

3.

$$\begin{aligned}\int_C z^2\,|\mathrm{d}z| &= \int_\pi^0 \,\mathrm{e}^{i2\theta}|i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta| \\ &= \int_\pi^0 -\mathrm{e}^{i2\theta}\,\mathrm{d}\theta \\ &= \left[\frac{i}{2}\,\mathrm{e}^{i2\theta}\right]_\pi^0 \\ &= \frac{i}{2}(1-1) \\ &= 0\end{aligned}$$

4.

$$\begin{aligned}\int_C |z^2|\,|\mathrm{d}z| &= \int_\pi^0 |\,\mathrm{e}^{i2\theta}||i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta| \\ &= \int_\pi^0 -\mathrm{d}\theta \\ &= [-\theta]_\pi^0 \\ &= \pi\end{aligned}$$

Solution 12.2

$$I = \int_{-\infty}^{\infty} \mathrm{e}^{-(ax^2+bx)} \, \mathrm{d}x$$

First we complete the square in the argument of the exponential.

$$I = \mathrm{e}^{b^2/(4a)} \int_{-\infty}^{\infty} \mathrm{e}^{-a(x+b/(2a))^2} \, \mathrm{d}x$$

Consider the parallelogram in the complex plane with corners at $\pm R$ and $\pm R + b/(2a)$. The integral of e^{-az^2} on this contour vanishes as it is an entire function. We can write this as

$$\int_{-R+b/(2a)}^{R+b/(2a)} \mathrm{e}^{-az^2} \, \mathrm{d}z = \left(\int_{-R+b/(2a)}^{-R} + \int_{-R}^{R} + \int_{R}^{R+b/(2a)} \right) \mathrm{e}^{-az^2} \, \mathrm{d}z.$$

The first and third integrals on the right side vanish as $R \to \infty$ because the integrand vanishes and the lengths of the paths of integration are finite. Taking the limit as $R \to \infty$ we have,

$$\int_{-\infty+b/(2a)}^{\infty+b/(2a)} \mathrm{e}^{-az^2} \, \mathrm{d}z \equiv \int_{-\infty}^{\infty} \mathrm{e}^{-a(x+b/(2a))^2} \, \mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{-ax^2} \, \mathrm{d}x.$$

Now we have

$$I = \mathrm{e}^{b^2/(4a)} \int_{-\infty}^{\infty} \mathrm{e}^{-ax^2} \, \mathrm{d}x.$$

We make the change of variables $\xi = \sqrt{a}x$.

$$I = \mathrm{e}^{b^2/(4a)} \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \mathrm{e}^{-\xi^2} \, \mathrm{d}x$$

$$\boxed{\int_{-\infty}^{\infty} \mathrm{e}^{-(ax^2+bx)} \, \mathrm{d}x = \sqrt{\frac{\pi}{a}} \, \mathrm{e}^{b^2/(4a)}}$$

Solution 12.3
Consider

$$I = 2 \int_0^\infty \mathrm{e}^{-ax^2} \cos(\omega x) \, \mathrm{d}x.$$

Since the integrand is an even function,

$$I = \int_{-\infty}^\infty \mathrm{e}^{-ax^2} \cos(\omega x) \, \mathrm{d}x.$$

Since $\mathrm{e}^{-ax^2} \sin(\omega x)$ is an odd function,

$$I = \int_{-\infty}^\infty \mathrm{e}^{-ax^2} \mathrm{e}^{i\omega x} \, \mathrm{d}x.$$

We evaluate this integral with the result of Exercise 12.2.

$$\boxed{2 \int_0^\infty \mathrm{e}^{-ax^2} \cos(\omega x) \, \mathrm{d}x = \sqrt{\frac{\pi}{a}} \, \mathrm{e}^{-\omega^2/(4a)}}$$

Consider

$$I = 2 \int_0^\infty x \, \mathrm{e}^{-ax^2} \sin(\omega x) \, \mathrm{d}x.$$

Since the integrand is an even function,

$$I = \int_{-\infty}^\infty x \, \mathrm{e}^{-ax^2} \sin(\omega x) \, \mathrm{d}x.$$

Since $x \, \mathrm{e}^{-ax^2} \cos(\omega x)$ is an odd function,

$$I = -i \int_{-\infty}^\infty x \, \mathrm{e}^{-ax^2} \mathrm{e}^{i\omega x} \, \mathrm{d}x.$$

We add a dash of integration by parts to get rid of the x factor.

$$I = -i\left[-\frac{1}{2a}\,\mathrm{e}^{-ax^2}\,\mathrm{e}^{i\omega x}\right]_{-\infty}^{\infty} + i\int_{-\infty}^{\infty}\left(-\frac{1}{2a}\,\mathrm{e}^{-ax^2}\,i\omega\,\mathrm{e}^{i\omega x}\right)\mathrm{d}x.$$

$$I = \frac{\omega}{2a}\int_{-\infty}^{\infty}\mathrm{e}^{-ax^2}\,\mathrm{e}^{i\omega x}\,\mathrm{d}x.$$

$$\boxed{2\int_{0}^{\infty} x\,\mathrm{e}^{-ax^2}\sin(\omega x)\,\mathrm{d}x. = \frac{\omega}{2a}\sqrt{\frac{\pi}{a}}\,\mathrm{e}^{-\omega^2/(4a)}}$$

Chapter 13

Cauchy's Integral Formula

If I were founding a university I would begin with a smoking room; next a dormitory; and then a decent reading room and a library. After that, if I still had more money that I couldn't use, I would hire a professor and get some text books.

- Stephen Leacock

13.1 Cauchy's Integral Formula

Result 13.1.1 Cauchy's Integral Formula. If $f(\zeta)$ is analytic in a compact, closed, connected domain D and z is a point in the interior of D then

$$f(z) = \frac{1}{i2\pi} \oint_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{i2\pi} \sum_k \oint_{C_k} \frac{f(\zeta)}{\zeta - z} \, d\zeta. \tag{13.1}$$

Here the set of contours $\{C_k\}$ make up the positively oriented boundary ∂D of the domain D. More generally, we have

$$f^{(n)}(z) = \frac{n!}{i2\pi} \oint_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta = \frac{n!}{i2\pi} \sum_k \oint_{C_k} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta. \tag{13.2}$$

Cauchy's Formula shows that the value of $f(z)$ and all its derivatives in a domain are determined by the value of $f(z)$ on the boundary of the domain. Consider the first formula of the result, Equation 13.1. We deform the contour to a circle of radius δ about the point $\zeta = z$.

$$\begin{aligned} \oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta &= \oint_{C_\delta} \frac{f(\zeta)}{\zeta - z} \, d\zeta \\ &= \oint_{C_\delta} \frac{f(z)}{\zeta - z} \, d\zeta + \oint_{C_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta \end{aligned}$$

We use the result of Example 12.5.1 to evaluate the first integral.

$$\oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta = i2\pi f(z) + \oint_{C_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta$$

The remaining integral along C_δ vanishes as $\delta \to 0$ because $f(\zeta)$ is continuous. We demonstrate this with the maximum modulus integral bound. The length of the path of integration is $2\pi\delta$.

$$\begin{aligned}
\lim_{\delta \to 0} \left| \oint_{C_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta \right| &\leq \lim_{\delta \to 0} \left((2\pi\delta) \frac{1}{\delta} \max_{|\zeta - z| = \delta} |f(\zeta) - f(z)| \right) \\
&\leq \lim_{\delta \to 0} \left(2\pi \max_{|\zeta - z| = \delta} |f(\zeta) - f(z)| \right) \\
&= 0
\end{aligned}$$

This gives us the desired result.

$$f(z) = \frac{1}{i2\pi} \oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

We derive the second formula, Equation 13.2, from the first by differentiating with respect to z. Note that the integral converges uniformly for z in any closed subset of the interior of C. Thus we can differentiate with respect to z and interchange the order of differentiation and integration.

$$\begin{aligned}
f^{(n)}(z) &= \frac{1}{i2\pi} \frac{d^n}{dz^n} \oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta \\
&= \frac{1}{i2\pi} \oint_C \frac{d^n}{dz^n} \frac{f(\zeta)}{\zeta - z} \, d\zeta \\
&= \frac{n!}{i2\pi} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta
\end{aligned}$$

Example 13.1.1 Consider the following integrals where C is the positive contour on the unit circle. For the third integral, the point $z = -1$ is removed from the contour.

1. $\displaystyle\oint_C \sin(\cos(z^5)) \, dz$

2. $\oint_C \frac{1}{(z-3)(3z-1)}\, dz$

3. $\int_C \sqrt{z}\, dz$

1. Since $\sin(\cos(z^5))$ is an analytic function inside the unit circle,

$$\oint_C \sin(\cos(z^5))\, dz = 0$$

2. $\frac{1}{(z-3)(3z-1)}$ has singularities at $z = 3$ and $z = 1/3$. Since $z = 3$ is outside the contour, only the singularity at $z = 1/3$ will contribute to the value of the integral. We will evaluate this integral using the Cauchy integral formula.

$$\oint_C \frac{1}{(z-3)(3z-1)}\, dz = i2\pi \left(\frac{1}{(1/3-3)3} \right) = -\frac{\pi i}{4}$$

3. Since the curve is not closed, we cannot apply the Cauchy integral formula. Note that $\sqrt{z}$ is single-valued and analytic in the complex plane with a branch cut on the negative real axis. Thus we use the Fundamental Theorem of Calculus.

$$\begin{aligned}\int_C \sqrt{z}\, dz &= \left[\frac{2}{3}\sqrt{z^3} \right]_{e^{-i\pi}}^{e^{i\pi}} \\ &= \frac{2}{3}\left(e^{i3\pi/2} - e^{-i3\pi/2} \right) \\ &= \frac{2}{3}(-i - i) \\ &= -i\frac{4}{3}\end{aligned}$$

Cauchy's Inequality. Suppose the $f(\zeta)$ is analytic in the closed disk $|\zeta - z| \leq r$. By Cauchy's integral formula,

$$f^{(n)}(z) = \frac{n!}{i2\pi} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta,$$

where C is the circle of radius r centered about the point z. We use this to obtain an upper bound on the modulus of $f^{(n)}(z)$.

$$\begin{aligned}
\left|f^{(n)}(z)\right| &= \frac{n!}{2\pi} \left| \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \right| \\
&\leq \frac{n!}{2\pi} 2\pi r \max_{|\zeta - z| = r} \left| \frac{f(\zeta)}{(\zeta - z)^{n+1}} \right| \\
&= \frac{n!}{r^n} \max_{|\zeta - z| = r} |f(\zeta)|
\end{aligned}$$

Result 13.1.2 Cauchy's Inequality. If $f(\zeta)$ is analytic in $|\zeta - z| \leq r$ then

$$\left|f^{(n)}(z)\right| \leq \frac{n!M}{r^n}$$

where $|f(\zeta)| \leq M$ for all $|\zeta - z| = r$.

Liouville's Theorem. Consider a function $f(z)$ that is analytic and bounded, $(f(z) \leq M)$, in the complex plane. From Cauchy's inequality,

$$|f'(z)| \leq \frac{M}{r}$$

for any positive r. By taking $r \to \infty$, we see that $f'(z)$ is identically zero for all z. Thus $f(z)$ is a constant.

Result 13.1.3 Liouville's Theorem. If $f(z)$ is analytic and bounded in the complex plane then $f(z)$ is a constant.

The Fundamental Theorem of Algebra. We will prove that every polynomial of degree $n \geq 1$ has exactly n roots, counting multiplicities. First we demonstrate that each such polynomial has at least one root. Suppose that an n^{th} degree polynomial $p(z)$ has no roots. Let the lower bound on the modulus of $p(z)$ be $0 < m \leq |p(z)|$. The function $f(z) = 1/p(z)$ is analytic, $(f'(z) = p'(z)/p^2(z))$, and bounded, $(|f(z)| \leq 1/m)$, in the extended complex plane. Using Liouville's theorem we conclude that $f(z)$ and hence $p(z)$ are constants, which yields a contradiction. Therefore every such polynomial $p(z)$ must have at least one root.

Now we show that we can factor the root out of the polynomial. Let

$$p(z) = \sum_{k=0}^{n} p_k z^k.$$

We note that

$$(z^n - c^n) = (z - c) \sum_{k=0}^{n-1} c^{n-1-k} z^k.$$

Suppose that the n^{th} degree polynomial $p(z)$ has a root at $z = c$.

$$\begin{aligned}
p(z) &= p(z) - p(c) \\
&= \sum_{k=0}^{n} p_k z^k - \sum_{k=0}^{n} p_k c^k \\
&= \sum_{k=0}^{n} p_k (z^k - c^k) \\
&= \sum_{k=0}^{n} p_k (z - c) \sum_{j=0}^{k-1} c^{k-1-j} z^j \\
&= (z - c) q(z)
\end{aligned}$$

Here $q(z)$ is a polynomial of degree $n - 1$. By induction, we see that $p(z)$ has exactly n roots.

Result 13.1.4 Fundamental Theorem of Algebra. Every polynomial of degree $n \geq 1$ has exactly n roots, counting multiplicities.

Gauss' Mean Value Theorem. Let $f(\zeta)$ be analytic in $|\zeta - z| \leq r$. By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where C is the circle $|\zeta - z| = r$. We parameterize the contour with $\zeta = z + r\,\mathrm{e}^{i\theta}$.

$$f(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + r\,\mathrm{e}^{i\theta})}{r\,\mathrm{e}^{i\theta}} ir\,\mathrm{e}^{i\theta} \, d\theta$$

Writing this in the form,

$$f(z) = \frac{1}{2\pi r} \int_0^{2\pi} f(z + r\,\mathrm{e}^{i\theta}) r \, d\theta,$$

we see that $f(z)$ is the average value of $f(\zeta)$ on the circle of radius r about the point z.

Result 13.1.5 Gauss' Average Value Theorem. If $f(\zeta)$ is analytic in $|\zeta - z| \leq r$ then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + r\,\mathrm{e}^{i\theta})\, d\theta.$$

That is, $f(z)$ is equal to its average value on a circle of radius r about the point z.

Extremum Modulus Theorem. Let $f(z)$ be analytic in closed, connected domain, D. The extreme values of the modulus of the function must occur on the boundary. If $|f(z)|$ has an interior extrema, then the function is a constant. We will show this with proof by contradiction. Assume that $|f(z)|$ has an interior maxima at the point $z = c$. This means that there exists an neighborhood of the point $z = c$ for which $|f(z)| \leq |f(c)|$. Choose an ϵ so that the set $|z - c| \leq \epsilon$ lies inside this neighborhood. First we use Gauss' mean value theorem.

$$f(c) = \frac{1}{2\pi} \int_0^{2\pi} f\left(c + \epsilon\,\mathrm{e}^{i\theta}\right)\, \mathrm{d}\theta$$

We get an upper bound on $|f(c)|$ with the maximum modulus integral bound.

$$|f(c)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(c + \epsilon\,\mathrm{e}^{i\theta}\right)\right|\, \mathrm{d}\theta$$

Since $z = c$ is a maxima of $|f(z)|$ we can get a lower bound on $|f(c)|$.

$$|f(c)| \geq \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(c + \epsilon\,\mathrm{e}^{i\theta}\right)\right|\, \mathrm{d}\theta$$

If $|f(z)| < |f(c)|$ for any point on $|z - c| = \epsilon$, then the continuity of $f(z)$ implies that $|f(z)| < |f(c)|$ in a neighborhood of that point which would make the value of the integral of $|f(z)|$ strictly less than $|f(c)|$. Thus we conclude that $|f(z)| = |f(c)|$ for all $|z - c| = \epsilon$. Since we can repeat the above procedure for any circle of radius smaller than ϵ, $|f(z)| = |f(c)|$ for all $|z - c| \leq \epsilon$, i.e. all the points in the disk of radius ϵ about $z = c$ are also maxima. By recursively repeating this procedure points in this disk, we see that $|f(z)| = |f(c)|$ for all $z \in D$. This implies that $f(z)$ is a constant in the domain. By reversing the inequalities in the above method we see that the minimum modulus of $f(z)$ must also occur on the boundary.

Result 13.1.6 Extremum Modulus Theorem. Let $f(z)$ be analytic in a closed, connected domain, D. The extreme values of the modulus of the function must occur on the boundary. If $|f(z)|$ has an interior extrema, then the function is a constant.

13.2 The Argument Theorem

Result 13.2.1 The Argument Theorem. Let $f(z)$ be analytic inside and on C except for isolated poles inside the contour. Let $f(z)$ be nonzero on C.

$$\frac{1}{i2\pi} \int_C \frac{f'(z)}{f(z)} \, dz = N - P$$

Here N is the number of zeros and P the number of poles, counting multiplicities, of $f(z)$ inside C.

First we will simplify the problem and consider a function $f(z)$ that has one zero or one pole. Let $f(z)$ be analytic and nonzero inside and on A except for a zero of order n at $z = a$. Then we can write $f(z) = (z - a)^n g(z)$

where $g(z)$ is analytic and nonzero inside and on A. The integral of $\frac{f'(z)}{f(z)}$ along A is

$$\begin{aligned}
\frac{1}{i2\pi}\int_A \frac{f'(z)}{f(z)}\,dz &= \frac{1}{i2\pi}\int_A \frac{\mathrm{d}}{\mathrm{d}z}\left(\log(f(z))\right)\,dz \\
&= \frac{1}{i2\pi}\int_A \frac{\mathrm{d}}{\mathrm{d}z}\left(\log((z-a)^n)+\log(g(z))\right)\,dz \\
&= \frac{1}{i2\pi}\int_A \frac{\mathrm{d}}{\mathrm{d}z}\left(\log((z-a)^n)\right)\,dz \\
&= \frac{1}{i2\pi}\int_A \frac{n}{z-a}\,dz \\
&= n
\end{aligned}$$

Now let $f(z)$ be analytic and nonzero inside and on B except for a pole of order p at $z=b$. Then we can write $f(z)=\frac{g(z)}{(z-b)^p}$ where $g(z)$ is analytic and nonzero inside and on B. The integral of $\frac{f'(z)}{f(z)}$ along B is

$$\begin{aligned}
\frac{1}{i2\pi}\int_B \frac{f'(z)}{f(z)}\,dz &= \frac{1}{i2\pi}\int_B \frac{\mathrm{d}}{\mathrm{d}z}\left(\log(f(z))\right)\,dz \\
&= \frac{1}{i2\pi}\int_B \frac{\mathrm{d}}{\mathrm{d}z}\big(\log((z-b)^{-p})+\log(g(z))\big)\,dz \\
&= \frac{1}{i2\pi}\int_B \frac{\mathrm{d}}{\mathrm{d}z}\big(\log((z-b)^{-p})+\big)\,dz \\
&= \frac{1}{i2\pi}\int_B \frac{-p}{z-b}\,dz \\
&= -p
\end{aligned}$$

Now consider a function $f(z)$ that is analytic inside an on the contour C except for isolated poles at the points $b_1,\ldots,b_p$. Let $f(z)$ be nonzero except at the isolated points $a_1,\ldots,a_n$. Let the contours A_k, $k=1,\ldots,n$, be simple, positive contours which contain the zero at a_k but no other poles or zeros of $f(z)$. Likewise, let the contours B_k, $k = 1, \dots, p$ be simple, positive contours which contain the pole at b_k but no other poles of zeros of $f(z)$. (See Figure 13.1.) By deforming the contour we obtain

$$\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} \int_{A_j} \frac{f'(z)}{f(z)}\,dz + \sum_{k=1}^{p} \int_{B_j} \frac{f'(z)}{f(z)}\,dz.$$

From this we obtain Result 13.2.1.

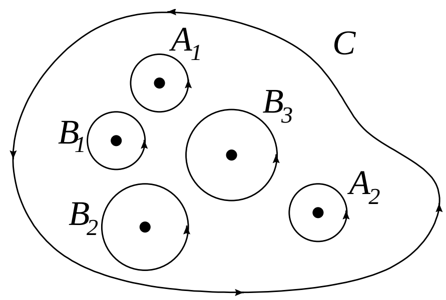


Figure 13.1: Deforming the contour C.

13.3 Rouche's Theorem

Result 13.3.1 Rouche's Theorem. Let $f(z)$ and (g) be analytic inside and on a simple, closed contour C. If $|f(z)| > |g(z)|$ on C then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C and no zeros on C.

First note that since $|f(z)| > |g(z)|$ on C, $f(z)$ is nonzero on C. The inequality implies that $|f(z) + g(z)| > 0$ on C so $f(z) + g(z)$ has no zeros on C. We well count the number of zeros of $f(z)$ and $g(z)$ using the Argument

Theorem, (Result 13.2.1). The number of zeros N of $f(z)$ inside the contour is

$$N = \frac{1}{i2\pi} \oint_C \frac{f'(z)}{f(z)} \, dz.$$

Now consider the number of zeros M of $f(z) + g(z)$. We introduce the function $h(z) = g(z)/f(z)$.

$$\begin{aligned}
M &= \frac{1}{i2\pi} \oint_C \frac{f'(z) + g'(z)}{f(z) + g(z)} \, dz \\
&= \frac{1}{i2\pi} \oint_C \frac{f'(z) + f'(z)h(z) + f(z)h'(z)}{f(z) + f(z)h(z)} \, dz \\
&= \frac{1}{i2\pi} \oint_C \frac{f'(z)}{f(z)} \, dz + \frac{1}{i2\pi} \oint_C \frac{h'(z)}{1 + h(z)} \, dz \\
&= N + \frac{1}{i2\pi} \left[\log(1 + h(z)) \right]_C \\
&= N
\end{aligned}$$

(Note that since $|h(z)| < 1$ on C, $\Re(1 + h(z)) > 0$ on C and the value of $\log(1 + h(z))$ does not not change in traversing the contour.) This demonstrates that $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C and proves the result.

13.4 Exercises

Exercise 13.1
What is

$$(\arg(\sin z))\Big|_C$$

where C is the unit circle?

Exercise 13.2
Let C be the circle of radius 2 centered about the origin and oriented in the positive direction. Evaluate the following integrals:

1. $\oint_C \frac{\sin z}{z^2+5} \, dz$
2. $\oint_C \frac{z}{z^2+1} \, dz$
3. $\oint_C \frac{z^2+1}{z} \, dz$

Exercise 13.3
Let $f(z)$ be analytic and bounded (i.e. $|f(z)| < M$) for $|z| > R$, but not necessarily analytic for $|z| \leq R$. Let the points α and β lie inside the circle $|z| = R$. Evaluate

$$\oint_C \frac{f(z)}{(z-\alpha)(z-\beta)} \mathrm{d}z$$

where C is any closed contour outside $|z| = R$, containing the circle $|z| = R$. [Hint: consider the circle at infinity] Now suppose that in addition $f(z)$ is analytic everywhere. Deduce that $f(\alpha) = f(\beta)$.

Exercise 13.4
Using Rouche's theorem show that all the roots of the equation $p(z) = z^6 - 5z^2 + 10 = 0$ lie in the annulus $1 < |z| < 2$.

Exercise 13.5
Evaluate as a function of t

$$\omega = \frac{1}{i2\pi} \oint_C \frac{\mathrm{e}^{zt}}{z^2(z^2 + a^2)} \mathrm{d}z,$$

where C is any positively oriented contour surrounding the circle $|z| = a$.

13.5 Hints

Hint 13.1
Use the argument theorem.

Hint 13.2

Hint 13.3
To evaluate the integral, consider the circle at infinity.

Hint 13.4

Hint 13.5

13.6 Solutions

Solution 13.1

Let $f(z)$ be analytic inside and on the contour C. Let $f(z)$ be nonzero on the contour. The argument theorem states that

$$\frac{1}{i2\pi}\int_C \frac{f'(z)}{f(z)}\,dz = N - P,$$

where N is the number of zeros and P is the number of poles, (counting multiplicities), of $f(z)$ inside C. The theorem is aptly named, as

$$\begin{aligned}\frac{1}{i2\pi}\int_C \frac{f'(z)}{f(z)}\,dz &= \frac{1}{i2\pi}\left[\log(f(z))\right]_C \\ &= \frac{1}{i2\pi}\left[\log|f(z)| + i\arg(f(z))\right]_C \\ &= \frac{1}{2\pi}\left[\arg(f(z))\right]_C.\end{aligned}$$

Thus we could write the argument theorem as

$$\frac{1}{i2\pi}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi}\left[\arg(f(z))\right]_C = N - P.$$

Since $\sin z$ has a single zero and no poles inside the unit circle, we have

$$\frac{1}{2\pi}\arg(\sin(z))\Big|_C = 1 - 0$$

$$\arg(\sin(z))\Big|_C = 2\pi$$

Solution 13.2

1. Since the integrand $\frac{\sin z}{z^2+5}$ is analytic inside and on the contour, (the only singularities are at $z = \pm i\sqrt{5}$ and at infinity), the integral is zero by Cauchy's Theorem.

2. First we expand the integrand in partial fractions.

$$\frac{z}{z^2+1} = \frac{a}{z-i} + \frac{b}{z+i}$$

$$a = \left.\frac{z}{z+i}\right|_{z=i} = \frac{1}{2}, \qquad b = \left.\frac{z}{z-i}\right|_{z=-i} = \frac{1}{2}$$

   Now we can do the integral with Cauchy's formula.

$$\begin{aligned}\int_C \frac{z}{z^2+1}\, dz &= \int_C \frac{1/2}{z-i}\, dz + \int_C \frac{1/2}{z+i}\, dz \\ &= \frac{1}{2} i2\pi + \frac{1}{2} i2\pi \\ &= i2\pi\end{aligned}$$

3.

$$\begin{aligned}\int_C \frac{z^2+1}{z}\, dz &= \int_C \left(z + \frac{1}{z}\right)\, dz \\ &= \int_C z\, dz + \int_C \frac{1}{z}\, dz \\ &= 0 + i2\pi \\ &= i2\pi\end{aligned}$$

Solution 13.3

Let C be the circle of radius r, $(r > R)$, centered at the origin. We get an upper bound on the integral with the Maximum Modulus Integral Bound, (Result 12.1.1).

$$\left| \oint_C \frac{f(z)}{(z-\alpha)(z-\beta)} \mathrm{d}z \right| \leq 2\pi r \max_{|z|=r} \left| \frac{f(z)}{(z-\alpha)(z-\beta)} \right| \leq 2\pi r \frac{M}{(r-|\alpha|)(r-|\beta|)}$$

By taking the limit as $r \to \infty$ we see that the modulus of the integral is bounded above by zero. Thus the integral vanishes.

Now we assume that $f(z)$ is analytic and evaluate the integral with Cauchy's Integral Formula. (We assume that $\alpha \neq \beta$.)

$$\oint_C \frac{f(z)}{(z-\alpha)(z-\beta)} \mathrm{d}z = 0$$
$$\oint_C \frac{f(z)}{(z-\alpha)(\alpha-\beta)} \mathrm{d}z + \oint_C \frac{f(z)}{(\beta-\alpha)(z-\beta)} \mathrm{d}z = 0$$
$$i2\pi \frac{f(\alpha)}{\alpha-\beta} + i2\pi \frac{f(\beta)}{\beta-\alpha} = 0$$
$$f(\alpha) = f(\beta)$$

Solution 13.4

Consider the circle $|z| = 2$. On this circle:

$$|z^6| = 64$$
$$|-5z^2 + 10| \leq |-5z^2| + |10| = 30$$

Since $|z^6| < |-5z^2 + 10|$ on $|z| = 2$, $p(z)$ has the same number of roots as z^6 in $|z| < 2$. $p(z)$ has 6 roots in $|z| < 2$.

Consider the circle $|z| = 1$. On this circle:

$$|10| = 10$$
$$|z^6 - 5z^2| \leq |z^6| + |-5z^2| = 6$$

Since $|z^6 - 5z^2| < |10|$ on $|z| = 1$, $p(z)$ has the same number of roots as 10 in $|z| < 1$. $p(z)$ has no roots in $|z| < 1$.

On the unit circle,

$$|p(z)| \geq |10| - |z^6| - |5z^2| = 4.$$

Thus $p(z)$ has no roots on the unit circle.

We conclude that $p(z)$ has exactly 6 roots in $1 < |z| < 2$.

Solution 13.5

We evaluate the integral with Cauchy's Integral Formula.

$$\omega = \frac{1}{2\pi i} \oint_C \frac{\mathrm{e}^{zt}}{z^2(z^2 + a^2)} \mathrm{d}z$$

$$\omega = \frac{1}{2\pi i} \oint_C \left(\frac{\mathrm{e}^{zt}}{a^2 z^2} + \frac{i\,\mathrm{e}^{zt}}{2a^3(z - ia)} - \frac{i\,\mathrm{e}^{zt}}{2a^3(z + ia)} \right) \mathrm{d}z$$

$$\omega = \left[\frac{\mathrm{d}}{\mathrm{d}z} \frac{\mathrm{e}^{zt}}{a^2} \right]_{z=0} + \frac{i\,\mathrm{e}^{iat}}{2a^3} - \frac{i\,\mathrm{e}^{-iat}}{2a^3}$$

$$\omega = \frac{t}{a^2} - \frac{\sin(at)}{a^3}$$

$$\boxed{\omega = \frac{at - \sin(at)}{a^3}}$$

Chapter 14

Series and Convergence

You are not thinking. You are merely being logical.

- Neils Bohr

14.1 Series of Constants

14.1.1 Definitions

Convergence of Sequences. The infinite sequence $\{a_n\}_{n=0}^{\infty} \equiv a_0, a_1, a_2, \ldots$ is said to converge if

$$\lim_{n \to \infty} a_n = a$$

for some constant a. If the limit does not exist, then the sequence diverges. Recall the definition of the limit in the above formula: For any $\epsilon > 0$ there exists an $N \in \mathbb{Z}$ such that $|a - a_n| < \epsilon$ for all $n > N$.

Example 14.1.1 The sequence $\{\sin(n)\}$ is divergent. The sequence is bounded above and below, but boundedness does not imply convergence.

Cauchy Convergence Criterion. Note that there is something a little fishy about the above definition. We should be able to say if a sequence converges without first finding the constant to which it converges. We fix this problem with the *Cauchy convergence criterion*. A sequence $\{a_n\}$ converges if and only if for any $\epsilon > 0$ there exists an N such that $|a_n - a_m| < \epsilon$ for all $n, m > N$. The Cauchy convergence criterion is equivalent to the definition we had before. For some problems it is handier to use. Now we don't need to know the limit of a sequence to show that it converges.

Convergence of Series. The series $\sum_{n=1}^{\infty} a_n$ converges if the sequence of *partial sums*, $S_N = \sum_{n=0}^{N-1} a_n$, converges. That is,

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=0}^{N-1} a_n = \text{constant}.$$

If the limit does not exist, then the series diverges. A necessary condition for the convergence of a series is that

$$\lim_{n\to\infty} a_n = 0.$$

Otherwise the sequence of partial sums would not converge.

Example 14.1.2 The series $\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + \cdots$ is divergent because the sequence of partial sums, $\{S_N\} = 1, 0, 1, 0, 1, 0, \ldots$ is divergent.

Tail of a Series. An infinite series, $\sum_{n=0}^{\infty} a_n$, converges or diverges with its tail. That is, for fixed N, $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges. This is because the sum of the first N terms of a series is just a number. Adding or subtracting a number to a series does not change its convergence.

Absolute Convergence. The series $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges. Absolute convergence implies convergence. If a series is convergent, but not absolutely convergent, then it is said to be *conditionally convergent*.

The terms of an absolutely convergent series can be rearranged in any order and the series will still converge to the same sum. This is not true of conditionally convergent series. Rearranging the terms of a conditionally convergent series may change the sum. In fact, the terms of a conditionally convergent series may be rearranged to obtain any desired sum.

Example 14.1.3 The alternating harmonic series,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

converges, (Exercise 14.2). Since

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges, (Exercise 14.3), the alternating harmonic series is not absolutely convergent. Thus the terms can be rearranged to obtain any sum, (Exercise 14.4).

Finite Series and Residuals. Consider the series $f(z) = \sum_{n=0}^{\infty} a_n(z)$. We will denote the sum of the first N terms in the series as

$$S_N(z) = \sum_{n=0}^{N-1} a_n(z).$$

We will denote the *residual* after N terms as

$$R_N(z) \equiv f(z) - S_N(z) = \sum_{n=N}^{\infty} a_n(z).$$

14.1.2 Special Series

Geometric Series. One of the most important series in mathematics is the *geometric series*, [1]

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots .$$

The series clearly diverges for $|z| \geq 1$ since the terms do not vanish as $n \to \infty$. Consider the partial sum, $S_N(z) \equiv \sum_{n=0}^{N-1} z^n$, for $|z| < 1$.

$$\begin{aligned}
(1 - z)S_N(z) &= (1 - z)\sum_{n=0}^{N-1} z^n \\
&= \sum_{n=0}^{N-1} z^n - \sum_{n=1}^{N} z^n \\
&= \left(1 + z + \cdots + z^{N-1}\right) - \left(z + z^2 + \cdots + z^N\right) \\
&= 1 - z^N
\end{aligned}$$

$$\sum_{n=0}^{N-1} z^n = \frac{1 - z^N}{1 - z} \to \frac{1}{1 - z} \quad \text{as } N \to \infty.$$

The limit of the partial sums is $\frac{1}{1-z}$.

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z} \quad \text{for } |z| < 1$$

[1] The series is so named because the terms grow or decay geometrically. Each term in the series is a constant times the previous term.

Harmonic Series. Another important series is the *harmonic series*,

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots .$$

The series is absolutely convergent for $\Re(\alpha) > 1$ and absolutely divergent for $\Re(\alpha) \leq 1$, (see the Exercise 14.6). The *Riemann zeta function* $\zeta(\alpha)$ is defined as the sum of the harmonic series.

$$\zeta(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

The *alternating harmonic series* is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^\alpha} = 1 - \frac{1}{2^\alpha} + \frac{1}{3^\alpha} - \frac{1}{4^\alpha} + \cdots .$$

Again, the series is absolutely convergent for $\Re(\alpha) > 1$ and absolutely divergent for $\Re(\alpha) \leq 1$.

14.1.3 Convergence Tests

The Comparison Test. The series of positive terms $\sum_{n=0}^{\infty} a_n$ converges if there exists a convergent series $\sum_{n=0}^{\infty} b_n$ such that $a_n \leq b_n$ for all n. Similarly, $\sum_{n=0}^{\infty} a_n$ diverges if there exists a divergent series $\sum_{n=0}^{\infty} b_n$ such that $a_n \geq b_n$ for all n.

Example 14.1.4 Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n^2}}.$$

We can rewrite this as

$$\sum_{\substack{n=1 \\ n \text{ a perfect square}}}^{\infty} \frac{1}{2^n}.$$

Then by comparing this series to the geometric series,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

we see that it is convergent.

Integral Test. If the coefficients a_n of a series $\sum_{n=0}^{\infty} a_n$ are monotonically decreasing and can be extended to a monotonically decreasing function of the continuous variable x,

$$a(x) = a_n \quad \text{for } x \in \mathbb{Z}^{0+},$$

then the series converges or diverges with the integral

$$\int_0^{\infty} a(x)\,\mathrm{d}x.$$

Example 14.1.5 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Define the functions $s_l(x)$ and $s_r(x)$, (left and right),

$$s_l(x) = \frac{1}{(\lceil x \rceil)^2}, \qquad s_r(x) = \frac{1}{(\lfloor x \rfloor)^2}.$$

Recall that $\lfloor x \rfloor$ is the greatest integer function, the greatest integer which is less than or equal to x. $\lceil x \rceil$ is the least integer function, the least integer greater than or equal to x. We can express the series as integrals of these functions.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \int_0^{\infty} s_l(x)\,dx = \int_1^{\infty} s_r(x)\,dx$$

In Figure 14.1 these functions are plotted against $y = 1/x^2$. From the graph, it is clear that we can obtain a lower and upper bound for the series.

$$\int_1^\infty \frac{1}{x^2}\,dx \leq \sum_{n=1}^\infty \frac{1}{n^2} \leq 1 + \int_1^\infty \frac{1}{x^2}\,dx$$

$$\boxed{1 \leq \sum_{n=1}^\infty \frac{1}{n^2} \leq 2}$$

Figure 14.1: Upper and Lower bounds to $\sum_{n=1}^\infty 1/n^2$.

In general, we have

$$\int_m^\infty a(x)\,dx \leq \sum_{n=m}^\infty a_n \leq a_m + \int_m^\infty a(x)\,dx.$$

Thus we see that the sum converges or diverges with the integral.

The Ratio Test. The series $\sum_{n=0}^\infty a_n$ converges absolutely if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

If the limit is greater than unity, then the series diverges. If the limit is unity, the test fails.

If the limit is greater than unity, then the terms are eventually increasing with n. Since the terms do not vanish, the sum is divergent. If the limit is less than unity, then there exists some N such that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq r < 1 \quad \text{for all } n \geq N.$$

From this we can show that $\sum_{n=0}^{\infty} a_n$ is absolutely convergent by comparing it to the geometric series.

$$\begin{aligned}\sum_{n=N}^{\infty} |a_n| &\leq |a_N| \sum_{n=0}^{\infty} r^n \\ &= |a_N| \frac{1}{1-r}\end{aligned}$$

Example 14.1.6 Consider the series,

$$\sum_{n=1}^{\infty} \frac{\mathrm{e}^n}{n!}.$$

We apply the ratio test to test for absolute convergence.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{\mathrm{e}^{n+1} n!}{\mathrm{e}^n (n+1)!} \\ &= \lim_{n\to\infty} \frac{e}{n+1} \\ &= 0\end{aligned}$$

The series is absolutely convergent.

Example 14.1.7 Consider the series,

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which we know to be absolutely convergent. We apply the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} \\
&= \lim_{n\to\infty} \frac{n^2}{n^2 + 2n + 1} \\
&= \lim_{n\to\infty} \frac{1}{1 + 2/n + 1/n^2} \\
&= 1
\end{aligned}$$

The test fails to predict the absolute convergence of the series.

The Root Test. The series $\sum_{n=0}^{\infty} a_n$ converges absolutely if

$$\lim_{n\to\infty} |a_n|^{1/n} < 1.$$

If the limit is greater than unity, then the series diverges. If the limit is unity, the test fails.

If the limit is greater than unity, then the terms in the series do not vanish as $n \to \infty$. This implies that the sum does not converge. If the limit is less than unity, then there exists some N such that

$$|a_n|^{1/n} \leq r < 1 \quad \text{for all } n \geq N.$$

We bound the tail of the series of $|a_n|$.

$$\begin{aligned}\sum_{n=N}^{\infty} |a_n| &= \sum_{n=N}^{\infty} (|a_n|^{1/n})^n \\ &\leq \sum_{n=N}^{\infty} r^n \\ &= \frac{r^N}{1-r}\end{aligned}$$

$\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

Example 14.1.8 Consider the series

$$\sum_{n=0}^{\infty} n^a b^n,$$

where a and b are real constants. We use the root test to check for absolute convergence.

$$\begin{gathered}\lim_{n\to\infty} |n^a b^n|^{1/n} < 1 \\ |b| \lim_{n\to\infty} n^{a/n} < 1 \\ |b| \exp\left(\lim_{n\to\infty} \frac{1 \log n}{n}\right) < 1 \\ |b| \, \mathrm{e}^0 < 1 \\ |b| < 1\end{gathered}$$

Thus we see that the series converges absolutely for $|b| < 1$. Note that the value of a does not affect the absolute convergence.

Example 14.1.9 Consider the absolutely convergent series,

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We aply the root test.

$$\begin{aligned}
\lim_{n\to\infty} |a_n|^{1/n} &= \lim_{n\to\infty} \left|\frac{1}{n^2}\right|^{1/n} \\
&= \lim_{n\to\infty} n^{-2/n} \\
&= \lim_{n\to\infty} \mathrm{e}^{-\frac{2}{n}\log n} \\
&= \mathrm{e}^0 \\
&= 1
\end{aligned}$$

It fails to predict the convergence of the series.

14.2 Uniform Convergence

Continuous Functions. A function $f(z)$ is continuous in a closed domain if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(\zeta)| < \epsilon$ for all $|z - \zeta| < \delta$ in the domain.

An equivalent definition is that $f(z)$ is continuous in a closed domain if

$$\lim_{\zeta\to z} f(\zeta) = f(z)$$

for all z in the domain.

Convergence. Consider a series in which the terms are functions of z, $\sum_{n=0}^{\infty} a_n(z)$. The series is convergent in a domain if the series converges for each point z in the domain. We can then define the function $f(z) = \sum_{n=0}^{\infty} a_n(z)$. We can state the convergence criterion as: For any given $\epsilon > 0$ there exists a function $N(z)$ such that

$$|f(z) - S_{N(z)}(z)| = \left| f(z) - \sum_{n=0}^{N(z)-1} a_n(z) \right| < \epsilon$$

for all z in the domain. Note that the rate of convergence, i.e. the number of terms, $N(z)$ required for for the absolute error to be less than ϵ, is a function of z.

Uniform Convergence. Consider a series $\sum_{n=0}^{\infty} a_n(z)$ that is convergent in some domain. If the rate of convergence is independent of z then the series is said to be uniformly convergent. Stating this a little more mathematically, the series is uniformly convergent in the domain if for any given $\epsilon > 0$ there exists an N, independent of z, such that

$$|f(z) - S_N(z)| = \left| f(z) - \sum_{n=1}^{N} a_n(z) \right| < \epsilon$$

for all z in the domain.

14.2.1 Tests for Uniform Convergence

Weierstrass M-test. The Weierstrass M-test is useful in determining if a series is uniformly convergent. The series $\sum_{n=0}^{\infty} a_n(z)$ is uniformly and absolutely convergent in a domain if there exists a convergent series of positive terms $\sum_{n=0}^{\infty} M_n$ such that $|a_n(z)| \leq M_n$ for all z in the domain. This condition first implies that the series is absolutely convergent for all z in the domain. The condition $|a_n(z)| \leq M_n$ also ensures that the rate of convergence is independent of z, which is the criterion for uniform convergence.

Note that absolute convergence and uniform convergence are independent. A series of functions may be absolutely convergent without being uniformly convergent or vice versa. The Weierstrass M-test is a sufficient but not a necessary condition for uniform convergence. The Weierstrass M-test can succeed only if the series is uniformly and absolutely convergent.

Example 14.2.1 The series

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin x}{n(n+1)}$$

is uniformly and absolutely convergent for all real x because $|\frac{\sin x}{n(n+1)}| < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Dirichlet Test. Consider a sequence of monotone decreasing, positive constants c_n with limit zero. If all the partial sums of $a_n(z)$ are bounded in some closed domain, that is

$$\left| \sum_{n=1}^{N} a_n(z) \right| < \text{constant}$$

for all N, then $\sum_{n=1}^{\infty} c_n a_n(z)$ is uniformly convergent in that closed domain. Note that the Dirichlet test does not imply that the series is absolutely convergent.

Example 14.2.2 Consider the series,

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}.$$

We cannot use the Weierstrass M-test to determine if the series is uniformly convergent on an interval. While it is easy to bound the terms with $|\sin(nx)/n| \leq 1/n$, the sum

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge. Thus we will try the Dirichlet test. Consider the sum $\sum_{n=1}^{N-1} \sin(nx)$. This sum can be evaluated in closed form. (See Exercise 14.7.)

$$\sum_{n=1}^{N-1} \sin(nx) = \begin{cases} 0 & \text{for } x = 2\pi k \\ \frac{\cos(x/2)-\cos((N-1/2)x)}{2\sin(x/2)} & \text{for } x \neq 2\pi k \end{cases}$$

The partial sums have infinite discontinuities at $x = 2\pi k$, $k \in \mathbb{Z}$. The partial sums are bounded on any closed interval that does not contain an integer multiple of 2π. By the Dirichlet test, the sum $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ is uniformly convergent on any such closed interval. The series may not be uniformly convergent in neighborhoods of $x = 2k\pi$.

14.2.2 Uniform Convergence and Continuous Functions.

Consider a series $f(z) = \sum_{n=1}^{\infty} a_n(z)$ that is uniformly convergent in some domain and whose terms $a_n(z)$ are continuous functions. Since the series is uniformly convergent, for any given $\epsilon > 0$ there exists an N such that $|R_N| < \epsilon$ for all z in the domain.

Since the finite sum S_N is continuous, for that ϵ there exists a $\delta > 0$ such that $|S_N(z) - S_N(\zeta)| < \epsilon$ for all ζ in the domain satisfying $|z - \zeta| < \delta$.

Combining these two results,

$$\begin{aligned} |f(z) - f(\zeta)| &= |S_N(z) + R_N(z) - S_N(\zeta) - R_N(\zeta)| \\ &\leq |S_N(z) - S_N(\zeta)| + |R_N(z)| + |R_N(\zeta)| \\ &< 3\epsilon \quad \text{for } |z - \zeta| < \delta. \end{aligned}$$

Thus $f(z)$ is continuous.

Result 14.2.1 A uniformly convergent series of continuous terms represents a continuous function.

Example 14.2.3 Again consider $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$. In Example 14.2.2 we showed that the convergence is uniform in any closed interval that does not contain an integer multiple of 2π. In Figure 14.2 is a plot of the first 10 and then 50 terms in the series and finally the function to which the series converges. We see that the function has jump discontinuities at $x = 2k\pi$ and is continuous on any closed interval not containing one of those points.

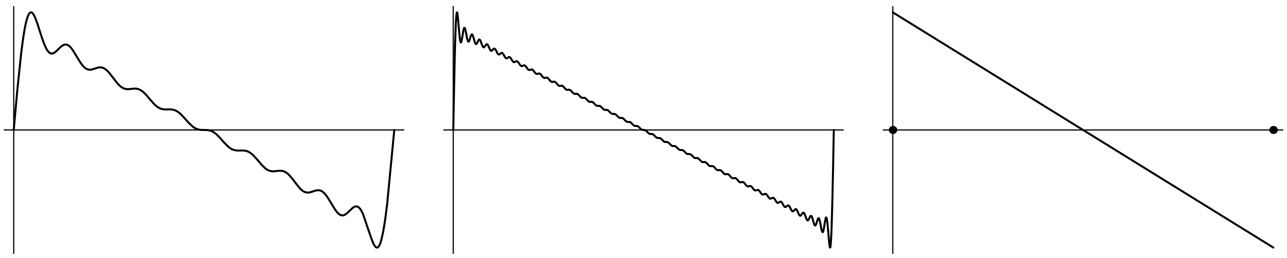


Figure 14.2: Ten, Fifty and all the Terms of $\sum_{n=1}^{\infty} \frac{sin(nx)}{n}$.

14.3 Uniformly Convergent Power Series

Power Series. Power series are series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Domain of Convergence of a Power Series Consider the series $\sum_{n=0}^{\infty} a_n z^n$. Let the series converge at some point z_0. Then $|a_n z_0^n|$ is bounded by some constant A for all n, so

$$|a_n z^n| = |a_n z_0^n| \left| \frac{z}{z_0} \right|^n < A \left| \frac{z}{z_0} \right|^n$$

This comparison test shows that the series converges absolutely for all z satisfying $|z| < |z_0|$.

Suppose that the series diverges at some point z_1. Then the series could not converge for any $|z| > |z_1|$ since this would imply convergence at z_1. Thus there exists some circle in the z plane such that the power series converges absolutely inside the circle and diverges outside the circle.

Result 14.3.1 The domain of convergence of a power series is a circle in the complex plane.

Radius of Convergence of Power Series. Consider a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Applying the ratio test, we see that the series converges if

$$\lim_{n\to\infty} \frac{|a_{n+1} z^{n+1}|}{|a_n z^n|} < l$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} |z| < 1$$

$$|z| < \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

Result 14.3.2 The radius of convergence of the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is

$$R = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|}$$

when the limit exists.

Result 14.3.3 Cauchy-Hadamard formula. The radius of convergence of the power series:

$$\sum_{n=0}^{\infty} a_n z^n$$

is

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}.$$

Absolute Convergence of Power Series. Consider a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

that converges for $z = z_0$. Let M be the value of the greatest term, $a_n z_0^n$. Consider any point z such that $|z| < |z_0|$. We can bound the residual of $\sum_{n=0}^{\infty} |a_n z^n|$,

$$\begin{aligned} R_N(z) &= \sum_{n=N}^{\infty} |a_n z^n| \\ &= \sum_{n=N}^{\infty} \left| \frac{a_n z^n}{a_n z_0^n} \right| |a_n z_0^n| \\ &\leq M \sum_{n=N}^{\infty} \left| \frac{z}{z_0} \right|^n \end{aligned}$$

Since $|z/z_0| < 1$, this is a convergent geometric series.

$$\begin{aligned} &= M \left| \frac{z}{z_0} \right|^N \frac{1}{1 - |z/z_0|} \\ &\to 0 \quad \text{as } N \to \infty \end{aligned}$$

Thus the power series is absolutely convergent for $|z| < |z_0|$.

Result 14.3.4 If the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for $z = z_0$, then the series converges absolutely for $|z| < |z_0|$.

Example 14.3.1 Find the radii of convergence of

$$1) \quad \sum_{n=1}^{\infty} n z^n, \qquad 2) \quad \sum_{n=1}^{\infty} n! z^n, \qquad 3) \quad \sum_{n=1}^{\infty} n! z^{n!}$$

1. Applying the formula for the radius of convergence,

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{n}{n+1} = 1$$

2. Applying the ratio test to the second series,

$$
\begin{aligned}
R &= \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right| \\
&= \lim_{n\to\infty} \frac{1}{n+1} \\
&= 0
\end{aligned}
$$

Thus we see that the second series has a vanishing radius of convergence.

3. The third series converges when

$$
\begin{gathered}
\lim_{n\to\infty} \left| \frac{(n+1)! z^{(n+1)!}}{n! z^{n!}} \right| < 1 \\
\lim_{n\to\infty} (n+1)|z|^{(n+1)!-n!} < 1 \\
\lim_{n\to\infty} (n+1)|z|^{(n)n!} < 1 \\
\lim_{n\to\infty} \left(\log(n+1) + (n)n! \log|z| \right) < 0 \\
\log|z| < \lim_{n\to\infty} \frac{-\log(n+1)}{(n)n!} \\
\log|z| < 0 \\
|z| < 1
\end{gathered}
$$

Thus the radius of convergence for the third series is 1.

Alternatively we could determine the radius of convergence of the third series with the comparison test. We know that

$$
\sum_{n=1}^{\infty} \left| n! z^{n!} \right| \le \sum_{n=1}^{\infty} |n z^n|
$$

$\sum_{n=1}^\infty nz^n$ has a radius of convergence of 1. Thus the third sum must have a radius of convergence of at least 1. Note that if $|z| > 1$ then the terms in the third series do not vanish as $n \to \infty$. Thus the series must diverge for all $|z| > 1$. We see that the radius of convergence is 1.

Uniform Convergence of Power Series. Consider a power series $\sum_{n=0}^\infty a_n z^n$ that converges in the disk $|z| < r_0$. The sum converges absolutely for z in the closed disk, $|z| \leq r < r_0$. Since $|a_n z^n| \leq |a_n r^n|$ and $\sum_{n=0}^\infty |a_n r^n|$ converges, the power series is uniformly convergent in $|z| \leq r < r_0$.

Result 14.3.5 If the power series $\sum_{n=0}^\infty a_n z^n$ converges for $|z| < r_0$ then the series converges uniformly for $|z| \leq r < r_0$.

Example 14.3.2 Convergence and Uniform Convergence. Consider the series

$$\log(1-z) = -\sum_{n=1}^\infty \frac{z^n}{n}.$$

This series converges for $|z| \leq 1, z \neq 1$. Is the series uniformly convergent in this domain? The residual after N terms R_N is

$$R_N(z) = \sum_{n=N+1}^\infty \frac{z^n}{n}.$$

We can get a lower bound on the absolute value of the residual for real, positive z.

$$\begin{aligned}
|R_N(x)| &= \sum_{n=N+1}^\infty \frac{x^n}{n} \\
&\leq \int_{N+1}^\infty \frac{x^\alpha}{\alpha}\, d\alpha \\
&= -\operatorname{Ei}\left((N+1)\log x\right)
\end{aligned}$$

The exponential integral function, $\mathrm{Ei}(z)$, is defined

$$\mathrm{Ei}(z) = -\int_{-z}^{\infty} \frac{\mathrm{e}^{-t}}{t}\, dt.$$

The exponential integral function is plotted in Figure 14.3. Since $\mathrm{Ei}(z)$ diverges as $z \to 0$, by choosing x sufficiently close to 1 the residual can be made arbitrarily large. Thus this series is not uniformly convergent in the domain $|z| \leq 1, z \neq 1$. The series is uniformly convergent for $|z| \leq r < 1$.

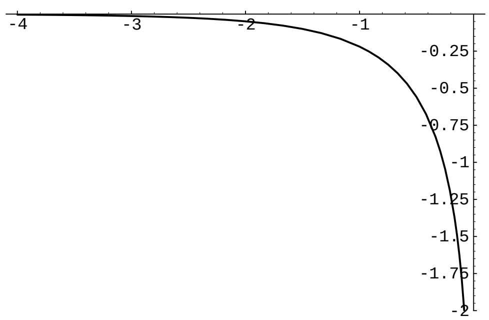


Figure 14.3: The Exponential Integral Function.

Analyticity. Recall that a sufficient condition for the analyticity of a function $f(z)$ in a domain is that $\oint_C f(z)\, dz = 0$ for all simple, closed contours in the domain.

Consider a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that is uniformly convergent in $|z| \leq r$. If C is any simple, closed contour in the domain then $\oint_C f(z)\, dz$ exists. Expanding $f(z)$ into a finite series and a residual,

$$\oint_C f(z)\, dz = \oint_C \left[S_N(z) + R_N(z)\right]\, dz.$$

Since the series is uniformly convergent, for any given $\epsilon > 0$ there exists an N_ϵ such that $|R_{N_\epsilon}| < \epsilon$ for all z in $|z| \leq r$. If L is the length of the contour C then

$$\left| \oint_C R_{N_\epsilon}(z)\, dz \right| \leq L\epsilon \to 0 \quad \text{as } N_\epsilon \to \infty.$$

$$\begin{aligned}
\oint_C f(z)\, dz &= \lim_{N\to\infty} \oint_C \left(\sum_{n=0}^{N-1} a_n z^n + R_N(z) \right) dz \\
&= \oint_C \sum_{n=0}^{\infty} a_n z^n \\
&= \sum_{n=0}^{\infty} a_n \oint_C z^n\, dz \\
&= 0.
\end{aligned}$$

Thus $f(z)$ is analytic for $|z| < r$.

Result 14.3.6 A power series is analytic in its domain of uniform convergence.

14.4 Integration and Differentiation of Power Series

Consider a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that is convergent in the disk $|z| < r_0$. Let C be any contour of finite length L lying entirely within the closed domain $|z| \leq r < r_0$. The integral of $f(z)$ along C is

$$\int_C f(z)\, dz = \int_C \left[S_N(z) + R_N(z)\right] dz.$$

Since the series is uniformly convergent in the closed disk, for any given $\epsilon > 0$, there exists an N_ϵ such that

$$|R_{N_\epsilon}(z)| < \epsilon \quad \text{for all } |z| \leq r.$$

Bounding the absolute value of the integral of $R_{N_\epsilon}(z)$,

$$\begin{aligned} \left| \int_C R_{N_\epsilon}(z)\, dz \right| &\leq \int_C |R_{N_\epsilon}(z)|\, dz \\ &< \epsilon L \\ &\to 0 \quad \text{as } N_\epsilon \to \infty \end{aligned}$$

Thus

$$\begin{aligned} \int_C f(z)\, dz &= \lim_{N\to\infty} \int_C \sum_{n=0}^{N} a_n z^n\, dz \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} a_n \int_C z^n\, dz \\ &= \sum_{n=0}^{\infty} a_n \int_C z^n\, dz \end{aligned}$$

Result 14.4.1 If C is a contour lying in the domain of uniform convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ then

$$\int_C \sum_{n=0}^{\infty} a_n z^n\, dz = \sum_{n=0}^{\infty} a_n \int_C z^n\, dz.$$

In the domain of uniform convergence of a series we can interchange the order of summation and a limit process. That is,

$$\lim_{z \to z_0} \sum_{n=0}^{\infty} a_n(z) = \sum_{n=0}^{\infty} \lim_{z \to z_0} a_n(z).$$

We can do this because the rate of convergence does not depend on z. Since differentiation is a limit process,

$$\frac{\mathrm{d}}{\mathrm{d}z} f(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$

we would expect that we could differentiate a uniformly convergent series.

Since we showed that a uniformly convergent power series is equal to an analytic function, we can differentiate a power series in it's domain of uniform convergence.

Result 14.4.2 In the domain of uniform convergence of a power series

$$\frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n.$$

Example 14.4.1 Differentiating a Series. Consider the series from Example 14.3.2

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Differentiating this series yields

$$-\frac{1}{1-z} = -\sum_{n=1}^{\infty} z^{n-1}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

We recognize this as the geometric series, which is convergent for $|z| < 1$ and uniformly convergent for $|z| \leq r < 1$. Note that the domain of convergence is different than the series for $\log(1 - z)$. The geometric series does not converge for $|z| = 1, z \neq 1$. However, the domain of uniform convergence has remained the same.

14.5 Taylor Series

Result 14.5.1 Taylor's Theorem. Let $f(z)$ be a function that is single-valued and analytic in $|z - z_0| < R$. For all z in this open disk, $f(z)$ has the convergent Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n. \tag{14.1}$$

We can also write this as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{i2\pi} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz, \tag{14.2}$$

where C is a simple, positive, closed contour in $0 < |z - z_0| < R$ that goes once around the point z_0.

Proof of Taylor's Theorem. Let's see why Result 14.5.1 is true. Consider a function $f(z)$ that is analytic in $|z| < R$. (Considering $z_0 \neq 0$ is only trivially more general as we can introduce the change of variables $\zeta = z - z_0$.) According to Cauchy's Integral Formula, (Result **??**),

$$f(z) = \frac{1}{i2\pi} \oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta, \tag{14.3}$$

where C is a positive, simple, closed contour in $0 < |\zeta - z| < R$ that goes once around z. We take this contour to be the circle about the origin of radius r where $|z| < r < R$. (See Figure 14.4.)

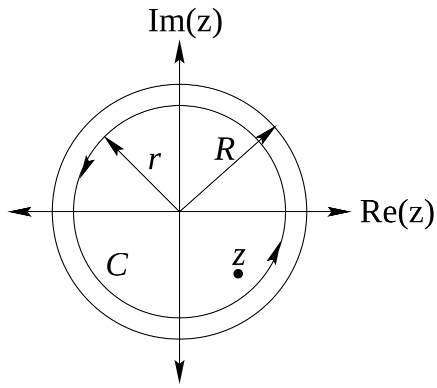


Figure 14.4: Graph of Domain of Convergence and Contour of Integration.

We expand $\frac{1}{\zeta - z}$ in a geometric series,

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1/\zeta}{1 - z/\zeta} \\ &= \frac{1}{\zeta} \sum_{n=0}^{\infty} \left(\frac{z}{\zeta} \right)^n, \quad \text{for } |z| < |\zeta| \\ &= \sum_{n=0}^{\infty} \frac{z^n}{\zeta^{n+1}}, \quad \text{for } |z| < |\zeta| \end{aligned}$$

We substitute this series into Equation 14.3.

$$f(z) = \frac{1}{i2\pi} \oint_C \left(\sum_{n=0}^{\infty} \frac{f(\zeta) z^n}{\zeta^{n+1}} \right) d\zeta$$

The series converges uniformly so we can interchange integration and summation.

$$= \sum_{n=0}^{\infty} \frac{1}{i2\pi} \oint_C \frac{f(\zeta)}{\zeta^{n+1}} \, d\zeta \, z^n$$

Now we have derived Equation 14.2. To obtain Equation 14.1, we apply Cauchy's Integral Formula.

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

There is a table of some commonly encountered Taylor series in Appendix H.

Example 14.5.1 Consider the Taylor series expansion of $1/(1-z)$ about $z = 0$. Previously, we showed that this function is the sum of the geometric series $\sum_{n=0}^{\infty} z^n$ and we used the ratio test to show that the series converged absolutely for $|z| < 1$. Now we find the series using Taylor's theorem. Since the nearest singularity of the function is at $z = 1$, the radius of convergence of the series is 1. The coefficients in the series are

$$\begin{aligned} a_n &= \frac{1}{n!} \left[\frac{d^n}{dz^n} \frac{1}{1-z} \right]_{z=0} \\ &= \frac{1}{n!} \left[\frac{n!}{(1-z)^n} \right]_{z=0} \\ &= 1 \end{aligned}$$

Thus we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad \text{for } |z| < 1.$$

14.5.1 Newton's Binomial Formula.

Result 14.5.2 For all $|z| < 1$, a complex:

$$(1+z)^a = 1 + \binom{a}{1} z + \binom{a}{2} z^2 + \binom{a}{3} z^3 + \cdots$$

where

$$\binom{a}{r} = \frac{a(a-1)(a-2)\cdots(a-r+1)}{r!}.$$

If a is complex, then the expansion is of the principle branch of $(1+z)^a$. We define

$$\binom{r}{0} = 1, \quad \binom{0}{r} = 0, \quad \text{for } r \neq 0, \qquad \binom{0}{0} = 1.$$

Example 14.5.2 Evaluate $\lim_{n\to\infty}(1+1/n)^n$.

First we expand $(1+1/n)^n$ using Newton's binomial formula.

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n &= \lim_{n\to\infty}\left(1+\binom{n}{1}1/n + \binom{n}{2}1/n^2 + \binom{n}{3}1/n^3 + \cdots\right) \\
&= \lim_{n\to\infty}\left(1+1+\frac{n(n-1)}{2!\,n^2} + \frac{n(n-1)(n-2)}{3!\,n^3} + \cdots\right) \\
&= \left(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)
\end{aligned}$$

We recognize this as the Taylor series expansion of e^1.

$$= e$$

We can also evaluate the limit using L'Hospital's rule.

$$\begin{aligned}
\operatorname{Log}\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) &= \lim_{x\to\infty} \operatorname{Log}\left((1+1/x)^x\right) \\
&= \lim_{x\to\infty} x \operatorname{Log}(1+1/x) \\
&= \lim_{x\to\infty} \frac{\operatorname{Log}(1+1/x)}{1/x} \\
&= \lim_{x\to\infty} \frac{\frac{-1/x^2}{1+1/x}}{-1/x^2} \\
&= 1
\end{aligned}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1$$

Example 14.5.3 Find the Taylor series expansion of $1/(1+z)$ about $z = 0$.

For $|z| < 1$,

$$\begin{aligned}
\frac{1}{1+z} &= 1 + \binom{-1}{1} z + \binom{-1}{2} z^2 + \binom{-1}{3} z^3 + \cdots \\
&= 1 + (-1)^1 z + (-1)^2 z^2 + (-1)^3 z^3 + \cdots \\
&= 1 - z + z^2 - z^3 + \cdots
\end{aligned}$$

Example 14.5.4 Find the first few terms in the Taylor series expansion of

$$\frac{1}{\sqrt{z^2 + 5z + 6}}$$

about the origin.

We factor the denominator and then apply Newton's binomial formula.

$$\begin{aligned}
\frac{1}{\sqrt{z^2 + 5z + 6}} &= \frac{1}{\sqrt{z+3}} \frac{1}{\sqrt{z+2}} \\
&= \frac{1}{\sqrt{3}\sqrt{1+z/3}} \frac{1}{\sqrt{2}\sqrt{1+z/2}} \\
&= \frac{1}{\sqrt{6}} \left[1 + \binom{-1/2}{1} \frac{z}{3} + \binom{-1/2}{2} \left(\frac{z}{3}\right)^2 + \cdots \right] \left[1 + \binom{-1/2}{1} \frac{z}{2} + \binom{-1/2}{2} \left(\frac{z}{2}\right)^2 + \cdots \right] \\
&= \frac{1}{\sqrt{6}} \left[1 - \frac{z}{6} + \frac{z^2}{24} + \cdots \right] \left[1 - \frac{z}{4} + \frac{3z^2}{32} + \cdots \right] \\
&= \frac{1}{\sqrt{6}} \left[1 - \frac{5}{12} z + \frac{17}{96} z^2 + \cdots \right]
\end{aligned}$$

14.6 Laurent Series

Result 14.6.1 Let $f(z)$ be single-valued and analytic in the annulus $R_1 < |z - z_0| < R_2$. For points in the annulus, the function has the convergent Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

and C is a positively oriented, closed contour around z_0 lying in the annulus.

To derive this result, consider a function $f(\zeta)$ that is analytic in the annulus $R_1 < |\zeta| < R_2$. Consider any point z in the annulus. Let C_1 be a circle of radius r_1 with $R_1 < r_1 < |z|$. Let C_2 be a circle of radius r_2 with $|z| < r_2 < R_2$. Let C_z be a circle around z, lying entirely between C_1 and C_2. (See Figure 14.5 for an illustration.)

Consider the integral of $\frac{f(\zeta)}{\zeta - z}$ around the C_2 contour. Since the the only singularities of $\frac{f(\zeta)}{\zeta - z}$ occur at $\zeta = z$ and at points outside the annulus,

$$\oint_{C_2} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \oint_{C_z} \frac{f(\zeta)}{\zeta - z} \, d\zeta + \oint_{C_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

By Cauchy's Integral Formula, the integral around C_z is

$$\oint_{C_z} \frac{f(\zeta)}{\zeta - z} \, d\zeta = i2\pi \, f(z).$$

This gives us an expression for $f(z)$.

$$f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta \tag{14.4}$$

On the C_2 contour, $|z| < |\zeta|$. Thus

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1/\zeta}{1 - z/\zeta} \\
&= \frac{1}{\zeta} \sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n, \quad \text{for } |z| < |\zeta| \\
&= \sum_{n=0}^{\infty} \frac{z^n}{\zeta^{n+1}}, \quad \text{for } |z| < |\zeta|
\end{aligned}$$

On the C_1 contour, $|\zeta| < |z|$. Thus

$$\begin{aligned}
-\frac{1}{\zeta - z} &= \frac{1/z}{1 - \zeta/z} \\
&= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{\zeta}{z}\right)^n, \quad \text{for } |\zeta| < |z| \\
&= \sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}}, \quad \text{for } |\zeta| < |z| \\
&= \sum_{n=-\infty}^{-1} \frac{z^n}{\zeta^{n+1}}, \quad \text{for } |\zeta| < |z|
\end{aligned}$$

We substitute these geometric series into Equation 14.4.

$$f(z) = \frac{1}{i2\pi} \oint_{C_2} \left(\sum_{n=0}^{\infty} \frac{f(\zeta) z^n}{\zeta^{n+1}} \right) d\zeta + \frac{1}{i2\pi} \oint_{C_1} \left(\sum_{n=-\infty}^{-1} \frac{f(\zeta) z^n}{\zeta^{n+1}} \right) d\zeta$$

Since the sums converge uniformly, we can interchange the order of integration and summation.

$$f(z) = \frac{1}{i2\pi} \sum_{n=0}^{\infty} \oint_{C_2} \frac{f(\zeta) z^n}{\zeta^{n+1}} \, d\zeta + \frac{1}{i2\pi} \sum_{n=-\infty}^{-1} \oint_{C_1} \frac{f(\zeta) z^n}{\zeta^{n+1}} \, d\zeta$$

Since the only singularities of the integrands lie outside of the annulus, the C_1 and C_2 contours can be deformed to any positive, closed contour C that lies in the annulus and encloses the origin. (See Figure 14.5.) Finally, we combine the two integrals to obtain the desired result.

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{i2\pi} \left(\oint_C \frac{f(\zeta)}{\zeta^{n+1}} \, d\zeta \right) z^n$$

For the case of arbitrary z_0, simply make the transformation $z \to z - z_0$.

Example 14.6.1 Find the Laurent series expansions of $1/(1+z)$.

For $|z| < 1$,

$$\begin{aligned} \frac{1}{1+z} &= 1 + \binom{-1}{1} z + \binom{-1}{2} z^2 + \binom{-1}{3} z^3 + \cdots \\ &= 1 + (-1)^1 z + (-1)^2 z^2 + (-1)^3 z^3 + \cdots \\ &= 1 - z + z^2 - z^3 + \cdots \end{aligned}$$

For $|z| > 1$,

$$\begin{aligned} \frac{1}{1+z} &= \frac{1/z}{1+1/z} \\ &= \frac{1}{z} \left(1 + \binom{-1}{1} z^{-1} + \binom{-1}{2} z^{-2} + \cdots \right) \\ &= z^{-1} - z^{-2} + z^{-3} - \cdots \end{aligned}$$

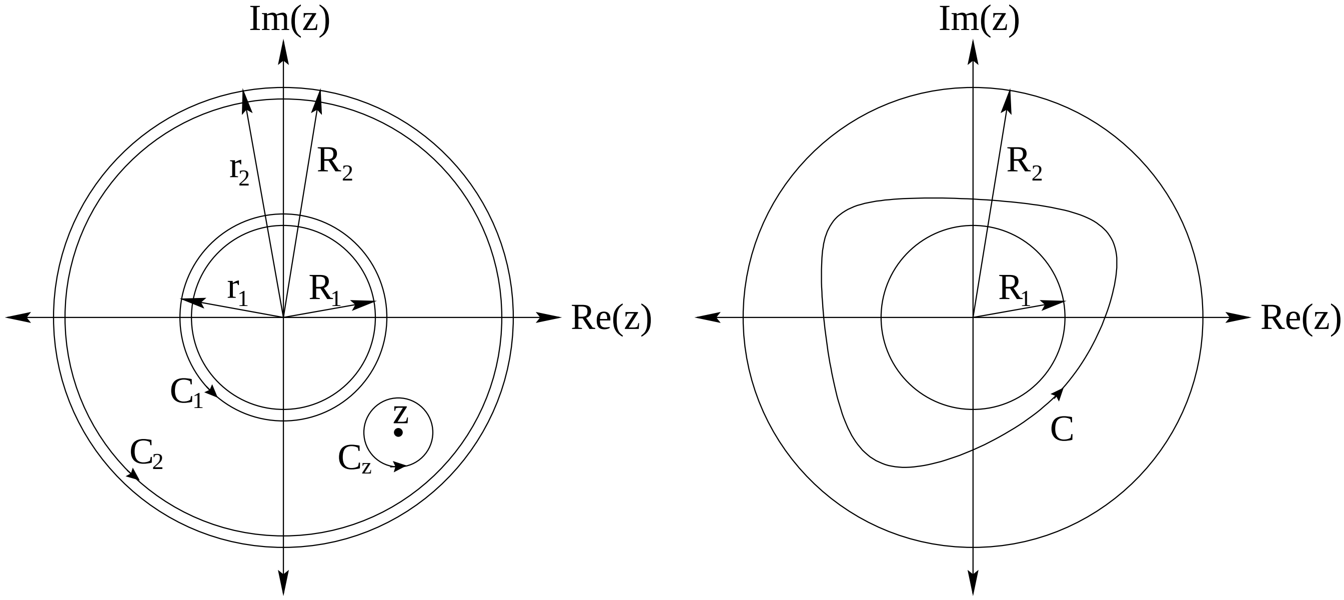


Figure 14.5: Contours for a Laurent Expansion in an Annulus.

14.7 Exercises

Exercise 14.1 (mathematica/fcv/series/constants.nb)
Does the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

converge?
Hint, Solution

Exercise 14.2 (mathematica/fcv/series/constants.nb)
Show that the alternating harmonic series,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

is convergent.
Hint, Solution

Exercise 14.3 (mathematica/fcv/series/constants.nb)
Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent with the Cauchy convergence criterion.
Hint, Solution

Exercise 14.4
The alternating harmonic series has the sum:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n} = \log(2).$$

Show that the terms in this series can be rearranged to sum to π.
Hint, Solution

Exercise 14.5 (mathematica/fcv/series/constants.nb)
Is the series,

$$\sum_{n=1}^{\infty} \frac{n!}{n^n},$$

convergent?
Hint, Solution

Exercise 14.6
Show that the harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots,$$

converges for $\alpha > 1$ and diverges for $\alpha \leq 1$.
Hint, Solution

Exercise 14.7
Evaluate $\sum_{n=1}^{N-1} \sin(nx)$.
Hint, Solution

Exercise 14.8
Using the geometric series, show that

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n, \quad \text{for } |z| < 1,$$

and

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \text{for } |z| < 1.$$

Hint, Solution

Exercise 14.9
Find the Taylor series of $\frac{1}{1+z^2}$ about the $z = 0$. Determine the radius of convergence of the Taylor series from the singularities of the function. Determine the radius of convergence with the ratio test.
Hint, Solution

Exercise 14.10
Use two methods to find the Taylor series expansion of $\log(1 + z)$ about $z = 0$ and determine the circle of convergence. First directly apply Taylor's theorem, then differentiate a geometric series.
Hint, Solution

Exercise 14.11
Find the Laurent series about $z = 0$ of $1/(z - i)$ for $|z| < 1$ and $|z| > 1$.
Hint, Solution

Exercise 14.12
Evaluate

$$\sum_{k=1}^{n} kz^k \quad \text{and} \quad \sum_{k=1}^{n} k^2 z^k$$

for $z \neq 1$.
Hint, Solution

Exercise 14.13
Find the circle of convergence of the following series.

1. $z + (\alpha - \beta)\frac{z^2}{2!} + (\alpha - \beta)(\alpha - 2\beta)\frac{z^3}{3!} + (\alpha - \beta)(\alpha - 2\beta)(\alpha - 3\beta)\frac{z^4}{4!} + \cdots$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}(z - i)^n$

3. $\displaystyle\sum_{n=1}^{\infty} n^n z^n$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$

5. $\displaystyle\sum_{n=1}^{\infty} [3 + (-1)^n]^n z^n$

6. $\displaystyle\sum_{n=1}^{\infty} (n + \alpha^n) z^n \quad (|\alpha| > 1)$

Hint, Solution

Exercise 14.14
Let $f(z) = (1 + z)^\alpha$ be the branch for which $f(0) = 1$. Find its Taylor series expansion about $z = 0$. What is the radius of convergence of the series? (α is an arbitrary complex number.)
Hint, Solution

Exercise 14.15
Obtain the Laurent expansion of

$$f(z) = \frac{1}{(z + 1)(z + 2)}$$

centered on $z = 0$ for the three regions:

1. $|z| < 1$
2. $1 < |z| < 2$
3. $2 < |z|$

Hint, Solution

Exercise 14.16

By comparing the Laurent expansion of $(z + 1/z)^m$, $m \in \mathbb{Z}^+$, with the binomial expansion of this quantity, show that

$$\int_0^{2\pi} (\cos\theta)^m \cos(n\theta)\, d\theta = \begin{cases} \frac{\pi}{2^{m-1}} \binom{m}{(m-n)/2} & -m \leq n \leq m \text{ and } m - n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Hint, Solution

Exercise 14.17

The function $f(z)$ is analytic in the entire z-plane, including ∞, except at the point $z = i/2$, where it has a simple pole, and at $z = 2$, where it has a pole of order 2. In addition

$$\oint_{|z|=1} f(z)\, dz = 2\pi i, \quad \oint_{|z|=3} f(z)\, dz = 0, \quad \oint_{|z|=3} (z-1) f(z)\, dz = 0.$$

Find $f(z)$ and its complete Laurent expansion about $z = 0$.
Hint, Solution

Exercise 14.18

Let $f(z) = \sum_{k=1}^{\infty} k^3 \left(\frac{z}{3}\right)^k$. Compute each of the following, giving justification in each case. The contours are circles of radius one about the origin.

1. $\displaystyle\int_{|z|=1} e^{iz} f(z)\, dz$

2. $\displaystyle\int_{|z|=1} \frac{f(z)}{z^4}\, dz$

3. $\displaystyle\int_{|z|=1} \frac{f(z)\, e^z}{z^2}\, dz$

Hint, Solution

14.8 Hints

Hint 14.1
Use the integral test.

Hint 14.2
Group the terms.

$$
\begin{aligned}
&1 - \frac{1}{2} = \frac{1}{2} \\
&\frac{1}{3} - \frac{1}{4} = \frac{1}{12} \\
&\frac{1}{5} - \frac{1}{6} = \frac{1}{30} \\
&\cdots
\end{aligned}
$$

Hint 14.3
Show that

$$|S_{2n} - S_n| > \frac{1}{2}.$$

Hint 14.4
The alternating harmonic series is conditionally convergent. Let $\{a_n\}$ and $\{b_n\}$ be the positive and negative terms in the sum, respectively, ordered in decreasing magnitude. Note that both $\sum_{n=1}^{\infty} a_n$ and $sumoinb_n$ are divergent. Devise a method for alternately taking terms from $\{a_n\}$ and $\{b_n\}$.

Hint 14.5
Use the ratio test.

Hint 14.6
Use the integral test.

Hint 14.7
Note that $\sin(nx) = \Im(\mathrm{e}^{inx})$. This substitute will yield a finite geometric series.

Hint 14.8
Differentiate the geometric series. Integrate the geometric series.

Hint 14.9
The Taylor series is a geometric series.

Hint 14.10

Hint 14.11

Hint 14.12
Let S_n be the sum. Consider $S_n - zS_n$. Use the finite geometric sum.

Hint 14.13

Hint 14.14

Hint 14.15

Hint 14.16

Hint 14.17

Hint 14.18

14.9 Solutions

Solution 14.1

Since $\sum_{n=2}^{\infty}$ is a series of positive, monotone decreasing terms, the sum converges or diverges with the integral,

$$\int_2^{\infty} \frac{1}{x \log x}\, dx = \int_{\log 2}^{\infty} \frac{1}{\xi}\, d\xi$$

Since the integral diverges, the series also diverges.

Solution 14.2

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n)} \\
&< \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \\
&< \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{\pi^2}{12}
\end{aligned}$$

Thus the series is convergent.

Solution 14.3
Since

$$
\begin{aligned}
|S_{2n} - S_n| &= \left| \sum_{j=n}^{2n-1} \frac{1}{j} \right| \\
&\geq \sum_{j=n}^{2n-1} \frac{1}{2n-1} \\
&= \frac{n}{2n-1} \\
&> \frac{1}{2}
\end{aligned}
$$

the series does not satisfy the Cauchy convergence criterion.

Solution 14.4
The alternating harmonic series is conditionally convergent. That is, the sum is convergent but not absolutely convergent. Let $\{a_n\}$ and $\{b_n\}$ be the positive and negative terms in the sum, respectively, ordered in decreasing magnitude. Note that both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent. Otherwise the alternating harmonic series would be absolutely convergent.

To sum the terms in the series to π we repeat the following two steps indefinitely:

1. Take terms from $\{a_n\}$ until the sum is greater than π.
2. Take terms from $\{b_n\}$ until the sum is less than π.

Each of these steps can always be accomplished because the sums, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both divergent. Hence the tails of the series are divergent. No matter how many terms we take, the remaining terms in each series are divergent. In each step a finite, nonzero number of terms from the respective series is taken. Thus all the terms will be used. Since the terms in each series vanish as $n \to \infty$, the running sum converges to π.

Solution 14.5
Applying the ratio test,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)!n^n}{n!(n+1)^{(n+1)}} \\ &= \lim_{n\to\infty}\frac{n^n}{(n+1)^n} \\ &= \lim_{n\to\infty}\left(\frac{n}{(n+1)}\right)^n \\ &= \frac{1}{e} \\ &< 1,\end{aligned}$$

we see that the series is absolutely convergent.

Solution 14.6
The harmonic series,

$$\sum_{n=1}^{\infty}\frac{1}{n^\alpha} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots,$$

converges or diverges absolutely with the integral,

$$\int_1^\infty \frac{1}{|x^\alpha|}\,\mathrm{d}x = \int_1^\infty \frac{1}{x^{\Re(\alpha)}}\,\mathrm{d}x = \begin{cases}[\log x]_1^\infty & \text{for } \Re(\alpha) = 1, \\ \left[\frac{x^{1-\Re(\alpha)}}{1-\Re(\alpha)}\right]_1^\infty & \text{for } \Re(\alpha) \neq 1.\end{cases}$$

The integral converges only for $\Re(\alpha) > 1$. Thus the harmonic series converges absolutely for $\Re(\alpha) > 1$ and diverges absolutely for $\Re(\alpha) \leq 1$.

Solution 14.7

$$
\begin{aligned}
\sum_{n=1}^{N-1} \sin(nx) &= \sum_{n=0}^{N-1} \sin(nx) \\
&= \sum_{n=0}^{N-1} \Im(\mathrm{e}^{inx}) \\
&= \Im\left(\sum_{n=0}^{N-1} (\mathrm{e}^{ix})^n \right) \\
&= \begin{cases} \Im(N) & \text{for } x = 2\pi k \\ \Im\left(\frac{1-\mathrm{e}^{iNx}}{1-\mathrm{e}^{ix}} \right) & \text{for } x \neq 2\pi k \end{cases} \\
&= \begin{cases} 0 & \text{for } x = 2\pi k \\ \Im\left(\frac{\mathrm{e}^{-ix/2}-\mathrm{e}^{i(N-1/2)x}}{\mathrm{e}^{-ix/2}-\mathrm{e}^{ix/2}} \right) & \text{for } x \neq 2\pi k \end{cases} \\
&= \begin{cases} 0 & \text{for } x = 2\pi k \\ \Im\left(\frac{\mathrm{e}^{-ix/2}-\mathrm{e}^{i(N-1/2)x}}{-2i\sin(x/2)} \right) & \text{for } x \neq 2\pi k \end{cases} \\
&= \begin{cases} 0 & \text{for } x = 2\pi k \\ \Re\left(\frac{\mathrm{e}^{-ix/2}-\mathrm{e}^{i(N-1/2)x}}{2\sin(x/2)} \right) & \text{for } x \neq 2\pi k \end{cases}
\end{aligned}
$$

$$
\boxed{\sum_{n=1}^{N-1} \sin(nx) = \begin{cases} 0 & \text{for } x = 2\pi k \\ \frac{\cos(x/2)-\cos((N-1/2)x)}{2\sin(x/2)} & \text{for } x \neq 2\pi k \end{cases}}
$$

Solution 14.8
The geometric series is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

This series is uniformly convergent in the domain, $|z| \leq r < 1$. Differentiating this equation yields,

$$\begin{aligned}\frac{1}{(1-z)^2} &= \sum_{n=1}^{\infty} n z^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1) z^n \quad \text{for } |z| < 1.\end{aligned}$$

Integrating the geometric series yields

$$-\log(1-z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$$

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \text{for } |z| < 1.$$

Solution 14.9

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

The function $\frac{1}{1+z^2} = \frac{1}{(1-iz)(1+iz)}$ has singularities at $z = \pm i$. Thus the radius of convergence is 1. Now we use the ratio test to corroborate that the radius of convergence is 1.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(z)}{a_n(z)} \right| < 1$$
$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} z^{2(n+1)}}{(-1)^n z^{2n}} \right| < 1$$
$$\lim_{n\to\infty} \left| z^2 \right| < 1$$
$$|z| < 1$$

Solution 14.10

Method 1.

$$\begin{aligned}
\log(1+z) &= [\log(1+z)]_{z=0} + \left[\frac{d}{dz}\log(1+z)\right]_{z=0} \frac{z}{1!} + \left[\frac{d^2}{dz^2}\log(1+z)\right]_{z=0} \frac{z^2}{2!} + \cdots \\
&= 0 + \left[\frac{1}{1+z}\right]_{z=0} \frac{z}{1!} + \left[\frac{-1}{(1+z)^2}\right]_{z=0} \frac{z^2}{2!} + \left[\frac{2}{(1+z)^3}\right]_{z=0} \frac{z^3}{3!} + \cdots \\
&= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}
\end{aligned}$$

Since the nearest singularity of $\log(1+z)$ is at $z = -1$, the radius of convergence is 1.

Method 2. We know the geometric series

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

converges for $|z| < 1$. Integrating this equation yields

$$\log(1 + z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n + 1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$$

for $|z| < 1$. We calculate the radius of convergence with the ratio test.

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \left| \frac{-(n + 1)}{n} \right| = 1$$

Thus the series converges absolutely for $|z| < 1$.

Solution 14.11
For $|z| < 1$:

$$\begin{aligned} \frac{1}{z - i} &= \frac{i}{1 + iz} \\ &= i \sum_{n=0}^{\infty} (-iz)^n \end{aligned}$$

(Note that $|z| < 1 \Leftrightarrow |-iz| < 1$.)

For $|z| > 1$:

$$\frac{1}{z - i} = \frac{1}{z} \frac{1}{(1 - i/z)}$$

(Note that $|z| > 1 \Leftrightarrow |-i/z| < 1$.)

$$
\begin{aligned}
&= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{i}{z} \right)^n \\
&= \frac{1}{z} \sum_{n=-\infty}^{0} i^{-n} z^n \\
&= \sum_{n=-\infty}^{0} (-i)^n z^{n-1} \\
&= \sum_{n=-\infty}^{-1} (-i)^{n+1} z^n
\end{aligned}
$$

Solution 14.12
Let

$$
S_n = \sum_{k=1}^{n} k z^k.
$$

$$
\begin{aligned}
S_n - z S_n &= \sum_{k=1}^{n} k z^k - \sum_{k=1}^{n} k z^{k+1} \\
&= \sum_{k=1}^{n} k z^k - \sum_{k=2}^{n+1} (k-1) z^k \\
&= \sum_{k=1}^{n} z^k - n z^{n+1} \\
&= \frac{z - z^{n+1}}{1 - z} - n z^{n+1}
\end{aligned}
$$

$$\boxed{\sum_{k=1}^{n} kz^k = \frac{z(1-(n+1)z^n + nz^{n+1})}{(1-z)^2}}$$

Let

$$S_n = \sum_{k=1}^{n} k^2 z^k.$$

$$\begin{aligned}
S_n - zS_n &= \sum_{k=1}^{n}(k^2-(k-1)^2)z^k - n^2 z^{n+1} \\
&= 2\sum_{k=1}^{n} kz^k - \sum_{k=1}^{n} z^k - n^2 z^{n+1} \\
&= 2\frac{z(1-(n+1)z^n + nz^{n+1})}{(1-z)^2} - \frac{z - z^{n+1}}{1-z} - n^2 z^{n+1}
\end{aligned}$$

$$\boxed{\sum_{k=1}^{n} k^2 z^k = \frac{z(1+z-z^n(1+z+n(n(z-1)-2)(z-1)))}{(1-z)^3}}$$

Solution 14.13

1. We assume that $\beta \neq 0$. We determine the radius of convergence with the ratio test.

$$
\begin{aligned}
R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(\alpha-\beta)\cdots(\alpha-(n-1)\beta)/n!}{(\alpha-\beta)\cdots(\alpha-n\beta)/(n+1)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{n+1}{\alpha-n\beta} \right| \\
&= \frac{1}{|\beta|}
\end{aligned}
$$

The series converges absolutely for $|z| < 1/|\beta|$.

2. By the ratio test formula, the radius of absolute convergence is

$$
\begin{aligned}
R &= \lim_{n\to\infty} \left| \frac{n/2^n}{(n+1)/2^{n+1}} \right| \\
&= 2 \lim_{n\to\infty} \left| \frac{n}{n+1} \right| \\
&= 2
\end{aligned}
$$

By the root test formula, the radius of absolute convergence is

$$
\begin{aligned}
R &= \frac{1}{\lim_{n\to\infty} \sqrt[n]{|n/2^n|}} \\
&= \frac{2}{\lim_{n\to\infty} \sqrt[n]{n}} \\
&= 2
\end{aligned}
$$

The series converges absolutely for $|z - i| < 2$.

3. We determine the radius of convergence with the Cauchy-Hadamard formula.

$$
\begin{aligned}
R &= \frac{1}{\limsup \sqrt[n]{|a_n|}} \\
&= \frac{1}{\limsup \sqrt[n]{|n^n|}} \\
&= \frac{1}{\limsup n} \\
&= 0
\end{aligned}
$$

The series converges only for $z = 0$.

4. By the ratio test formula, the radius of absolute convergence is

$$
\begin{aligned}
R &= \lim_{n\to\infty} \left| \frac{n!/n^n}{(n+1)!/(n+1)^{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^n}{n^n} \right| \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n \\
&= \exp\left(\lim_{n\to\infty} \log\left(\left(\frac{n+1}{n} \right)^n \right) \right) \\
&= \exp\left(\lim_{n\to\infty} n \log\left(\frac{n+1}{n} \right) \right) \\
&= \exp\left(\lim_{n\to\infty} \frac{\log(n+1) - \log(n)}{1/n} \right) \\
&= \exp\left(\lim_{n\to\infty} \frac{1/(n+1) - 1/n}{-1/n^2} \right) \\
&= \exp\left(\lim_{n\to\infty} \frac{n}{n+1} \right) \\
&= \mathrm{e}^1
\end{aligned}
$$

The series converges absolutely in the circle, $|z| < \mathrm{e}$.

5. By the Cauchy-Hadamard formula, the radius of absolute convergence is

$$
\begin{aligned}
R &= \frac{1}{\limsup \sqrt[n]{|\,(3 + (-1)^n)^n\,|}} \\
&= \frac{1}{\limsup\,(3 + (-1)^n)} \\
&= \frac{1}{4}
\end{aligned}
$$

Thus the series converges absolutely for $|z| < 1/4$.

6. By the Cauchy-Hadamard formula, the radius of absolute convergence is

$$\begin{aligned} R &= \frac{1}{\limsup \sqrt[n]{|n + \alpha^n|}} \\ &= \frac{1}{\limsup |\alpha| \sqrt[n]{|1 + n/\alpha^n|}} \\ &= \frac{1}{|\alpha|} \end{aligned}$$

Thus the sum converges absolutely for $|z| < 1/|\alpha|$.

Solution 14.14
The Taylor series expansion of $f(z)$ about $z = 0$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

The derivatives of $f(z)$ are

$$f^{(n)}(z) = \left(\prod_{k=0}^{n-1} (\alpha - k) \right) (1 + z)^{\alpha - n}.$$

Thus $f^{(n)}(0)$ is

$$f^{(n)}(0) = \prod_{k=0}^{n-1} (\alpha - k).$$

If $\alpha = m$ is a non-negative integer, then only the first $m+1$ terms are nonzero. The Taylor series is a polynomial and the series has an infinite radius of convergence.

$$(1+z)^m = \sum_{n=0}^{m} \frac{\prod_{k=0}^{n-1}(\alpha - k)}{n!} z^n$$

If α is not a non-negative integer, then all of the terms in the series are non-zero.

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n-1}(\alpha - k)}{n!} z^n$$

The radius of convergence of the series is the distance to the nearest singularity of $(1+z)^\alpha$. This occurs at $z = 1$. Thus the series converges for $|z| < 1$. We can corroborate this with the ratio test. The radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{\left(\prod_{k=0}^{n-1}(\alpha - k)\right)/n!}{\left(\prod_{k=0}^{n}(\alpha - k)\right)/(n+1)!} \right| = \lim_{n\to\infty} \left| \frac{n+1}{\alpha - n} \right| = |-1| = 1.$$

If we define the binomial coefficient,

$$\binom{\alpha}{n} \equiv \frac{\prod_{k=0}^{n-1}(\alpha - k)}{n!},$$

then we can write the series as

$$\boxed{(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n.}$$

Solution 14.15

We expand the function in partial fractions.

$$f(z) = \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

The Taylor series about $z = 0$ for $1/(z + 1)$ is

$$\begin{aligned}
\frac{1}{1+z} &= \frac{1}{1-(-z)} \\
&= \sum_{n=0}^{\infty} (-z)^n, \quad \text{for } |z| < 1 \\
&= \sum_{n=0}^{\infty} (-1)^n z^n, \quad \text{for } |z| < 1
\end{aligned}$$

The series about $z = \infty$ for $1/(z + 1)$ is

$$\begin{aligned}
\frac{1}{1+z} &= \frac{1/z}{1+1/z} \\
&= \frac{1}{z} \sum_{n=0}^{\infty} (-1/z)^n, \quad \text{for } |1/z| < 1 \\
&= \sum_{n=0}^{\infty} (-1)^n z^{-n-1}, \quad \text{for } |z| > 1 \\
&= \sum_{n=-\infty}^{-1} (-1)^{n+1} z^n, \quad \text{for } |z| > 1
\end{aligned}$$

The Taylor series about $z = 0$ for $1/(z + 2)$ is

$$\begin{aligned}
\frac{1}{2+z} &= \frac{1/2}{1+z/2} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} (-z/2)^n, \quad \text{for } |z/2| < 1 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n, \quad \text{for } |z| < 2
\end{aligned}$$

The series about $z = \infty$ for $1/(z+2)$ is

$$\begin{aligned}
\frac{1}{2+z} &= \frac{1/z}{1+2/z} \\
&= \frac{1}{z}\sum_{n=0}^{\infty}(-2/z)^n, \quad \text{for } |2/z| < 1 \\
&= \sum_{n=0}^{\infty}(-1)^n 2^n z^{-n-1}, \quad \text{for } |z| > 2 \\
&= \sum_{n=-\infty}^{-1}\frac{(-1)^{n+1}}{2^{n+1}} z^n, \quad \text{for } |z| > 2
\end{aligned}$$

To find the expansions in the three regions, we just choose the appropriate series.

1.

$$\begin{aligned}
f(z) &= \frac{1}{1+z} - \frac{1}{2+z} \\
&= \sum_{n=0}^{\infty}(-1)^n z^n - \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}} z^n, \quad \text{for } |z| < 1 \\
&= \sum_{n=0}^{\infty}(-1)^n\left(1 - \frac{1}{2^{n+1}}\right) z^n, \quad \text{for } |z| < 1
\end{aligned}$$

$$\boxed{f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{2^{n+1}-1}{2^{n+1}} z^n, \quad \text{for } |z| < 1}$$

2.

$$f(z) = \frac{1}{1+z} - \frac{1}{2+z}$$

$$\boxed{f(z) = \sum_{n=-\infty}^{-1} (-1)^{n+1} z^n - \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n, \quad \text{for } 1 < |z| < 2}$$

3.

$$\begin{aligned} f(z) &= \frac{1}{1+z} - \frac{1}{2+z} \\ &= \sum_{n=-\infty}^{-1} (-1)^{n+1} z^n - \sum_{n=-\infty}^{-1} \frac{(-1)^{n+1}}{2^{n+1}} z^n, \quad \text{for } 2 < |z| \end{aligned}$$

$$\boxed{f(z) = \sum_{n=-\infty}^{-1} (-1)^{n+1} \frac{2^{n+1}-1}{2^{n+1}} z^n, \quad \text{for } 2 < |z|}$$

Solution 14.16

Laurent Series. We assume that m is a non-negative integer and that n is an integer. The Laurent series about the point $z = 0$ of

$$f(z) = \left(z + \frac{1}{z}\right)^m$$

is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

where

$$a_n = \frac{1}{i2\pi} \oint_C \frac{f(z)}{z^{n+1}}\, dz$$

and C is a contour going around the origin once in the positive direction. We manipulate the coefficient integral into the desired form.

$$\begin{aligned}
a_n &= \frac{1}{i2\pi} \oint_C \frac{(z + 1/z)^m}{z^{n+1}}\, dz \\
&= \frac{1}{i2\pi} \int_0^{2\pi} \frac{(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta})^m}{\mathrm{e}^{i(n+1)\theta}} i\, \mathrm{e}^{i\theta}\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} 2^m \cos^m\theta\, \mathrm{e}^{-in\theta}\, d\theta \\
&= \frac{2^{m-1}}{\pi} \int_0^{2\pi} \cos^m\theta (\cos(n\theta) - i\sin(n\theta))\, d\theta
\end{aligned}$$

Note that $\cos^m\theta$ is even and $\sin(n\theta)$ is odd about $\theta = \pi$.

$$= \frac{2^{m-1}}{\pi} \int_0^{2\pi} \cos^m\theta \cos(n\theta)\, d\theta$$

Binomial Series. Now we find the binomial series expansion of $f(z)$.

$$\begin{aligned}
\left(z + \frac{1}{z}\right)^m &= \sum_{n=0}^{m} \binom{m}{n} z^{m-n} \left(\frac{1}{z}\right)^n \\
&= \sum_{n=0}^{m} \binom{m}{n} z^{m-2n} \\
&= \sum_{\substack{n=-m \\ m-n \text{ even}}}^{m} \binom{m}{(m-n)/2} z^n
\end{aligned}$$

The coefficients in the series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ are

$$a_n = \begin{cases} \binom{m}{(m-n)/2} & -m \leq n \leq m \text{ and } m-n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

By equating the coefficients found by the two methods, we evaluate the desired integral.

$$\boxed{\int_0^{2\pi} (\cos\theta)^m \cos(n\theta)\, d\theta = \begin{cases} \frac{\pi}{2^{m-1}} \binom{m}{(m-n)/2} & -m \leq n \leq m \text{ and } m-n \text{ even} \\ 0 & \text{otherwise} \end{cases}}$$

Solution 14.17

First we write $f(z)$ in the form

$$f(z) = \frac{g(z)}{(z - i/2)(z-2)^2}.$$

$g(z)$ is an entire function which grows no faster that z^3 at infinity. By expanding $g(z)$ in a Taylor series about the origin, we see that it is a polynomial of degree no greater than 3.

$$f(z) = \frac{\alpha z^3 + \beta z^2 + \gamma z + \delta}{(z - i/2)(z-2)^2}.$$

Since $f(z)$ is a rational function we expand it in partial fractions to obtain a form that is convenient to integrate.

$$f(z) = \frac{a}{z - i/2} + \frac{b}{z-2} + \frac{c}{(z-2)^2} + d$$

We use the value of the integrals of $f(z)$ to determine the constants, a, b, c and d.

$$\oint_{|z|=1} \left(\frac{a}{z - i/2} + \frac{b}{z-2} + \frac{c}{(z-2)^2} + d \right) dz = i2\pi$$

$$i2\pi a = i2\pi$$

$$a = 1$$

$$\oint_{|z|=3} \left(\frac{1}{z - i/2} + \frac{b}{z-2} + \frac{c}{(z-2)^2} + d \right) dz = 0$$
$$i2\pi(1 + b) = 0$$
$$b = -1$$

Note that by applying the second constraint, we can change the third constraint to

$$\oint_{|z|=3} z f(z)\, dz = 0.$$

$$\oint_{|z|=3} z \left(\frac{1}{z - i/2} - \frac{1}{z-2} + \frac{c}{(z-2)^2} + d \right) dz = 0$$
$$\oint_{|z|=3} \left(\frac{(z - i/2) + i/2}{z - i/2} - \frac{(z-2) + 2}{z-2} + \frac{c(z-2) + 2c}{(z-2)^2} \right) dz = 0$$
$$i2\pi \left(\frac{i}{2} - 2 + c \right) = 0$$
$$c = 2 - \frac{i}{2}$$

Thus we see that the function is

$$\boxed{f(z) = \frac{1}{z - i/2} - \frac{1}{z-2} + \frac{2 - i/2}{(z-2)^2} + d.}$$

where d is an arbitrary constant. We can also write the function in the form:

$$\boxed{f(z) = \frac{dz^3 + 15 - i8}{4(z - i/2)(z-2)^2}.}$$

Complete Laurent Series. We find the complete Laurent series about $z = 0$ for each of the terms in the partial fraction expansion of $f(z)$.

$$
\begin{aligned}
\frac{1}{z - i/2} &= \frac{i2}{1 + i2z} \\
&= i2 \sum_{n=0}^{\infty} (-i2z)^n, \quad \text{for } |-i2z| < 1 \\
&= -\sum_{n=0}^{\infty} (-i2)^{n+1} z^n, \quad \text{for } |z| < 1/2
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{z - i/2} &= \frac{1/z}{1 - i/(2z)} \\
&= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{i}{2z} \right)^n, \quad \text{for } |i/(2z)| < 1 \\
&= \sum_{n=0}^{\infty} \left(\frac{i}{2} \right)^n z^{-n-1}, \quad \text{for } |z| < 2 \\
&= \sum_{n=-\infty}^{-1} \left(\frac{i}{2} \right)^{-n-1} z^n, \quad \text{for } |z| < 2 \\
&= \sum_{n=-\infty}^{-1} (-i2)^{n+1} z^n, \quad \text{for } |z| < 2
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{z-2} &= \frac{1/2}{1-z/2} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n, \quad \text{for } |z/2| < 1 \\
&= \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}, \quad \text{for } |z| < 2
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{z-2} &= -\frac{1/z}{1-2/z} \\
&= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n, \quad \text{for } |2/z| < 1 \\
&= -\sum_{n=0}^{\infty}2^n z^{-n-1}, \quad \text{for } |z| > 2 \\
&= -\sum_{n=-\infty}^{-1}2^{-n-1} z^n, \quad \text{for } |z| > 2
\end{aligned}
$$

$$
\begin{aligned}
\frac{2-i/2}{(z-2)^2} &= (2-i/2)\frac{1}{4}(1-z/2)^{-2} \\
&= \frac{4-i}{8}\sum_{n=0}^{\infty}\binom{-2}{n}\left(-\frac{z}{2}\right)^n, \quad \text{for } |z/2|<1 \\
&= \frac{4-i}{8}\sum_{n=0}^{\infty}(-1)^n(n+1)(-1)^n 2^{-n} z^n, \quad \text{for } |z|<2 \\
&= \frac{4-i}{8}\sum_{n=0}^{\infty}\frac{n+1}{2^n} z^n, \quad \text{for } |z|<2
\end{aligned}
$$

$$
\begin{aligned}
\frac{2-i/2}{(z-2)^2} &= \frac{2-i/2}{z^2}\left(1-\frac{2}{z}\right)^{-2} \\
&= \frac{2-i/2}{z^2}\sum_{n=0}^{\infty}\binom{-2}{n}\left(-\frac{2}{z}\right)^n, \quad \text{for } |2/z|<1 \\
&= (2-i/2)\sum_{n=0}^{\infty}(-1)^n(n+1)(-1)^n 2^n z^{-n-2}, \quad \text{for } |z|>2 \\
&= (2-i/2)\sum_{n=-\infty}^{-2}(-n-1)2^{-n-2} z^n, \quad \text{for } |z|>2 \\
&= -(2-i/2)\sum_{n=-\infty}^{-2}\frac{n+1}{2^{n+2}} z^n, \quad \text{for } |z|>2
\end{aligned}
$$

We take the appropriate combination of these series to find the Laurent series expansions in the regions:

$|z| < 1/2$, $1/2 < |z| < 2$ and $2 < |z|$. For $|z| < 1/2$, we have

$$f(z) = -\sum_{n=0}^{\infty}(-i2)^{n+1}z^n + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \frac{4-i}{8}\sum_{n=0}^{\infty}\frac{n+1}{2^n}z^n + d$$

$$f(z) = \sum_{n=0}^{\infty}\left(-(-i2)^{n+1} + \frac{1}{2^{n+1}} + \frac{4-i}{8}\frac{n+1}{2^n}\right)z^n + d$$

$$\boxed{f(z) = \sum_{n=0}^{\infty}\left(-(-i2)^{n+1} + \frac{1}{2^{n+1}}\left(1 + \frac{4-i}{4}(n+1)\right)\right)z^n + d, \quad \text{for } |z| < 1/2}$$

For $1/2 < |z| < 2$, we have

$$f(z) = \sum_{n=-\infty}^{-1}(-i2)^{n+1}z^n + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \frac{4-i}{8}\sum_{n=0}^{\infty}\frac{n+1}{2^n}z^n + d$$

$$\boxed{f(z) = \sum_{n=-\infty}^{-1}(-i2)^{n+1}z^n + \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}}\left(1 + \frac{4-i}{4}(n+1)\right)\right)z^n + d, \quad \text{for } 1/2 < |z| < 2}$$

For $2 < |z|$, we have

$$f(z) = \sum_{n=-\infty}^{-1}(-i2)^{n+1}z^n - \sum_{n=-\infty}^{-1}2^{-n-1}z^n - (2-i/2)\sum_{n=-\infty}^{-2}\frac{n+1}{2^{n+2}}z^n + d$$

$$\boxed{f(z) = \sum_{n=-\infty}^{-2}\left((-i2)^{n+1} - \frac{1}{2^{n+1}}\left(1 + (1-i/4)(n+1)\right)\right)z^n + d, \quad \text{for } 2 < |z|}$$

Solution 14.18

The radius of convergence of the series for $f(z)$ is

$$R = \lim_{n\to\infty} \left| \frac{k^3/3^k}{(k+1)^3/3^{k+1}} \right| = 3 \lim_{n\to\infty} \left| \frac{k^3}{(k+1)^3} \right| = 3.$$

Thus $f(z)$ is a function which is analytic inside the circle of radius 3.

1. The integrand is analytic. Thus by Cauchy's theorem the value of the integral is zero.

$$\boxed{\oint_{|z|=1} \mathrm{e}^{iz} f(z)\, dz = 0}$$

2. We use Cauchy's integral formula to evaluate the integral.

$$\oint_{|z|=1} \frac{f(z)}{z^4}\, dz = \frac{i2\pi}{3!} f^{(3)}(0) = \frac{i2\pi}{3!} \frac{3!\, 3^3}{3^3} = i2\pi$$

$$\boxed{\oint_{|z|=1} \frac{f(z)}{z^4}\, dz = i2\pi.}$$

3. We use Cauchy's integral formula to evaluate the integral.

$$\oint_{|z|=1} \frac{f(z)\,\mathrm{e}^z}{z^2}\, dz = \frac{i2\pi}{1!} \frac{\mathrm{d}}{\mathrm{d}z}\left(f(z)\,\mathrm{e}^z\right)\Big|_{z=0} = i2\pi \frac{1!\, 1^3}{3^1}$$

$$\boxed{\oint_{|z|=1} \frac{f(z)\,\mathrm{e}^z}{z^2}\, dz = \frac{i2\pi}{3}}$$

Chapter 15

The Residue Theorem

Man will occasionally stumble over the truth, but most of the time he will pick himself up and continue on.

- Winston Churchill

15.1 The Residue Theorem

We will find that many integrals on closed contours may be evaluated in terms of the *residues* of a function. We first define residues and then prove the Residue Theorem.

Result 15.1.1 Residues. Let $f(z)$ be single-valued an analytic in a deleted neighborhood of z_0. Then $f(z)$ has the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

The residue of $f(z)$ at $z = z_0$ is the coefficient of the $\frac{1}{z-z_0}$ term:

$$\operatorname{Res}(f(z), z_0) = a_{-1}.$$

The residue at a branch point or non-isolated singularity is undefined as the Laurent series does not exist. If $f(z)$ has a pole of order n at $z = z_0$ then we can use the Residue Formula:

$$\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} \left(\frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z - z_0)^n f(z) \right] \right).$$

See Exercise 15.1 for a proof of the Residue Formula.

Example 15.1.1 In Example 10.4.5 we showed that $f(z) = z/\sin z$ has first order poles at $z = n\pi$, $n \in \mathbb{Z} \setminus \{0\}$.

Now we find the residues at these isolated singularities.

$$\begin{aligned}
\operatorname{Res}\left(\frac{z}{\sin z}, z = n\pi\right) &= \lim_{z \to n\pi}\left((z - n\pi)\frac{z}{\sin z}\right) \\
&= n\pi \lim_{z \to n\pi} \frac{z - n\pi}{\sin z} \\
&= n\pi \lim_{z \to n\pi} \frac{1}{\cos z} \\
&= n\pi \frac{1}{(-1)^n} \\
&= (-1)^n n\pi
\end{aligned}$$

Residue Theorem. We can evaluate many integrals in terms of the residues of a function. Suppose $f(z)$ has only one singularity, (at $z = z_0$), inside the simple, closed, positively oriented contour C. $f(z)$ has a convergent Laurent series in some deleted disk about z_0. We deform C to lie in the disk. See Figure 15.1. We now evaluate $\int_C f(z)\,\mathrm{d}z$ by deforming the contour and using the Laurent series expansion of the function.

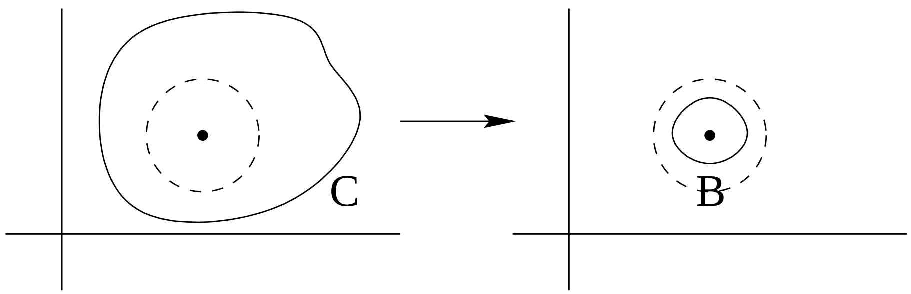


Figure 15.1: Deform the contour to lie in the deleted disk.

$$
\begin{aligned}
\int_C f(z)\,\mathrm{d}z &= \int_B f(z)\,\mathrm{d}z \\
&= \int_B \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \,\mathrm{d}z \\
&= \sum_{\substack{n=-\infty \\ n\neq -1}}^{\infty} a_n \left[\frac{(z-z_0)^{n+1}}{n+1}\right]_{r\,\mathrm{e}^{i\theta}}^{r\,\mathrm{e}^{i(\theta+2\pi)}} + a_{-1}\left[\log(z-z_0)\right]_{r\,\mathrm{e}^{i\theta}}^{r\,\mathrm{e}^{i(\theta+2\pi)}} \\
&= a_{-1} i2\pi
\end{aligned}
$$

$$
\int_C f(z)\,\mathrm{d}z = i2\pi \operatorname{Res}\left(f(z), z_0\right)
$$

Now assume that $f(z)$ has n singularities at $\{z_1, \ldots, z_n\}$. We deform C to n contours $C_1, \ldots, C_n$ which enclose the singularities and lie in deleted disks about the singularities in which $f(z)$ has convergent Laurent series. See Figure 15.2. We evaluate $\int_C f(z)\,\mathrm{d}z$ by deforming the contour.

$$
\int_C f(z)\,\mathrm{d}z = \sum_{k=1}^{n} \int_{C_k} f(z)\,\mathrm{d}z = i2\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z), z_k\right)
$$

Now instead let $f(z)$ be analytic *outside* and on C except for isolated singularities at $\{\zeta_n\}$ in the domain outside C and perhaps an isolated singularity at infinity. Let a be any point in the interior of C. To evaluate $\int_C f(z)\,\mathrm{d}z$ we make the change of variables $\zeta = 1/(z-a)$. This maps the contour C to C'. (Note that C' is negatively oriented.) All the points outside C are mapped to points inside C' and vice versa. We can then evaluate the integral in terms of the singularities inside C'.

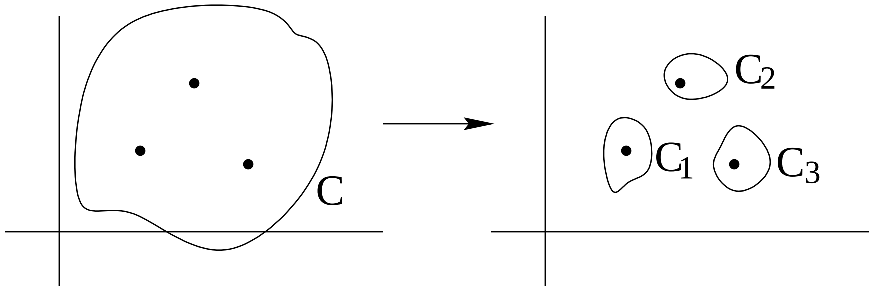


Figure 15.2: Deform the contour n contours which enclose the n singularities.

$$
\begin{aligned}
\oint_C f(z)\,\mathrm{d}z &= \oint_{C'} f\left(\frac{1}{\zeta}+a\right)\frac{-1}{\zeta^2}\,\mathrm{d}\zeta \\
&= \oint_{-C'} \frac{1}{z^2} f\left(\frac{1}{z}+a\right)\,\mathrm{d}z \\
&= i2\pi \sum_n \operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}+a\right), \frac{1}{\zeta_n - a}\right) + i2\pi \operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}+a\right), 0\right).
\end{aligned}
$$

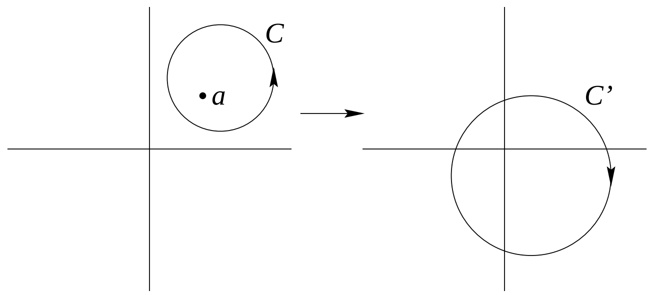


Figure 15.3: The change of variables $\zeta = 1/(z-a)$.

Result 15.1.2 Residue Theorem. If $f(z)$ is analytic in a compact, closed, connected domain D except for isolated singularities at $\{z_n\}$ in the interior of D then

$$\int_{\partial D} f(z)\,\mathrm{d}z = \sum_k \oint_{C_k} f(z)\,\mathrm{d}z = \imath 2\pi \sum_n \operatorname{Res}(f(z), z_n).$$

Here the set of contours $\{C_k\}$ make up the positively oriented boundary ∂D of the domain D. If the boundary of the domain is a single contour C then the formula simplifies.

$$\oint_C f(z)\,\mathrm{d}z = \imath 2\pi \sum_n \operatorname{Res}(f(z), z_n)$$

If instead $f(z)$ is analytic outside and on C except for isolated singularities at $\{\zeta_n\}$ in the domain outside C and perhaps an isolated singularity at infinity then

$$\oint_C f(z)\,\mathrm{d}z = \imath 2\pi \sum_n \operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}+a\right), \frac{1}{\zeta_n - a}\right) + \imath 2\pi \operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}+a\right), 0\right).$$

Here a is a any point in the interior of C.

Example 15.1.2 Consider

$$\frac{1}{2\pi i} \int_C \frac{\sin z}{z(z-1)} \, \mathrm{d}z$$

where C is the positively oriented circle of radius 2 centered at the origin. Since the integrand is single-valued with only isolated singularities, the Residue Theorem applies. The value of the integral is the sum of the residues from singularities inside the contour.

The only places that the integrand could have singularities are $z = 0$ and $z = 1$. Since

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \frac{\cos z}{1} = 1,$$

there is a removable singularity at the point $z = 0$. There is no residue at this point.

Now we consider the point $z = 1$. Since $\sin(z)/z$ is analytic and nonzero at $z = 1$, that point is a first order pole of the integrand. The residue there is

$$\operatorname{Res}\left(\frac{\sin z}{z(z-1)}, z = 1\right) = \lim_{z \to 1} (z-1) \frac{\sin z}{z(z-1)} = \sin(1).$$

There is only one singular point with a residue inside the path of integration. The residue at this point is $\sin(1)$. Thus the value of the integral is

$$\frac{1}{2\pi i} \int_C \frac{\sin z}{z(z-1)} \, \mathrm{d}z = \sin(1)$$

Example 15.1.3 Evaluate the integral

$$\int_C \frac{\cot z \coth z}{z^3} \, \mathrm{d}z$$

where C is the unit circle about the origin in the positive direction.

The integrand is

$$\frac{\cot z \coth z}{z^3} = \frac{\cos z \cosh z}{z^3 \sin z \sinh z}$$

$\sin z$ has zeros at $n\pi$. $\sinh z$ has zeros at $in\pi$. Thus the only pole inside the contour of integration is at $z = 0$. Since $\sin z$ and $\sinh z$ both have simple zeros at $z = 0$,

$$\sin z = z + \mathcal{O}(z^3), \qquad \sinh z = z + \mathcal{O}(z^3)$$

the integrand has a pole of order 5 at the origin. The residue at $z = 0$ is

$$\begin{aligned}
\lim_{z\to 0} \frac{1}{4!} \frac{\mathrm{d}^4}{\mathrm{d}z^4} \left(z^5 \frac{\cot z \coth z}{z^3} \right) &= \lim_{z\to 0} \frac{1}{4!} \frac{\mathrm{d}^4}{\mathrm{d}z^4} \left(z^2 \cot z \coth z \right) \\
&= \frac{1}{4!} \lim_{z\to 0} \Big(24 \cot(z) \coth(z) \csc(z)^2 - 32 z \coth(z) \csc(z)^4 \\
&\quad - 16 z \cos(2 z) \coth(z) \csc(z)^4 + 22 z^2 \cot(z) \coth(z) \csc(z)^4 \\
&\quad + 2 z^2 \cos(3 z) \coth(z) \csc(z)^5 + 24 \cot(z) \coth(z) \operatorname{csch}(z)^2 \\
&\quad + 24 \csc(z)^2 \operatorname{csch}(z)^2 - 48 z \cot(z) \csc(z)^2 \operatorname{csch}(z)^2 \\
&\quad - 48 z \coth(z) \csc(z)^2 \operatorname{csch}(z)^2 + 24 z^2 \cot(z) \coth(z) \csc(z)^2 \operatorname{csch}(z)^2 \\
&\quad + 16 z^2 \csc(z)^4 \operatorname{csch}(z)^2 + 8 z^2 \cos(2 z) \csc(z)^4 \operatorname{csch}(z)^2 \\
&\quad - 32 z \cot(z) \operatorname{csch}(z)^4 - 16 z \cosh(2 z) \cot(z) \operatorname{csch}(z)^4 \\
&\quad + 22 z^2 \cot(z) \coth(z) \operatorname{csch}(z)^4 + 16 z^2 \csc(z)^2 \operatorname{csch}(z)^4 \\
&\quad + 8 z^2 \cosh(2 z) \csc(z)^2 \operatorname{csch}(z)^4 + 2 z^2 \cosh(3 z) \cot(z) \operatorname{csch}(z)^5 \Big) \\
&= \frac{1}{4!} \left(-\frac{56}{15} \right) \\
&= -\frac{7}{45}
\end{aligned}$$

Since taking the fourth derivative of $z^2 \cot z \coth z$ really sucks, we would like a more elegant way of finding the residue. We expand the functions in the integrand in Taylor series about the origin.

$$
\begin{aligned}
\frac{\cos z \cosh z}{z^3 \sin z \sinh z} &= \frac{\left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots\right)\left(1 + \frac{z^2}{2} + \frac{z^4}{24} + \cdots\right)}{z^3 \left(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\right)\left(z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots\right)} \\
&= \frac{1 - \frac{z^4}{6} + \cdots}{z^3\left(z^2 + z^6\left(\frac{-1}{36} + \frac{1}{90}\right) + \cdots\right)} \\
&= \frac{1}{z^5} \frac{1 - \frac{z^4}{6} + \cdots}{1 - \frac{z^4}{90} + \cdots} \\
&= \frac{1}{z^5}\left(1 - \frac{z^4}{6} + \cdots\right)\left(1 + \frac{z^4}{90} + \cdots\right) \\
&= \frac{1}{z^5}\left(1 - \frac{7}{45} z^4 + \cdots\right) \\
&= \frac{1}{z^5} - \frac{7}{45}\frac{1}{z} + \cdots
\end{aligned}
$$

Thus we see that the residue is $-\frac{7}{45}$. Now we can evaluate the integral.

$$
\int_C \frac{\cot z \coth z}{z^3} \, \mathrm{d}z = -i \frac{14}{45} \pi
$$

15.2 Cauchy Principal Value for Real Integrals

15.2.1 The Cauchy Principal Value

First we recap improper integrals. If $f(x)$ has a singularity at $x_0 \in (a \dots b)$ then

$$
\int_a^b f(x) \, \mathrm{d}x \equiv \lim_{\epsilon \to 0^+} \int_a^{x_0 - \epsilon} f(x) \, \mathrm{d}x + \lim_{\delta \to 0^+} \int_{x_0 + \delta}^b f(x) \, \mathrm{d}x.
$$

For integrals on $(-\infty \ldots \infty)$,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \equiv \lim_{a\to-\infty,\ b\to\infty} \int_a^b f(x)\,\mathrm{d}x.$$

Example 15.2.1 $\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x$ is divergent. We show this with the definition of improper integrals.

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x &= \lim_{\epsilon\to0^+} \int_{-1}^{-\epsilon} \frac{1}{x}\,\mathrm{d}x + \lim_{\delta\to0^+} \int_{\delta}^{1} \frac{1}{x}\,\mathrm{d}x \\
&= \lim_{\epsilon\to0^+} [\ln|x|]_{-1}^{-\epsilon} + \lim_{\delta\to0^+} [\ln|x|]_{\delta}^{1} \\
&= \lim_{\epsilon\to0^+} \ln\epsilon - \lim_{\delta\to0^+} \ln\delta
\end{aligned}$$

The integral diverges because ϵ and δ approach zero independently.

Since $1/x$ is an odd function, it appears that the area under the curve is zero. Consider what would happen if ϵ and δ were not independent. If they approached zero symmetrically, $\delta = \epsilon$, then the value of the integral would be zero.

$$\lim_{\epsilon\to0^+} \left(\int_{-1}^{-\epsilon} + \int_{\epsilon}^{1} \right) \frac{1}{x}\,\mathrm{d}x = \lim_{\epsilon\to0^+} (\ln\epsilon - \ln\epsilon) = 0$$

We could make the integral have any value we pleased by choosing $\delta = c\epsilon$. [1]

$$\lim_{\epsilon\to0^+} \left(\int_{-1}^{-\epsilon} + \int_{c\epsilon}^{1} \right) \frac{1}{x}\,\mathrm{d}x = \lim_{\epsilon\to0^+} (\ln\epsilon - \ln(c\epsilon)) = -\ln c$$

We have seen it is reasonable that

$$\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x$$

[1] This may remind you of conditionally convergent series. You can rearrange the terms to make the series sum to any number.

has some meaning, and if we could evaluate the integral, the most reasonable value would be zero. The *Cauchy principal value* provides us with a way of evaluating such integrals. If $f(x)$ is continuous on (a, b) except at the point $x_0 \in (a, b)$ then the Cauchy principal value of the integral is defined

$$\fint_a^b f(x)\,\mathrm{d}x = \lim_{\epsilon \to 0^+} \left(\int_a^{x_0-\epsilon} f(x)\,\mathrm{d}x + \int_{x_0+\epsilon}^b f(x)\,\mathrm{d}x \right).$$

The Cauchy principal value is obtained by approaching the singularity symmetrically. The principal value of the integral may exist when the integral diverges. If the integral exists, it is equal to the principal value of the integral.

The Cauchy principal value of $\int_{-1}^1 \frac{1}{x}\,\mathrm{d}x$ is defined

$$\begin{aligned}
\fint_{-1}^1 \frac{1}{x}\,\mathrm{d}x &\equiv \lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} \frac{1}{x}\,\mathrm{d}x + \int_{\epsilon}^1 \frac{1}{x}\,\mathrm{d}x \right) \\
&= \lim_{\epsilon \to 0^+} \left([\log |x|]_{-1}^{-\epsilon} [\log |x|]_{\epsilon}^{1} \right) \\
&= \lim_{\epsilon \to 0^+} \left(\log |-\epsilon| - \log |\epsilon| \right) \\
&= 0.
\end{aligned}$$

(Another notation for the principal value of an integral is $\mathrm{PV} \int f(x)\,\mathrm{d}x$.) Since the limits of integration approach zero symmetrically, the two halves of the integral cancel. If the limits of integration approached zero independently, (the definition of the integral), then the two halves would both diverge.

Example 15.2.2 $\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,\mathrm{d}x$ is divergent. We show this with the definition of improper integrals.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,\mathrm{d}x &= \lim_{a \to -\infty,\ b \to \infty} \int_a^b \frac{x}{x^2+1}\,\mathrm{d}x \\
&= \lim_{a \to -\infty,\ b \to \infty} \left[\frac{1}{2} \ln(x^2+1) \right]_a^b \\
&= \frac{1}{2} \lim_{a \to -\infty,\ b \to \infty} \ln \left(\frac{b^2+1}{a^2+1} \right)
\end{aligned}$$

The integral diverges because a and b approach infinity independently. Now consider what would happen if a and b were not independent. If they approached zero symmetrically, $a = -b$, then the value of the integral would be zero.

$$\frac{1}{2} \lim_{b\to\infty} \ln\left(\frac{b^2+1}{b^2+1}\right) = 0$$

We could make the integral have any value we pleased by choosing $a = -cb$.

We can assign a meaning to divergent integrals of the form $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ with the Cauchy principal value. The Cauchy principal value of the integral is defined

$$⨍_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{a\to\infty} \int_{-a}^{a} f(x)\,\mathrm{d}x.$$

The Cauchy principal value is obtained by approaching infinity symmetrically.

The Cauchy principal value of $\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,\mathrm{d}x$ is defined

$$\begin{aligned} ⨍_{-\infty}^{\infty} \frac{x}{x^2+1}\,\mathrm{d}x &= \lim_{a\to\infty} \int_{-a}^{a} \frac{x}{x^2+1}\,\mathrm{d}x \\ &= \lim_{a\to\infty} \left[\frac{1}{2}\ln\left(x^2+1\right)\right]_{-a}^{a} \\ &= 0. \end{aligned}$$

Result 15.2.1 Cauchy Principal Value. If $f(x)$ is continuous on (a, b) except at the point $x_0 \in (a, b)$ then the integral of $f(x)$ is defined

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\epsilon \to 0^+} \int_a^{x_0-\epsilon} f(x)\,\mathrm{d}x + \lim_{\delta \to 0^+} \int_{x_0+\delta}^b f(x)\,\mathrm{d}x.$$

The Cauchy principal value of the integral is defined

$$\fint_a^b f(x)\,\mathrm{d}x = \lim_{\epsilon \to 0^+} \left(\int_a^{x_0-\epsilon} f(x)\,\mathrm{d}x + \int_{x_0+\epsilon}^b f(x)\,\mathrm{d}x \right).$$

If $f(x)$ is continuous on $(-\infty, \infty)$ then the integral of $f(x)$ is defined

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{a \to -\infty,\ b \to \infty} \int_a^b f(x)\,\mathrm{d}x.$$

The Cauchy principal value of the integral is defined

$$\fint_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{a \to \infty} \int_{-a}^a f(x)\,\mathrm{d}x.$$

The principal value of the integral may exist when the integral diverges. If the integral exists, it is equal to the principal value of the integral.

Example 15.2.3 Clearly $\int_{-\infty}^{\infty} x\,\mathrm{d}x$ diverges, however the Cauchy principal value exists.

$$\fint_{-\infty}^{\infty} x\,\mathrm{d}x = \lim_{a \to \infty} \left[\frac{x^2}{2} \right]_{-a} a = 0$$

In general, if $f(x)$ is an odd function with no singularities on the finite real axis then

$$\-\!\!\!\!\!\!\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 0.$$

15.3 Cauchy Principal Value for Contour Integrals

Example 15.3.1 Consider the integral

$$\int_{C_r} \frac{1}{z-1}\,\mathrm{d}z,$$

where C_r is the positively oriented circle of radius r and center at the origin. From the residue theorem, we know that the integral is

$$\int_{C_r} \frac{1}{z-1}\,\mathrm{d}z = \begin{cases} 0 & \text{for } r < 1, \\ 2\pi i & \text{for } r > 1. \end{cases}$$

When $r = 1$, the integral diverges, as there is a first order pole on the path of integration. However, the principal value of the integral exists.

$$\begin{aligned} \-\!\!\!\!\!\!\int_{C_r} \frac{1}{z-1}\,\mathrm{d}z &= \lim_{\epsilon\to 0^+} \int_{\epsilon}^{2\pi-\epsilon} \frac{1}{e^{i\theta}-1} i e^{i\theta}\,\mathrm{d}\theta \\ &= \lim_{\epsilon\to 0^+} \left[\log(e^{i\theta}-1)\right]_{\epsilon}^{2\pi-\epsilon} \end{aligned}$$

We choose the branch of the logarithm with a branch cut on the positive real axis and $\arg \log z \in (0, 2\pi)$.

$$
\begin{aligned}
&= \lim_{\epsilon \to 0^+} \left(\log \left(e^{i(2\pi-\epsilon)} - 1 \right) - \log \left(e^{i\epsilon} - 1 \right) \right) \\
&= \lim_{\epsilon \to 0^+} \left(\log \left(\left(1 - i\epsilon + O(\epsilon^2)\right) - 1 \right) - \log \left(\left(1 + i\epsilon + O(\epsilon^2)\right) - 1 \right) \right) \\
&= \lim_{\epsilon \to 0^+} \left(\log \left(-i\epsilon + O(\epsilon^2) \right) - \log \left(i\epsilon + O(\epsilon^2) \right) \right) \\
&= \lim_{\epsilon \to 0^+} \left(\operatorname{Log} \left(\epsilon + O(\epsilon^2) \right) + i \arg \left(-i\epsilon + O(\epsilon^2) \right) - \operatorname{Log} \left(\epsilon + O(\epsilon^2) \right) - i \arg \left(i\epsilon + O(\epsilon^2) \right) \right) \\
&= i\frac{3\pi}{2} - i\frac{\pi}{2} \\
&= i\pi
\end{aligned}
$$

Thus we obtain

$$
\fint_{C_r} \frac{1}{z-1} \,\mathrm{d}z = \begin{cases} 0 & \text{for } r < 1, \\ \pi i & \text{for } r = 1, \\ 2\pi i & \text{for } r > 1. \end{cases}
$$

In the above example we evaluated the contour integral by parameterizing the contour. This approach is only feasible when the integrand is simple. We would like to use the residue theorem to more easily evaluate the principal value of the integral. But before we do that, we will need a preliminary result.

Result 15.3.1 Let $f(z)$ have a first order pole at $z = z_0$ and let $(z - z_0)f(z)$ be analytic in some neighborhood of z_0. Let the contour C_ϵ be a circular arc from $z_0 + \epsilon e^{i\alpha}$ to $z_0 + \epsilon e^{i\beta}$. (We assume that $\beta > \alpha$ and $\beta - \alpha < 2\pi$.)

$$
\lim_{\epsilon \to 0^+} \int_{C_\epsilon} f(z) \,\mathrm{d}z = i(\beta - \alpha) \operatorname{Res}\left(f(z), z_0\right)
$$

The contour is shown in Figure 15.4. (See Exercise 15.6 for a proof of this result.)

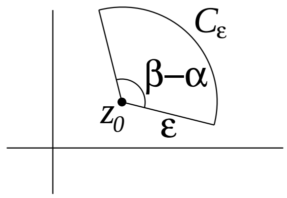


Figure 15.4: The C_ϵ Contour

Example 15.3.2 Consider

$$\int_C \frac{1}{z-1} \, \mathrm{d}z$$

where C is the unit circle. Let C_p be the circular arc of radius 1 that starts and ends a distance of ϵ from $z = 1$. Let C_ϵ be the positive, circular arc of radius ϵ with center at $z = 1$ that joins the endpoints of C_p. Let C_i, be the union of C_p and C_ϵ. (C_p stands for Principal value Contour; C_i stands for Indented Contour.) C_i is an indented contour that avoids the first order pole at $z = 1$. Figure 15.5 shows the three contours.

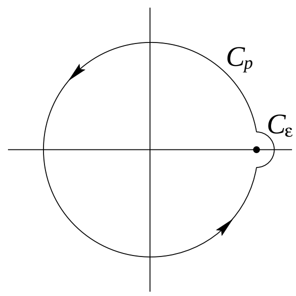


Figure 15.5: The Indented Contour.

Note that the principal value of the integral is

$$-\!\!\!\!\!\!\int_C \frac{1}{z-1} \, \mathrm{d}z = \lim_{\epsilon \to 0^+} \int_{C_p} \frac{1}{z-1} \, \mathrm{d}z.$$

We can calculate the integral along C_i with the residue theorem.

$$\int_{C_i} \frac{1}{z-1} \, \mathrm{d}z = 2\pi i$$

We can calculate the integral along C_ϵ using Result 15.3.1. Note that as $\epsilon \to 0^+$, the contour becomes a semi-circle, a circular arc of π radians.

$$\lim_{\epsilon \to 0^+} \int_{C_\epsilon} \frac{1}{z-1} \, \mathrm{d}z = i\pi \operatorname{Res}\left(\frac{1}{z-1}, 1\right) = i\pi$$

Now we can write the principal value of the integral along C in terms of the two known integrals.

$$\begin{aligned} -\!\!\!\!\!\!\int_C \frac{1}{z-1} \, \mathrm{d}z &= \int_{C_i} \frac{1}{z-1} \, \mathrm{d}z - \int_{C_\epsilon} \frac{1}{z-1} \, \mathrm{d}z \\ &= i2\pi - i\pi \\ &= i\pi \end{aligned}$$

In the previous example, we formed an indented contour that included the first order pole. You can show that if we had indented the contour to exclude the pole, we would obtain the same result. (See Exercise 15.8.)

We can extend the residue theorem to principal values of integrals. (See Exercise 15.7.)

Result 15.3.2 Residue Theorem for Principal Values. Let $f(z)$ be analytic inside and on a simple, closed, positive contour C, except for isolated singularities at $z_1, \ldots, z_m$ inside the contour and first order poles at $\zeta_1, \ldots, \zeta_n$ on the contour. Further, let the contour be C^1 at the locations of these first order poles. (i.e., the contour does not have a corner at any of the first order poles.) Then the principal value of the integral of $f(z)$ along C is

$$-\!\!\!\!\!\!\int_C f(z)\,\mathrm{d}z = i2\pi \sum_{j=1}^{m} \operatorname{Res}(f(z), z_j) + i\pi \sum_{j=1}^{n} \operatorname{Res}(f(z), \zeta_j).$$

15.4 Integrals on the Real Axis

Example 15.4.1 We wish to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x.$$

We can evaluate this integral directly using calculus.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x &= [\arctan x]_{-\infty}^{\infty} \\ &= \pi\end{aligned}$$

Now we will evaluate the integral using contour integration. Let C_R be the semicircular arc from R to $-R$ in the upper half plane. Let C be the union of C_R and the interval $[-R, R]$.

We can evaluate the integral along C with the residue theorem. The integrand has first order poles at $z = \pm i$.

For $R > 1$, we have

$$\begin{aligned}\int_C \frac{1}{z^2+1}\,\mathrm{d}z &= i2\pi \operatorname{Res}\left(\frac{1}{z^2+1}, i\right)\\ &= i2\pi\frac{1}{2i}\\ &= \pi.\end{aligned}$$

Now we examine the integral along C_R. We use the maximum modulus integral bound to show that the value of the integral vanishes as $R \to \infty$.

$$\begin{aligned}\left|\int_{C_R} \frac{1}{z^2+1}\,\mathrm{d}z\right| &\leq \pi R \max_{z\in C_R}\left|\frac{1}{z^2+1}\right|\\ &= \pi R\frac{1}{R^2-1}\\ &\to 0 \quad \text{as } R \to \infty.\end{aligned}$$

Now we are prepared to evaluate the original real integral.

$$\int_C \frac{1}{z^2+1}\,\mathrm{d}z = \pi$$

$$\int_{-R}^{R} \frac{1}{x^2+1}\,\mathrm{d}x + \int_{C_R} \frac{1}{z^2+1}\,\mathrm{d}z = \pi$$

We take the limit as $R \to \infty$.

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = \pi}$$

We would get the same result by closing the path of integration in the lower half plane. Note that in this case the closed contour would be in the negative direction.

If you are really observant, you may have noticed that we did something a little funny in evaluating

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x.$$

The definition of this improper integral is

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = \lim_{a\to+\infty} \int_{-a}^{0} \frac{1}{x^2+1}\,\mathrm{d}x + = \lim_{b\to+\infty} \int_{0}^{b} \frac{1}{x^2+1}\,\mathrm{d}x.$$

In the above example we instead computed

$$\lim_{R\to+\infty} \int_{-R}^{R} \frac{1}{x^2+1}\,\mathrm{d}x.$$

Note that for some integrands, the former and latter are not the same. Consider the integral of $\frac{x}{x^2+1}$.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,\mathrm{d}x &= \lim_{a\to+\infty} \int_{-a}^{0} \frac{x}{x^2+1}\,\mathrm{d}x + \lim_{b\to+\infty} \int_{0}^{b} \frac{x}{x^2+1}\,\mathrm{d}x \\ &= \lim_{a\to+\infty} \left(\frac{1}{2}\log|a^2+1|\right) + \lim_{b\to+\infty} \left(-\frac{1}{2}\log|b^2+1|\right)\end{aligned}$$

Note that the limits do not exist and hence the integral diverges. We get a different result if the limits of integration approach infinity symmetrically.

$$\begin{aligned}\lim_{R\to+\infty} \int_{-R}^{R} \frac{x}{x^2+1}\,\mathrm{d}x &= \lim_{R\to+\infty} \left(\frac{1}{2}(\log|R^2+1| - \log|R^2+1|)\right) \\ &= 0\end{aligned}$$

(Note that the integrand is an odd function, so the integral from $-R$ to R is zero.) We call this the *principal value* of the integral and denote it by writing "PV" in front of the integral sign or putting a dash through the integral.

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \equiv ⨍_{-\infty}^{\infty} f(x)\,\mathrm{d}x \equiv \lim_{R\to+\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x$$

The principal value of an integral may exist when the integral diverges. If the integral does converge, then it is equal to its principal value.

We can use the method of Example 15.4.1 to evaluate the principal value of integrals of functions that vanish fast enough at infinity.

Result 15.4.1 Let $f(z)$ be analytic except for isolated singularities, with only first order poles on the real axis. Let C_R be the semi-circle from R to $-R$ in the upper half plane. If

$$\lim_{R \to \infty} \left(R \max_{z \in C_R} |f(z)| \right) = 0$$

then

$$⨍_{-\infty}^{\infty} f(x) \, \mathrm{d}x = i2\pi \sum_{k=1}^{m} \operatorname{Res}\,(f(z), z_k) + i\pi \sum_{k=1}^{n} \operatorname{Res}\,(f(z), x_k)$$

where $z_1, \ldots z_m$ are the singularities of $f(z)$ in the upper half plane and $x_1, \ldots, x_n$ are the first order poles on the real axis.
Now let C_R be the semi-circle from R to $-R$ in the lower half plane. If

$$\lim_{R \to \infty} \left(R \max_{z \in C_R} |f(z)| \right) = 0$$

then

$$⨍_{-\infty}^{\infty} f(x) \, \mathrm{d}x = -i2\pi \sum_{k=1}^{m} \operatorname{Res}\,(f(z), z_k) - i\pi \sum_{k=1}^{n} \operatorname{Res}\,(f(z), x_k)$$

where $z_1, \ldots z_m$ are the singularities of $f(z)$ in the lower half plane and $x_1, \ldots, x_n$ are the first order poles on the real axis.

This result is proved in Exercise 15.9. Of course we can use this result to evaluate the integrals of the form

$$\int_0^\infty f(z)\,\mathrm{d}z,$$

where $f(x)$ is an even function.

15.5 Fourier Integrals

In order to do Fourier transforms, which are useful in solving differential equations, it is necessary to be able to calculate Fourier integrals. Fourier integrals have the form

$$\int_{-\infty}^\infty \mathrm{e}^{i\omega x} f(x)\,\mathrm{d}x.$$

We evaluate these integrals by closing the path of integration in the lower or upper half plane and using techniques of contour integration.

Consider the integral

$$\int_0^{\pi/2} \mathrm{e}^{-R\sin\theta}\,\mathrm{d}\theta.$$

Since $2\theta/\pi \leq \sin\theta$ for $0 \leq \theta \leq \pi/2$,

$$\mathrm{e}^{-R\sin\theta} \leq \mathrm{e}^{-R2\theta/\pi} \quad \text{for } 0 \leq \theta \leq \pi/2$$

$$
\begin{aligned}
\int_0^{\pi/2} \mathrm{e}^{-R\sin\theta}\,\mathrm{d}\theta &\leq \int_0^{\pi/2} \mathrm{e}^{-R2\theta/\pi}\,\mathrm{d}\theta \\
&= \left[-\frac{\pi}{2R}\,\mathrm{e}^{-R2\theta/\pi}\right]_0^{\pi/2} \\
&= -\frac{\pi}{2R}(\,\mathrm{e}^{-R} - 1) \\
&\leq \frac{\pi}{2R} \\
&\to 0 \quad \text{as } R \to \infty
\end{aligned}
$$

We can use this to prove the following Result 15.5.1. (See Exercise 15.13.)

Result 15.5.1 Jordan's Lemma.

$$\int_0^{\pi} \mathrm{e}^{-R\sin\theta}\,\mathrm{d}\theta < \frac{\pi}{R}.$$

Suppose that $f(z)$ vanishes as $|z| \to \infty$. If ω is a (positive/negative) real number and C_R is a semi-circle of radius R in the (upper/lower) half plane then the integral

$$\int_{C_R} f(z)\,\mathrm{e}^{i\omega z}\,\mathrm{d}z$$

vanishes as $R \to \infty$.

We can use Jordan's Lemma and the Residue Theorem to evaluate many Fourier integrals. Consider $\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{i\omega x}\,\mathrm{d}x$ where ω is a positive real number. Let $f(z)$ be analytic except for isolated singularities, with only first order poles on the real axis. Let C be the contour from $-R$ to R on the real axis and then back to $-R$ along a semi-circle in the upper half plane. If R is large enough so that C encloses all the singularities of $f(z)$ in the upper half plane then

$$\int_C f(z)\,\mathrm{e}^{i\omega z}\,\mathrm{d}z = i2\pi \sum_{k=1}^{m} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, z_k\right) + i\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, x_k\right)$$

where $z_1, \dots z_m$ are the singularities of $f(z)$ in the upper half plane and $x_1, \dots, x_n$ are the first order poles on the real axis. If $f(z)$ vanishes as $|z| \to \infty$ then the integral on C_R vanishes as $R \to \infty$ by Jordan's Lemma.

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{i\omega x}\,\mathrm{d}x = i2\pi \sum_{k=1}^{m} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, z_k\right) + i\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, x_k\right)$$

For negative ω we close the path of integration in the lower half plane. Note that the contour is then in the negative direction.

Result 15.5.2 Fourier Integrals. Let $f(z)$ be analytic except for isolated singularities, with only first order poles on the real axis. Suppose that $f(z)$ vanishes as $|z| \to \infty$. If ω is a positive real number then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{i\omega x}\,\mathrm{d}x = i2\pi \sum_{k=1}^{m} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, z_k\right) + i\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, x_k\right)$$

where $z_1, \dots z_m$ are the singularities of $f(z)$ in the upper half plane and $x_1, \dots, x_n$ are the first order poles on the real axis. If ω is a negative real number then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{i\omega x}\,\mathrm{d}x = -i2\pi \sum_{k=1}^{m} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, z_k\right) - i\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, x_k\right)$$

where $z_1, \dots z_m$ are the singularities of $f(z)$ in the lower half plane and $x_1, \dots, x_n$ are the first order poles on the real axis.

15.6 Fourier Cosine and Sine Integrals

Fourier cosine and sine integrals have the form,

$$\int_0^\infty f(x) \cos(\omega x) \, \mathrm{d}x \quad \text{and} \quad \int_0^\infty f(x) \sin(\omega x) \, \mathrm{d}x.$$

If $f(x)$ is even/odd then we can evaluate the cosine/sine integral with the method we developed for Fourier integrals.

Let $f(z)$ be analytic except for isolated singularities, with only first order poles on the real axis. Suppose that $f(x)$ is an even function and that $f(z)$ vanishes as $|z| \to \infty$. We consider real $\omega > 0$.

$$⨍_0^\infty f(x) \cos(\omega x) \, \mathrm{d}x = \frac{1}{2} ⨍_{-\infty}^\infty f(x) \cos(\omega x) \, \mathrm{d}x$$

Since $f(x) \sin(\omega x)$ is an odd function,

$$\frac{1}{2} ⨍_{-\infty}^\infty f(x) \sin(\omega x) \, \mathrm{d}x = 0.$$

Thus

$$⨍_0^\infty f(x) \cos(\omega x) \, \mathrm{d}x = \frac{1}{2} ⨍_{-\infty}^\infty f(x) \, \mathrm{e}^{i\omega x} \, \mathrm{d}x$$

Now we apply Result 15.5.2.

$$⨍_0^\infty f(x) \cos(\omega x) \, \mathrm{d}x = i\pi \sum_{k=1}^m \operatorname{Res}\left(f(z) \, \mathrm{e}^{i\omega z}, z_k\right) + \frac{i\pi}{2} \sum_{k=1}^n \operatorname{Res}\left(f(z) \, \mathrm{e}^{i\omega z}, x_k\right)$$

where $z_1, \ldots z_m$ are the singularities of $f(z)$ in the upper half plane and $x_1, \ldots, x_n$ are the first order poles on the real axis.

If $f(x)$ is an odd function, we note that $f(x)\cos(\omega x)$ is an odd function to obtain the analogous result for Fourier sine integrals.

Result 15.6.1 Fourier Cosine and Sine Integrals. Let $f(z)$ be analytic except for isolated singularities, with only first order poles on the real axis. Suppose that $f(x)$ is an even function and that $f(z)$ vanishes as $|z| \to \infty$. We consider real $\omega > 0$.

$$\int_0^\infty f(x)\cos(\omega x)\,\mathrm{d}x = i\pi \sum_{k=1}^{m} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, z_k\right) + \frac{i\pi}{2}\sum_{k=1}^{n} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, x_k\right)$$

where $z_1, \dots z_m$ are the singularities of $f(z)$ in the upper half plane and $x_1, \dots, x_n$ are the first order poles on the real axis. If $f(x)$ is an odd function then,

$$\int_0^\infty f(x)\sin(\omega x)\,\mathrm{d}x = \pi \sum_{k=1}^{\mu} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, \zeta_k\right) + \frac{\pi}{2}\sum_{k=1}^{n} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, x_k\right)$$

where $\zeta_1, \dots \zeta_\mu$ are the singularities of $f(z)$ in the lower half plane and $x_1, \dots, x_n$ are the first order poles on the real axis.

Now suppose that $f(x)$ is neither even nor odd. We can evaluate integrals of the form:

$$\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\sin(\omega x)\,\mathrm{d}x$$

by writing them in terms of Fourier integrals

$$\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{i\omega x}\,\mathrm{d}x + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-i\omega x}\,\mathrm{d}x$$
$$\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,\mathrm{d}x = -\frac{i}{2}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{i\omega x}\,\mathrm{d}x + \frac{i}{2}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-i\omega x}\,\mathrm{d}x$$

15.7 Contour Integration and Branch Cuts

Example 15.7.1 Consider

$$\int_0^{\infty} \frac{x^{-a}}{x+1}\,\mathrm{d}x, \quad 0 < a < 1,$$

where x^{-a} denotes $\exp(-a\ln(x))$. We choose the branch of the function

$$f(z) = \frac{z^{-a}}{z+1} \quad |z| > 0,\ 0 < \arg z < 2\pi$$

with a branch cut on the positive real axis.

Let C_ϵ and C_R denote the circular arcs of radius ϵ and R where $\epsilon < 1 < R$. C_ϵ is negatively oriented; C_R is positively oriented. Consider the closed contour C that is traced by a point moving from C_ϵ to C_R above the branch cut, next around C_R, then below the cut to C_ϵ, and finally around C_ϵ. (See Figure 15.10.)

We write $f(z)$ in polar coordinates.

$$f(z) = \frac{\exp(-a\ \log z)}{z+1} = \frac{\exp(-a(\log r + i\theta))}{r\,\mathrm{e}^{i\theta}+1}$$

We evaluate the function above, $(z = r\,\mathrm{e}^{i0})$, and below, $(z = r\,\mathrm{e}^{i2\pi})$, the branch cut.

$$f(r\,\mathrm{e}^{i0}) = \frac{\exp[-a(\log r + i0)]}{r+1} = \frac{r^{-a}}{r+1}$$
$$f(r\,\mathrm{e}^{i2\pi}) = \frac{\exp[-a(\log r + i2\pi)]}{r+1} = \frac{r^{-a}\,\mathrm{e}^{-i2a\pi}}{r+1}.$$

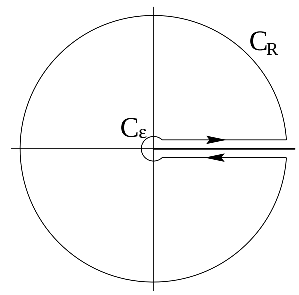


Figure 15.6:

We use the residue theorem to evaluate the integral along C.

$$\oint_C f(z)\,\mathrm{d}z = i2\pi \operatorname{Res}(f(z), -1)$$

$$\int_\epsilon^R \frac{r^{-a}}{r+1} dr + \int_{C_R} f(z)\,\mathrm{d}z - \int_\epsilon^R \frac{r^{-a}\,\mathrm{e}^{-i2a\pi}}{r+1} dr + \int_{C_\epsilon} f(z)\,\mathrm{d}z = i2\pi \operatorname{Res}(f(z), -1)$$

The residue is

$$\operatorname{Res}(f(z), -1) = \exp(-a\log(-1)) = \exp(-a(\log 1 + i\pi)) = \mathrm{e}^{-ia\pi}.$$

We bound the integrals along C_ϵ and C_R with the maximum modulus integral bound.

$$\left|\int_{C_\epsilon} f(z)\,\mathrm{d}z\right| \leq 2\pi\epsilon \frac{\epsilon^{-a}}{1-\epsilon} = 2\pi\frac{\epsilon^{1-a}}{1-\epsilon}$$

$$\left|\int_{C_R} f(z)\,\mathrm{d}z\right| \leq 2\pi R \frac{R^{-a}}{R-1} = 2\pi\frac{R^{1-a}}{R-1}$$

Since $0 < a < 1$, the values of the integrals tend to zero as $\epsilon \to 0$ and $R \to \infty$. Thus we have

$$\int_0^\infty \frac{r^{-a}}{r+1} dr = i2\pi \frac{\mathrm{e}^{-ia\pi}}{1 - \mathrm{e}^{-i2a\pi}}$$

$$\int_0^{\infty} \frac{x^{-a}}{x+1}\,\mathrm{d}x = \frac{\pi}{\sin a\pi}$$

Result 15.7.1 Integrals from Zero to Infinity. Let $f(z)$ be a single-valued analytic function with only isolated singularities and no singularities on the positive, real axis, $[0, \infty)$. Let $a \notin \mathbb{Z}$. If the integrals exist then,

$$\int_0^\infty f(x)\,\mathrm{d}x = -\sum_{k=1}^{n} \operatorname{Res}\left(f(z)\log z, z_k\right),$$

$$\int_0^\infty x^a f(x)\,\mathrm{d}x = \frac{i2\pi}{1-\mathrm{e}^{i2\pi a}} \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z), z_k\right),$$

$$\int_0^\infty f(x)\log x\,\mathrm{d}x = -\frac{1}{2}\sum_{k=1}^{n} \operatorname{Res}\left(f(z)\log^2 z, z_k\right) + i\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z)\log z, z_k\right),$$

$$\int_0^\infty x^a f(x)\log x\,\mathrm{d}x = \frac{i2\pi}{1-\mathrm{e}^{i2\pi a}} \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z)\log z, z_k\right) + \frac{\pi^2 a}{\sin^2(\pi a)} \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z), z_k\right),$$

$$\int_0^\infty x^a f(x)\log^m x\,\mathrm{d}x = \frac{\partial^m}{\partial a^m}\left(\frac{i2\pi}{1-\mathrm{e}^{i2\pi a}} \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z), z_k\right)\right),$$

where $z_1, \ldots, z_n$ are the singularities of $f(z)$ and there is a branch cut on the positive real axis with $0 < \arg(z) < 2\pi$.

15.8 Exploiting Symmetry

We have already used symmetry of the integrand to evaluate certain integrals. For $f(x)$ an even function we were able to evaluate $\int_0^\infty f(x)\,\mathrm{d}x$ by extending the range of integration from $-\infty$ to ∞. For

$$\int_0^\infty x^\alpha f(x)\,\mathrm{d}x$$

we put a branch cut on the positive real axis and noted that the value of the integrand below the branch cut is a constant multiple of the value of the function above the branch cut. This enabled us to evaluate the real integral with contour integration. In this section we will use other kinds of symmetry to evaluate integrals. We will discover that periodicity of the integrand will produce this symmetry.

15.8.1 Wedge Contours

We note that $z^n = r^n\,\mathrm{e}^{in\theta}$ is periodic in θ with period $2\pi/n$. The real and imaginary parts of z^n are odd periodic in θ with period π/n. This observation suggests that certain integrals on the positive real axis may be evaluated by closing the path of integration with a wedge contour.

Example 15.8.1 Consider

$$\int_0^\infty \frac{1}{1+x^n}\,\mathrm{d}x$$

where $n \in \mathbb{N}$, $n \geq 2$. We can evaluate this integral using Result 15.7.1.

$$\begin{aligned}
\int_0^\infty \frac{1}{1+x^n} \, \mathrm{d}x &= -\sum_{k=0}^{n-1} \operatorname{Res}\left(\frac{\log z}{1+z^n}, \mathrm{e}^{i\pi(1+2k)/n} \right) \\
&= -\sum_{k=0}^{n-1} \lim_{z \to \mathrm{e}^{i\pi(1+2k)/n}} \left(\frac{\left(z - \mathrm{e}^{i\pi(1+2k)/n}\right) \log z}{1+z^n} \right) \\
&= -\sum_{k=0}^{n-1} \lim_{z \to \mathrm{e}^{i\pi(1+2k)/n}} \left(\frac{\log z + \left(z - \mathrm{e}^{i\pi(1+2k)/n}\right)/z}{n z^{n-1}} \right) \\
&= -\sum_{k=0}^{n-1} \left(\frac{i\pi(1+2k)/n}{n \, \mathrm{e}^{i\pi(1+2k)(n-1)/n}} \right) \\
&= -\frac{i\pi}{n^2 \, \mathrm{e}^{i\pi(n-1)/n}} \sum_{k=0}^{n-1} (1+2k) \, \mathrm{e}^{i2\pi k/n} \\
&= \frac{i2\pi \, \mathrm{e}^{i\pi/n}}{n^2} \sum_{k=1}^{n-1} k \, \mathrm{e}^{i2\pi k/n} \\
&= \frac{i2\pi \, \mathrm{e}^{i\pi/n}}{n^2} \frac{n}{\mathrm{e}^{i2\pi/n} - 1} \\
&= \frac{\pi}{n \sin(\pi/n)}
\end{aligned}$$

This is a bit grungy. To find a spiffier way to evaluate the integral we note that if we write the integrand as a function of r and θ, it is periodic in θ with period $2\pi/n$.

$$\frac{1}{1+z^n} = \frac{1}{1+r^n \, \mathrm{e}^{in\theta}}$$

The integrand along the rays $\theta = 2\pi/n, 4\pi/n, 6\pi/n, \ldots$ has the same value as the integrand on the real axis. Consider the contour C that is the boundary of the wedge $0 < r < R$, $0 < \theta < 2\pi/n$. There is one singularity inside the contour. We evaluate the residue there.

$$\begin{aligned}
\operatorname{Res}\left(\frac{1}{1+z^n}, \mathrm{e}^{i\pi/n}\right) &= \lim_{z \to \mathrm{e}^{i\pi/n}} \frac{z - \mathrm{e}^{i\pi/n}}{1+z^n} \\
&= \lim_{z \to \mathrm{e}^{i\pi/n}} \frac{1}{nz^{n-1}} \\
&= -\frac{\mathrm{e}^{i\pi/n}}{n}
\end{aligned}$$

We evaluate the integral along C with the residue theorem.

$$\int_C \frac{1}{1+z^n} \,\mathrm{d}z = \frac{-i2\pi\,\mathrm{e}^{i\pi/n}}{n}$$

Let C_R be the circular arc. The integral along C_R vanishes as $R \to \infty$.

$$\begin{aligned}
\left|\int_{C_R} \frac{1}{1+z^n} \,\mathrm{d}z\right| &\leq \frac{2\pi R}{n} \max_{z \in C_R} \left|\frac{1}{1+z^n}\right| \\
&\leq \frac{2\pi R}{n} \frac{1}{R^n - 1} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

We parametrize the contour to evaluate the desired integral.

$$\int_0^\infty \frac{1}{1+x^n} \,\mathrm{d}x + \int_\infty^0 \frac{1}{1+x^n} \mathrm{e}^{i2\pi/n} \,\mathrm{d}x = \frac{-i2\pi\,\mathrm{e}^{i\pi/n}}{n}$$

$$\int_0^\infty \frac{1}{1+x^n} \,\mathrm{d}x = \frac{-i2\pi\,\mathrm{e}^{i\pi/n}}{n(1 - \mathrm{e}^{i2\pi/n})}$$

$$\boxed{\int_0^\infty \frac{1}{1+x^n} \,\mathrm{d}x = \frac{\pi}{n \sin(\pi/n)}}$$

15.8.2 Box Contours

Recall that $e^z = e^{x+iy}$ is periodic in y with period 2π. This implies that the hyperbolic trigonometric functions $\cosh z$, $\sinh z$ and $\tanh z$ are periodic in y with period 2π and odd periodic in y with period π. We can exploit this property to evaluate certain integrals on the real axis by closing the path of integration with a box contour.

Example 15.8.2 Consider the integral

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{\cosh x}\, dx &= \left[i \log\left(\tanh\left(\frac{i\pi}{4} + \frac{x}{2} \right) \right) \right]_{-\infty}^{\infty} \\ &= i\log(1) - i\log(-1) \\ &= \pi.\end{aligned}$$

We will evaluate this integral using contour integration. Note that

$$\cosh(x + i\pi) = \frac{e^{x+i\pi} + e^{-x-i\pi}}{2} = -\cosh(x).$$

Consider the box contour C that is the boundary of the region $-R < x < R$, $0 < y < \pi$. The only singularity of the integrand inside the contour is a first order pole at $z = i\pi/2$. We evaluate the integral along C with the residue theorem.

$$\begin{aligned}\oint_C \frac{1}{\cosh z}\, dz &= i2\pi \operatorname{Res}\left(\frac{1}{\cosh z}, \frac{i\pi}{2} \right) \\ &= i2\pi \lim_{z \to i\pi/2} \frac{z - i\pi/2}{\cosh z} \\ &= i2\pi \lim_{z \to i\pi/2} \frac{1}{\sinh z} \\ &= 2\pi\end{aligned}$$

The integrals along the sides of the box vanish as $R \to \infty$.

$$\begin{aligned}
\left| \int_{\pm R}^{\pm R + i\pi} \frac{1}{\cosh z} \, \mathrm{d}z \right| &\leq \pi \max_{z \in [\pm R \ldots \pm R + i\pi]} \left| \frac{1}{\cosh z} \right| \\
&\leq \pi \max_{y \in [0 \ldots \pi]} \left| \frac{2}{\mathrm{e}^{\pm R + iy} + \mathrm{e}^{\mp R - iy}} \right| \\
&= \frac{2}{\mathrm{e}^R - \mathrm{e}^{-R}} \\
&\leq \frac{\pi}{\sinh R} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

The value of the integrand on the top of the box is the negative of its value on the bottom. We take the limit as $R \to \infty$.

$$\int_{-\infty}^{\infty} \frac{1}{\cosh x} \, \mathrm{d}x + \int_{\infty}^{-\infty} \frac{1}{-\cosh x} \, \mathrm{d}x = 2\pi$$

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{\cosh x} \, \mathrm{d}x = \pi}$$

15.9 Definite Integrals Involving Sine and Cosine

Example 15.9.1 For real-valued a, evaluate the integral:

$$f(a) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + a \sin \theta}.$$

What is the value of the integral for complex-valued a.

Real-Valued a. For $-1 < a < 1$, the integrand is bounded, hence the integral exists. For $|a| = 1$, the integrand has a second order pole on the path of integration. For $|a| > 1$ the integrand has two first order poles on the path of integration. The integral is divergent for these two cases. Thus we see that the integral exists for $-1 < a < 1$.

For $a = 0$, the value of the integral is 2π. Now consider $a \neq 0$. We make the change of variables $z = \mathrm{e}^{i\theta}$. The real integral from $\theta = 0$ to $\theta = 2\pi$ becomes a contour integral along the unit circle, $|z| = 1$. We write the sine, cosine and the differential in terms of z.

$$\sin\theta = \frac{z - z^{-1}}{2i}, \quad \cos\theta = \frac{z + z^{-1}}{2}, \quad \mathrm{d}z = i\,\mathrm{e}^{i\theta}\mathrm{d}\theta, \quad \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$$

We write $f(a)$ as an integral along C, the positively oriented unit circle $|z| = 1$.

$$f(a) = \oint_C \frac{1/(iz)}{1 + a(z - z^{-1})/(2i)}\,\mathrm{d}z = \oint_C \frac{2/a}{z^2 + (2i/a)z - 1}\,\mathrm{d}z$$

We factor the denominator of the integrand.

$$f(a) = \oint_C \frac{2/a}{(z - z_1)(z - z_2)}\,\mathrm{d}z$$

$$z_1 = i\left(\frac{-1 + \sqrt{1 - a^2}}{a}\right), \quad z_2 = i\left(\frac{-1 - \sqrt{1 - a^2}}{a}\right)$$

Because $|a| < 1$, the second root is outside the unit circle.

$$|z_2| = \frac{1 + \sqrt{1 - a^2}}{|a|} > 1.$$

Since $|z_1 z_2| = 1$, $|z_1| < 1$. Thus the pole at z_1 is inside the contour and the pole at z_2 is outside. We evaluate the contour integral with the residue theorem.

$$\begin{aligned}
f(a) &= \oint_C \frac{2/a}{z^2 + (2i/a)z - 1}\,\mathrm{d}z \\
&= i2\pi \frac{2/a}{z_1 - z_2} \\
&= i2\pi \frac{1}{i\sqrt{1 - a^2}}
\end{aligned}$$

$$\boxed{f(a) = \frac{2\pi}{\sqrt{1-a^2}}}$$

Complex-Valued a. We note that the integral converges except for real-valued a satisfying $|a| \geq 1$. On any closed subset of $\mathbb{C} \setminus \{a \in \mathbb{R} \,|\, |a| \geq 1\}$ the integral is uniformly convergent. Thus except for the values $\{a \in \mathbb{R} \,|\, |a| \geq 1\}$, we can differentiate the integral with respect to a. $f(a)$ is analytic in the complex plane except for the set of points on the real axis: $a \in (-\infty \ldots -1]$ and $a \in [1 \ldots \infty)$. The value of the analytic function $f(a)$ on the real axis for the interval $(-1 \ldots 1)$ is

$$f(a) = \frac{2\pi}{\sqrt{1-a^2}}.$$

By analytic continuation we see that the value of $f(a)$ in the complex plane is the branch of the function

$$f(a) = \frac{2\pi}{(1-a^2)^{1/2}}$$

where $f(a)$ is positive, real-valued for $a \in (-1 \ldots 1)$ and there are branch cuts on the real axis on the intervals: $(-\infty \ldots -1]$ and $[1 \ldots \infty)$.

Result 15.9.1 For evaluating integrals of the form

$$\int_a^{a+2\pi} F(\sin\theta, \cos\theta)\,\mathrm{d}\theta$$

it may be useful to make the change of variables $z = \mathrm{e}^{i\theta}$. This gives us a contour integral along the unit circle about the origin. We can write the sine, cosine and differential in terms of z.

$$\sin\theta = \frac{z - z^{-1}}{2i}, \quad \cos\theta = \frac{z + z^{-1}}{2}, \quad d\theta = \frac{\mathrm{d}z}{iz}$$

15.10 Infinite Sums

The function $g(z) = \pi \cot(\pi z)$ has simple poles at $z = n \in \mathbb{Z}$. The residues at these points are all unity.

$$\begin{aligned}
\operatorname{Res}\left(\pi \cot(\pi z), n\right) &= \lim_{z \to n} \frac{\pi(z-n)\cos(\pi z)}{\sin(\pi z)} \\
&= \lim_{z \to n} \frac{\pi \cos(\pi z) - \pi(z-n)\sin(\pi z)}{\pi \cos(\pi z)} \\
&= 1
\end{aligned}$$

Let C_n be the square contour with corners at $z = (n + 1/2)(\pm 1 \pm i)$. Recall that

$$\cos z = \cos x \cosh y - i \sin x \sinh y \quad \text{and} \quad \sin z = \sin x \cosh y + i \cos x \sinh y.$$

First we bound the modulus of $\cot(z)$.

$$\begin{aligned}
|\cot(z)| &= \left| \frac{\cos x \cosh y - i \sin x \sinh y}{\sin x \cosh y + i \cos x \sinh y} \right| \\
&= \sqrt{\frac{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}} \\
&\leq \sqrt{\frac{\cosh^2 y}{\sinh^2 y}} \\
&= |\coth(y)|
\end{aligned}$$

The hyperbolic cotangent, $\coth(y)$, has a simple pole at $y = 0$ and tends to ± 1 as $y \to \pm\infty$.

Along the top and bottom of C_n, $(z = x \pm i(n + 1/2))$, we bound the modulus of $g(z) = \pi \cot(\pi z)$.

$$|\pi \cot(\pi z)| \leq \pi \left| \coth(\pi(n + 1/2)) \right|$$

Along the left and right sides of C_n, $(z = \pm(n+1/2)+iy)$, the modulus of the function is bounded by a constant.

$$\begin{aligned}
|g(\pm(n+1/2)+iy)| &= \left|\pi \frac{\cos(\pi(n+1/2))\cosh(\pi y) \mp i \sin(\pi(n+1/2))\sinh(\pi y)}{\sin(\pi(n+1/2))\cosh(\pi y) + i\cos(\pi(n+1/2))\sinh(\pi y)}\right| \\
&= |\mp i\pi \tanh(\pi y)| \\
&\leq \pi
\end{aligned}$$

Thus the modulus of $\pi\cot(\pi z)$ can be bounded by a constant M on C_n.

Let $f(z)$ be analytic except for isolated singularities. Consider the integral,

$$\oint_{C_n} \pi \cot(\pi z) f(z) \, \mathrm{d}z.$$

We use the maximum modulus integral bound.

$$\left|\oint_{C_n} \pi \cot(\pi z) f(z) \, \mathrm{d}z\right| \leq (8n+4)M \max_{z \in C_n} |f(z)|$$

Note that if

$$\lim_{|z|\to\infty} |z f(z)| = 0,$$

then

$$\lim_{n\to\infty} \oint_{C_n} \pi \cot(\pi z) f(z) \, \mathrm{d}z = 0.$$

This implies that the sum of all residues of $\pi\cot(\pi z) f(z)$ is zero. Suppose further that $f(z)$ is analytic at $z = n \in \mathbb{Z}$. The residues of $\pi\cot(\pi z) f(z)$ at $z = n$ are $f(n)$. This means

$$\sum_{n=-\infty}^{\infty} f(n) = -(\text{ sum of the residues of } \pi\cot(\pi z) f(z) \text{ at the poles of } f(z)\ ).$$

Result 15.10.1 If

$$\lim_{|z|\to\infty} |z f(z)| = 0,$$

then the sum of all the residues of $\pi \cot(\pi z) f(z)$ is zero. If in addition $f(z)$ is analytic at $z = n \in \mathbb{Z}$ then

$$\sum_{n=-\infty}^{\infty} f(n) = -(\text{ sum of the residues of } \pi \cot(\pi z) f(z) \text{ at the poles of } f(z)\text{ }).$$

Example 15.10.1 Consider the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2}, \quad a \notin \mathbb{Z}.$$

By Result 15.10.1 with $f(z) = 1/(z+a)^2$ we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} &= -\operatorname{Res}\left(\pi \cot(\pi z) \frac{1}{(z+a)^2}, -a\right) \\
&= -\pi \lim_{z\to -a} \frac{\mathrm{d}}{\mathrm{d}z} \cot(\pi z) \\
&= -\pi \frac{-\pi \sin^2(\pi z) - \pi \cos^2(\pi z)}{\sin^2(\pi z)}.
\end{aligned}$$

$$\boxed{\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2(\pi a)}}$$

Example 15.10.2 Derive $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \cdots$.

Consider the integral

$$I_n = \frac{1}{2\pi i} \int_{C_n} \frac{\mathrm{d}w}{w(w-z)\sin w}$$

where C_n is the square with corners at $w = (n + 1/2)(\pm 1 \pm i)\pi$, $n \in \mathbb{Z}^+$. With the substitution $w = x + iy$,

$$|\sin w|^2 = \sin^2 x + \sinh^2 y,$$

we see that $|1/\sin w| \leq 1$ on C_n. Thus $I_n \to 0$ as $n \to \infty$. We use the residue theorem and take the limit $n \to \infty$.

$$0 = \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n\pi(n\pi - z)} + \frac{(-1)^n}{n\pi(n\pi + z)} \right] + \frac{1}{z \sin z} - \frac{1}{z^2}$$

$$\begin{aligned} \frac{1}{\sin z} &= \frac{1}{z} - 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2 - z^2} \\ &= \frac{1}{z} - \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n\pi - z} - \frac{(-1)^n}{n\pi + z} \right] \end{aligned}$$

We substitute $z = \pi/2$ into the above expression to obtain

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \cdots$$

15.11 Exercises

The Residue Theorem

Exercise 15.1
Let $f(z)$ have a pole of order n at $z = z_0$. Prove the Residue Formula:

$$\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} \left(\frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z - z_0)^n f(z) \right] \right).$$

Hint, Solution

Exercise 15.2
Consider the function

$$f(z) = \frac{z^4}{z^2 + 1}.$$

Classify the singularities of $f(z)$ in the extended complex plane. Calculate the residue at each pole and at infinity. Find the Laurent series expansions and their domains of convergence about the points $z = 0$, $z = i$ and $z = \infty$.
Hint, Solution

Exercise 15.3
Let $P(z)$ be a polynomial none of whose roots lie on the closed contour Γ. Show that

$$\frac{1}{i2\pi} \int \frac{P'(z)}{P(z)} \, \mathrm{d}z = \text{number of roots of } P(z) \text{ which lie inside } \Gamma.$$

where the roots are counted according to their multiplicity.

Hint: From the fundamental theorem of algebra, it is always possible to factor $P(z)$ in the form $P(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$. Using this form of $P(z)$ the integrand $P'(z)/P(z)$ reduces to a very simple expression.
Hint, Solution

Exercise 15.4
Find the value of

$$\oint_C \frac{e^z}{(z-\pi)\tan z}\,dz$$

where C is the positively-oriented circle

1. $|z| = 2$
2. $|z| = 4$

Hint, Solution

Cauchy Principal Value for Real Integrals

Solution 15.1
Show that the integral

$$\int_{-1}^{1} \frac{1}{x}\,dx.$$

is divergent. Evaluate the integral

$$\int_{-1}^{1} \frac{1}{x - i\alpha}\,dx, \quad \alpha \in \mathbb{R},\ \alpha \neq 0.$$

Evaluate

$$\lim_{\alpha \to 0^+} \int_{-1}^{1} \frac{1}{x - i\alpha}\,dx$$

and

$$\lim_{\alpha \to 0^-} \int_{-1}^{1} \frac{1}{x - i\alpha}\,dx.$$

The integral exists for α arbitrarily close to zero, but diverges when $\alpha = 0$. Plot the real and imaginary part of the integrand. If one were to assign meaning to the integral for $\alpha = 0$, what would the value of the integral be?

Exercise 15.5
Do the principal values of the following integrals exist?

1. $\int_{-1}^{1} \frac{1}{x^2} \, \mathrm{d}x$,

2. $\int_{-1}^{1} \frac{1}{x^3} \, \mathrm{d}x$,

3. $\int_{-1}^{1} \frac{f(x)}{x^3} \, \mathrm{d}x$.

Assume that $f(x)$ is real analytic on the interval $(-1, 1)$.
Hint, Solution

Cauchy Principal Value for Contour Integrals

Exercise 15.6
Let $f(z)$ have a first order pole at $z = z_0$ and let $(z - z_0)f(z)$ be analytic in some neighborhood of z_0. Let the contour C_ϵ be a circular arc from $z_0 + \epsilon e^{i\alpha}$ to $z_0 + \epsilon e^{i\beta}$. (Assume that $\beta > \alpha$ and $\beta - \alpha < 2\pi$.) Show that

$$\lim_{\epsilon \to 0^+} \int_{C_\epsilon} f(z) \, \mathrm{d}z = i(\beta - \alpha) \operatorname{Res}(f(z), z_0)$$

Hint, Solution

Exercise 15.7
Let $f(z)$ be analytic inside and on a simple, closed, positive contour C, except for isolated singularities at $z_1, \ldots, z_m$ inside the contour and first order poles at $\zeta_1, \ldots, \zeta_n$ on the contour. Further, let the contour be C^1 at

the locations of these first order poles. (i.e., the contour does not have a corner at any of the first order poles.) Show that the principal value of the integral of $f(z)$ along C is

$$\int_C f(z)\,dz = i2\pi \sum_{j=1}^{m} \operatorname{Res}\left(f(z), z_j\right) + i\pi \sum_{j=1}^{n} \operatorname{Res}\left(f(z), \zeta_j\right).$$

Hint, Solution

Exercise 15.8
Let C be the unit circle. Evaluate

$$\int_C \frac{1}{z-1}\,dz$$

by indenting the contour to exclude the first order pole at $z = 1$.
Hint, Solution

Integrals from $-\infty$ to ∞

Exercise 15.9
Prove Result 15.4.1.
Hint, Solution

Exercise 15.10
Evaluate

$$\int_{-\infty}^{\infty} \frac{2x}{x^2+x+1}.$$

Hint, Solution

Exercise 15.11
Use contour integration to evaluate the integrals

1. $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^4}$,
2. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,\mathrm{d}x}{(1+x^2)^2}$,
3. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\,\mathrm{d}x$.

Hint, Solution

Exercise 15.12
Evaluate by contour integration

$$\int_0^{\infty} \frac{x^6}{(x^4+1)^2}\,\mathrm{d}x.$$

Hint, Solution

Fourier Integrals

Exercise 15.13
Suppose that $f(z)$ vanishes as $|z| \to \infty$. If ω is a (positive / negative) real number and C_R is a semi-circle of radius R in the (upper / lower) half plane then show that the integral

$$\int_{C_R} f(z)\,\mathrm{e}^{i\omega z}\,\mathrm{d}z$$

vanishes as $R \to \infty$.
Hint, Solution

Exercise 15.14
Evaluate by contour integration

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x - i\pi}\, dx.$$

Hint, Solution

Fourier Cosine and Sine Integrals

Exercise 15.15
Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx.$$

Hint, Solution

Exercise 15.16
Evaluate

$$\int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2}\, dx.$$

Hint, Solution

Exercise 15.17
Evaluate

$$\int_{0}^{\infty} \frac{\sin(\pi x)}{x(1 - x^2)}\, dx.$$

Hint, Solution

Contour Integration and Branch Cuts

Exercise 15.18
Show that

1. $\displaystyle\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,\mathrm{d}x = \frac{\pi^3}{8}$,
2. $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,\mathrm{d}x = 0.$

Hint, Solution

Exercise 15.19
By methods of contour integration find

$$\int_0^\infty \frac{\mathrm{d}x}{x^2+5x+6}$$

[Recall the trick of considering $\int_\Gamma f(z)\log z\,\mathrm{d}z$ with a suitably chosen contour Γ and branch for $\log z$.]
Hint, Solution

Exercise 15.20
Show that

$$\int_0^\infty \frac{x^a}{(x+1)^2}\,\mathrm{d}x = \frac{\pi a}{\sin(\pi a)} \quad \text{for } -1 < \Re(a) < 1.$$

From this derive that

$$\int_0^\infty \frac{\log x}{(x+1)^2}\,\mathrm{d}x = 0, \qquad \int_0^\infty \frac{\log^2 x}{(x+1)^2}\,\mathrm{d}x = \frac{\pi^2}{3}.$$

Hint, Solution

Exercise 15.21
Consider the integral

$$I(a) = \int_0^\infty \frac{x^a}{1+x^2}\,\mathrm{d}x.$$

1. For what values of a does the integral exist?

2. Evaluate the integral. Show that

$$I(a) = \frac{\pi}{2\cos(\pi a/2)}$$

3. Deduce from your answer in part (b) the results

$$\int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x = 0, \qquad \int_0^\infty \frac{\log^2 x}{1+x^2}\,\mathrm{d}x = \frac{\pi^3}{8}.$$

   You may assume that it is valid to differentiate under the integral sign.

Hint, Solution

Exercise 15.22
Let $f(z)$ be a single-valued analytic function with only isolated singularities and no singularities on the positive real axis, $[0, \infty)$. Give sufficient conditions on $f(x)$ for absolute convergence of the integral

$$\int_0^\infty x^a f(x)\,\mathrm{d}x.$$

Assume that a is not an integer. Evaluate the integral by considering the integral of $z^a f(z)$ on a suitable contour. (Consider the branch of z^a on which $1^a = 1$.)
Hint, Solution

Exercise 15.23
Using the solution to Exercise 15.22, evaluate

$$\int_0^\infty x^a f(x) \log x \, dx,$$

and

$$\int_0^\infty x^a f(x) \log^m x \, dx,$$

where m is a positive integer.
Hint, Solution

Exercise 15.24
Using the solution to Exercise 15.22, evaluate

$$\int_0^\infty f(x) \, dx,$$

i.e. examine $a = 0$. The solution will suggest a way to evaluate the integral with contour integration. Do the contour integration to corroborate the value of $\int_0^\infty f(x) \, dx$.
Hint, Solution

Exercise 15.25
Let $f(z)$ be an analytic function with only isolated singularities and no singularities on the positive real axis, $[0, \infty)$. Give sufficient conditions on $f(x)$ for absolute convergence of the integral

$$\int_0^\infty f(x) \log x \, dx$$

Evaluate the integral with contour integration.
Hint, Solution

Exercise 15.26
For what values of a does the following integral exist?

$$\int_0^\infty \frac{x^a}{1+x^4}\,\mathrm{d}x.$$

Evaluate the integral. (Consider the branch of x^a on which $1^a = 1$.)
Hint, Solution

Exercise 15.27
By considering the integral of $f(z) = z^{1/2}\log z/(z+1)^2$ on a suitable contour, show that

$$\int_0^\infty \frac{x^{1/2}\log x}{(x+1)^2}\,\mathrm{d}x = \pi, \qquad \int_0^\infty \frac{x^{1/2}}{(x+1)^2}\,\mathrm{d}x = \frac{\pi}{2}.$$

Hint, Solution

Exploiting Symmetry

Exercise 15.28
Evaluate by contour integration, the principal value integral

$$I(a) = ⨍_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{\mathrm{e}^{x} - \mathrm{e}^{-x}}\,\mathrm{d}x$$

for a real and $|a| < 1$. [Hint: Consider the contour that is the boundary of the box, $-R < x < R$, $0 < y < \pi$, but indented around $z = 0$ and $z = i\pi$.
Hint, Solution

Exercise 15.29
Evaluate the following integrals.

1. $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^2}$,

2. $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{1+x^3}$.

Hint, Solution

Exercise 15.30
Find the value of the integral I

$$I = \int_0^\infty \frac{\mathrm{d}x}{1+x^6}$$

by considering the contour integral

$$\int_\Gamma \frac{\mathrm{d}z}{1+z^6}$$

with an appropriately chosen contour Γ.
Hint, Solution

Exercise 15.31
Let C be the boundary of the sector $0 < r < R$, $0 < \theta < \pi/4$. By integrating e^{-z^2} on C and letting $R \to \infty$ show that

$$\int_0^\infty \cos(x^2)\,\mathrm{d}x = \int_0^\infty \sin(x^2)\,\mathrm{d}x = \frac{1}{\sqrt{2}} \int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x.$$

Hint, Solution

Exercise 15.32
Evaluate

$$\int_{-\infty}^{\infty} \frac{x}{\sinh x} \, \mathrm{d}x$$

using contour integration.
Hint, Solution

Exercise 15.33
Show that

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{\mathrm{e}^{x} + 1} \, \mathrm{d}x = \frac{\pi}{\sin(\pi a)} \quad \text{for } 0 < a < 1.$$

Use this to derive that

$$\int_{-\infty}^{\infty} \frac{\cosh(bx)}{\cosh x} \, \mathrm{d}x = \frac{\pi}{\cos(\pi b/2)} \quad \text{for } -1 < b < 1.$$

Hint, Solution

Exercise 15.34
Using techniques of contour integration find for real a and b:

$$F(a, b) = \int_0^{\pi} \frac{d\theta}{(a + b\cos\theta)^2}$$

What are the restrictions on a and b if any? Can the result be applied for complex a, b? How?
Hint, Solution

Exercise 15.35
Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{\mathrm{e}^x + \mathrm{e}^{-x}} \, \mathrm{d}x = \frac{\pi}{\mathrm{e}^{\pi/2} + \mathrm{e}^{-\pi/2}}$$

[Hint: Begin by considering the integral of $\mathrm{e}^{iz}/(\mathrm{e}^z + \mathrm{e}^{-z})$ around a rectangle with vertices: $\pm R$, $\pm R + i\pi$.]
Hint, Solution

Definite Integrals Involving Sine and Cosine

Exercise 15.36
Use contour integration to evaluate the integrals

1. $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{2 + \sin(\theta)}$,

2. $\displaystyle\int_{-\pi}^{\pi} \frac{\cos(n\theta)}{1 - 2a\cos(\theta) + a^2} \, \mathrm{d}\theta \quad \text{for } |a| < 1,\ n \in \mathbb{Z}^{0+}$.

Hint, Solution

Exercise 15.37
By integration around the unit circle, suitably indented, show that

$$⨍_0^{\pi} \frac{\cos(n\theta)}{\cos\theta - \cos\alpha} \, \mathrm{d}\theta = \pi \frac{\sin(n\alpha)}{\sin\alpha}.$$

Hint, Solution

Exercise 15.38
Evaluate

$$\int_0^1 \frac{x^2}{(1+x^2)\sqrt{1-x^2}}\,\mathrm{d}x.$$

Hint, Solution

Infinite Sums

Exercise 15.39
Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Hint, Solution

Exercise 15.40
Sum the following series using contour integration:

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2-\alpha^2}$$

Hint, Solution

15.12 Hints

The Residue Theorem

Hint 15.1
Substitute the Laurent series into the formula and simplify.

Hint 15.2
Use that the sum of all residues of the function in the extended complex plane is zero in calculating the residue at infinity. To obtain the Laurent series expansion about $z = i$, write the function as a proper rational function, (numerator has a lower degree than the denominator) and expand in partial fractions.

Hint 15.3

Hint 15.4

Cauchy Principal Value for Real Integrals

Hint 15.5

Hint 15.6
For the third part, does the integrand have a term that behaves like $1/x^2$?

Cauchy Principal Value for Contour Integrals

Hint 15.7
Expand $f(z)$ in a Laurent series. Only the first term will make a contribution to the integral in the limit as $\epsilon \to 0^+$.

Hint 15.8
Use the result of Exercise 15.6.

Hint 15.9
Look at Example 15.3.2.

Integrals from $-\infty$ to ∞

Hint 15.10
Close the path of integration in the upper or lower half plane with a semi-circle. Use the maximum modulus integral bound, (Result 12.1.1), to show that the integral along the semi-circle vanishes.

Hint 15.11
Make the change of variables $x = 1/\xi$.

Hint 15.12
Use Result 15.4.1.

Hint 15.13

Fourier Integrals

Hint 15.14
Use

$$\int_0^\pi \mathrm{e}^{-R\sin\theta}\,\mathrm{d}\theta < \frac{\pi}{R}.$$

Hint 15.15

Fourier Cosine and Sine Integrals

Hint 15.16
Consider the integral of $\frac{\mathrm{e}^{ix}}{ix}$.

Hint 15.17
Show that

$$\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,\mathrm{d}x = ⨍_{-\infty}^{\infty} \frac{1-\mathrm{e}^{ix}}{x^2}\,\mathrm{d}x.$$

Hint 15.18
Show that

$$\int_0^{\infty} \frac{\sin(\pi x)}{x(1-x^2)}\,\mathrm{d}x = -\frac{i}{2} ⨍_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{x(1-x^2)}\,\mathrm{d}x.$$

Contour Integration and Branch Cuts

Hint 15.19
Integrate a branch of $(\log z)^2/(1+z^2)$ along the boundary of the domain $\epsilon < r < R$, $0 < \theta < \pi$.

Hint 15.20

Hint 15.21
Note that

$$\int_0^1 x^a \, \mathrm{d}x$$

converges for $\Re(a) > -1$; and

$$\int_1^\infty x^a \, \mathrm{d}x$$

converges for $\Re(a) < 1$.

Consider $f(z) = z^a/(z+1)^2$ with a branch cut along the positive real axis and the contour in Figure 15.10 in the limit as $\rho \to 0$ and $R \to \infty$.

To derive the last two integrals, differentiate with respect to a.

Hint 15.22

Hint 15.23
Consider the integral of $z^a f(z)$ on the contour in Figure 15.10.

Hint 15.24
Differentiate with respect to a.

Hint 15.25
Take the limit as $a \to 0$. Use L'Hospital's rule. To corroborate the result, consider the integral of $f(z) \log z$ on an appropriate contour.

Hint 15.26
Consider the integral of $f(z) \log^2 z$ on the contour in Figure 15.10.

Hint 15.27
Consider the integral of

$$f(z) = \frac{z^a}{1 + z^4}$$

on the boundary of the region $\epsilon < r < R$, $0 < \theta < \pi/2$. Take the limits as $\epsilon \to 0$ and $R \to \infty$.

Hint 15.28
Consider the branch of $f(z) = z^{1/2} \log z/(z+1)^2$ with a branch cut on the positive real axis and $0 < \arg z < 2\pi$. Integrate this function on the contour in Figure 15.10.

Exploiting Symmetry

Hint 15.29

Hint 15.30
For the second part, consider the integral along the boundary of the region, $0 < r < R$, $0 < \theta < 2\pi/3$.

Hint 15.31

Hint 15.32
To show that the integral on the quarter-circle vanishes as $R \to \infty$ establish the inequality,

$$\cos 2\theta \geq 1 - \frac{4}{\pi}\theta, \quad 0 \leq \theta \leq \frac{\pi}{4}.$$

Hint 15.33
Consider the box contour C this is the boundary of the rectangle, $-R \leq x \leq R$, $0 \leq y \leq \pi$. The value of the integral is $\pi^2/2$.

Hint 15.34
Consider the rectangular contour with corners at $\pm R$ and $\pm R + i2\pi$. Let $R \to \infty$.

Hint 15.35

Hint 15.36

Definite Integrals Involving Sine and Cosine

Hint 15.37

Hint 15.38

Hint 15.39
Make the changes of variables $x = \sin\xi$ and then $z = e^{i\xi}$.

Infinite Sums

Hint 15.40
Use Result 15.10.1.

Hint 15.41

15.13 Solutions

The Residue Theorem

Solution 15.2
Since $f(z)$ has an isolated pole of order n at $z = z_0$, it has a Laurent series that is convergent in a deleted neighborhood about that point. We substitute this Laurent series into the Residue Formula to verify it.

$$\begin{aligned}
\operatorname{Res}(f(z), z_0) &= \lim_{z \to z_0} \left(\frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z-z_0)^n f(z) \right] \right) \\
&= \lim_{z \to z_0} \left(\frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z-z_0)^n \sum_{k=-n}^{\infty} a_k (z-z_0)^k \right] \right) \\
&= \lim_{z \to z_0} \left(\frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[\sum_{k=0}^{\infty} a_{k-n} (z-z_0)^k \right] \right) \\
&= \lim_{z \to z_0} \left(\frac{1}{(n-1)!} \sum_{k=n-1}^{\infty} a_{k-n} \frac{k!}{(k-n+1)!} (z-z_0)^{k-n+1} \right) \\
&= \lim_{z \to z_0} \left(\frac{1}{(n-1)!} \sum_{k=0}^{\infty} a_{k-1} \frac{(k+n-1)!}{k!} (z-z_0)^k \right) \\
&= \frac{1}{(n-1)!} a_{-1} \frac{(n-1)!}{0!} \\
&= a_{-1}
\end{aligned}$$

This proves the Residue Formula.

Solution 15.3
Classify Singularities.

$$f(z) = \frac{z^4}{z^2+1} = \frac{z^4}{(z-i)(z+i)}.$$

There are first order poles at $z = \pm i$. Since the function behaves like z^2 at infinity, there is a second order pole there. To see this more slowly, we can make the substitution $z = 1/\zeta$ and examine the point $\zeta = 0$.

$$f\left(\frac{1}{\zeta}\right) = \frac{\zeta^{-4}}{\zeta^{-2}+1} = \frac{1}{\zeta^2+\zeta^4} = \frac{1}{\zeta^2(1+\zeta^2)}$$

$f(1/\zeta)$ has a second order pole at $\zeta = 0$, which implies that $f(z)$ has a second order pole at infinity.

Residues. The residues at $z = \pm i$ are,

$$\operatorname{Res}\left(\frac{z^4}{z^2+1}, i\right) = \lim_{z \to i} \frac{z^4}{z+i} = -\frac{i}{2},$$

$$\operatorname{Res}\left(\frac{z^4}{z^2+1}, -i\right) = \lim_{z \to -i} \frac{z^4}{z-i} = \frac{i}{2}.$$

The residue at infinity is

$$\begin{aligned}
\operatorname{Res}(f(z), \infty) &= \operatorname{Res}\left(\frac{-1}{\zeta^2} f\left(\frac{1}{\zeta}\right), \zeta = 0\right) \\
&= \operatorname{Res}\left(\frac{-1}{\zeta^2} \frac{\zeta^{-4}}{\zeta^{-2}+1}, \zeta = 0\right) \\
&= \operatorname{Res}\left(-\frac{\zeta^{-4}}{1+\zeta^2}, \zeta = 0\right)
\end{aligned}$$

Here we could use the residue formula, but it's easier to find the Laurent expansion.

$$\begin{aligned} &= \operatorname{Res}\left(-\zeta^{-4}\sum_{n=0}^{\infty}(-1)^n\zeta^{2n}, \zeta = 0\right) \\ &= 0 \end{aligned}$$

We could also calculate the residue at infinity by recalling that the sum of all residues of this function in the extended complex plane is zero.

$$\frac{-i}{2} + \frac{i}{2} + \operatorname{Res}\left(f(z), \infty\right) = 0$$

$$\operatorname{Res}\left(f(z), \infty\right) = 0$$

Laurent Series about z = 0. Since the nearest singularities are at $z = \pm i$, the Taylor series will converge in the disk $|z| < 1$.

$$\begin{aligned} \frac{z^4}{z^2+1} &= z^4 \frac{1}{1-(-z)^2} \\ &= z^4 \sum_{n=0}^{\infty}(-z^2)^n \\ &= z^4 \sum_{n=0}^{\infty}(-1)^n z^{2n} \\ &= \sum_{n=2}^{\infty}(-1)^n z^{2n} \end{aligned}$$

This geometric series converges for $|-z^2| < 1$, or $|z| < 1$. The series expansion of the function is

$$\boxed{\frac{z^4}{z^2+1} = \sum_{n=2}^{\infty}(-1)^n z^{2n} \quad \text{for } |z| < 1}$$

Laurent Series about $\mathbf{z = i}$. We expand $f(z)$ in partial fractions. First we write the function as a proper rational function, (i.e. the numerator has lower degree than the denominator). By polynomial division, we see that

$$f(z) = z^2 - 1 + \frac{1}{z^2 + 1}.$$

Now we expand the last term in partial fractions.

$$f(z) = z^2 - 1 + \frac{-i/2}{z - i} + \frac{i/2}{z + i}$$

Since the nearest singularity is at $z = -i$, the Laurent series will converge in the annulus $0 < |z - i| < 2$.

$$\begin{aligned} z^2 - 1 &= ((z - i) + i)^2 - 1 \\ &= (z - i)^2 + i2(z - i) - 2 \end{aligned}$$

$$\begin{aligned} \frac{i/2}{z + i} &= \frac{i/2}{i2 + (z - i)} \\ &= \frac{1/4}{1 - i(z - i)/2} \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{i(z - i)}{2} \right)^n \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \frac{i^n}{2^n} (z - i)^n \end{aligned}$$

This geometric series converges for $|i(z - i)/2| < 1$, or $|z - i| < 2$. The series expansion of $f(z)$ is

$$f(z) = \frac{-i/2}{z - i} - 2 + i2(z - i) + (z - i)^2 + \frac{1}{4} \sum_{n=0}^{\infty} \frac{i^n}{2^n} (z - i)^n.$$

$$\boxed{\frac{z^4}{z^2+1} = \frac{-i/2}{z-i} - 2 + i2(z-i) + (z-i)^2 + \frac{1}{4}\sum_{n=0}^{\infty} \frac{i^n}{2^n}(z-i)^n \quad \text{for } |z-i| < 2}$$

Laurent Series about $\mathbf{z} = \infty$. Since the nearest singularities are at $z = \pm i$, the Laurent series will converge in the annulus $1 < |z| < \infty$.

$$\begin{aligned}
\frac{z^4}{z^2+1} &= \frac{z^2}{1+1/z^2} \\
&= z^2 \sum_{n=0}^{\infty} \left(-\frac{1}{z^2}\right)^n \\
&= \sum_{n=-\infty}^{0} (-1)^n z^{2(n+1)} \\
&= \sum_{n=-\infty}^{1} (-1)^{n+1} z^{2n}
\end{aligned}$$

This geometric series converges for $|-1/z^2| < 1$, or $|z| > 1$. The series expansion of $f(z)$ is

$$\boxed{\frac{z^4}{z^2+1} = \sum_{n=-\infty}^{1} (-1)^{n+1} z^{2n} \quad \text{for } 1 < |z| < \infty}$$

Solution 15.4

Method 1: Residue Theorem. We factor $P(z)$. Let m be the number of roots, counting multiplicities, that lie inside the contour Γ. We find a simple expression for $P'(z)/P(z)$.

$$P(z) = c\prod_{k=1}^{n}(z - z_k)$$
$$P'(z) = c\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}(z - z_j)$$

$$\begin{aligned}
\frac{P'(z)}{P(z)} &= \frac{c\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}(z - z_j)}{c\prod_{k=1}^{n}(z - z_k)} \\
&= \sum_{k=1}^{n}\frac{\prod_{\substack{j=1\\ j\neq k}}^{n}(z - z_j)}{\prod_{j=1}^{n}(z - z_j)} \\
&= \sum_{k=1}^{n}\frac{1}{z - z_k}
\end{aligned}$$

Now we do the integration using the residue theorem.

$$\begin{aligned}
\frac{1}{i2\pi}\int_{\Gamma}\frac{P'(z)}{P(z)}\,\mathrm{d}z &= \frac{1}{i2\pi}\int_{\Gamma}\sum_{k=1}^{n}\frac{1}{z - z_k}\,\mathrm{d}z \\
&= \sum_{k=1}^{n}\frac{1}{i2\pi}\int_{\Gamma}\frac{1}{z - z_k}\,\mathrm{d}z \\
&= \sum_{z_k \text{ inside } \Gamma}\frac{1}{i2\pi}\int_{\Gamma}\frac{1}{z - z_k}\,\mathrm{d}z \\
&= \sum_{z_k \text{ inside } \Gamma} 1 \\
&= m
\end{aligned}$$

Method 2: Fundamental Theorem of Calculus. We factor the polynomial, $P(z) = c\prod_{k=1}^n (z - z_k)$. Let m be the number of roots, counting multiplicities, that lie inside the contour Γ.

$$
\begin{aligned}
\frac{1}{i2\pi}\int_\Gamma \frac{P'(z)}{P(z)}\,\mathrm{d}z &= \frac{1}{i2\pi}\left[\log P(z)\right]_C \\
&= \frac{1}{i2\pi}\left[\log \prod_{k=1}^n (z - z_k)\right]_C \\
&= \frac{1}{i2\pi}\left[\sum_{k=1}^n \log(z - z_k)\right]_C
\end{aligned}
$$

The value of the logarithm changes by $i2\pi$ for the terms in which z_k is inside the contour. Its value does not change for the terms in which z_k is outside the contour.

$$
\begin{aligned}
&= \frac{1}{i2\pi}\left[\sum_{z_k \text{ inside } \Gamma} \log(z - z_k)\right]_C \\
&= \frac{1}{i2\pi}\sum_{z_k \text{ inside } \Gamma} i2\pi \\
&= m
\end{aligned}
$$

Solution 15.5

1.

$$
\oint_C \frac{\mathrm{e}^z}{(z-\pi)\tan z}\,\mathrm{d}z = \oint_C \frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z}\,\mathrm{d}z
$$

The integrand has first order poles at $z = n\pi$, $n \in \mathbb{Z}$, $n \neq 1$ and a double pole at $z = \pi$. The only pole inside the contour occurs at $z = 0$. We evaluate the integral with the residue theorem.

$$\begin{aligned}
\oint_C \frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z}\,\mathrm{d}z &= i2\pi \operatorname{Res}\left(\frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z}, z=0\right) \\
&= i2\pi \lim_{z=0} z \frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z} \\
&= -i2 \lim_{z=0} \frac{z}{\sin z} \\
&= -i2 \lim_{z=0} \frac{1}{\cos z} \\
&= -i2
\end{aligned}$$

$$\boxed{\oint_C \frac{\mathrm{e}^z}{(z-\pi)\tan z}\,\mathrm{d}z = -i2}$$

2. The integrand has a first order poles at $z = 0, -\pi$ and a second order pole at $z = \pi$ inside the contour. The value of the integral is $i2\pi$ times the sum of the residues at these points. From the previous part we know that residue at $z = 0$.

$$\operatorname{Res}\left(\frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z}, z=0\right) = -\frac{1}{\pi}$$

We find the residue at $z = -\pi$ with the residue formula.

$$\begin{aligned}
\operatorname{Res}\left(\frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z}, z=-\pi\right) &= \lim_{z\to-\pi} (z+\pi)\frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z} \\
&= \frac{\mathrm{e}^{-\pi}(-1)}{-2\pi} \lim_{z\to-\pi} \frac{z+\pi}{\sin z} \\
&= \frac{\mathrm{e}^{-\pi}}{2\pi} \lim_{z\to-\pi} \frac{1}{\cos z} \\
&= -\frac{\mathrm{e}^{-\pi}}{2\pi}
\end{aligned}$$

We find the residue at $z = \pi$ by finding the first few terms in the Laurent series of the integrand.

$$\begin{aligned}
\frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z} &= \frac{\left(\mathrm{e}^\pi + \mathrm{e}^\pi (z-\pi) + \mathcal{O}\left((z-\pi)^2\right)\right)\left(1 + \mathcal{O}\left((z-\pi)^2\right)\right)}{(z-\pi)\left(-(z-\pi) + \mathcal{O}\left((z-\pi)^3\right)\right)} \\
&= \frac{-\mathrm{e}^\pi - \mathrm{e}^\pi (z-\pi) + \mathcal{O}\left((z-\pi)^2\right)}{-(z-\pi)^2 + \mathcal{O}\left((z-\pi)^4\right)} \\
&= \frac{\frac{\mathrm{e}^\pi}{(z-\pi)^2} + \frac{\mathrm{e}^\pi}{z-\pi} + \mathcal{O}(1)}{1 + \mathcal{O}\left((z-\pi)^2\right)} \\
&= \left(\frac{\mathrm{e}^\pi}{(z-\pi)^2} + \frac{\mathrm{e}^\pi}{z-\pi} + \mathcal{O}(1)\right)\left(1 + \mathcal{O}\left((z-\pi)^2\right)\right) \\
&= \frac{\mathrm{e}^\pi}{(z-\pi)^2} + \frac{\mathrm{e}^\pi}{z-\pi} + \mathcal{O}(1)
\end{aligned}$$

With this we see that

$$\operatorname{Res}\left(\frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z}, z = \pi\right) = \mathrm{e}^\pi.$$

The integral is

$$\begin{aligned}
\oint_C \frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z}\,\mathrm{d}z = i2\pi \Bigg(&\operatorname{Res}\left(\frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z}, z = -\pi\right) + \operatorname{Res}\left(\frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z}, z = 0\right) \\
&+ \operatorname{Res}\left(\frac{\mathrm{e}^z \cos z}{(z-\pi)\sin z}, z = \pi\right)\Bigg) \\
= i2\pi &\left(-\frac{1}{\pi} - \frac{\mathrm{e}^{-\pi}}{2\pi} + \mathrm{e}^\pi\right)
\end{aligned}$$

$$\boxed{\oint_C \frac{\mathrm{e}^z}{(z-\pi)\tan z}\,\mathrm{d}z = i\left(2\pi\,\mathrm{e}^\pi - 2 - \mathrm{e}^{-\pi}\right)}$$

Cauchy Principal Value for Real Integrals

Solution 15.6
Consider the integral

$$\int_{-1}^{1} \frac{1}{x} \,\mathrm{d}x.$$

By the definition of improper integrals we have

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{x} \,\mathrm{d}x &= \lim_{\epsilon \to 0^+} \int_{-1}^{-\epsilon} \frac{1}{x} \,\mathrm{d}x + \lim_{\delta \to 0^+} \int_{\delta}^{1} \frac{1}{x} \,\mathrm{d}x \\
&= \lim_{\epsilon \to 0^+} [\log |x|]_{-1}^{-\epsilon} + \lim_{\delta \to 0^+} [\log |x|]_{\delta}^{1} \\
&= \lim_{\epsilon \to 0^+} \log \epsilon - \lim_{\delta \to 0^+} \log \delta
\end{aligned}$$

This limit diverges. Thus the integral diverges.

Now consider the integral

$$\int_{-1}^{1} \frac{1}{x - i\alpha} \,\mathrm{d}x$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Since the integrand is bounded, the integral exists.

$$
\begin{aligned}
\int_{-1}^{1} \frac{1}{x - i\alpha} \, \mathrm{d}x &= \int_{-1}^{1} \frac{x + i\alpha}{x^2 + \alpha^2} \, \mathrm{d}x \\
&= \int_{-1}^{1} \frac{i\alpha}{x^2 + \alpha^2} \, \mathrm{d}x \\
&= 2i \int_{0}^{1} \frac{\alpha}{x^2 + \alpha^2} \, \mathrm{d}x \\
&= 2i \int_{0}^{1/\alpha} \frac{1}{\xi^2 + 1} \, \mathrm{d}\xi \\
&= 2i \left[\arctan \xi\right]_0^{1/\alpha} \\
&= 2i \arctan\left(\frac{1}{\alpha}\right)
\end{aligned}
$$

Note that the integral exists for all nonzero real α and that

$$
\lim_{\alpha \to 0^+} \int_{-1}^{1} \frac{1}{x - i\alpha} \, \mathrm{d}x = i\pi
$$

and

$$
\lim_{\alpha \to 0^-} \int_{-1}^{1} \frac{1}{x - i\alpha} \, \mathrm{d}x = -i\pi.
$$

The integral exists for α arbitrarily close to zero, but diverges when $\alpha = 0$. The real part of the integrand is an odd function with two humps that get thinner and taller with decreasing α. The imaginary part of the integrand is an even function with a hump that gets thinner and taller with decreasing α. (See Figure 15.7.)

$$
\Re\left(\frac{1}{x - i\alpha}\right) = \frac{x}{x^2 + \alpha^2}, \qquad \Im\left(\frac{1}{x - i\alpha}\right) = \frac{\alpha}{x^2 + \alpha^2}
$$

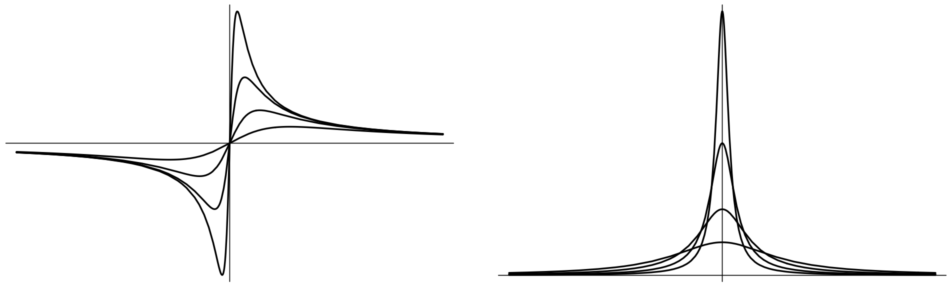


Figure 15.7: The real and imaginary part of the integrand for several values of α.

Note that

$$\Re \int_0^1 \frac{1}{x - i\alpha} \, \mathrm{d}x \to +\infty \text{ as } \alpha \to 0^+$$

and

$$\Re \int_{-1}^0 \frac{1}{x - i\alpha} \, \mathrm{d}x \to -\infty \text{ as } \alpha \to 0^-.$$

However,

$$\lim_{\alpha \to 0} \Re \int_{-1}^1 \frac{1}{x - i\alpha} \, \mathrm{d}x = 0$$

because the two integrals above cancel each other.

Now note that when $\alpha = 0$, the integrand is real. Of course the integral doesn't converge for this case, but if we could assign some value to

$$\int_{-1}^1 \frac{1}{x} \, \mathrm{d}x$$

it would be a real number. Since

$$\lim_{\alpha \to 0} \int_{-1}^{1} \Re\left[\frac{1}{x - i\alpha}\right] \mathrm{d}x = 0,$$

This number should be zero.

Solution 15.7

1.

$$\begin{aligned}
⨍_{-1}^{1} \frac{1}{x^2} \, \mathrm{d}x &= \lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} \frac{1}{x^2} \, \mathrm{d}x + \int_{\epsilon}^{1} \frac{1}{x^2} \, \mathrm{d}x \right) \\
&= \lim_{\epsilon \to 0^+} \left(\left[-\frac{1}{x} \right]_{-1}^{-\epsilon} + \left[-\frac{1}{x} \right]_{\epsilon}^{1} \right) \\
&= \lim_{\epsilon \to 0^+} \left(\frac{1}{\epsilon} - 1 - 1 + \frac{1}{\epsilon} \right)
\end{aligned}$$

The principal value of the integral does not exist.

2.

$$\begin{aligned}
⨍_{-1}^{1} \frac{1}{x^3} \, \mathrm{d}x &= \lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} \frac{1}{x^3} \, \mathrm{d}x + \int_{\epsilon}^{1} \frac{1}{x^3} \, \mathrm{d}x \right) \\
&= \lim_{\epsilon \to 0^+} \left(\left[-\frac{1}{2x^2} \right]_{-1}^{-\epsilon} + \left[-\frac{1}{2x^2} \right]_{\epsilon}^{1} \right) \\
&= \lim_{\epsilon \to 0^+} \left(-\frac{1}{2(-\epsilon)^2} + \frac{1}{2(-1)^2} - \frac{1}{2(1)^2} + \frac{1}{2\epsilon^2} \right) \\
&= 0
\end{aligned}$$

3. Since $f(x)$ is real analytic,

$$f(x) = \sum_{n=1}^{\infty} f_n x^n \quad \text{for } x \in (-1, 1).$$

We can rewrite the integrand as

$$\frac{f(x)}{x^3} = \frac{f_0}{x^3} + \frac{f_1}{x^2} + \frac{f_2}{x} + \frac{f(x) - f_0 - f_1 x - f_2 x^2}{x^3}.$$

Note that the final term is real analytic on $(-1, 1)$. Thus the principal value of the integral exists if and only if $f_2 = 0$.

Cauchy Principal Value for Contour Integrals

Solution 15.8

We can write $f(z)$ as

$$f(z) = \frac{f_0}{z - z_0} + \frac{(z - z_0) f(z) - f_0}{z - z_0}.$$

Note that the second term is analytic in a neighborhood of z_0. Thus it is bounded on the contour. Let M_ϵ be the maximum modulus of $\frac{(z-z_0)f(z)-f_0}{z-z_0}$ on C_ϵ. By using the maximum modulus integral bound, we have

$$\begin{aligned} \left| \int_{C_\epsilon} \frac{(z - z_0) f(z) - f_0}{z - z_0} \, \mathrm{d}z \right| &\leq (\beta - \alpha) \epsilon M_\epsilon \\ &\to 0 \quad \text{as } \epsilon \to 0^+. \end{aligned}$$

Thus we see that

$$\lim_{\epsilon \to 0^+} \int_{C_\epsilon} f(z) \, \mathrm{d}z \lim_{\epsilon \to 0^+} \int_{C_\epsilon} \frac{f_0}{z - z_0} \, \mathrm{d}z.$$

We parameterize the path of integration with

$$z = z_0 + \epsilon e^{i\theta}, \quad \theta \in (\alpha, \beta).$$

Now we evaluate the integral.

$$\begin{aligned}
\lim_{\epsilon \to 0^+} \int_{C_\epsilon} \frac{f_0}{z - z_0} \, \mathrm{d}z &= \lim_{\epsilon \to 0^+} \int_\alpha^\beta \frac{f_0}{\epsilon e^{i\theta}} i \epsilon e^{i\theta} \, \mathrm{d}\theta \\
&= \lim_{\epsilon \to 0^+} \int_\alpha^\beta i f_0 \, \mathrm{d}\theta \\
&= i(\beta - \alpha) f_0 \\
&\equiv i(\beta - \alpha) \operatorname{Res}(f(z), z_0)
\end{aligned}$$

This proves the result.

Solution 15.9

Let C_i be the contour that is indented with circular arcs or radius ϵ at each of the first order poles on C so as to enclose these poles. Let $A_1, \ldots, A_n$ be these circular arcs of radius ϵ centered at the points $\zeta_1, \ldots, \zeta_n$. Let C_p be the contour, (not necessarily connected), obtained by subtracting each of the A_j's from C_i.

Since the curve is C^1, (or continuously differentiable), at each of the first order poles on C, the A_j's becomes semi-circles as $\epsilon \to 0^+$. Thus

$$\int_{A_j} f(z) \, \mathrm{d}z = i\pi \operatorname{Res}(f(z), \zeta_j) \quad \text{for } j = 1, \ldots, n.$$

The principal value of the integral along C is

$$
\begin{aligned}
-\!\!\!\!\!\!\int_C f(z)\,\mathrm{d}z &= \lim_{\epsilon\to 0^+} \int_{C_p} f(z)\,\mathrm{d}z \\
&= \lim_{\epsilon\to 0^+} \left(\int_{C_i} f(z)\,\mathrm{d}z - \sum_{j=1}^{n} \int_{A_j} f(z)\,\mathrm{d}z \right) \\
&= i2\pi \left(\sum_{j=1}^{m} \operatorname{Res}\left(f(z), z_j\right) + \sum_{j=1}^{n} \operatorname{Res}\left(f(z), \zeta_j\right) \right) - i\pi \sum_{j=1}^{n} \operatorname{Res}\left(f(z), \zeta_j\right)
\end{aligned}
$$

$$
\boxed{-\!\!\!\!\!\!\int_C f(z)\,\mathrm{d}z = i2\pi \sum_{j=1}^{m} \operatorname{Res}\left(f(z), z_j\right) + i\pi \sum_{j=1}^{n} \operatorname{Res}\left(f(z), \zeta_j\right).}
$$

Solution 15.10
Consider

$$
-\!\!\!\!\!\!\int_C \frac{1}{z-1}\,\mathrm{d}z
$$

where C is the unit circle. Let C_p be the circular arc of radius 1 that starts and ends a distance of ϵ from $z = 1$. Let C_ϵ be the negative, circular arc of radius ϵ with center at $z = 1$ that joins the endpoints of C_p. Let C_i, be the union of C_p and C_ϵ. (C_p stands for Principal value Contour; C_i stands for Indented Contour.) C_i is an indented contour that avoids the first order pole at $z = 1$. Figure 15.8 shows the three contours.

Figure 15.8: The Indented Contour.

Note that the principal value of the integral is

$$⨍_C \frac{1}{z-1}\,\mathrm{d}z = \lim_{\epsilon\to 0^+} \int_{C_p} \frac{1}{z-1}\,\mathrm{d}z.$$

We can calculate the integral along C_i with Cauchy's theorem. The integrand is analytic inside the contour.

$$\int_{C_i} \frac{1}{z-1}\,\mathrm{d}z = 0$$

We can calculate the integral along C_ϵ using Result 15.3.1. Note that as $\epsilon \to 0^+$, the contour becomes a semi-circle, a circular arc of π radians in the negative direction.

$$\lim_{\epsilon\to 0^+} \int_{C_\epsilon} \frac{1}{z-1}\,\mathrm{d}z = -\imath\pi \operatorname{Res}\left(\frac{1}{z-1}, 1\right) = -\imath\pi$$

Now we can write the principal value of the integral along C in terms of the two known integrals.

$$\begin{aligned} ⨍_C \frac{1}{z-1}\,\mathrm{d}z &= \int_{C_i} \frac{1}{z-1}\,\mathrm{d}z - \int_{C_\epsilon} \frac{1}{z-1}\,\mathrm{d}z \\ &= 0 - (-\imath\pi) \\ &= \imath\pi \end{aligned}$$

Integrals from $-\infty$ to ∞

Solution 15.11

Let C_R be the semicircular arc from R to $-R$ in the upper half plane. Let C be the union of C_R and the interval $[-R, R]$. We can evaluate the principal value of the integral along C with Result 15.3.2.

$$⨍_C f(x)\,\mathrm{d}x = \imath 2\pi \sum_{k=1}^{m} \operatorname{Res}\left(f(z), z_k\right) + \imath\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z), x_k\right)$$

We examine the integral along C_R as $R \to \infty$.

$$\left| \int_{C_R} f(z) \, \mathrm{d}z \right| \leq \pi R \max_{z \in C_R} |f(z)|$$
$$\to 0 \quad \text{as } R \to \infty.$$

Now we are prepared to evaluate the real integral.

$$\begin{aligned}
⨍_{-\infty}^{\infty} f(x) \, \mathrm{d}x &= \lim_{R \to \infty} ⨍_{-R}^{R} f(x) \, \mathrm{d}x \\
&= \lim_{R \to \infty} ⨍_{C} f(z) \, \mathrm{d}z \\
&= i2\pi \sum_{k=1}^{m} \operatorname{Res}\left(f(z), z_k\right) + i\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z), x_k\right)
\end{aligned}$$

If we close the path of integration in the lower half plane, the contour will be in the negative direction.

$$⨍_{-\infty}^{\infty} f(x) \, \mathrm{d}x = -i2\pi \sum_{k=1}^{m} \operatorname{Res}\left(f(z), z_k\right) - i\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z), x_k\right)$$

Solution 15.12

We consider

$$⨍_{-\infty}^{\infty} \frac{2x}{x^2 + x + 1} \, \mathrm{d}x.$$

With the change of variables $x = 1/\xi$, this becomes

$$⨍_{\infty}^{-\infty} \frac{2\xi^{-1}}{\xi^{-2} + \xi^{-1} + 1} \left(\frac{-1}{\xi^2} \right) \mathrm{d}\xi,$$

$$\fint_{-\infty}^{\infty} \frac{2\xi^{-1}}{\xi^2+\xi+1} \, \mathrm{d}\xi$$

There are first order poles at $\xi = 0$ and $\xi = -1/2 \pm i\sqrt{3}/2$. We close the path of integration in the upper half plane with a semi-circle. Since the integrand decays like ξ^{-3} the integrand along the semi-circle vanishes as the radius tends to infinity. The value of the integral is thus

$$i\pi \operatorname{Res}\left(\frac{2z^{-1}}{z^2+z+1}, z = 0\right) + i2\pi \operatorname{Res}\left(\frac{2z^{-1}}{z^2+z+1}, z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$i\pi \lim_{z\to 0}\left(\frac{2}{z^2+z+1}\right) + i2\pi \lim_{z\to(-1+i\sqrt{3})/2}\left(\frac{2z^{-1}}{z+(1+i\sqrt{3})/2}\right)$$

$$\boxed{\fint_{-\infty}^{\infty} \frac{2x}{x^2+x+1} \, \mathrm{d}x = -\frac{2\pi}{\sqrt{3}}}$$

Solution 15.13

1. Consider

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} \, \mathrm{d}x.$$

The integrand $\frac{1}{z^4+1}$ is analytic on the real axis and has isolated singularities at the points $z = \{ e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4},$
Let C_R be the semi-circle of radius R in the upper half plane. Since

$$\lim_{R\to\infty}\left(R \max_{z\in C_R}\left|\frac{1}{z^4+1}\right|\right) = \lim_{R\to\infty}\left(R\frac{1}{R^4-1}\right) = 0,$$

we can apply Result 15.4.1.

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,\mathrm{d}x = i2\pi \left(\operatorname{Res}\left(\frac{1}{z^4+1},\, \mathrm{e}^{i\pi/4}\right) + \operatorname{Res}\left(\frac{1}{z^4+1},\, \mathrm{e}^{i3\pi/4}\right)\right)$$

The appropriate residues are,

$$\begin{aligned}
\operatorname{Res}\left(\frac{1}{z^4+1},\, \mathrm{e}^{i\pi/4}\right) &= \lim_{z\to \mathrm{e}^{i\pi/4}} \frac{z-\mathrm{e}^{i\pi/4}}{z^4+1} \\
&= \lim_{z\to \mathrm{e}^{i\pi/4}} \frac{1}{4z^3} \\
&= \frac{1}{4}\,\mathrm{e}^{-i3\pi/4} \\
&= \frac{-1-i}{4\sqrt{2}},
\end{aligned}$$

$$\begin{aligned}
\operatorname{Res}\left(\frac{1}{z^4+1},\, \mathrm{e}^{i3\pi/4}\right) &= \frac{1}{4(\mathrm{e}^{i3\pi/4})^3} \\
&= \frac{1}{4}\,\mathrm{e}^{-i\pi/4} \\
&= \frac{1-i}{4\sqrt{2}},
\end{aligned}$$

We evaluate the integral with the residue theorem.

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,\mathrm{d}x = i2\pi \left(\frac{-1-i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}}\right)$$

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}}}$$

2. Now consider

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} \, \mathrm{d}x.$$

The integrand is analytic on the real axis and has second order poles at $z = \pm i$. Since the integrand decays sufficiently fast at infinity,

$$\lim_{R \to \infty} \left(R \max_{z \in C_R} \left| \frac{z^2}{(z^2+1)^2} \right| \right) = \lim_{R \to \infty} \left(R \frac{R^2}{(R^2-1)^2} \right) = 0$$

we can apply Result 15.4.1.

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} \, \mathrm{d}x = i2\pi \operatorname{Res} \left(\frac{z^2}{(z^2+1)^2}, z = i \right)$$

$$\begin{aligned}
\operatorname{Res} \left(\frac{z^2}{(z^2+1)^2}, z = i \right) &= \lim_{z \to i} \frac{\mathrm{d}}{\mathrm{d}z} \left((z-i)^2 \frac{z^2}{(z^2+1)^2} \right) \\
&= \lim_{z \to i} \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{z^2}{(z+i)^2} \right) \\
&= \lim_{z \to i} \left(\frac{(z+i)^2 2z - z^2 2(z+i)}{(z+i)^4} \right) \\
&= -\frac{i}{4}
\end{aligned}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} \, \mathrm{d}x = \frac{\pi}{2}}$$

3. Since

$$\frac{\sin(x)}{1+x^2}$$

is an odd function,

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{1+x^2}\,\mathrm{d}x$$

Since $\mathrm{e}^{iz}/(1+z^2)$ is analytic except for simple poles at $z = \pm i$ and the integrand decays sufficiently fast in the upper half plane,

$$\lim_{R\to\infty}\left(R \max_{z\in C_R}\left|\frac{\mathrm{e}^{iz}}{1+z^2}\right|\right) = \lim_{R\to\infty}\left(R\frac{1}{R^2-1}\right) = 0$$

we can apply Result 15.4.1.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{1+x^2}\,\mathrm{d}x &= i2\pi \operatorname{Res}\left(\frac{\mathrm{e}^{iz}}{(z-i)(z+i)}, z = i\right)\\ &= i2\pi\frac{\mathrm{e}^{-1}}{i2}\end{aligned}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\,\mathrm{d}x = \frac{\pi}{e}}$$

Solution 15.14
Consider the function

$$f(z) = \frac{z^6}{(z^4+1)^2}.$$

The value of the function on the imaginary axis:

$$\frac{-y^6}{(y^4+1)^2}$$

is a constant multiple of the value of the function on the real axis:

$$\frac{x^6}{(x^4+1)^2}.$$

Thus to evaluate the real integral we consider the path of integration, C, which starts at the origin, follows the real axis to R, follows a circular path to iR and then follows the imaginary axis back down to the origin. $f(z)$ has second order poles at the fourth roots of -1: $(\pm 1 \pm i)/\sqrt{2}$. Of these only $(1+i)/\sqrt{2}$ lies inside the path of integration. We evaluate the contour integral with the Residue Theorem. For $R > 1$:

$$\begin{aligned}
\int_C \frac{z^6}{(z^4+1)^2}\,\mathrm{d}z &= i2\pi \operatorname{Res}\left(\frac{z^6}{(z^4+1)^2}, z = \mathrm{e}^{i\pi/4}\right) \\
&= i2\pi \lim_{z\to \mathrm{e}^{i\pi/4}} \frac{\mathrm{d}}{\mathrm{d}z}\left((z-\mathrm{e}^{i\pi/4})^2 \frac{z^6}{(z^4+1)^2}\right) \\
&= i2\pi \lim_{z\to \mathrm{e}^{i\pi/4}} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^6}{(z-\mathrm{e}^{i3\pi/4})^2(z-\mathrm{e}^{i5\pi/4})^2(z-\mathrm{e}^{i7\pi/4})^2}\right) \\
&= i2\pi \lim_{z\to \mathrm{e}^{i\pi/4}} \Bigg(\frac{z^6}{(z-\mathrm{e}^{i3\pi/4})^2(z-\mathrm{e}^{i5\pi/4})^2(z-\mathrm{e}^{i7\pi/4})^2} \\
&\qquad \left(\frac{6}{z} - \frac{2}{z-\mathrm{e}^{i3\pi/4}} - \frac{2}{z-\mathrm{e}^{i5\pi/4}} - \frac{2}{z-\mathrm{e}^{i7\pi/4}}\right)\Bigg) \\
&= i2\pi \frac{-i}{(2)(4i)(-2)}\left(\frac{6\sqrt{2}}{1+i} - \frac{2}{\sqrt{2}} - \frac{2\sqrt{2}}{2+i2} - \frac{2}{i\sqrt{2}}\right) \\
&= i2\pi \frac{3}{32}(1-i)\sqrt{2} \\
&= \frac{3\pi}{8\sqrt{2}}(1+i)
\end{aligned}$$

The integral along the circular part of the contour, C_R, vanishes as $R \to \infty$. We demonstrate this with the maximum modulus integral bound.

$$\begin{aligned}\left|\int_{C_R} \frac{z^6}{(z^4+1)^2}\,\mathrm{d}z\right| &\leq \frac{\pi R}{4} \max_{z\in C_R}\left(\frac{z^6}{(z^4+1)^2}\right)\\ &= \frac{\pi R}{4}\frac{R^6}{(R^4-1)^2}\\ &\to 0 \text{ as } R\to\infty\end{aligned}$$

Taking the limit $R \to \infty$, we have:

$$\begin{gathered}\int_0^\infty \frac{x^6}{(x^4+1)^2}\,\mathrm{d}x + \int_\infty^0 \frac{(iy)^6}{((iy)^4+1)^2} i\,\mathrm{d}y = \frac{3\pi}{8\sqrt{2}}(1+i)\\ \int_0^\infty \frac{x^6}{(x^4+1)^2}\,\mathrm{d}x + i\int_0^\infty \frac{y^6}{(y^4+1)^2}\,\mathrm{d}y = \frac{3\pi}{8\sqrt{2}}(1+i)\\ (1+i)\int_0^\infty \frac{x^6}{(x^4+1)^2}\,\mathrm{d}x = \frac{3\pi}{8\sqrt{2}}(1+i)\\ \boxed{\int_0^\infty \frac{x^6}{(x^4+1)^2}\,\mathrm{d}x = \frac{3\pi}{8\sqrt{2}}}\end{gathered}$$

Fourier Integrals

Solution 15.15
We know that

$$\int_0^\pi \mathrm{e}^{-R\sin\theta}\,\mathrm{d}\theta < \frac{\pi}{R}.$$

First take the case that ω is positive and the semi-circle is in the upper half plane.

$$\begin{aligned}
\left| \int_{C_R} f(z)\, \mathrm{e}^{i\omega z}\, \mathrm{d}z \right| &\leq \left| \int_{C_R} \mathrm{e}^{i\omega z}\, \mathrm{d}z \right| \max_{z \in C_R} |f(z)| \\
&\leq \int_0^\pi \left| \mathrm{e}^{i\omega R\, \mathrm{e}^{i\theta}} R\, \mathrm{e}^{i\theta} \right| \, \mathrm{d}\theta \max_{z \in C_R} |f(z)| \\
&= R \int_0^\pi \left| \mathrm{e}^{-\omega R \sin\theta} \right| \, \mathrm{d}\theta \max_{z \in C_R} |f(z)| \\
&< R \frac{\pi}{\omega R} \max_{z \in C_R} |f(z)| \\
&= \frac{\pi}{\omega} \max_{z \in C_R} |f(z)| \\
&\to 0 \quad \text{as } R \to \infty
\end{aligned}$$

The procedure is almost the same for negative ω.

Solution 15.16

First we write the integral in terms of Fourier integrals.

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x - i\pi} \, \mathrm{d}x = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i2x}}{2(x - i\pi)} \, \mathrm{d}x + \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-i2x}}{2(x - i\pi)} \, \mathrm{d}x$$

Note that $\frac{1}{2(z - i\pi)}$ vanishes as $|z| \to \infty$. We close the former Fourier integral in the upper half plane and the latter in the lower half plane. There is a first order pole at $z = i\pi$ in the upper half plane.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{e}^{i2x}}{2(x - i\pi)} \, \mathrm{d}x &= i2\pi \operatorname{Res}\left(\frac{\mathrm{e}^{i2z}}{2(z - i\pi)}, z = i\pi \right) \\
&= i2\pi \frac{\mathrm{e}^{-2\pi}}{2}
\end{aligned}$$

There are no singularities in the lower half plane.

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-i2x}}{2(x - i\pi)} \, \mathrm{d}x = 0$$

Thus the value of the original real integral is

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos 2x}{x - i\pi} \, \mathrm{d}x = i\pi \, \mathrm{e}^{-2\pi}}$$

Fourier Cosine and Sine Integrals

Solution 15.17
We are considering the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x.$$

The integrand is an entire function. So it doesn't appear that the residue theorem would directly apply. Also the integrand is unbounded as $x \to +i\infty$ and $x \to -i\infty$, so closing the integral in the upper or lower half plane is not directly applicable. In order to proceed, we must write the integrand in a different form. Note that

$$⨍_{-\infty}^{\infty} \frac{\cos x}{x} \, \mathrm{d}x = 0$$

since the integrand is odd and has only a first order pole at $x = 0$. Thus

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x = ⨍_{-\infty}^{\infty} \frac{e^{ix}}{ix} \, \mathrm{d}x.$$

Let C_R be the semicircular arc in the upper half plane from R to $-R$. Let C be the closed contour that is the union of C_R and the real interval $[-R, R]$. If we close the path of integration with a semicircular arc in the upper half plane, we have

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x = \lim_{R \to \infty} \left(⨍_C \frac{e^{iz}}{iz} \, \mathrm{d}z - \int_{C_R} \frac{e^{iz}}{iz} \, \mathrm{d}z \right),$$

provided that all the integrals exist.

The integral along C_R vanishes as $R \to \infty$ by Jordan's lemma. By the residue theorem for principal values we have

$$-\!\!\!\!\!\!\int \frac{e^{iz}}{iz} \, dz = i\pi \operatorname{Res}\left(\frac{e^{iz}}{iz}, 0\right) = \pi.$$

Combining these results,

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \pi.}$$

Solution 15.18

Note that $(1 - \cos x)/x^2$ has a removable singularity at $x = 0$. The integral decays like $\frac{1}{x^2}$ at infinity, so the integral exists. Since $(\sin x)/x^2$ is a odd function with a simple pole at $x = 0$, the principal value of its integral vanishes.

$$-\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{\sin x}{x^2} \, dx = 0$$

$$\int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} \, dx = -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{1 - \cos x - i \sin x}{x^2} \, dx = -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{1 - e^{ix}}{x^2} \, dx$$

Let C_R be the semi-circle of radius R in the upper half plane. Since

$$\lim_{R \to \infty} \left(R \max_{z \in C_R} \left| \frac{1 - e^{iz}}{z^2} \right| \right) = \lim_{R \to \infty} R \frac{2}{R^2} = 0$$

the integral along C_R vanishes as $R \to \infty$.

$$\int_{C_R} \frac{1 - e^{iz}}{z^2} \, dz \to 0 \text{ as } R \to \infty$$

We can apply Result 15.4.1.

$$\oint_{-\infty}^{\infty} \frac{1 - \mathrm{e}^{ix}}{x^2} \, \mathrm{d}x = i\pi \operatorname{Res}\left(\frac{1 - \mathrm{e}^{iz}}{z^2}, z = 0 \right) = i\pi \lim_{z \to 0} \frac{1 - \mathrm{e}^{iz}}{z} = i\pi \lim_{z \to 0} \frac{-i \, \mathrm{e}^{iz}}{1}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} \, \mathrm{d}x = \pi}$$

Solution 15.19
Consider

$$\int_0^{\infty} \frac{\sin(\pi x)}{x(1 - x^2)} \, \mathrm{d}x.$$

Note that the integrand has removable singularities at the points $x = 0, \pm 1$ and is an even function.

$$\int_0^{\infty} \frac{\sin(\pi x)}{x(1 - x^2)} \, \mathrm{d}x = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(\pi x)}{x(1 - x^2)} \, \mathrm{d}x.$$

Note that $\dfrac{\cos(\pi x)}{x(1 - x^2)}$ is an odd function with first order poles at $x = 0, \pm 1$.

$$\oint_{-\infty}^{\infty} \frac{\cos(\pi x)}{x(1 - x^2)} \, \mathrm{d}x = 0$$

$$\int_0^{\infty} \frac{\sin(\pi x)}{x(1 - x^2)} \, \mathrm{d}x = -\frac{i}{2} \oint_{-\infty}^{\infty} \frac{\mathrm{e}^{i\pi x}}{x(1 - x^2)} \, \mathrm{d}x.$$

Let C_R be the semi-circle of radius R in the upper half plane. Since

$$\lim_{R \to \infty} \left(R \max_{z \in C_R} \left| \frac{\mathrm{e}^{i\pi z}}{z(1 - z^2)} \right| \right) = \lim_{R \to \infty} R \frac{1}{R(R^2 - 1)} = 0$$

the integral along C_R vanishes as $R \to \infty$.

$$\int_{C_R} \frac{e^{i\pi z}}{z(1-z^2)} \, dz \to 0 \text{ as } R \to \infty$$

We can apply Result 15.4.1.

$$\begin{aligned}
-\frac{i}{2} \fint_{-\infty}^{\infty} \frac{e^{i\pi x}}{x(1-x^2)} \, dx &= i\pi \frac{-i}{2} \left(\operatorname{Res}\left(\frac{e^{iz}}{z(1-z^2)}, z=0 \right) + \operatorname{Res}\left(\frac{e^{iz}}{z(1-z^2)}, z=1 \right) \right. \\
&\qquad \left. + \operatorname{Res}\left(\frac{e^{iz}}{z(1-z^2)}, z=-1 \right) \right) \\
&= \frac{\pi}{2} \left(\lim_{z\to 0} \frac{e^{i\pi z}}{1-z^2} - \lim_{z\to 0} \frac{e^{i\pi z}}{z(1+z)} + \lim_{z\to 0} \frac{e^{i\pi z}}{z(1-z)} \right) \\
&= \frac{\pi}{2} \left(1 - \frac{-1}{2} + \frac{-1}{-2} \right)
\end{aligned}$$

$$\boxed{\int_0^{\infty} \frac{\sin(\pi x)}{x(1-x^2)} \, dx = \pi}$$

Contour Integration and Branch Cuts

Solution 15.20

Let C be the boundary of the region $\epsilon < r < R$, $0 < \theta < \pi$. Choose the branch of the logarithm with a branch cut on the negative imaginary axis and the angle range $-\pi/2 < \theta < 3\pi/2$. We consider the integral of $(\log z)^2/(1+z^2)$

on this contour.

$$\begin{aligned}
\oint_C \frac{(\log z)^2}{1+z^2}\,\mathrm{d}z &= i2\pi \operatorname{Res}\left(\frac{(\log z)^2}{1+z^2}, z=i\right) \\
&= i2\pi \lim_{z\to i} \frac{(\log z)^2}{z+i} \\
&= i2\pi \frac{(i\pi/2)^2}{2i} \\
&= -\frac{\pi^3}{4}
\end{aligned}$$

Let C_R be the semi-circle from R to $-R$ in the upper half plane. We show that the integral along C_R vanishes as $R\to\infty$ with the maximum modulus integral bound.

$$\begin{aligned}
\left|\int_{C_R} \frac{(\log z)^2}{1+z^2}\,\mathrm{d}z\right| &\leq \pi R \max_{z\in C_R}\left|\frac{(\log z)^2}{1+z^2}\right| \\
&\leq \pi R \frac{(\ln R)^2 + 2\pi \ln R + \pi^2}{R^2-1} \\
&\to 0 \text{ as } R\to\infty
\end{aligned}$$

Let C_ϵ be the semi-circle from $-\epsilon$ to ϵ in the upper half plane. We show that the integral along C_ϵ vanishes as $\epsilon\to 0$ with the maximum modulus integral bound.

$$\begin{aligned}
\left|\int_{C_\epsilon} \frac{(\log z)^2}{1+z^2}\,\mathrm{d}z\right| &\leq \pi\epsilon \max_{z\in C_\epsilon}\left|\frac{(\log z)^2}{1+z^2}\right| \\
&\leq \pi\epsilon \frac{(\ln \epsilon)^2 - 2\pi \ln \epsilon + \pi^2}{1-\epsilon^2} \\
&\to 0 \text{ as } \epsilon\to 0
\end{aligned}$$

Now we take the limit as $\epsilon \to 0$ and $R \to \infty$ for the integral along C.

$$\oint_C \frac{(\log z)^2}{1+z^2} \, \mathrm{d}z = -\frac{\pi^3}{4}$$

$$\int_0^\infty \frac{(\ln r)^2}{1+r^2} \, \mathrm{d}r + \int_\infty^0 \frac{(\ln r + i\pi)^2}{1+r^2} \, \mathrm{d}r = -\frac{\pi^3}{4}$$

$$2\int_0^\infty \frac{(\ln x)^2}{1+x^2} \, \mathrm{d}x + i2\pi \int_0^\infty \frac{\ln x}{1+x^2} \, \mathrm{d}x = \pi^2 \int_0^\infty \frac{1}{1+x^2} \, \mathrm{d}x - \frac{\pi^3}{4} \tag{15.1}$$

We evaluate the integral of $1/(1+x^2)$ by extending the path of integration to $(-\infty \ldots \infty)$ and closing the path of integration in the upper half plane. Since

$$\lim_{R\to\infty} \left(R \max_{z \in C_R} \left| \frac{1}{1+z^2} \right| \right) \leq \lim_{R\to\infty} \left(R \frac{1}{R^2 - 1} \right) = 0,$$

the integral of $1/(1+z^2)$ along C_R vanishes as $R \to \infty$. We evaluate the integral with the Residue Theorem.

$$\begin{aligned}
\pi^2 \int_0^\infty \frac{1}{1+x^2} \, \mathrm{d}x &= \frac{\pi^2}{2} \int_{-\infty}^\infty \frac{1}{1+x^2} \, \mathrm{d}x \\
&= \frac{\pi^2}{2} i2\pi \operatorname{Res}\left(\frac{1}{1+z^2}, z = i \right) \\
&= i\pi^3 \lim_{z \to i} \frac{1}{z+i} \\
&= \frac{\pi^3}{2}
\end{aligned}$$

Now we return to Equation 15.1.

$$2\int_0^\infty \frac{(\ln x)^2}{1+x^2} \, \mathrm{d}x + i2\pi \int_0^\infty \frac{\ln x}{1+x^2} \, \mathrm{d}x = \frac{\pi^3}{4}$$

We equate the real and imaginary parts to solve for the desired integrals.

$$\boxed{\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,\mathrm{d}x = \frac{\pi^3}{8}}$$

$$\boxed{\int_0^\infty \frac{\ln x}{1+x^2}\,\mathrm{d}x = 0}$$

Solution 15.21

We consider the branch of the function

$$f(z) = \frac{\log z}{z^2 + 5z + 6}$$

with a branch cut on the real axis and $0 < \arg(z) < 2\pi$.

Let C_ϵ and C_R denote the circles of radius ϵ and R where $\epsilon < 1 < R$. C_ϵ is negatively oriented; C_R is positively oriented. Consider the closed contour, C, that is traced by a point moving from ϵ to R above the branch cut, next around C_R back to R, then below the cut to ϵ, and finally around C_ϵ back to ϵ. (See Figure 15.10.)

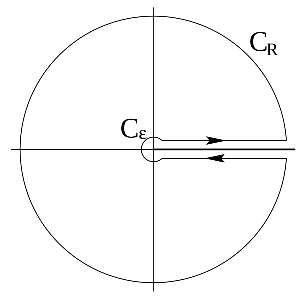


Figure 15.9: The path of integration.

We can evaluate the integral of $f(z)$ along C with the residue theorem. For $R > 3$, there are first order poles inside the path of integration at $z = -2$ and $z = -3$.

$$\begin{aligned}
\int_C \frac{\log z}{z^2 + 5z + 6} \, \mathrm{d}z &= i2\pi \left(\operatorname{Res}\left(\frac{\log z}{z^2 + 5z + 6}, z = -2 \right) + \operatorname{Res}\left(\frac{\log z}{z^2 + 5z + 6}, z = -3 \right) \right) \\
&= i2\pi \left(\lim_{z \to -2} \frac{\log z}{z + 3} + \lim_{z \to -3} \frac{\log z}{z + 2} \right) \\
&= i2\pi \left(\frac{\log(-2)}{1} + \frac{\log(-3)}{-1} \right) \\
&= i2\pi \left(\log(2) + i\pi - \log(3) - i\pi \right) \\
&= i2\pi \log\left(\frac{2}{3} \right)
\end{aligned}$$

In the limit as $\epsilon \to 0$, the integral along C_ϵ vanishes. We demonstrate this with the maximum modulus theorem.

$$\begin{aligned}
\left| \int_{C_\epsilon} \frac{\log z}{z^2 + 5z + 6} \, \mathrm{d}z \right| &\leq 2\pi\epsilon \max_{z \in C_\epsilon} \left| \frac{\log z}{z^2 + 5z + 6} \right| \\
&\leq 2\pi\epsilon \frac{2\pi - \log \epsilon}{6 - 5\epsilon - \epsilon^2} \\
&\to 0 \text{ as } \epsilon \to 0
\end{aligned}$$

In the limit as $R \to \infty$, the integral along C_R vanishes. We again demonstrate this with the maximum modulus theorem.

$$\begin{aligned}
\left| \int_{C_R} \frac{\log z}{z^2 + 5z + 6} \, \mathrm{d}z \right| &\leq 2\pi R \max_{z \in C_R} \left| \frac{\log z}{z^2 + 5z + 6} \right| \\
&\leq 2\pi R \frac{\log R + 2\pi}{R^2 - 5R - 6} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

Taking the limit as $\epsilon \to 0$ and $R \to \infty$, the integral along C is:

$$\begin{aligned}\int_C \frac{\log z}{z^2+5z+6}\,\mathrm{d}z &= \int_0^\infty \frac{\log x}{x^2+5x+6}\,\mathrm{d}x + \int_\infty^0 \frac{\log x + i2\pi}{x^2+5x+6}\,\mathrm{d}x \\ &= -i2\pi \int_0^\infty \frac{\log x}{x^2+5x+6}\,\mathrm{d}x\end{aligned}$$

Now we can evaluate the real integral.

$$-i2\pi \int_0^\infty \frac{\log x}{x^2+5x+6}\,\mathrm{d}x = i2\pi \log\left(\frac{2}{3}\right)$$

$$\boxed{\int_0^\infty \frac{\log x}{x^2+5x+6}\,\mathrm{d}x = \log\left(\frac{3}{2}\right)}$$

Solution 15.22

We consider the integral

$$I(a) = \int_0^\infty \frac{x^a}{(x+1)^2}\,\mathrm{d}x.$$

To examine convergence, we split the domain of integration.

$$\int_0^\infty \frac{x^a}{(x+1)^2}\,\mathrm{d}x = \int_0^1 \frac{x^a}{(x+1)^2}\,\mathrm{d}x + \int_1^\infty \frac{x^a}{(x+1)^2}\,\mathrm{d}x$$

First we work with the integral on $(0 \ldots 1)$.

$$\begin{aligned}\left|\int_0^1 \frac{x^a}{(x+1)^2}\,\mathrm{d}x\right| &\le \int_0^1 \left|\frac{x^a}{(x+1)^2}\right| |dx| \\ &= \int_0^1 \frac{x^{\Re(a)}}{(x+1)^2}\,\mathrm{d}x \\ &\le \int_0^1 x^{\Re(a)}\,\mathrm{d}x\end{aligned}$$

This integral converges for $\Re(a) > -1$.

Next we work with the integral on $(1 \dots \infty)$.

$$\begin{aligned}\left| \int_1^\infty \frac{x^a}{(x+1)^2} \, \mathrm{d}x \right| &\leq \int_1^\infty \left| \frac{x^a}{(x+1)^2} \right| |dx| \\ &= \int_1^\infty \frac{x^{\Re(a)}}{(x+1)^2} \, \mathrm{d}x \\ &\leq \int_1^\infty x^{\Re(a)-2} \, \mathrm{d}x\end{aligned}$$

This integral converges for $\Re(a) < 1$.

Thus we see that the integral defining $I(a)$ converges in the strip, $-1 < \Re(a) < 1$. The integral converges uniformly in any closed subset of this domain. Uniform convergence means that we can differentiate the integral with respect to a and interchange the order of integration and differentiation.

$$I'(a) = \int_0^\infty \frac{x^a \log x}{(x+1)^2} \, \mathrm{d}x$$

Thus we see that $I(a)$ is analytic for $-1 < \Re(a) < 1$.

For $-1 < \Re(a) < 1$ and $a \neq 0$, z^a is multi-valued. Consider the branch of the function $f(z) = z^a/(z+1)^2$ with a branch cut on the positive real axis and $0 < \arg(z) < 2\pi$. We integrate along the contour in Figure 15.10.

The integral on C_ϵ vanishes as $\epsilon \to 0$. We show this with the maximum modulus integral bound. First we write z^a in modulus-argument form, $z = \epsilon \, \mathrm{e}^{i\theta}$, where $a = \alpha + i\beta$.

$$\begin{aligned}z^a &= \mathrm{e}^{a \log z} \\ &= \mathrm{e}^{(\alpha + i\beta)(\ln \epsilon + i\theta)} \\ &= \mathrm{e}^{\alpha \ln \epsilon - \beta\theta + i(\beta \ln \epsilon + \alpha\theta)} \\ &= \epsilon^\alpha \, \mathrm{e}^{-\beta\theta} \, \mathrm{e}^{i(\beta \log \epsilon + \alpha\theta)}\end{aligned}$$

Now we bound the integral.

$$\begin{aligned}\left|\int_{C_\epsilon} \frac{z^a}{(z+1)^2}\,\mathrm{d}z\right| &\leq 2\pi\epsilon \max_{z\in C_\epsilon}\left|\frac{z^a}{(z+1)^2}\right| \\ &\leq 2\pi\epsilon \frac{\epsilon^\alpha\,\mathrm{e}^{2\pi|\beta|}}{(1-\epsilon)^2} \\ &\to 0 \text{ as } \epsilon \to 0\end{aligned}$$

The integral on C_R vanishes as $R \to \infty$.

$$\begin{aligned}\left|\int_{C_R} \frac{z^a}{(z+1)^2}\,\mathrm{d}z\right| &\leq 2\pi R \max_{z\in C_R}\left|\frac{z^a}{(z+1)^2}\right| \\ &\leq 2\pi R \frac{R^\alpha\,\mathrm{e}^{2\pi|\beta|}}{(R-1)^2} \\ &\to 0 \text{ as } R \to \infty\end{aligned}$$

Above the branch cut, $(z = r\,\mathrm{e}^{i0})$, the integrand is

$$f(r\,\mathrm{e}^{i0}) = \frac{r^a}{(r+1)^2}.$$

Below the branch cut, $(z = r\,\mathrm{e}^{i2\pi})$, we have,

$$f(r\,\mathrm{e}^{i2\pi}) = \frac{\mathrm{e}^{i2\pi a} r^a}{(r+1)^2}.$$

Now we use the residue theorem.

$$\int_0^\infty \frac{r^a}{(r+1)^2}\,\mathrm{d}r + \int_\infty^0 \frac{\mathrm{e}^{i2\pi a} r^a}{(r+1)^2}\,\mathrm{d}r = i2\pi \operatorname{Res}\left(\frac{z^a}{(z+1)^2}, -1\right)$$

$$\left(1-\mathrm{e}^{i2\pi a}\right)\int_0^\infty \frac{r^a}{(r+1)^2}\,\mathrm{d}r = i2\pi \lim_{z\to -1}\frac{\mathrm{d}}{\mathrm{d}z}(z^a)$$

$$\int_0^\infty \frac{r^a}{(r+1)^2}\,\mathrm{d}r = i2\pi\frac{a\,\mathrm{e}^{i\pi(a-1)}}{1-\mathrm{e}^{i2\pi a}}$$

$$\int_0^\infty \frac{r^a}{(r+1)^2}\,\mathrm{d}r = \frac{-i2\pi a}{\mathrm{e}^{-i\pi a}-\mathrm{e}^{i\pi a}}$$

$$\int_0^\infty \frac{x^a}{(x+1)^2}\,\mathrm{d}x = \frac{\pi a}{\sin(\pi a)} \quad \text{for } -1 < \Re(a) < 1,\ a \neq 0$$

The right side has a removable singularity at $a=0$. We use analytic continuation to extend the answer to $a=0$.

$$\boxed{I(a) = \int_0^\infty \frac{x^a}{(x+1)^2}\,\mathrm{d}x = \begin{cases} \frac{\pi a}{\sin(\pi a)} & \text{for } -1 < \Re(a) < 1,\ a \neq 0 \\ 1 & \text{for } a = 0 \end{cases}}$$

We can derive the last two integrals by differentiating this formula with respect to a and taking the limit $a \to 0$.

$$I'(a) = \int_0^\infty \frac{x^a \log x}{(x+1)^2}\,\mathrm{d}x, \qquad I''(a) = \int_0^\infty \frac{x^a \log^2 x}{(x+1)^2}\,\mathrm{d}x$$

$$I'(0) = \int_0^\infty \frac{\log x}{(x+1)^2}\,\mathrm{d}x, \qquad I''(0) = \int_0^\infty \frac{\log^2 x}{(x+1)^2}\,\mathrm{d}x$$

We can find $I'(0)$ and $I''(0)$ either by differentiating the expression for $I(a)$ or by finding the first few terms in the Taylor series expansion of $I(a)$ about $a=0$. The latter approach is a little easier.

$$I(a) = \sum_{n=0}^\infty \frac{I^{(n)}(0)}{n!} a^n$$

$$
\begin{aligned}
I(a) &= \frac{\pi a}{\sin(\pi a)} \\
&= \frac{\pi a}{\pi a - (\pi a)^3/6 + \mathcal{O}(a^5)} \\
&= \frac{1}{1 - (\pi a)^2/6 + \mathcal{O}(a^4)} \\
&= 1 + \frac{\pi^2 a^2}{6} + \mathcal{O}(a^4)
\end{aligned}
$$

$$\boxed{I'(0) = \int_0^\infty \frac{\log x}{(x+1)^2}\,\mathrm{d}x = 0}$$

$$\boxed{I''(0) = \int_0^\infty \frac{\log^2 x}{(x+1)^2}\,\mathrm{d}x = \frac{\pi^2}{3}}$$

Solution 15.23

1. We consider the integral

$$I(a) = \int_0^\infty \frac{x^a}{1+x^2}\,\mathrm{d}x.$$

To examine convergence, we split the domain of integration.

$$\int_0^\infty \frac{x^a}{1+x^2}\,\mathrm{d}x = \int_0^1 \frac{x^a}{1+x^2}\,\mathrm{d}x + \int_1^\infty \frac{x^a}{1+x^2}\,\mathrm{d}x$$

First we work with the integral on $(0 \dots 1)$.

$$\begin{aligned}\left| \int_0^1 \frac{x^a}{1+x^2} \, \mathrm{d}x \right| &\leq \int_0^1 \left| \frac{x^a}{1+x^2} \right| |dx| \\ &= \int_0^1 \frac{x^{\Re(a)}}{1+x^2} \, \mathrm{d}x \\ &\leq \int_0^1 x^{\Re(a)} \, \mathrm{d}x\end{aligned}$$

This integral converges for $\Re(a) > -1$.

Next we work with the integral on $(1 \dots \infty)$.

$$\begin{aligned}\left| \int_1^\infty \frac{x^a}{1+x^2} \, \mathrm{d}x \right| &\leq \int_1^\infty \left| \frac{x^a}{1+x^2} \right| |dx| \\ &= \int_1^\infty \frac{x^{\Re(a)}}{1+x^2} \, \mathrm{d}x \\ &\leq \int_1^\infty x^{\Re(a)-2} \, \mathrm{d}x\end{aligned}$$

This integral converges for $\Re(a) < 1$.

Thus we see that the integral defining $I(a)$ converges in the strip, $-1 < \Re(a) < 1$. The integral converges uniformly in any closed subset of this domain. Uniform convergence means that we can differentiate the integral with respect to a and interchange the order of integration and differentiation.

$$I'(a) = \int_0^\infty \frac{x^a \log x}{1+x^2} \, \mathrm{d}x$$

Thus we see that $I(a)$ is analytic for $-1 < \Re(a) < 1$.

2. For $-1 < \Re(a) < 1$ and $a \neq 0$, z^a is multi-valued. Consider the branch of the function $f(z) = z^a/(1+z^2)$ with a branch cut on the positive real axis and $0 < \arg(z) < 2\pi$. We integrate along the contour in Figure 15.10.

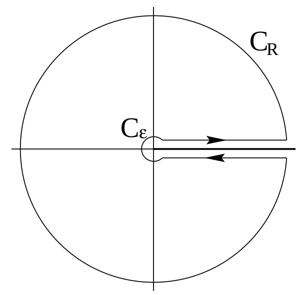


Figure 15.10:

The integral on C_ρ vanishes are $\rho \to 0$. We show this with the maximum modulus integral bound. First we write z^a in modulus-argument form, where $z = \rho\, \mathrm{e}^{i\theta}$ and $a = \alpha + i\beta$.

$$\begin{aligned} z^a &= \mathrm{e}^{a \log z} \\ &= \mathrm{e}^{(\alpha+i\beta)(\log \rho + i\theta)} \\ &= \mathrm{e}^{\alpha \log \rho - \beta\theta + i(\beta \log \rho + \alpha\theta)} \\ &= \rho^a\, \mathrm{e}^{-\beta\theta}\, \mathrm{e}^{i(\beta \log \rho + \alpha\theta)} \end{aligned}$$

Now we bound the integral.

$$\begin{aligned} \left| \int_{C_\rho} \frac{z^a}{1+z^2}\, \mathrm{d}z \right| &\leq 2\pi\rho \max_{z \in C_\rho} \left| \frac{z^a}{1+z^2} \right| \\ &\leq 2\pi\rho \frac{\rho^\alpha\, \mathrm{e}^{2\pi|\beta|}}{1-\rho^2} \\ &\to 0 \text{ as } \rho \to 0 \end{aligned}$$

The integral on C_R vanishes as $R \to \infty$.

$$\begin{aligned}\left| \int_{C_R} \frac{z^a}{1+z^2} \, \mathrm{d}z \right| &\leq 2\pi R \max_{z \in C_R} \left| \frac{z^a}{1+z^2} \right| \\ &\leq 2\pi R \frac{R^\alpha \, \mathrm{e}^{2\pi|\beta|}}{R^2-1} \\ &\to 0 \text{ as } R \to \infty\end{aligned}$$

Above the branch cut, $(z = r\,\mathrm{e}^{i0})$, the integrand is

$$f(r\,\mathrm{e}^{i0}) = \frac{r^a}{1+r^2}.$$

Below the branch cut, $(z = r\,\mathrm{e}^{i2\pi})$, we have,

$$f(r\,\mathrm{e}^{i2\pi}) = \frac{\mathrm{e}^{i2\pi a} r^a}{1+r^2}.$$

Now we use the residue theorem.

$$\int_0^\infty \frac{r^a}{1+r^2}\,\mathrm{d}r + \int_\infty^0 \frac{\mathrm{e}^{i2\pi a} r^a}{1+r^2}\,\mathrm{d}r = i2\pi\left(\operatorname{Res}\left(\frac{z^a}{1+z^2}, i\right) + \operatorname{Res}\left(\frac{z^a}{1+z^2}, -i\right)\right)$$

$$\left(1 - \mathrm{e}^{i2\pi a}\right)\int_0^\infty \frac{x^a}{1+x^2}\,\mathrm{d}x = i2\pi\left(\lim_{z\to i}\frac{z^a}{z+i} + \lim_{z\to -i}\frac{z^a}{z-i}\right)$$

$$\left(1 - \mathrm{e}^{i2\pi a}\right)\int_0^\infty \frac{x^a}{1+x^2}\,\mathrm{d}x = i2\pi\left(\frac{\mathrm{e}^{ia\pi/2}}{2i} + \frac{\mathrm{e}^{ia3\pi/2}}{-2i}\right)$$

$$\int_0^\infty \frac{x^a}{1+x^2}\,\mathrm{d}x = \pi\frac{\mathrm{e}^{ia\pi/2} - \mathrm{e}^{ia3\pi/2}}{1 - \mathrm{e}^{i2a\pi}}$$

$$\int_0^\infty \frac{x^a}{1+x^2}\,\mathrm{d}x = \pi\frac{\mathrm{e}^{ia\pi/2}(1 - \mathrm{e}^{ia\pi})}{(1 + \mathrm{e}^{ia\pi})(1 - \mathrm{e}^{ia\pi})}$$

$$\int_0^\infty \frac{x^a}{1+x^2}\,\mathrm{d}x = \frac{\pi}{\mathrm{e}^{-ia\pi/2} + \mathrm{e}^{ia\pi/2}}$$

$$\int_0^\infty \frac{x^a}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2\cos(\pi a/2)} \quad \text{for } -1 < \Re(a) < 1,\ a \neq 0$$

We use analytic continuation to extend the answer to $a = 0$.

$$\boxed{I(a) = \int_0^\infty \frac{x^a}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2\cos(\pi a/2)} \quad \text{for } -1 < \Re(a) < 1}$$

3. We can derive the last two integrals by differentiating this formula with respect to a and taking the limit $a \to 0$.

$$I'(a) = \int_0^\infty \frac{x^a \log x}{1+x^2}\,\mathrm{d}x, \qquad I''(a) = \int_0^\infty \frac{x^a \log^2 x}{1+x^2}\,\mathrm{d}x$$

$$I'(0) = \int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x, \qquad I''(0) = \int_0^\infty \frac{\log^2 x}{1+x^2}\,\mathrm{d}x$$

We can find $I'(0)$ and $I''(0)$ either by differentiating the expression for $I(a)$ or by finding the first few terms in the Taylor series expansion of $I(a)$ about $a = 0$. The latter approach is a little easier.

$$I(a) = \sum_{n=0}^{\infty} \frac{I^{(n)}(0)}{n!} a^n$$

$$\begin{aligned} I(a) &= \frac{\pi}{2\cos(\pi a/2)} \\ &= \frac{\pi}{2} \frac{1}{1 - (\pi a/2)^2/2 + \mathcal{O}(a^4)} \\ &= \frac{\pi}{2} \left(1 + (\pi a/2)^2/2 + \mathcal{O}(a^4)\right) \\ &= \frac{\pi}{2} + \frac{\pi^3/8}{2} a^2 + \mathcal{O}(a^4) \end{aligned}$$

$$\boxed{I'(0) = \int_0^\infty \frac{\log x}{1+x^2} \, \mathrm{d}x = 0}$$

$$\boxed{I''(0) = \int_0^\infty \frac{\log^2 x}{1+x^2} \, \mathrm{d}x = \frac{\pi^3}{8}}$$

Solution 15.24

Convergence. If $x^a f(x) \ll x^\alpha$ as $x \to 0$ for some $\alpha > -1$ then the integral

$$\int_0^1 x^a f(x) \, \mathrm{d}x$$

will converge absolutely. If $x^a f(x) \ll x^\beta$ as $x \to \infty$ for some $\beta < -1$ then the integral

$$\int_1^\infty x^a f(x)$$

will converge absolutely. These are sufficient conditions for the absolute convergence of

$$\int_0^\infty x^a f(x)\,\mathrm{d}x.$$

Contour Integration. We put a branch cut on the positive real axis and choose $0 < \arg(z) < 2\pi$. We consider the integral of $z^a f(z)$ on the contour in Figure 15.10. Let the singularities of $f(z)$ occur at $z_1, \ldots, z_n$. By the residue theorem,

$$\int_C z^a f(z)\,\mathrm{d}z = i2\pi \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z), z_k\right).$$

On the circle of radius ϵ, the integrand is $o(\epsilon^{-1})$. Since the length of C_ϵ is $2\pi\epsilon$, the integral on C_ϵ vanishes as $\epsilon \to 0$. On the circle of radius R, the integrand is $o(R^{-1})$. Since the length of C_R is $2\pi R$, the integral on C_R vanishes as $R \to \infty$.

The value of the integrand below the branch cut, $z = x\,\mathrm{e}^{i2\pi}$, is

$$f(x\,\mathrm{e}^{i2\pi}) = x^a\,\mathrm{e}^{i2\pi a} f(x)$$

In the limit as $\epsilon \to 0$ and $R \to \infty$ we have

$$\int_0^\infty x^a f(x)\,\mathrm{d}x + \int_{-\infty}^0 x^a\,\mathrm{e}^{i2\pi a} f(x)\,\mathrm{d}x = i2\pi \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z), z_k\right).$$

$$\boxed{\int_0^\infty x^a f(x)\,\mathrm{d}x = \frac{i2\pi}{1 - \mathrm{e}^{i2\pi a}} \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z), z_k\right).}$$

Solution 15.25

In the interval of uniform convergence of th integral, we can differentiate the formula

$$\int_0^\infty x^a f(x)\,\mathrm{d}x = \frac{i2\pi}{1-\mathrm{e}^{i2\pi a}} \sum_{k=1}^n \operatorname{Res}\left(z^a f(z), z_k\right),$$

with respect to a to obtain,

$$\int_0^\infty x^a f(x) \log x\,\mathrm{d}x = \frac{i2\pi}{1-\mathrm{e}^{i2\pi a}} \sum_{k=1}^n \operatorname{Res}\left(z^a f(z)\log z, z_k\right), - \frac{4\pi^2 a\,\mathrm{e}^{i2\pi a}}{(1-\mathrm{e}^{i2\pi a})^2} \sum_{k=1}^n \operatorname{Res}\left(z^a f(z), z_k\right).$$

$$\boxed{\int_0^\infty x^a f(x) \log x\,\mathrm{d}x = \frac{i2\pi}{1-\mathrm{e}^{i2\pi a}} \sum_{k=1}^n \operatorname{Res}\left(z^a f(z)\log z, z_k\right), + \frac{\pi^2 a}{\sin^2(\pi a)} \sum_{k=1}^n \operatorname{Res}\left(z^a f(z), z_k\right),}$$

Differentiating the solution of Exercise 15.22 m times with respect to a yields

$$\boxed{\int_0^\infty x^a f(x) \log^m x\,\mathrm{d}x = \frac{\partial^m}{\partial a^m}\left(\frac{i2\pi}{1-\mathrm{e}^{i2\pi a}} \sum_{k=1}^n \operatorname{Res}\left(z^a f(z), z_k\right)\right),}$$

Solution 15.26

Taking the limit as $a \to 0 \in \mathbb{Z}$ in the solution of Exercise 15.22 yields

$$\int_0^\infty f(x)\,\mathrm{d}x = i2\pi \lim_{a\to 0}\left(\frac{\sum_{k=1}^n \operatorname{Res}\left(z^a f(z), z_k\right)}{1-\mathrm{e}^{i2\pi a}}\right)$$

The numerator vanishes because the sum of all residues of $z^n f(z)$ is zero. Thus we can use L'Hospital's rule.

$$\int_0^\infty f(x)\,\mathrm{d}x = i2\pi \lim_{a\to 0}\left(\frac{\sum_{k=1}^n \operatorname{Res}\left(z^a f(z)\log z, z_k\right)}{-i2\pi\,\mathrm{e}^{i2\pi a}}\right)$$

$$\boxed{\int_0^\infty f(x)\,\mathrm{d}x = -\sum_{k=1}^{n} \mathrm{Res}\,\left(f(z)\log z, z_k\right)}$$

This suggests that we could have derived the result directly by considering the integral of $f(z)\log z$ on the contour in Figure 15.10. We put a branch cut on the positive real axis and choose the branch $\arg z = 0$. Recall that we have assumed that $f(z)$ has only isolated singularities and no singularities on the positive real axis, $[0, \infty)$. By the residue theorem,

$$\int_C f(z)\log z\,\mathrm{d}z = i2\pi \sum_{k=1}^{n} \mathrm{Res}\,\left(f(z)\log z, z = z_k\right).$$

By assuming that $f(z) \ll z^\alpha$ as $z \to 0$ where $\alpha > -1$ the integral on C_ϵ will vanish as $\epsilon \to 0$. By assuming that $f(z) \ll z^\beta$ as $z \to \infty$ where $\beta < -1$ the integral on C_R will vanish as $R \to \infty$. The value of the integrand below the branch cut, $z = x\,\mathrm{e}^{i2\pi}$ is $f(x)(\log x + i2\pi)$. Taking the limit as $\epsilon \to 0$ and $R \to \infty$, we have

$$\int_0^\infty f(x)\log x\,\mathrm{d}x + \int_\infty^0 f(x)(\log x + i2\pi)\,\mathrm{d}x = i2\pi \sum_{k=1}^{n} \mathrm{Res}\,\left(f(z)\log z, z_k\right).$$

Thus we corroborate the result.

$$\int_0^\infty f(x)\,\mathrm{d}x = -\sum_{k=1}^{n} \mathrm{Res}\,\left(f(z)\log z, z_k\right)$$

Solution 15.27

Consider the integral of $f(z)\log^2 z$ on the contour in Figure 15.10. We put a branch cut on the positive real axis and choose the branch $0 < \arg z < 2\pi$. Let $z_1, \ldots z_n$ be the singularities of $f(z)$. By the residue theorem,

$$\int_C f(z)\log^2 z\,\mathrm{d}z = i2\pi \sum_{k=1}^{n} \mathrm{Res}\,\left(f(z)\log^2 z, z_k\right).$$

If $f(z) \ll z^\alpha$ as $z \to 0$ for some $\alpha > -1$ then the integral on C_ϵ will vanish as $\epsilon \to 0$. $f(z) \ll z^\beta$ as $z \to \infty$ for some $\beta < -1$ then the integral on C_R will vanish as $R \to \infty$. Below the branch cut the integrand is $f(x)(\log x + i2\pi)^2$. Thus we have

$$\int_0^\infty f(x) \log^2 x \,\mathrm{d}x + \int_\infty^0 f(x)(\log^2 x + i4\pi \log x - 4\pi^2) \,\mathrm{d}x = i2\pi \sum_{k=1}^n \operatorname{Res}\left(f(z) \log^2 z, z_k\right).$$

$$-i4\pi \int_0^\infty f(x) \log x \,\mathrm{d}x + 4\pi^2 \int_0^\infty f(x) \,\mathrm{d}x = i2\pi \sum_{k=1}^n \operatorname{Res}\left(f(z) \log^2 z, z_k\right).$$

$$\boxed{\int_0^\infty f(x) \log x \,\mathrm{d}x = -\frac{1}{2} \sum_{k=1}^n \operatorname{Res}\left(f(z) \log^2 z, z_k\right) + i\pi \sum_{k=1}^n \operatorname{Res}\left(f(z) \log z, z_k\right)}$$

Solution 15.28

Convergence. We consider

$$\int_0^\infty \frac{x^a}{1 + x^4} \,\mathrm{d}x.$$

Since the integrand behaves like x^a near $x = 0$ we must have $\Re(a) > -1$. Since the integrand behaves like x^{a-4} at infinity we must have $\Re(a - 4) < -1$. The integral converges for $-1 < \Re(a) < 3$.

Contour Integration. The function

$$f(z) = \frac{z^a}{1 + z^4}$$

has first order poles at $z = (\pm 1 \pm i)/\sqrt{2}$ and a branch point at $z = 0$. We could evaluate the real integral by putting a branch cut on the positive real axis with $0 < \arg(z) < 2\pi$ and integrating $f(z)$ on the contour in Figure 15.11.

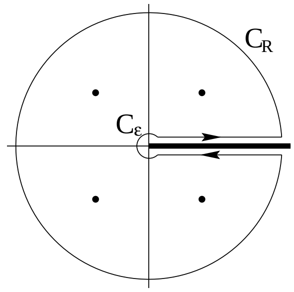


Figure 15.11: Possible Path of Integration for $f(z) = \frac{z^a}{1+z^4}$

Integrating on this contour would work because the value of the integrand below the branch cut is a constant times the value of the integrand above the branch cut. After demonstrating that the integrals along C_ϵ and C_R vanish in the limits as $\epsilon \to 0$ and $R \to \infty$ we would see that the value of the integral is a constant times the sum of the residues at the four poles. However, this is not the only, (and not the best), contour that can be used to evaluate the real integral. Consider the value of the integral on the line $\arg(z) = \theta$.

$$f(r\,\mathrm{e}^{i\theta}) = \frac{r^a\,\mathrm{e}^{ia\theta}}{1 + r^4\,\mathrm{e}^{i4\theta}}$$

If θ is a integer multiple of $\pi/2$ then the integrand is a constant multiple of

$$f(x) = \frac{r^a}{1 + r^4}.$$

Thus any of the contours in Figure 15.12 can be used to evaluate the real integral. The only difference is how many residues we have to calculate. Thus we choose the first contour in Figure 15.12. We put a branch cut on the negative real axis and choose the branch $-\pi < \arg(z) < \pi$ to satisfy $f(1) = 1$.

We evaluate the integral along C with the Residue Theorem.

$$\int_C \frac{z^a}{1+z^4}\,\mathrm{d}z = i2\pi \operatorname{Res}\left(\frac{z^a}{1+z^4}, z = \frac{1+i}{\sqrt{2}}\right)$$

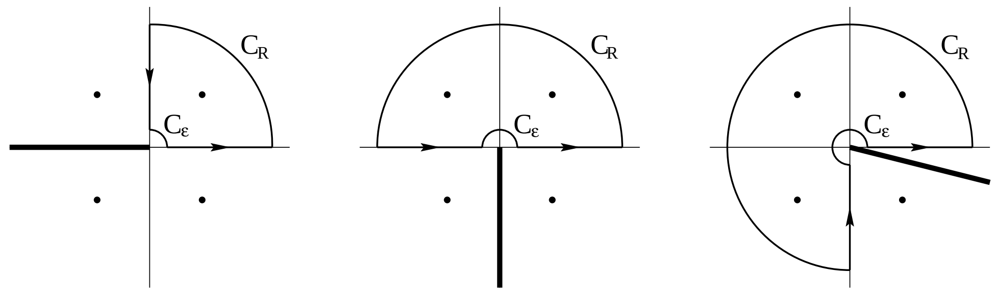


Figure 15.12: Possible Paths of Integration for $f(z) = \frac{z^a}{1+z^4}$

Let $a = \alpha + i\beta$ and $z = r\,\mathrm{e}^{i\theta}$. Note that

$$|z^a| = |(r\,\mathrm{e}^{i\theta})^{\alpha+i\beta}| = r^\alpha\,\mathrm{e}^{-\beta\theta}.$$

The integral on C_ϵ vanishes as $\epsilon \to 0$. We demonstrate this with the maximum modulus integral bound.

$$\begin{aligned}
\left|\int_{C_\epsilon} \frac{z^a}{1+z^4}\,\mathrm{d}z\right| &\leq \frac{\pi\epsilon}{2} \max_{z\in C_\epsilon} \left|\frac{z^a}{1+z^4}\right| \\
&\leq \frac{\pi\epsilon}{2} \frac{\epsilon^\alpha\,\mathrm{e}^{\pi|\beta|/2}}{1-\epsilon^4} \\
&\to 0 \quad \text{as } \epsilon \to 0
\end{aligned}$$

The integral on C_R vanishes as $R \to \infty$.

$$\begin{aligned}
\left|\int_{C_R} \frac{z^a}{1+z^4}\,\mathrm{d}z\right| &\leq \frac{\pi R}{2} \max_{z\in C_R} \left|\frac{z^a}{1+z^4}\right| \\
&\leq \frac{\pi R}{2} \frac{R^\alpha\,\mathrm{e}^{\pi|\beta|/2}}{R^4-1} \\
&\to 0 \quad \text{as } R \to \infty
\end{aligned}$$

The value of the integrand on the positive imaginary axis, $z = x\,\mathrm{e}^{i\pi/2}$, is

$$\frac{(x\,\mathrm{e}^{i\pi/2})^a}{1 + (x\,\mathrm{e}^{i\pi/2})^4} = \frac{x^a\,\mathrm{e}^{i\pi a/2}}{1 + x^4}.$$

We take the limit as $\epsilon \to 0$ and $R \to \infty$.

$$\int_0^\infty \frac{x^a}{1+x^4}\,\mathrm{d}x + \int_\infty^0 \frac{x^a\,\mathrm{e}^{i\pi a/2}}{1+x^4}\,\mathrm{e}^{i\pi/2}\,\mathrm{d}x = i2\pi \operatorname{Res}\left(\frac{z^a}{1+z^4},\ \mathrm{e}^{i\pi/4}\right)$$
$$\left(1 - \mathrm{e}^{i\pi(a+1)/2}\right)\int_0^\infty \frac{x^a}{1+x^4}\,\mathrm{d}x = i2\pi \lim_{z\to\mathrm{e}^{i\pi/4}}\left(\frac{z^a(z - \mathrm{e}^{i\pi/2})}{1+z^4}\right)$$
$$\int_0^\infty \frac{x^a}{1+x^4}\,\mathrm{d}x = \frac{i2\pi}{1 - \mathrm{e}^{i\pi(a+1)/2}} \lim_{z\to\mathrm{e}^{i\pi/4}}\left(\frac{az^a(z - \mathrm{e}^{i\pi/2}) + z^a}{4z^3}\right)$$
$$\int_0^\infty \frac{x^a}{1+x^4}\,\mathrm{d}x = \frac{i2\pi}{1 - \mathrm{e}^{i\pi(a+1)/2}}\,\frac{\mathrm{e}^{i\pi a/4}}{4\,\mathrm{e}^{i3\pi/4}}$$
$$\int_0^\infty \frac{x^a}{1+x^4}\,\mathrm{d}x = \frac{-i\pi}{2(\mathrm{e}^{-i\pi(a+1)/4} - \mathrm{e}^{i\pi(a+1)/4})}$$
$$\boxed{\int_0^\infty \frac{x^a}{1+x^4}\,\mathrm{d}x = \frac{\pi}{4}\csc\left(\frac{\pi(a+1)}{4}\right)}$$

Solution 15.29

Consider the branch of $f(z) = z^{1/2}\log z/(z+1)^2$ with a branch cut on the positive real axis and $0 < \arg z < 2\pi$. We integrate this function on the contour in Figure 15.10.

We use the maximum modulus integral bound to show that the integral on C_ρ vanishes as $\rho \to 0$.

$$
\begin{aligned}
\left| \int_{C_\rho} \frac{z^{1/2} \log z}{(z+1)^2} \, \mathrm{d}z \right| &\leq 2\pi\rho \max_{C_\rho} \left| \frac{z^{1/2} \log z}{(z+1)^2} \right| \\
&= 2\pi\rho \frac{\rho^{1/2}(2\pi - \log \rho)}{(1-\rho)^2} \\
&\to 0 \text{ as } \rho \to 0
\end{aligned}
$$

The integral on C_R vanishes as $R \to \infty$.

$$
\begin{aligned}
\left| \int_{C_R} \frac{z^{1/2} \log z}{(z+1)^2} \, \mathrm{d}z \right| &\leq 2\pi R \max_{C_R} \left| \frac{z^{1/2} \log z}{(z+1)^2} \right| \\
&= 2\pi R \frac{R^{1/2}(\log R + 2\pi)}{(R-1)^2} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}
$$

Above the branch cut, $(z = x\,\mathrm{e}^{i0})$, the integrand is,

$$
f(x\,\mathrm{e}^{i0}) = \frac{x^{1/2} \log x}{(x+1)^2}.
$$

Below the branch cut, $(z = x\,\mathrm{e}^{i2\pi}\ )$, we have,

$$
f(x\,\mathrm{e}^{i2\pi}) = \frac{-x^{1/2}(\log x + i\pi)}{(x+1)^2}.
$$

Taking the limit as $\rho \to 0$ and $R \to \infty$, the residue theorem gives us

$$
\int_0^\infty \frac{x^{1/2} \log x}{(x+1)^2} \, \mathrm{d}x + \int_\infty^0 \frac{-x^{1/2}(\log x + i2\pi)}{(x+1)^2} \, \mathrm{d}x = i2\pi \operatorname{Res}\left(\frac{z^{1/2} \log z}{(z+1)^2}, -1 \right).
$$

$$2\int_0^\infty \frac{x^{1/2}\log x}{(x+1)^2}\,dx + i2\pi\int_0^\infty \frac{x^{1/2}}{(x+1)^2}\,dx = i2\pi \lim_{z\to -1}\frac{d}{dz}\left(z^{1/2}\log z\right)$$

$$2\int_0^\infty \frac{x^{1/2}\log x}{(x+1)^2}\,dx + i2\pi\int_0^\infty \frac{x^{1/2}}{(x+1)^2}\,dx = i2\pi \lim_{z\to -1}\left(\frac{1}{2}z^{-1/2}\log z + z^{1/2}\frac{1}{z}\right)$$

$$2\int_0^\infty \frac{x^{1/2}\log x}{(x+1)^2}\,dx + i2\pi\int_0^\infty \frac{x^{1/2}}{(x+1)^2}\,dx = i2\pi\left(\frac{1}{2}(-i)(i\pi) - i\right)$$

$$2\int_0^\infty \frac{x^{1/2}\log x}{(x+1)^2}\,dx + i2\pi\int_0^\infty \frac{x^{1/2}}{(x+1)^2}\,dx = 2\pi + i\pi^2$$

Equating real and imaginary parts,

$$\boxed{\int_0^\infty \frac{x^{1/2}\log x}{(x+1)^2}\,dx = \pi, \qquad \int_0^\infty \frac{x^{1/2}}{(x+1)^2}\,dx = \frac{\pi}{2}.}$$

Exploiting Symmetry

Solution 15.30
Convergence. The integrand,

$$\frac{e^{az}}{e^z - e^{-z}} = \frac{e^{az}}{2\sinh(z)},$$

has first order poles at $z = in\pi$, $n \in \mathbb{Z}$. To study convergence, we split the domain of integration.

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{\infty}$$

The principal value integral

$$⨍_{-1}^{1} \frac{e^{ax}}{e^x - e^{-x}} \, dx$$

exists for any a because the integrand has only a first order pole on the path of integration.

Now consider the integral on $(1 \ldots \infty)$.

$$\begin{aligned}
\left| \int_1^\infty \frac{e^{ax}}{e^x - e^{-x}} \, dx \right| &= \int_1^\infty \frac{e^{(a-1)x}}{1 - e^{-2x}} \, dx \\
&\leq \frac{1}{1 - e^{-2}} \int_1^\infty e^{(a-1)x} \, dx
\end{aligned}$$

This integral converges for $a - 1 < 0$; $a < 1$.

Finally consider the integral on $(-\infty \ldots -1)$.

$$\begin{aligned}
\left| \int_{-\infty}^{-1} \frac{e^{ax}}{e^x - e^{-x}} \, dx \right| &= \int_{-\infty}^{-1} \frac{e^{(a+1)x}}{1 - e^{2x}} \, dx \\
&\leq \frac{1}{1 - e^{-2}} \int_{-\infty}^{-1} e^{(a+1)x} \, dx
\end{aligned}$$

This integral converges for $a + 1 > 0$; $a > -1$.

Thus we see that the integral for $I(a)$ converges for real a, $|a| < 1$.

Choice of Contour. Consider the contour C that is the boundary of the region: $-R < x < R$, $0 < y < \pi$. The integrand has no singularities inside the contour. There are first order poles on the contour at $z = 0$ and $z = i\pi$. The value of the integral along the contour is $i\pi$ times the sum of these two residues.

The integrals along the vertical sides of the contour vanish as $R \to \infty$.

$$\begin{aligned}
\left| \int_R^{R+i\pi} \frac{e^{az}}{e^z - e^{-z}} \, dz \right| &\leq \pi \max_{z \in (R \ldots R+i\pi)} \left| \frac{e^{az}}{e^z - e^{-z}} \right| \\
&\leq \pi \frac{e^{aR}}{e^R - e^{-R}} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

$$\left| \int_{-R}^{-R+i\pi} \frac{\mathrm{e}^{az}}{\mathrm{e}^{z} - \mathrm{e}^{-z}} \,\mathrm{d}z \right| \leq \pi \max_{z \in (-R \ldots -R+i\pi)} \left| \frac{\mathrm{e}^{az}}{\mathrm{e}^{z} - \mathrm{e}^{-z}} \right|$$
$$\leq \pi \frac{\mathrm{e}^{-aR}}{\mathrm{e}^{-R} - \mathrm{e}^{R}}$$
$$\to 0 \text{ as } R \to \infty$$

Evaluating the Integral. We take the limit as $R \to \infty$ and apply the residue theorem.

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{\mathrm{e}^{x} - \mathrm{e}^{-x}} \,\mathrm{d}x + \int_{\infty+i\pi}^{-\infty+i\pi} \frac{\mathrm{e}^{az}}{\mathrm{e}^{z} - \mathrm{e}^{-z}} \,\mathrm{d}z$$
$$= i\pi \operatorname{Res}\left(\frac{\mathrm{e}^{az}}{\mathrm{e}^{z} - \mathrm{e}^{-z}}, z = 0 \right) + i\pi \operatorname{Res}\left(\frac{\mathrm{e}^{az}}{\mathrm{e}^{z} - \mathrm{e}^{-z}}, z = i\pi \right)$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{\mathrm{e}^{x} - \mathrm{e}^{-x}} \,\mathrm{d}x + \int_{\infty}^{-\infty} \frac{\mathrm{e}^{a(x+i\pi}}{\mathrm{e}^{x+i\pi} - \mathrm{e}^{-x-i\pi}} \,\mathrm{d}z = i\pi \lim_{z\to 0} \frac{z\,\mathrm{e}^{az}}{2\sinh(z)} + i\pi \lim_{z \to i\pi} \frac{(z - i\pi)\,\mathrm{e}^{az}}{2\sinh(z)}$$
$$(1 + \mathrm{e}^{ia\pi}) \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{\mathrm{e}^{x} - \mathrm{e}^{-x}} \,\mathrm{d}x = i\pi \lim_{z\to 0} \frac{\mathrm{e}^{az} + az\,\mathrm{e}^{az}}{2\cosh(z)} + i\pi \lim_{z \to i\pi} \frac{\mathrm{e}^{az} + a(z - i\pi)\,\mathrm{e}^{az}}{2\cosh(z)}$$
$$(1 + \mathrm{e}^{ia\pi}) \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{\mathrm{e}^{x} - \mathrm{e}^{-x}} \,\mathrm{d}x = i\pi \frac{1}{2} + i\pi \frac{\mathrm{e}^{ia\pi}}{-2}$$
$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{\mathrm{e}^{x} - \mathrm{e}^{-x}} \,\mathrm{d}x = \frac{i\pi(1 - \mathrm{e}^{ia\pi})}{2(1 + \mathrm{e}^{ia\pi})}$$
$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{\mathrm{e}^{x} - \mathrm{e}^{-x}} \,\mathrm{d}x = \frac{\pi}{2} \frac{i(\,\mathrm{e}^{-ia\pi/2} - \mathrm{e}^{ia\pi/2})}{\mathrm{e}^{ia\pi/2} + \mathrm{e}^{ia\pi/2}}$$
$$\boxed{\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{\mathrm{e}^{x} - \mathrm{e}^{-x}} \,\mathrm{d}x = \frac{\pi}{2} \tan\left(\frac{a\pi}{2} \right)}$$

Solution 15.31

1.
$$\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{\mathrm{d}x}{(1+x^2)^2}$$

We apply Result 15.4.1 to the integral on the real axis. First we verify that the integrand vanishes fast enough in the upper half plane.

$$\lim_{R\to\infty}\left(R\max_{z\in C_R}\left|\frac{1}{(1+z^2)^2}\right|\right) = \lim_{R\to\infty}\left(R\frac{1}{(R^2-1)^2}\right) = 0$$

Then we evaluate the integral with the residue theorem.

$$\begin{aligned}\int_{-\infty}^\infty \frac{\mathrm{d}x}{(1+x^2)^2} &= i2\pi \operatorname{Res}\left(\frac{1}{(1+z^2)^2}, z=i\right)\\ &= i2\pi \operatorname{Res}\left(\frac{1}{(z-i)^2(z+i)^2}, z=i\right)\\ &= i2\pi \lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{(z+i)^2}\\ &= i2\pi \lim_{z\to i}\frac{-2}{(z+i)^3}\\ &= \frac{\pi}{2}\end{aligned}$$

$$\boxed{\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^2} = \frac{\pi}{4}}$$

2. We wish to evaluate

$$\int_0^\infty \frac{\mathrm{d}x}{x^3+1}.$$

Let the contour C be the boundary of the region $0 < r < R$, $0 < \theta < 2\pi/3$. We factor the denominator of the integrand to see that the contour encloses the simple pole at $\mathrm{e}^{i\pi/3}$ for $R > 1$.

$$z^3 + 1 = (z - \mathrm{e}^{i\pi/3})(z + 1)(z - \mathrm{e}^{-i\pi/3}),$$

We calculate the residue at that point.

$$\begin{aligned}
\operatorname{Res}\left(\frac{1}{z^3+1}, z = \mathrm{e}^{i\pi/3}\right) &= \lim_{z \to \mathrm{e}^{i\pi/3}} \left((z - \mathrm{e}^{i\pi/3})\frac{1}{z^3+1}\right) \\
&= \lim_{z \to \mathrm{e}^{i\pi/3}} \left(\frac{1}{(z+1)(z - \mathrm{e}^{-i\pi/3})}\right) \\
&= \frac{1}{(\mathrm{e}^{i\pi/3} + 1)(\mathrm{e}^{i\pi/3} - \mathrm{e}^{-i\pi/3})}
\end{aligned}$$

We use the residue theorem to evaluate the integral.

$$\oint_C \frac{\mathrm{d}z}{z^3+1} = \frac{i2\pi}{(\mathrm{e}^{i\pi/3} + 1)(\mathrm{e}^{i\pi/3} - \mathrm{e}^{-i\pi/3})}$$

Let C_R be the circular arc portion of the contour.

$$\begin{aligned}
\int_C \frac{\mathrm{d}z}{z^3+1} &= \int_0^R \frac{\mathrm{d}x}{x^3+1} + \int_{C_R} \frac{\mathrm{d}z}{z^3+1} - \int_0^R \frac{\mathrm{e}^{2i\pi/3}\,dx}{x^3+1} \\
&= (1 + \mathrm{e}^{-i\pi/3}) \int_0^R \frac{\mathrm{d}x}{x^3+1} + \int_{C_R} \frac{\mathrm{d}z}{z^3+1}
\end{aligned}$$

We show that the integral along C_R vanishes as $R \to \infty$ with the maximum modulus integral bound.

$$\left| \int_{C_R} \frac{\mathrm{d}z}{z^3+1} \right| \leq \frac{2\pi R}{3} \frac{1}{R^3 - 1} \to 0 \quad \text{as } R \to \infty$$

We take $R \to \infty$ and solve for the desired integral.

$$
\begin{aligned}
\left(1 + \mathrm{e}^{-i\pi/3}\right) \int_0^\infty \frac{\mathrm{d}x}{x^3+1} &= \frac{i2\pi}{(1+\mathrm{e}^{i\pi/3})(\mathrm{e}^{i\pi/3} - \mathrm{e}^{-i\pi/3})} \\
\int_0^\infty \frac{\mathrm{d}x}{x^3+1} &= \frac{i2\pi}{(1+\mathrm{e}^{-i\pi/3})(1+\mathrm{e}^{i\pi/3})(\mathrm{e}^{i\pi/3} - \mathrm{e}^{-i\pi/3})} \\
\int_0^\infty \frac{\mathrm{d}x}{x^3+1} &= \frac{i2\pi}{\frac{1}{2}(3+i\sqrt{3})\frac{1}{2}(3-i\sqrt{3})(i\sqrt{3})} \\
\int_0^\infty \frac{\mathrm{d}x}{x^3+1} &= \frac{2\pi}{3\sqrt{3}}
\end{aligned}
$$

Solution 15.32

Method 1: Semi-Circle Contour. We wish to evaluate the integral

$$
I = \int_0^\infty \frac{\mathrm{d}x}{1+x^6}.
$$

We note that the integrand is an even function and express I as an integral over the whole real axis.

$$
I = \frac{1}{2} \int_{-\infty}^\infty \frac{\mathrm{d}x}{1+x^6}
$$

Now we will evaluate the integral using contour integration. We close the path of integration in the upper half plane. Let Γ_R be the semicircular arc from R to $-R$ in the upper half plane. Let Γ be the union of Γ_R and the interval $[-R, R]$. (See Figure 15.13.)

We can evaluate the integral along Γ with the residue theorem. The integrand has first order poles at

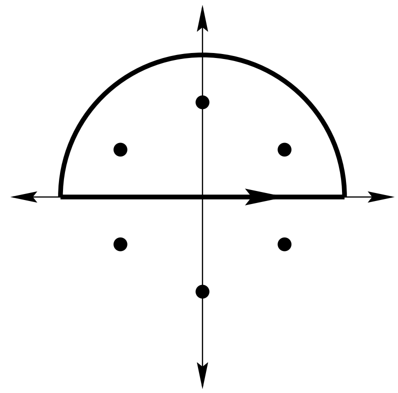


Figure 15.13: The semi-circle contour.

$z = \mathrm{e}^{i\pi(1+2k)/6}$, $k = 0, 1, 2, 3, 4, 5$. Three of these poles are in the upper half plane. For $R > 1$, we have

$$\begin{aligned}\int_\Gamma \frac{1}{z^6+1}\,\mathrm{d}z &= i2\pi \sum_{k=0}^{2} \operatorname{Res}\left(\frac{1}{z^6+1}, \mathrm{e}^{i\pi(1+2k)/6}\right) \\ &= i2\pi \sum_{k=0}^{2} \lim_{z\to \mathrm{e}^{i\pi(1+2k)/6}} \frac{z - \mathrm{e}^{i\pi(1+2k)/6}}{z^6+1}\end{aligned}$$

Since the numerator and denominator vanish, we apply L'Hospital's rule.

$$
\begin{aligned}
&= i2\pi \sum_{k=0}^{2} \lim_{z \to \mathrm{e}^{i\pi(1+2k)/6}} \frac{1}{6z^5} \\
&= \frac{i\pi}{3} \sum_{k=0}^{2} \mathrm{e}^{-i\pi 5(1+2k)/6} \\
&= \frac{i\pi}{3} \left(\mathrm{e}^{-i\pi 5/6} + \mathrm{e}^{-i\pi 15/6} + \mathrm{e}^{-i\pi 25/6} \right) \\
&= \frac{i\pi}{3} \left(\mathrm{e}^{-i\pi 5/6} + \mathrm{e}^{-i\pi/2} + \mathrm{e}^{-i\pi/6} \right) \\
&= \frac{i\pi}{3} \left(\frac{-\sqrt{3} - i}{2} - i + \frac{\sqrt{3} - i}{2} \right) \\
&= \frac{2\pi}{3}
\end{aligned}
$$

Now we examine the integral along Γ_R. We use the maximum modulus integral bound to show that the value of the integral vanishes as $R \to \infty$.

$$
\begin{aligned}
\left| \int_{\Gamma_R} \frac{1}{z^6 + 1} \, \mathrm{d}z \right| &\leq \pi R \max_{z \in \Gamma_R} \left| \frac{1}{z^6 + 1} \right| \\
&= \pi R \frac{1}{R^6 - 1} \\
&\to 0 \quad \text{as } R \to \infty.
\end{aligned}
$$

Now we are prepared to evaluate the original real integral.

$$
\int_{\Gamma} \frac{1}{z^6 + 1} \, \mathrm{d}z = \frac{2\pi}{3}
$$

$$
\int_{-R}^{R} \frac{1}{x^6 + 1} \, \mathrm{d}x + \int_{\Gamma_R} \frac{1}{z^6 + 1} \, \mathrm{d}z = \frac{2\pi}{3}
$$

We take the limit as $R \to \infty$.

$$\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,\mathrm{d}x = \frac{2\pi}{3}$$
$$\int_{0}^{\infty} \frac{1}{x^6+1}\,\mathrm{d}x = \frac{\pi}{3}$$

We would get the same result by closing the path of integration in the lower half plane. Note that in this case the closed contour would be in the negative direction.

Method 2: Wedge Contour. Consider the contour Γ, which starts at the origin, goes to the point R along the real axis, then to the point $R\,\mathrm{e}^{i\pi/3}$ along a circle of radius R and then back to the origin along the ray $\theta = \pi/3$. (See Figure 15.14.)

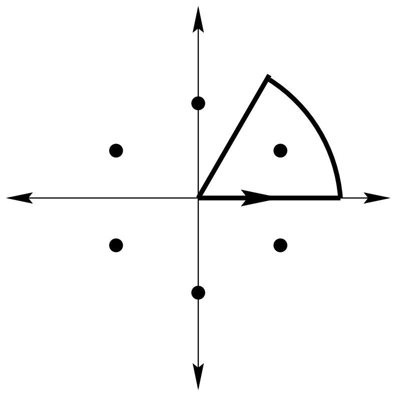


Figure 15.14: The wedge contour.

We can evaluate the integral along Γ with the residue theorem. The integrand has one first order pole inside the contour at $z = \mathrm{e}^{i\pi/6}$. For $R > 1$, we have

$$\begin{aligned}
\int_\Gamma \frac{1}{z^6+1}\,\mathrm{d}z &= i2\pi \operatorname{Res}\left(\frac{1}{z^6+1}, \mathrm{e}^{i\pi/6}\right) \\
&= i2\pi \lim_{z\to \mathrm{e}^{i\pi/6}} \frac{z - \mathrm{e}^{i\pi/6}}{z^6+1}
\end{aligned}$$

Since the numerator and denominator vanish, we apply L'Hospital's rule.

$$\begin{aligned}
&= i2\pi \lim_{z\to \mathrm{e}^{i\pi/6}} \frac{1}{6z^5} \\
&= \frac{i\pi}{3}\,\mathrm{e}^{-i\pi 5/6} \\
&= \frac{\pi}{3}\,\mathrm{e}^{-i\pi/3}
\end{aligned}$$

Now we examine the integral along the circular arc, Γ_R. We use the maximum modulus integral bound to show that the value of the integral vanishes as $R \to \infty$.

$$\begin{aligned}
\left| \int_{\Gamma_R} \frac{1}{z^6+1}\,\mathrm{d}z \right| &\leq \frac{\pi R}{3} \max_{z\in\Gamma_R} \left| \frac{1}{z^6+1} \right| \\
&= \frac{\pi R}{3} \frac{1}{R^6 - 1} \\
&\to 0 \quad \text{as } R \to \infty.
\end{aligned}$$

Now we are prepared to evaluate the original real integral.

$$\int_\Gamma \frac{1}{z^6+1}\,\mathrm{d}z = \frac{\pi}{3}\,\mathrm{e}^{-i\pi/3}$$

$$\int_0^R \frac{1}{x^6+1}\,\mathrm{d}x + \int_{\Gamma_R} \frac{1}{z^6+1}\,\mathrm{d}z + \int_{R\,\mathrm{e}^{i\pi/3}}^0 \frac{1}{z^6+1}\,\mathrm{d}z = \frac{\pi}{3}\,\mathrm{e}^{-i\pi/3}$$

$$\int_0^R \frac{1}{x^6+1}\,\mathrm{d}x + \int_{\Gamma_R} \frac{1}{z^6+1}\,\mathrm{d}z + \int_R^0 \frac{1}{x^6+1}\,\mathrm{e}^{i\pi/3}\,\mathrm{d}x = \frac{\pi}{3}\,\mathrm{e}^{-i\pi/3}$$

We take the limit as $R \to \infty$.

$$\left(1 - e^{i\pi/3}\right) \int_0^\infty \frac{1}{x^6 + 1} \, dx = \frac{\pi}{3} e^{-i\pi/3}$$
$$\int_0^\infty \frac{1}{x^6 + 1} \, dx = \frac{\pi}{3} \frac{e^{-i\pi/3}}{1 - e^{i\pi/3}}$$
$$\int_0^\infty \frac{1}{x^6 + 1} \, dx = \frac{\pi}{3} \frac{(1 - i\sqrt{3})/2}{1 - (1 + i\sqrt{3})/2}$$
$$\int_0^\infty \frac{1}{x^6 + 1} \, dx = \frac{\pi}{3}$$

Solution 15.33
First note that

$$\cos(2\theta) \geq 1 - \frac{4}{\pi}\theta, \quad 0 \leq \theta \leq \frac{\pi}{4}.$$

These two functions are plotted in Figure 15.15. To prove this inequality analytically, note that the two functions are equal at the endpoints of the interval and that $\cos(2\theta)$ is concave downward on the interval,

$$\frac{d^2}{d\theta^2} \cos(2\theta) = -4\cos(2\theta) \leq 0 \quad \text{for } 0 \leq \theta \leq \frac{\pi}{4},$$

while $1 - 4\theta/\pi$ is linear.

Let C_R be the quarter circle of radius R from $\theta = 0$ to $\theta = \pi/4$. The integral along this contour vanishes as

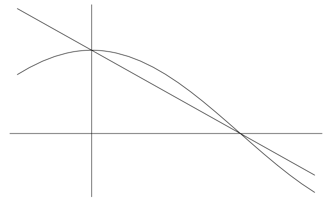


Figure 15.15: $\cos(2\theta)$ and $1 - \frac{4}{\pi}\theta$

$R \to \infty$.

$$
\begin{aligned}
\left| \int_{C_R} \mathrm{e}^{-z^2} \,\mathrm{d}z \right| &\leq \int_0^{\pi/4} \left| \mathrm{e}^{-(R\,\mathrm{e}^{i\theta})^2} \right| \left| Ri\,\mathrm{e}^{i\theta} \right| \,\mathrm{d}\theta \\
&\leq \int_0^{\pi/4} R\,\mathrm{e}^{-R^2\cos(2\theta)} \,\mathrm{d}\theta \\
&\leq \int_0^{\pi/4} R\,\mathrm{e}^{-R^2(1-4\theta/\pi)} \,\mathrm{d}\theta \\
&= \left[R\frac{\pi}{4R^2}\,\mathrm{e}^{-R^2(1-4\theta/\pi)} \right]_0^{\pi/4} \\
&= \frac{\pi}{4R}\left(1 - \mathrm{e}^{-R^2}\right) \\
&\to 0 \text{ as } R \to \infty
\end{aligned}
$$

Let C be the boundary of the domain $0 < r < R$, $0 < \theta < \pi/4$. Since the integrand is analytic inside C the integral along C is zero. Taking the limit as $R \to \infty$, the integral from $r = 0$ to ∞ along $\theta = 0$ is equal to the integral from $r = 0$ to ∞ along $\theta = \pi/4$.

$$
\int_0^\infty \mathrm{e}^{-x^2} \,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-\left(\frac{1+i}{\sqrt{2}}x\right)^2} \frac{1+i}{\sqrt{2}} \,\mathrm{d}x
$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{1+i}{\sqrt{2}} \int_0^\infty e^{-ix^2}\,dx$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{1+i}{\sqrt{2}} \int_0^\infty \left(\cos(x^2) - i\sin(x^2)\right)\,dx$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{\sqrt{2}} \left(\int_0^\infty \cos(x^2)\,dx + \int_0^\infty \sin(x^2)\,dx \right) + \frac{i}{\sqrt{2}} \left(\int_0^\infty \cos(x^2)\,dx - \int_0^\infty \sin(x^2)\,dx \right)$$

We equate the imaginary part of this equation to see that the integrals of $\cos(x^2)$ and $\sin(x^2)$ are equal.

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx$$

The real part of the equation then gives us the desired identity.

$$\boxed{\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{\sqrt{2}} \int_0^\infty e^{-x^2}\,dx}$$

Solution 15.34

Consider the box contour C that is the boundary of the rectangle $-R \leq x \leq R$, $0 \leq y \leq \pi$. There is a removable singularity at $z = 0$ and a first order pole at $z = i\pi$. By the residue theorem,

$$\begin{aligned}
\oint_C \frac{z}{\sinh z}\,dz &= i\pi \operatorname{Res}\left(\frac{z}{\sinh z}, i\pi\right) \\
&= i\pi \lim_{z \to i\pi} \frac{z(z - i\pi)}{\sinh z} \\
&= i\pi \lim_{z \to i\pi} \frac{2z - i\pi}{\cosh z} \\
&= \pi^2
\end{aligned}$$

The integrals along the side of the box vanish as $R \to \infty$.

$$\begin{aligned}
\left| \int_{\pm R}^{\pm R + i\pi} \frac{z}{\sinh z} \, \mathrm{d}z \right| &\leq \pi \max_{z \in [\pm R, \pm R + i\pi]} \left| \frac{z}{\sinh z} \right| \\
&\leq \pi \frac{R + \pi}{\sinh R} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

The value of the integrand on the top of the box is

$$\frac{x + i\pi}{\sinh(x + i\pi)} = -\frac{x + i\pi}{\sinh x}.$$

Taking the limit as $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{x}{\sinh x} \, \mathrm{d}x + \fint_{-\infty}^{\infty} -\frac{x + i\pi}{\sinh x} \, \mathrm{d}x = \pi^2.$$

Note that

$$\fint_{-\infty}^{\infty} \frac{1}{\sinh x} \, \mathrm{d}x = 0$$

as there is a first order pole at $x = 0$ and the integrand is odd.

$$\boxed{\int_{-\infty}^{\infty} \frac{x}{\sinh x} \, \mathrm{d}x = \frac{\pi^2}{2}}$$

Solution 15.35

First we evaluate

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{\mathrm{e}^x + 1} \, \mathrm{d}x.$$

Consider the rectangular contour in the positive direction with corners at $\pm R$ and $\pm R + i2\pi$. With the maximum modulus integral bound we see that the integrals on the vertical sides of the contour vanish as $R \to \infty$.

$$\left| \int_{R}^{R+i2\pi} \frac{\mathrm{e}^{az}}{\mathrm{e}^z + 1} \, \mathrm{d}z \right| \leq 2\pi \frac{\mathrm{e}^{aR}}{\mathrm{e}^R - 1} \to 0 \text{ as } R \to \infty$$
$$\left| \int_{-R+i2\pi}^{-R} \frac{\mathrm{e}^{az}}{\mathrm{e}^z + 1} \, \mathrm{d}z \right| \leq 2\pi \frac{\mathrm{e}^{-aR}}{1 - \mathrm{e}^{-R}} \to 0 \text{ as } R \to \infty$$

In the limit as R tends to infinity, the integral on the rectangular contour is the sum of the integrals along the top and bottom sides.

$$\int_C \frac{\mathrm{e}^{az}}{\mathrm{e}^z + 1} \, \mathrm{d}z = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{\mathrm{e}^x + 1} \, \mathrm{d}x + \int_{\infty}^{-\infty} \frac{\mathrm{e}^{a(x+i2\pi)}}{\mathrm{e}^{x+i2\pi} + 1} \, \mathrm{d}x$$
$$\int_C \frac{\mathrm{e}^{az}}{\mathrm{e}^z + 1} \, \mathrm{d}z = \left(1 - \mathrm{e}^{-i2a\pi}\right) \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{\mathrm{e}^x + 1} \, \mathrm{d}x$$

The only singularity of the integrand inside the contour is a first order pole at $z = i\pi$. We use the residue theorem to evaluate the integral.

$$\begin{aligned}
\int_C \frac{\mathrm{e}^{az}}{\mathrm{e}^z + 1} \, \mathrm{d}z &= i2\pi \operatorname{Res}\left(\frac{\mathrm{e}^{az}}{\mathrm{e}^z + 1}, i\pi \right) \\
&= i2\pi \lim_{z \to i\pi} \frac{(z - i\pi)\, \mathrm{e}^{az}}{\mathrm{e}^z + 1} \\
&= i2\pi \lim_{z \to i\pi} \frac{a(z - i\pi)\, \mathrm{e}^{az} + \mathrm{e}^{az}}{\mathrm{e}^z} \\
&= -i2\pi \, \mathrm{e}^{ia\pi}
\end{aligned}$$

We equate the two results for the value of the contour integral.

$$\left(1 - e^{-i2a\pi}\right) \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} \, dx = -i2\pi \, e^{ia\pi}$$

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} \, dx = \frac{i2\pi}{e^{ia\pi} - e^{-ia\pi}}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} \, dx = \frac{\pi}{\sin(\pi a)}}$$

Now we derive the value of,

$$\int_{-\infty}^{\infty} \frac{\cosh(bx)}{\cosh x} \, dx.$$

First make the change of variables $x \to 2x$ in the previous result.

$$\int_{-\infty}^{\infty} \frac{e^{2ax}}{e^{2x} + 1} 2 \, dx = \frac{\pi}{\sin(\pi a)}$$

$$\int_{-\infty}^{\infty} \frac{e^{(2a-1)x}}{e^x + e^{-x}} \, dx = \frac{\pi}{\sin(\pi a)}$$

Now we set $b = 2a - 1$.

$$\int_{-\infty}^{\infty} \frac{e^{bx}}{\cosh x} \, dx = \frac{\pi}{\sin(\pi(b+1)/2)} = \frac{\pi}{\cos(\pi b/2)} \quad \text{for } -1 < b < 1$$

Since the cosine is an even function, we also have,

$$\int_{-\infty}^{\infty} \frac{e^{-bx}}{\cosh x} \, dx = \frac{\pi}{\cos(\pi b/2)} \quad \text{for } -1 < b < 1$$

Adding these two equations and dividing by 2 yields the desired result.

$$\boxed{\int_{-\infty}^{\infty} \frac{\cosh(bx)}{\cosh x} \, dx = \frac{\pi}{\cos(\pi b/2)} \quad \text{for } -1 < b < 1}$$

Solution 15.36
Real-Valued Parameters. For $b = 0$, the integral has the value: π/a^2. If b is nonzero, then we can write the integral as

$$F(a, b) = \frac{1}{b^2} \int_0^{\pi} \frac{d\theta}{(a/b + \cos\theta)^2}.$$

We define the new parameter $c = a/b$ and the function,

$$G(c) = b^2 F(a, b) = \int_0^{\pi} \frac{d\theta}{(c + \cos\theta)^2}.$$

If $-1 \leq c \leq 1$ then the integrand has a double pole on the path of integration. The integral diverges. Otherwise the integral exists. To evaluate the integral, we extend the range of integration to $(0..2\pi)$ and make the change of variables, $z = \mathrm{e}^{i\theta}$ to integrate along the unit circle in the complex plane.

$$G(c) = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{(c + \cos\theta)^2}$$

For this change of variables, we have,

$$\cos\theta = \frac{z + z^{-1}}{2}, \qquad d\theta = \frac{\mathrm{d}z}{iz}.$$

$$\begin{aligned} G(c) &= \frac{1}{2} \int_C \frac{dz/(iz)}{(c + (z + z^{-1})/2)^2} \\ &= -i2 \int_C \frac{z}{(2cz + z^2 + 1)^2} \,\mathrm{d}z \\ &= -i2 \int_C \frac{z}{(z + c + \sqrt{c^2 - 1})^2 (z + c - \sqrt{c^2 - 1})^2} \,\mathrm{d}z \end{aligned}$$

If $c > 1$, then $-c - \sqrt{c^2 - 1}$ is outside the unit circle and $-c + \sqrt{c^2 - 1}$ is inside the unit circle. The integrand has a second order pole inside the path of integration. We evaluate the integral with the residue theorem.

$$
\begin{aligned}
G(c) &= -i2i2\pi \operatorname{Res}\left(\frac{z}{(z + c + \sqrt{c^2 - 1})^2 (z + c - \sqrt{c^2 - 1})^2}, z = -c + \sqrt{c^2 - 1} \right) \\
&= 4\pi \lim_{z \to -c + \sqrt{c^2 - 1}} \frac{\mathrm{d}}{\mathrm{d}z} \frac{z}{(z + c + \sqrt{c^2 - 1})^2} \\
&= 4\pi \lim_{z \to -c + \sqrt{c^2 - 1}} \left(\frac{1}{(z + c + \sqrt{c^2 - 1})^2} - \frac{2z}{(z + c + \sqrt{c^2 - 1})^3} \right) \\
&= 4\pi \lim_{z \to -c + \sqrt{c^2 - 1}} \frac{c + \sqrt{c^2 - 1} - z}{(z + c + \sqrt{c^2 - 1})^3} \\
&= 4\pi \frac{2c}{(2\sqrt{c^2 - 1})^3} \\
&= \frac{\pi c}{\sqrt{(c^2 - 1)^3}}
\end{aligned}
$$

If $c < 1$, then $-c - \sqrt{c^2-1}$ is inside the unit circle and $-c + \sqrt{c^2-1}$ is outside the unit circle.

$$\begin{aligned}
G(c) &= -i2i2\pi \operatorname{Res}\left(\frac{z}{(z+c+\sqrt{c^2-1})^2(z+c-\sqrt{c^2-1})^2}, z = -c-\sqrt{c^2-1}\right) \\
&= 4\pi \lim_{z \to -c-\sqrt{c^2-1}} \frac{\mathrm{d}}{\mathrm{d}z} \frac{z}{(z+c-\sqrt{c^2-1})^2} \\
&= 4\pi \lim_{z \to -c-\sqrt{c^2-1}} \left(\frac{1}{(z+c-\sqrt{c^2-1})^2} - \frac{2z}{(z+c-\sqrt{c^2-1})^3}\right) \\
&= 4\pi \lim_{z \to -c-\sqrt{c^2-1}} \frac{c-\sqrt{c^2-1}-z}{(z+c-\sqrt{c^2-1})^3} \\
&= 4\pi \frac{2c}{(-2\sqrt{c^2-1})^3} \\
&= -\frac{\pi c}{\sqrt{(c^2-1)^3}}
\end{aligned}$$

Thus we see that

$$G(c) \begin{cases} = \frac{\pi c}{\sqrt{(c^2-1)^3}} & \text{for } c > 1, \\ = -\frac{\pi c}{\sqrt{(c^2-1)^3}} & \text{for } c < 1, \\ \text{is divergent} & \text{for } -1 \leq c \leq 1. \end{cases}$$

In terms of $F(a,b)$, this is

$$F(a,b) \begin{cases} = \frac{a\pi}{\sqrt{(a^2-b^2)^3}} & \text{for } a/b > 1, \\ = -\frac{a\pi}{\sqrt{(a^2-b^2)^3}} & \text{for } a/b < 1, \\ \text{is divergent} & \text{for } -1 \leq a/b \leq 1. \end{cases}$$

Complex-Valued Parameters. Consider

$$G(c) = \int_0^\pi \frac{d\theta}{(c + \cos\theta)^2},$$

for complex c. Except for real-valued c between -1 and 1, the integral converges uniformly. We can interchange differentiation and integration. The derivative of $G(c)$ is

$$\begin{aligned} G'(c) &= \frac{\mathrm{d}}{\mathrm{d}c} \int_0^\pi \frac{d\theta}{(c + \cos\theta)^2} \\ &= \int_0^\pi \frac{-2}{(c + \cos\theta)^3} \, \mathrm{d}\theta \end{aligned}$$

Thus we see that $G(c)$ is analytic in the complex plane with a cut on the real axis from -1 to 1. The value of the function on the positive real axis for $c > 1$ is

$$G(c) = \frac{\pi c}{\sqrt{(c^2 - 1)^3}}.$$

We use analytic continuation to determine $G(c)$ for complex c. By inspection we see that $G(c)$ is the branch of

$$\frac{\pi c}{(c^2 - 1)^{3/2}},$$

with a branch cut on the real axis from -1 to 1 and which is real-valued and positive for real $c > 1$. Using $F(a, b) = G(c)/b^2$ we can determine F for complex-valued a and b.

Solution 15.37
First note that

$$\int_{-\infty}^{\infty} \frac{\cos x}{\mathrm{e}^x + \mathrm{e}^{-x}} \, \mathrm{d}x = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{\mathrm{e}^x + \mathrm{e}^{-x}} \, \mathrm{d}x$$

since $\sin x/(\mathrm{e}^x + \mathrm{e}^{-x})$ is an odd function. For the function

$$f(z) = \frac{\mathrm{e}^{iz}}{\mathrm{e}^z + \mathrm{e}^{-z}}$$

we have

$$f(x + i\pi) = \frac{\mathrm{e}^{ix-\pi}}{\mathrm{e}^{x+i\pi} + \mathrm{e}^{-x-i\pi}} = -\,\mathrm{e}^{-\pi}\frac{\mathrm{e}^{ix}}{\mathrm{e}^x + \mathrm{e}^{-x}} = -\,\mathrm{e}^{-\pi} f(x).$$

Thus we consider the integral

$$\int_C \frac{\mathrm{e}^{iz}}{\mathrm{e}^z + \mathrm{e}^{-z}}\,\mathrm{d}z$$

where C is the box contour with corners at $\pm R$ and $\pm R + i\pi$. We can evaluate this integral with the residue theorem. We can write the integrand as

$$\frac{\mathrm{e}^{iz}}{2\cosh z}.$$

We see that the integrand has first order poles at $z = i\pi(n + 1/2)$. The only pole inside the path of integration is at $z = i\pi/2$.

$$\begin{aligned}
\int_C \frac{\mathrm{e}^{iz}}{\mathrm{e}^z + \mathrm{e}^{-z}}\,\mathrm{d}z &= i2\pi \operatorname{Res}\left(\frac{\mathrm{e}^{iz}}{\mathrm{e}^z + \mathrm{e}^{-z}}, z = \frac{i\pi}{2}\right) \\
&= i2\pi \lim_{z\to i\pi/2} \frac{(z - i\pi/2)\,\mathrm{e}^{iz}}{\mathrm{e}^z + \mathrm{e}^{-z}} \\
&= i2\pi \lim_{z\to i\pi/2} \frac{\mathrm{e}^{iz} + i(z - i\pi/2)\,\mathrm{e}^{iz}}{\mathrm{e}^z - \mathrm{e}^{-z}} \\
&= i2\pi \frac{\mathrm{e}^{-\pi/2}}{\mathrm{e}^{i\pi/2} - \mathrm{e}^{-i\pi/2}} \\
&= \pi\,\mathrm{e}^{-\pi/2}
\end{aligned}$$

The integrals along the vertical sides of the box vanish as $R \to \infty$.

$$
\begin{aligned}
\left| \int_{\pm R}^{\pm R + i\pi} \frac{\mathrm{e}^{iz}}{\mathrm{e}^z + \mathrm{e}^{-z}} \, \mathrm{d}z \right| &\leq \pi \max_{z \in [\pm R \ldots \pm R + i\pi]} \left| \frac{\mathrm{e}^{iz}}{\mathrm{e}^z + \mathrm{e}^{-z}} \right| \\
&\leq \pi \max_{y \in [0 \ldots \pi]} \left| \frac{1}{\mathrm{e}^{R+iy} + \mathrm{e}^{-R-iy}} \right| \\
&\leq \pi \max_{y \in [0 \ldots \pi]} \left| \frac{1}{\mathrm{e}^{R} + \mathrm{e}^{-R-i2y}} \right| \\
&= \pi \frac{1}{2 \sinh R} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}
$$

Taking the limit as $R \to \infty$, we have

$$
\begin{gathered}
\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{\mathrm{e}^x + \mathrm{e}^{-x}} \, \mathrm{d}x + \int_{\infty + i\pi}^{-\infty + i\pi} \frac{\mathrm{e}^{iz}}{\mathrm{e}^z + \mathrm{e}^{-z}} \, \mathrm{d}z = \pi \, \mathrm{e}^{-\pi/2} \\
(1 + \mathrm{e}^{-\pi}) \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{\mathrm{e}^x + \mathrm{e}^{-x}} \, \mathrm{d}x = \pi \, \mathrm{e}^{-\pi/2} \\
\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{\mathrm{e}^x + \mathrm{e}^{-x}} \, \mathrm{d}x = \frac{\pi}{\mathrm{e}^{\pi/2} + \mathrm{e}^{-\pi/2}}
\end{gathered}
$$

Finally we have,

$$
\boxed{\int_{-\infty}^{\infty} \frac{\cos x}{\mathrm{e}^x + \mathrm{e}^{-x}} \, \mathrm{d}x = \frac{\pi}{\mathrm{e}^{\pi/2} + \mathrm{e}^{-\pi/2}}.}
$$

Definite Integrals Involving Sine and Cosine

Solution 15.38

1. Let C be the positively oriented unit circle about the origin. We parametrize this contour.

$$
z = \mathrm{e}^{i\theta}, \quad \mathrm{d}z = i \, \mathrm{e}^{i\theta} \mathrm{d}\theta, \quad \theta \in (0 \ldots 2\pi)
$$

We write $\sin\theta$ and the differential $\mathrm{d}\theta$ in terms of z. Then we evaluate the integral with the Residue theorem.

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{2+\sin\theta}\,\mathrm{d}\theta &= \oint_C \frac{1}{2+(z-1/z)/(2i)}\frac{\mathrm{d}z}{iz} \\
&= \oint_C \frac{2}{z^2+i4z-1}\,\mathrm{d}z \\
&= \oint_C \frac{2}{\left(z+i\left(2+\sqrt{3}\right)\right)\left(z+i\left(2-\sqrt{3}\right)\right)}\,\mathrm{d}z \\
&= i2\pi \operatorname{Res}\left(\left(z+i\left(2+\sqrt{3}\right)\right)\left(z+i\left(2-\sqrt{3}\right)\right), z=i\left(-2+\sqrt{3}\right)\right) \\
&= i2\pi\frac{2}{i2\sqrt{3}} \\
&= \frac{2\pi}{\sqrt{3}}
\end{aligned}$$

2. First consider the case $a=0$.

$$\int_{-\pi}^{\pi}\cos(n\theta)\,\mathrm{d}\theta = \begin{cases} 0 & \text{for } n\in\mathbb{Z}^+ \\ 2\pi & \text{for } n=0 \end{cases}$$

Now we consider $|a|<1$, $a\neq 0$. Since

$$\frac{\sin(n\theta)}{1-2a\cos\theta+a^2}$$

is an even function,

$$\int_{-\pi}^{\pi}\frac{\cos(n\theta)}{1-2a\cos\theta+a^2}\,\mathrm{d}\theta = \int_{-\pi}^{\pi}\frac{\mathrm{e}^{in\theta}}{1-2a\cos\theta+a^2}\,\mathrm{d}\theta$$

Let C be the positively oriented unit circle about the origin. We parametrize this contour.

$$z=\mathrm{e}^{i\theta},\quad \mathrm{d}z = i\,\mathrm{e}^{i\theta}\mathrm{d}\theta,\quad \theta\in(-\pi\ldots\pi)$$

We write the integrand and the differential $\mathrm{d}\theta$ in terms of z. Then we evaluate the integral with the Residue theorem.

$$
\begin{aligned}
\int_{-\pi}^{\pi} \frac{e^{in\theta}}{1 - 2a\cos\theta + a^2} \, \mathrm{d}\theta &= \oint_C \frac{z^n}{1 - a(z + 1/z) + a^2} \frac{\mathrm{d}z}{iz} \\
&= -i \oint_C \frac{z^n}{-az^2 + (1 + a^2)z - a} \, \mathrm{d}z \\
&= \frac{i}{a} \oint_C \frac{z^n}{z^2 - (a + 1/a)z + 1} \, \mathrm{d}z \\
&= \frac{i}{a} \oint_C \frac{z^n}{(z - a)(z - 1/a)} \, \mathrm{d}z \\
&= i2\pi \frac{i}{a} \operatorname{Res}\left(\frac{z^n}{(z - a)(z - 1/a)}, z = a \right) \\
&= -\frac{2\pi}{a} \frac{a^n}{a - 1/a} \\
&= \frac{2\pi a^n}{1 - a^2}
\end{aligned}
$$

We write the value of the integral for $|a| < 1$ and $n \in \mathbb{Z}^{0+}$.

$$
\boxed{\int_{-\pi}^{\pi} \frac{\cos(n\theta)}{1 - 2a\cos\theta + a^2} \, \mathrm{d}\theta = \begin{cases} 2\pi & \text{for } a = 0, n = 0 \\ \frac{2\pi a^n}{1 - a^2} & \text{otherwise} \end{cases}}
$$

Solution 15.39

Convergence. We consider the integral

$$
I(\alpha) = -\!\!\!\!\!\!\int_0^{\pi} \frac{\cos(n\theta)}{\cos\theta - \cos\alpha} \, \mathrm{d}\theta = \pi \frac{\sin(n\alpha)}{\sin\alpha}.
$$

We assume that α is real-valued. If α is an integer, then the integrand has a second order pole on the path of integration, the principal value of the integral does not exist. If α is real, but not an integer, then the integrand has a first order pole on the path of integration. The integral diverges, but its principal value exists.

Contour Integration. We will evaluate the integral for real, non-integer α.

$$\begin{aligned} I(\alpha) &= \fint_0^{\pi} \frac{\cos(n\theta)}{\cos\theta - \cos\alpha} \, \mathrm{d}\theta \\ &= \frac{1}{2} \fint_0^{2\pi} \frac{\cos(n\theta)}{\cos\theta - \cos\alpha} \, \mathrm{d}\theta \\ &= \frac{1}{2} \Re \fint_0^{2\pi} \frac{\mathrm{e}^{in\theta}}{\cos\theta - \cos\alpha} \, \mathrm{d}\theta \end{aligned}$$

We make the change of variables: $z = \mathrm{e}^{i\theta}$.

$$\begin{aligned} I(\alpha) &= \frac{1}{2} \Re \fint_C \frac{z^n}{(z + 1/z)/2 - \cos\alpha} \frac{\mathrm{d}z}{iz} \\ &= \Re \fint_C \frac{-iz^n}{(z - \mathrm{e}^{i\alpha})(z - \mathrm{e}^{-i\alpha})} \, \mathrm{d}z \end{aligned}$$

Now we use the residue theorem.

$$
\begin{aligned}
&= \Re\bigg(i\pi(-i)\bigg(\operatorname{Res}\left(\frac{z^n}{(z-\mathrm{e}^{i\alpha})(z-\mathrm{e}^{-i\alpha})}, z=\mathrm{e}^{i\alpha}\right) \\
&\qquad + \operatorname{Res}\left(\frac{z^n}{(z-\mathrm{e}^{i\alpha})(z-\mathrm{e}^{-i\alpha})}, z=\mathrm{e}^{-i\alpha}\right)\bigg)\bigg) \\
&= \pi\Re\left(\lim_{z\to\mathrm{e}^{i\alpha}} \frac{z^n}{z-\mathrm{e}^{-i\alpha}} + \lim_{z\to\mathrm{e}^{-i\alpha}} \frac{z^n}{z-\mathrm{e}^{i\alpha}}\right) \\
&= \pi\Re\left(\frac{\mathrm{e}^{in\alpha}}{\mathrm{e}^{i\alpha}-\mathrm{e}^{-i\alpha}} + \frac{\mathrm{e}^{-in\alpha}}{\mathrm{e}^{-i\alpha}-\mathrm{e}^{i\alpha}}\right) \\
&= \pi\Re\left(\frac{\mathrm{e}^{in\alpha}-\mathrm{e}^{-in\alpha}}{\mathrm{e}^{i\alpha}-\mathrm{e}^{-i\alpha}}\right) \\
&= \pi\Re\left(\frac{\sin(n\alpha)}{\sin(\alpha)}\right)
\end{aligned}
$$

$$
\boxed{I(\alpha) = -\!\!\!\!\!\!\int_0^\pi \frac{\cos(n\theta)}{\cos\theta - \cos\alpha}\,\mathrm{d}\theta = \pi\frac{\sin(n\alpha)}{\sin\alpha}.}
$$

Solution 15.40

Consider the integral

$$
\int_0^1 \frac{x^2}{(1+x^2)\sqrt{1-x^2}}\,\mathrm{d}x.
$$

We make the change of variables $x = \sin\xi$ to obtain,

$$
\int_0^{\pi/2} \frac{\sin^2\xi}{(1+\sin^2\xi)\sqrt{1-\sin^2\xi}} \cos\xi\,\mathrm{d}\xi
$$

$$\int_0^{\pi/2} \frac{\sin^2 \xi}{1 + \sin^2 \xi} \, \mathrm{d}\xi$$

$$\int_0^{\pi/2} \frac{1 - \cos(2\xi)}{3 - \cos(2\xi)} \, \mathrm{d}\xi$$

$$\frac{1}{4} \int_0^{2\pi} \frac{1 - \cos \xi}{3 - \cos \xi} \, \mathrm{d}\xi$$

Now we make the change of variables $z = \mathrm{e}^{i\xi}$ to obtain a contour integral on the unit circle.

$$\frac{1}{4} \int_C \frac{1 - (z + 1/z)/2}{3 - (z + 1/z)/2} \left(\frac{-i}{z} \right) \mathrm{d}z$$

$$\frac{-i}{4} \int_C \frac{(z-1)^2}{z(z - 3 + 2\sqrt{2})(z - 3 - 2\sqrt{2})} \, \mathrm{d}z$$

There are two first order poles inside the contour. The value of the integral is

$$i2\pi \frac{-i}{4} \left(\operatorname{Res} \left(\frac{(z-1)^2}{z(z - 3 + 2\sqrt{2})(z - 3 - 2\sqrt{2})}, 0 \right) + \operatorname{Res} \left(\frac{(z-1)^2}{z(z - 3 + 2\sqrt{2})(z - 3 - 2\sqrt{2})}, z = 3 - 2\sqrt{2} \right) \right)$$

$$\frac{\pi}{2} \left(\lim_{z \to 0} \left(\frac{(z-1)^2}{(z - 3 + 2\sqrt{2})(z - 3 - 2\sqrt{2})} \right) + \lim_{z \to 3 - 2\sqrt{2}} \left(\frac{(z-1)^2}{z(z - 3 - 2\sqrt{2})} \right) \right).$$

$$\boxed{\int_0^1 \frac{x^2}{(1 + x^2)\sqrt{1 - x^2}} \, \mathrm{d}x = \frac{(2 - \sqrt{2})\pi}{4}}$$

Solution 15.41

From Result 15.10.1 we see that the sum of the residues of $\pi \cot(\pi z)/z^4$ is zero. This function has simples poles at nonzero integers $z = n$ with residue $1/n^4$. There is a fifth order pole at $z = 0$. Finding the residue with the formula

$$\frac{1}{4!} \lim_{z \to 0} \frac{\mathrm{d}^4}{\mathrm{d}z^4} (\pi z \cot(\pi z))$$

would be a real pain. After doing the differentiation, we would have to apply L'Hospital's rule multiple times. A better way of finding the residue is with the Laurent series expansion of the function. Note that

$$\begin{aligned}
\frac{1}{\sin(\pi z)} &= \frac{1}{\pi z - (\pi z)^3/6 + (\pi z)^5/120 - \cdots} \\
&= \frac{1}{\pi z} \frac{1}{1 - (\pi z)^2/6 + (\pi z)^4/120 - \cdots} \\
&= \frac{1}{\pi z} \left(1 + \left(\frac{\pi^2}{6} z^2 - \frac{\pi^4}{120} z^4 + \cdots \right) + \left(\frac{\pi^2}{6} z^2 - \frac{\pi^4}{120} z^4 + \cdots \right)^2 + \cdots \right).
\end{aligned}$$

Now we find the z^{-1} term in the Laurent series expansion of $\pi \cot(\pi z)/z^4$.

$$\begin{aligned}
\frac{\pi \cos(\pi z)}{z^4 \sin(\pi z)} &= \frac{\pi}{z^4} \left(1 - \frac{\pi^2}{2} z^2 + \frac{\pi^4}{24} z^4 - \cdots \right) \frac{1}{\pi z} \left(1 + \left(\frac{\pi^2}{6} z^2 - \frac{\pi^4}{120} z^4 + \cdots \right) + \left(\frac{\pi^2}{6} z^2 - \frac{\pi^4}{120} z^4 + \cdots \right)^2 + \cdots \right) \\
&= \frac{1}{z^5} \left(\cdots + \left(-\frac{\pi^4}{120} + \frac{\pi^4}{36} - \frac{\pi^4}{12} + \frac{\pi^4}{24} \right) z^4 + \cdots \right) \\
&= \cdots - \frac{\pi^4}{45} \frac{1}{z} + \cdots
\end{aligned}$$

Thus the residue at $z = 0$ is $-\pi^4/45$. Summing the residues,

$$\sum_{n=-\infty}^{-1} \frac{1}{n^4} - \frac{\pi^4}{45} + \sum_{n=1}^{\infty} \frac{1}{n^4} = 0.$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$

Solution 15.42
For this problem we will use the following result: If

$$\lim_{|z|\to\infty} |z f(z)| = 0,$$

then the sum of all the residues of $\pi \cot(\pi z) f(z)$ is zero. If in addition, $f(z)$ is analytic at $z = n \in \mathbb{Z}$ then

$$\sum_{n=-\infty}^{\infty} f(n) = -(\text{ sum of the residues of } \pi \cot(\pi z) f(z) \text{ at the poles of } f(z) \text{ }).$$

We assume that α is not an integer, otherwise the sum is not defined. Consider $f(z) = 1/(z^2 - \alpha^2)$. Since

$$\lim_{|z|\to\infty} \left| z \frac{1}{z^2 - \alpha^2} \right| = 0,$$

and $f(z)$ is analytic at $z = n$, $n \in \mathbb{Z}$, we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 - \alpha^2} = -(\text{ sum of the residues of } \pi \cot(\pi z) f(z) \text{ at the poles of } f(z) \text{ }).$$

$f(z)$ has first order poles at $z = \pm\alpha$.

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 - \alpha^2} &= -\operatorname{Res}\left(\frac{\pi \cot(\pi z)}{z^2 - \alpha^2}, z = \alpha \right) - \operatorname{Res}\left(\frac{\pi \cot(\pi z)}{z^2 - \alpha^2}, z = -\alpha \right) \\
&= -\lim_{z\to\alpha} \frac{\pi \cot(\pi z)}{z + \alpha} - \lim_{z\to-\alpha} \frac{\pi \cot(\pi z)}{z - \alpha} \\
&= -\frac{\pi \cot(\pi \alpha)}{2\alpha} - \frac{\pi \cot(-\pi \alpha)}{-2\alpha}
\end{aligned}$$

$$\boxed{\sum_{n=-\infty}^{\infty} \frac{1}{n^2 - \alpha^2} = -\frac{\pi \cot(\pi\alpha)}{\alpha}}$$

Part IV

Ordinary Differential Equations

Chapter 16

First Order Differential Equations

Don't show me your technique. Show me your heart.

-Tetsuyasu Uekuma

16.1 Notation

A *differential equation* is an equation involving a function, it's derivatives, and independent variables. If there is only one independent variable, then it is an *ordinary differential equation.* Identities such as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f^2(x)\right) = 2f(x)f'(x), \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y} = 1$$

are not differential equations.

The *order* of a differential equation is the order of the highest derivative. The following equations are first, second and third order, respectively.

- $y' = xy^2$

- $y'' + 3xy' + 2y = x^2$
- $y''' = y''y$

The *degree* of a differential equation is the highest power of the highest derivative in the equation. The following equations are first, second and third degree, respectively.

- $y' - 3y = \sin x$
- $(y'')^2 + 2xy = \mathrm{e}^x$
- $(y')^3 + y^5 = 0$

An equation is said to be *linear* if it is linear in the dependent variable.

- $y'' \cos x + x^2 y = 0$ is a linear differential equation.
- $y' + xy^2 = 0$ is a nonlinear differential equation.

A differential equation is *homogeneous* if it has no terms that are functions of the independent variable alone. Thus an *inhomogeneous* equation is one in which there are terms that are functions of the independent variables alone.

- $y'' + xy + y = 0$ is a homogeneous equation.
- $y' + y + x^2 = 0$ is an inhomogeneous equation.

A first order differential equation may be written in terms of differentials. Recall that for the function $y(x)$ the differential $\mathrm{d}y$ is defined $\mathrm{d}y = y'(x)\,\mathrm{d}x$. Thus the differential equations

$$y' = x^2 y \quad \text{and} \quad y' + xy^2 = \sin(x)$$

can be denoted:

$$\mathrm{d}y = x^2 y\,\mathrm{d}x \quad \text{and} \quad \mathrm{d}y + xy^2\,\mathrm{d}x = \sin(x)\,\mathrm{d}x.$$

A *solution* of a differential equation is a function which when substituted into the equation yields an identity. For example, $y = x \log x$ is a solution of

$$y' - \frac{y}{x} = 1$$

and $y = c\,\mathrm{e}^x$ is a solution of

$$y'' - y = 0$$

for any value of the parameter c.

16.2 One Parameter Families of Functions

Consider the equation

$$F(x, y(x); c) = 0, \tag{16.1}$$

which implicitly defines a one-parameter family of functions $y(x)$. (We assume that F has a non-trivial dependence on y, that is $F_y \neq 0$.) Differentiating this equation with respect to x yields

$$F_x + F_y y' = 0.$$

This gives us two equations involving the independent variable x, the dependent variable $y(x)$ and its derivative and the parameter c. If we algebraically eliminate c between the two equations, the eliminant will be a first order differential equation for $y(x)$. Thus we see that every equation of the form (16.1) defines a one-parameter family of functions $y(x)$ which satisfy a first order differential equation. This $y(x)$ is the *primitive* of the differential equation. Later we will discuss why $y(x)$ is the *general solution* of the differential equation.

Example 16.2.1 Consider the family of circles of radius c centered about the origin,

$$x^2 + y^2 = c^2.$$

Differentiating this yields,

$$2x + 2yy' = 0.$$

It is trivial to eliminate the parameter and obtain a differential equation for the family of circles.

$$x + yy' = 0.$$

We can see the geometric meaning in this equation by writing it in the form

$$y' = -\frac{x}{y}.$$

The slope of the tangent to a circle at a point is the negative of the cotangent of the angle.

Example 16.2.2 Consider the one-parameter family of functions,

$$y(x) = f(x) + cg(x),$$

where $f(x)$ and $g(x)$ are known functions. The derivative is

$$y' = f' + cg'.$$

Eliminating the parameter yields

$$gy' - g'y = gf' - g'f$$
$$y' - \frac{g'}{g}y = f' - \frac{g'f}{g}.$$

Thus we see that $y(x) = f(x) + cg(x)$ satisfies a first order *linear* differential equation.

We know that every one-parameter family of functions satisfies a first order differential equation. The converse is true as well.

Result 16.2.1 Every first order differential equation has a one-parameter family of solutions, $y(x)$, defined by an equation of the form:

$$F(x, y(x); c) = 0.$$

This $y(x)$ is called the *general solution*. If the equation is linear then the general solution expresses the totality of solutions of the differential equation. If the equation is nonlinear, there may be other special *singular solutions*, which do not depend on a parameter.

This is strictly an existence result. It does not say that the general solution of a first order differential equation can be determined by some method, it just says that it exists. There is no method for solving the general first order differential equation. However, there are some special forms that are soluble. We will devote the rest of this chapter to studying these forms.

16.3 Exact Equations

Any first order ordinary differential equation of the first degree can be written as the total differential equation,

$$P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = 0.$$

If this equation can be integrated directly, that is if there is a primitive, $u(x, y)$, such that

$$du = P\,\mathrm{d}x + Q\,\mathrm{d}y,$$

then this equation is called *exact*. The (implicit) solution of the differential equation is

$$u(x, y) = c,$$

where c is an arbitrary constant. Since the differential of a function, $u(x, y)$, is

$$du \equiv \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y,$$

P and Q are the partial derivatives of u:

$$P(x, y) = \frac{\partial u}{\partial x}, \qquad Q(x, y) = \frac{\partial u}{\partial y}.$$

In an alternate notation, the differential equation

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \tag{16.2}$$

is exact if there is a primitive $u(x, y)$ such that

$$\frac{\mathrm{d}u}{\mathrm{d}x} \equiv \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

The solution of the differential equation is $u(x, y) = c$.

Example 16.3.1

$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is an exact differential equation since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}(x^2 + y^2)\right) = x + y\frac{\mathrm{d}y}{\mathrm{d}x}$$

The solution of the differential equation is

$$\frac{1}{2}(x^2 + y^2) = c.$$

Example 16.3.2 , Let $f(x)$ and $g(x)$ be known functions.

$$g(x)y' + g'(x)y = f(x)$$

is an exact differential equation since

$$\frac{\mathrm{d}}{\mathrm{d}x}(g(x)y(x)) = gy' + g'y.$$

The solution of the differential equation is

$$g(x)y(x) = \int f(x)\,\mathrm{d}x + c$$
$$y(x) = \frac{1}{g(x)}\int f(x)\,\mathrm{d}x + \frac{c}{g(x)}.$$

A necessary condition for exactness. The solution of the exact equation $P + Qy' = 0$ is $u = c$ where u is the primitive of the equation, $\frac{\mathrm{d}u}{\mathrm{d}x} = P + Qy'$. At present the only method we have for determining the primitive is guessing. This is fine for simple equations, but for more difficult cases we would like a method more concrete than divine inspiration. As a first step toward this goal we determine a criterion for determining if an equation is exact.

Consider the exact equation,

$$P + Qy' = 0,$$

with primitive u, where we assume that the functions P and Q are continuously differentiable. Since the mixed partial derivatives of u are equal,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x},$$

a necessary condition for exactness is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

A sufficient condition for exactness. This necessary condition for exactness is also a sufficient condition. We demonstrate this by deriving the general solution of (16.2). Assume that $P + Qy' = 0$ is not necessarily exact, but satisfies the condition $P_y = Q_x$. If the equation has a primitive,

$$\frac{\mathrm{d}u}{\mathrm{d}x} \equiv \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x},$$

then it satisfies

$$\frac{\partial u}{\partial x} = P, \qquad \frac{\partial u}{\partial y} = Q. \tag{16.3}$$

Integrating the first equation of (16.3), we see that the primitive has the form

$$u(x, y) = \int_{x_0}^{x} P(\xi, y)\,\mathrm{d}\xi + f(y),$$

for some $f(y)$. Now we substitute this form into the second equation of (16.3).

$$\frac{\partial u}{\partial y} = Q(x, y)$$

$$\int_{x_0}^{x} P_y(\xi, y)\,\mathrm{d}\xi + f'(y) = Q(x, y)$$

Now we use the condition $P_y = Q_x$.

$$\int_{x_0}^{x} Q_x(\xi, y)\,\mathrm{d}\xi + f'(y) = Q(x, y)$$

$$Q(x, y) - Q(x_0, y) + f'(y) = Q(x, y)$$

$$f'(y) = Q(x_0, y)$$

$$f(y) = \int_{y_0}^{y} Q(x_0, \eta)\,\mathrm{d}\eta$$

Thus we see that

$$u = \int_{x_0}^{x} P(\xi, y)\,\mathrm{d}\xi + \int_{y_0}^{y} Q(x_0, \eta)\,\mathrm{d}\eta$$

is a primitive of the derivative; the equation is exact. The solution of the differential equation is

$$\int_{x_0}^{x} P(\xi, y)\,\mathrm{d}\xi + \int_{y_0}^{y} Q(x_0, \eta)\,\mathrm{d}\eta = c.$$

Even though there are three arbitrary constants: x_0, y_0 and c, the solution is a one-parameter family. This is because changing x_0 or y_0 only changes the left side by an additive constant.

Result 16.3.1 Any first order differential equation of the first degree can be written in the form

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

This equation is exact if and only if

$$P_y = Q_x.$$

In this case the solution of the differential equation is given by

$$\int_{x_0}^{x} P(\xi, y)\,\mathrm{d}\xi + \int_{y_0}^{y} Q(x_0, \eta)\,\mathrm{d}\eta = c.$$

16.3.1 Separable Equations

Any differential equation that can written in the form

$$P(x) + Q(y)y' = 0$$

is a *separable equation*, (because the dependent and independent variables are separated). We can obtain an implicit solution by integrating with respect to x.

$$\int P(x)\,\mathrm{d}x + \int Q(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = c$$
$$\int P(x)\,\mathrm{d}x + \int Q(y)\,\mathrm{d}y = c$$

Result 16.3.2 The general solution to the separable equation $P(x) + Q(y)y' = 0$ is

$$\int P(x)\,\mathrm{d}x + \int Q(y)\,\mathrm{d}y = c$$

Example 16.3.3 Consider the equation $y' = xy^2$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2$$
$$y^{-2}\,\mathrm{d}y = x\,\mathrm{d}x$$
$$\int y^{-2}\,\mathrm{d}y = \int x\,\mathrm{d}x + c$$
$$-y^{-1} = \frac{1}{2}x^2 + c$$
$$\boxed{y = \frac{-1}{\frac{1}{2}x^2 + c}}$$

Example 16.3.4 The equation

$$y' = y - y^2,$$

is separable.

$$\frac{y'}{y - y^2} = 1$$

We expand in partial fractions and integrate.

$$\left(\frac{1}{y} - \frac{1}{y-1}\right) y' = 1$$
$$\log(y) - \log(y-1) = x + c$$

Then we solve for $y(x)$.

$$\log\left(\frac{y}{y-1}\right) = x + c$$
$$\frac{y}{y-1} = \mathrm{e}^{x+c}$$
$$y = \frac{\mathrm{e}^{x+c}}{\mathrm{e}^{x+c} - 1}$$

Finally we substitute $a = \mathrm{e}^{-c}$ to write the solution in a nice form.

$$\boxed{y = \frac{1}{1 - a\,\mathrm{e}^{-x}}}$$

16.3.2 Homogeneous Coefficient Equations

Euler's Theorem on Homogeneous Functions. The function $F(x, y)$ is *homogeneous of degree* n if

$$F(\lambda x, \lambda y) = \lambda^n F(x, y).$$

From this definition we see that

$$F(x,y) = x^n F\left(1, \frac{y}{x}\right).$$

(Just formally substitute $1/x$ for λ.) For example,

$$xy^2, \qquad \frac{x^2 y + 2y^3}{x+y}, \qquad x\cos(y/x)$$

are homogeneous functions of orders 3, 2 and 1, respectively.

Euler's theorem for a homogeneous function of order n is:

$$xF_x + yF_y = nF.$$

To prove this, we define $\xi = \lambda x$, $\eta = \lambda y$. From the definition of homogeneous functions, we have

$$F(\xi,\eta) = \lambda^n F(x,y).$$

We differentiate this equation with respect to λ.

$$\frac{\partial F(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial \lambda} + \frac{\partial F(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial \lambda} = n\lambda^{n-1} F(x,y)$$
$$xF_\xi + yF_\eta = n\lambda^{n-1} F(x,y)$$

Setting $\lambda = 1$, (and hence $\xi = x$, $\eta = y$), proves Euler's theorem.

Result 16.3.3 Euler's Theorem. If $F(x,y)$ is a homogeneous function of degree n, then

$$xF_x + yF_y = nF.$$

Homogeneous Coefficient Differential Equations. If the coefficient functions $P(x,y)$ and $Q(x,y)$ are homogeneous of degree n then the differential equation,

$$Q(x,y) + P(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

is called a *homogeneous coefficient equation.* They are often referred to as simply *homogeneous equations.* We can write the equation in the form,

$$x^n Q\left(1, \frac{y}{x}\right) + x^n P\left(1, \frac{y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$
$$Q\left(1, \frac{y}{x}\right) + P\left(1, \frac{y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

This suggests the change of dependent variable $u(x) = \frac{y(x)}{x}$.

$$Q(1,u) + P(1,u)\left(u + x\frac{\mathrm{d}u}{\mathrm{d}x}\right) = 0$$

This equation is separable.

$$Q(1,u) + uP(1,u) + xP(1,u)\frac{\mathrm{d}u}{\mathrm{d}x} = 0$$
$$\frac{1}{x} + \frac{P(1,u)}{Q(1,u) + uP(1,u)}\frac{\mathrm{d}u}{\mathrm{d}x} = 0$$
$$\log x + \int \frac{1}{u + Q(1,u)/P(1,u)}\,\mathrm{d}u = c$$

By substituting $\log c$ for c, we can write this in the form,

$$\int \frac{1}{u + Q(1,u)/P(1,u)}\,\mathrm{d}u = \log\left(\frac{c}{x}\right).$$

Example 16.3.5 Consider the homogeneous coefficient equation

$$x^2 - y^2 + xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

The solution for $u(x) = y(x)/x$ is determined by

$$\int \frac{1}{u + \frac{1-u^2}{u}} \, \mathrm{d}u = \log\left(\frac{c}{x}\right)$$
$$\int u \, \mathrm{d}u = \log\left(\frac{c}{x}\right)$$
$$\frac{1}{2}u^2 = \log\left(\frac{c}{x}\right)$$
$$u = \pm\sqrt{2\log(c/x)}$$

Thus the solution of the differential equation is

$$\boxed{y = \pm x\sqrt{2\log(c/x)}}$$

Result 16.3.4 Homogeneous Coefficient Differential Equations. If $P(x, y)$ and $Q(x, y)$ are homogeneous functions of degree n, then the equation

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is made separable by the change of independent variable $u(x) = \frac{y(x)}{x}$. The solution is determined by

$$\int \frac{1}{u + Q(1, u)/P(1, u)} \, \mathrm{d}u = \log\left(\frac{c}{x}\right).$$

16.4 The First Order, Linear Differential Equation

16.4.1 Homogeneous Equations

The first order, linear, homogeneous equation has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = 0.$$

Note that this equation is separable.

$$\begin{aligned} \frac{y'}{y} &= -p(x) \\ \log(y) &= -\int p(x)\,\mathrm{d}x + a \\ y &= \mathrm{e}^{-\int p(x)\,\mathrm{d}x + a} \\ y &= c\,\mathrm{e}^{-\int p(x)\,\mathrm{d}x} \end{aligned}$$

Example 16.4.1 Consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = 0.$$

$$y(x) = c\,\mathrm{e}^{-\int 1/x\,\mathrm{d}x}$$

$$y(x) = c\,\mathrm{e}^{-\log x}$$

$$\boxed{y(x) = \frac{c}{x}}$$

16.4.2 Inhomogeneous Equations

The first order, linear, inhomogeneous differential equation has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x). \tag{16.4}$$

This equation is not separable. Note that it is similar to the exact equation we solved in Example 16.3.2,

$$g(x)y'(x) + g'(x)y(x) = f(x).$$

To solve Equation 16.4, we multiply by an *integrating factor*. Multiplying a differential equation by its integrating factor changes it to an exact equation. Multiplying Equation 16.4 by the function, $I(x)$, yields,

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)I(x)y = f(x)I(x).$$

In order that $I(x)$ be an integrating factor, it must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}I(x) = p(x)I(x).$$

This is a first order, linear, homogeneous equation with the solution

$$I(x) = c\,\mathrm{e}^{\int p(x)\,\mathrm{d}x}.$$

This is an integrating factor for any constant c. For simplicity we will choose $c = 1$.

To solve Equation 16.4 we multiply by the integrating factor and integrate. Let $P(x) = \int p(x)\,\mathrm{d}x$.

$$\begin{aligned}
\mathrm{e}^{P(x)}\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)\,\mathrm{e}^{P(x)}y &= \mathrm{e}^{P(x)}f(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{P(x)}y\right) &= \mathrm{e}^{P(x)}f(x) \\
y &= \mathrm{e}^{-P(x)}\int \mathrm{e}^{P(x)}f(x)\,\mathrm{d}x + c\,\mathrm{e}^{-P(x)} \\
y &\equiv y_p + c\,y_h
\end{aligned}$$

Note that the *general solution* is the sum of a *particular solution*, y_p, that satisfies $y' + p(x)y = f(x)$, and an arbitrary constant times a *homogeneous solution*, y_h, that satisfies $y' + p(x)y = 0$.

Example 16.4.2 Consider the differential equation

$$y' + \frac{1}{x} y = x^2.$$

The integrating factor is

$$I(x) = \exp\left(\int \frac{1}{x}\,\mathrm{d}x\right) = \mathrm{e}^{\log x} = x.$$

Multiplying by the integrating factor and integrating,

$$\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3 \\
xy = \frac{1}{4}x^4 + c \\
\boxed{y = \frac{1}{4}x^3 + \frac{c}{x}.}
\end{gathered}$$

We see that the particular and homogeneous solutions are

$$y_p = \frac{1}{4}x^3 \qquad \text{and} \qquad y_h = \frac{1}{x}.$$

Note that the general solution to the differential equation is a one-parameter family of functions. The general solution is plotted in Figure 16.1 for various values of c.

16.4.3 Variation of Parameters.

We could also have found the particular solution with the method of variation of parameters. Although we can solve first order equations without this method, it will become important in the study of higher order

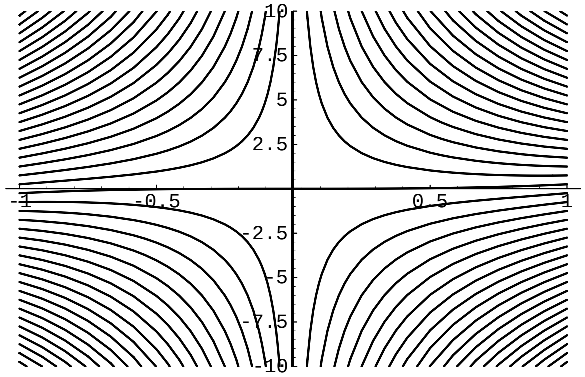


Figure 16.1: Solutions to $y' + y/x = x^2$.

inhomogeneous equations. We begin by assuming that the particular solution has the form $y_p = u(x)y_h(x)$ where $u(x)$ is an unknown function. We substitute this into the differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}x} y_p + p(x) y_p = f(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x} (u y_h) + p(x) u y_h = f(x)$$
$$u' y_h + u(y_h' + p(x) y_h) = f(x)$$

Since y_h is a homogeneous solution, $y_h' + p(x)y_h = 0$.

$$u' = \frac{f(x)}{y_h}$$
$$u = \int \frac{f(x)}{y_h(x)} \, \mathrm{d}x$$

Recall that the homogeneous solution is $y_h = \mathrm{e}^{-P(x)}$.

$$u = \int \mathrm{e}^{P(x)} f(x) \, \mathrm{d}x$$

Thus the particular solution is

$$y_p = \mathrm{e}^{-P(x)} \int \mathrm{e}^{P(x)} f(x) \, \mathrm{d}x.$$

16.5 Initial Conditions

In physical problems involving first order differential equations, the solution satisfies both the differential equation and a constraint which we call the *initial condition*. Consider a first order linear differential equation subject to the initial condition $y(x_0) = y_0$. The general solution is

$$y = y_p + c y_h = \mathrm{e}^{-P(x)} \int \mathrm{e}^{P(x)} f(x) \, \mathrm{d}x + c \, \mathrm{e}^{-P(x)}.$$

For the moment, we will assume that this problem is *well-posed*. A problem is well-posed if there is a unique solution to the differential equation that satisfies the constraint(s). Recall that $\int \mathrm{e}^{P(x)} f(x) \, \mathrm{d}x$ denotes any integral of $\mathrm{e}^{P(x)} f(x)$. For convenience, we choose $\int_{x_0}^{x} \mathrm{e}^{P(\xi)} f(\xi) \, \mathrm{d}\xi$. The initial condition requires that

$$y(x_0) = y_0 = \mathrm{e}^{-P(x_0)} \int_{x_0}^{x_0} \mathrm{e}^{P(\xi)} f(\xi) \, \mathrm{d}\xi + c \, \mathrm{e}^{-P(x_0)} = c \, \mathrm{e}^{-P(x_0)}.$$

Thus $c = y_0 \, \mathrm{e}^{P(x_0)}$. The solution subject to the initial condition is

$$y = \mathrm{e}^{-P(x)} \int_{x_0}^{x} \mathrm{e}^{P(\xi)} f(\xi) \, \mathrm{d}\xi + y_0 \, \mathrm{e}^{P(x_0)-P(x)}.$$

Example 16.5.1 Consider the problem

$$y' + (\cos x) y = x, \qquad y(0) = 2.$$

From Result 16.5.1, the solution subject to the initial condition is

$$\boxed{y = \mathrm{e}^{-\sin x} \int_0^x \xi \, \mathrm{e}^{\sin \xi} \, \mathrm{d}\xi + 2 \, \mathrm{e}^{-\sin x}.}$$

16.5.1 Piecewise Continuous Coefficients and Inhomogeneities

If the coefficient function $p(x)$ and the inhomogeneous term $f(x)$ in the first order linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x) y = f(x)$$

are continuous, then the solution is continuous and has a continuous first derivative. To see this, we note that the solution

$$y = \mathrm{e}^{-P(x)} \int \mathrm{e}^{P(x)} f(x) \, \mathrm{d}x + c \, \mathrm{e}^{-P(x)}$$

is continuous since the integral of a piecewise continuous function is continuous. The first derivative of the solution can be found directly from the differential equation.

$$y' = -p(x) y + f(x)$$

Since $p(x)$, y, and $f(x)$ are continuous, y' is continuous.

If $p(x)$ or $f(x)$ is only piecewise continuous, then the solution will be continuous since the integral of a piecewise continuous function is continuous. The first derivative of the solution will be piecewise continuous.

Example 16.5.2 Consider the problem

$$y' - y = H(x - 1), \qquad y(0) = 1,$$

where $H(x)$ is the Heaviside function.

$$H(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

To solve this problem, we divide it into two equations on separate domains.

$$\begin{aligned} y_1' - y_1 &= 0, \qquad y_1(0) = 1, \qquad && \text{for } x < 1 \\ y_2' - y_2 &= 1, \qquad y_2(1) = y_1(1), \qquad && \text{for } x > 1 \end{aligned}$$

With the condition $y_2(1) = y_1(1)$ on the second equation, we demand that the solution be continuous. The solution to the first equation is $y = \mathrm{e}^x$. The solution for the second equation is

$$y = \mathrm{e}^x \int_1^x \mathrm{e}^{-\xi}\,\mathrm{d}\xi + \mathrm{e}^1\,\mathrm{e}^{x-1} = -1 + \mathrm{e}^{x-1} + \mathrm{e}^x.$$

Thus the solution over the whole domain is

$$\boxed{y = \begin{cases} \mathrm{e}^x & \text{for } x < 1, \\ (1 + \mathrm{e}^{-1})\,\mathrm{e}^x - 1 & \text{for } x > 1. \end{cases}}$$

The solution is graphed in Figure 16.2.

Example 16.5.3 Consider the problem,

$$y' + \operatorname{sign}(x) y = 0, \qquad y(1) = 1.$$

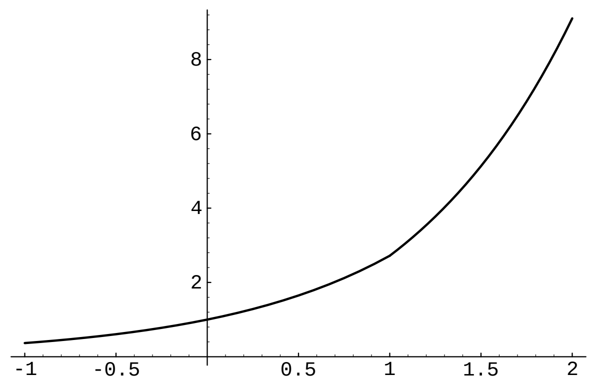


Figure 16.2: Solution to $y' - y = H(x - 1)$.

Recall that

$$\operatorname{sign} x = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0. \end{cases}$$

Since $\operatorname{sign} x$ is piecewise defined, we solve the two problems,

$$\begin{aligned} y_+' + y_+ &= 0, \qquad y_+(1) = 1, \qquad && \text{for } x > 0 \\ y_-' - y_- &= 0, \qquad y_-(0) = y_+(0), \qquad && \text{for } x < 0, \end{aligned}$$

and define the solution, y, to be

$$y(x) = \begin{cases} y_+(x), & \text{for } x \geq 0, \\ y_-(x), & \text{for } x \leq 0. \end{cases}$$

The initial condition for y_- demands that the solution be continuous.

Solving the two problems for positive and negative x, we obtain

$$y(x) = \begin{cases} \mathrm{e}^{1-x}, & \text{for } x > 0, \\ \mathrm{e}^{1+x}, & \text{for } x < 0. \end{cases}$$

This can be simplified to

$$\boxed{y(x) = \mathrm{e}^{1-|x|}.}$$

This solution is graphed in Figure 16.3.

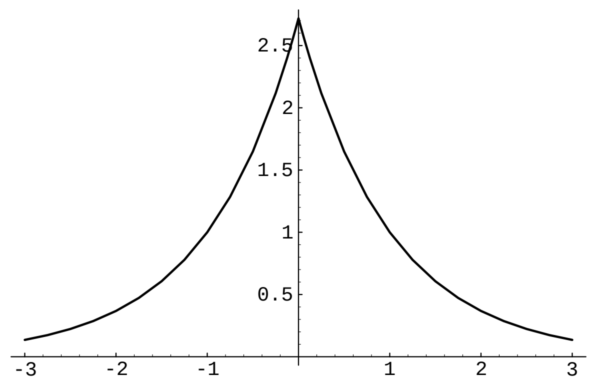


Figure 16.3: Solution to $y' + \operatorname{sign}(x)y = 0$.

Result 16.5.1 Existence, Uniqueness Theorem. Let $p(x)$ and $f(x)$ be piecewise continuous on the interval $[a, b]$ and let $x_0 \in [a, b]$. Consider the problem,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x), \qquad y(x_0) = y_0.$$

The general solution of the differential equation is

$$y = \mathrm{e}^{-P(x)} \int \mathrm{e}^{P(x)} f(x)\,\mathrm{d}x + c\,\mathrm{e}^{-P(x)}.$$

The unique, continuous solution of the differential equation subject to the initial condition is

$$y = \mathrm{e}^{-P(x)} \int_{x_0}^{x} \mathrm{e}^{P(\xi)} f(\xi)\,\mathrm{d}\xi + y_0\,\mathrm{e}^{P(x_0)-P(x)},$$

where $P(x) = \int p(x)\,\mathrm{d}x$.

16.6 Well-Posed Problems

Example 16.6.1 Consider the problem,

$$y' - \frac{1}{x} y = 0, \qquad y(0) = 1.$$

The general solution is $y = cx$. Applying the initial condition demands that $1 = c \cdot 0$, which cannot be satisfied. The general solution for various values of c is plotted in Figure 16.4.

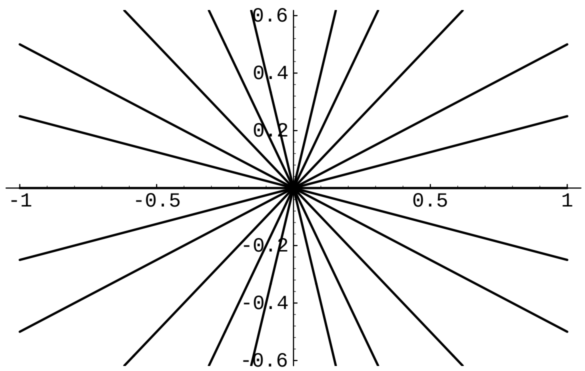


Figure 16.4: Solutions to $y' - y/x = 0$.

Example 16.6.2 Consider the problem

$$y' - \frac{1}{x}y = -\frac{1}{x}, \qquad y(0) = 1.$$

The general solution is

$$y = 1 + cx.$$

The initial condition is satisfied for any value of c so there are an infinite number of solutions.

Example 16.6.3 Consider the problem

$$y' + \frac{1}{x}y = 0, \qquad y(0) = 1.$$

The general solution is $y = \frac{c}{x}$. Depending on whether c is nonzero, the solution is either singular or zero at the origin and cannot satisfy the initial condition.

The above problems in which there were either no solutions or an infinite number of solutions are said to be *ill-posed*. If there is a unique solution that satisfies the initial condition, the problem is said to be *well-posed*. We should have suspected that we would run into trouble in the above examples as the initial condition was given at a singularity of the coefficient function, $p(x) = 1/x$.

Consider the problem,

$$y' + p(x)y = f(x), \qquad y(x_0) = y_0.$$

We assume that $f(x)$ bounded in a neighborhood of $x = x_0$. The differential equation has the general solution,

$$y = \mathrm{e}^{-P(x)} \int \mathrm{e}^{P(x)} f(x)\, \mathrm{d}x + c\,\mathrm{e}^{-P(x)}.$$

If the homogeneous solution, $\mathrm{e}^{-P(x)}$, is nonzero and finite at $x = x_0$, then there is a unique value of c for which the initial condition is satisfied. If the homogeneous solution vanishes at $x = x_0$ then either the initial condition cannot be satisfied or the initial condition is satisfied for all values of c. The homogeneous solution can vanish or be infinite only if $P(x) \to \pm\infty$ as $x \to x_0$. This can occur only if the coefficient function, $p(x)$, is unbounded at that point.

Result 16.6.1 If the initial condition is given where the homogeneous solution to a first order, linear differential equation is zero or infinite then the problem may be ill-posed. This may occur only if the coefficient function, $p(x)$, is unbounded at that point.

16.7 Equations in the Complex Plane

16.7.1 Ordinary Points

Consider the first order homogeneous equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0,$$

where $p(z)$, a function of a complex variable, is analytic in some domain D. The integrating factor,

$$I(z) = \exp\left(\int p(z)\,\mathrm{d}z\right),$$

is an analytic function in that domain. As with the case of real variables, multiplying by the integrating factor and integrating yields the solution,

$$w(z) = c\exp\left(-\int p(z)\,\mathrm{d}z\right).$$

We see that the solution is analytic in D.

Example 16.7.1 It does not make sense to pose the equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + |z|w = 0.$$

For the solution to exist, w and hence $w'(z)$ must be analytic. Since $p(z) = |z|$ is not analytic anywhere in the complex plane, the equation has no solution.

Any point at which $p(z)$ is analytic is called an *ordinary point* of the differential equation. Since the solution is analytic we can expand it in a Taylor series about an ordinary point. The radius of convergence of the series will be at least the distance to the nearest singularity of $p(z)$ in the complex plane.

Example 16.7.2 Consider the equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{1}{1-z}w = 0.$$

The general solution is $w = \frac{c}{1-z}$. Expanding this solution about the origin,

$$w = \frac{c}{1-z} = c\sum_{n=0}^{\infty} z^n.$$

The radius of convergence of the series is,

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 1,$$

which is the distance from the origin to the nearest singularity of $p(z) = \frac{1}{1-z}$.

We do not need to solve the differential equation to find the Taylor series expansion of the homogeneous solution. We could substitute a general Taylor series expansion into the differential equation and solve for the coefficients. Since we can always solve first order equations, this method is of limited usefulness. However, when we consider higher order equations in which we cannot solve the equations exactly, this will become an important method.

Example 16.7.3 Again consider the equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{1}{1-z}w = 0.$$

Since we know that the solution has a Taylor series expansion about $z = 0$, we substitute $w = \sum_{n=0}^{\infty} a_n z^n$ into the differential equation.

$$(1-z)\frac{\mathrm{d}}{\mathrm{d}z}\sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n z^n = 0$$
$$\sum_{n=1}^{\infty} n a_n z^{n-1} - \sum_{n=1}^{\infty} n a_n z^n - \sum_{n=0}^{\infty} a_n z^n = 0$$
$$\sum_{n=0}^{\infty} (n+1) a_{n+1} z^n - \sum_{n=0}^{\infty} n a_n z^n - \sum_{n=0}^{\infty} a_n z^n = 0$$
$$\sum_{n=0}^{\infty} \left((n+1)a_{n+1} - (n+1)a_n\right) z^n = 0.$$

Now we equate powers of z to zero. For z^n, the equation is $(n+1)a_{n+1} - (n+1)a_n = 0$, or $a_{n+1} = a_n$. Thus we have that $a_n = a_0$ for all $n \geq 1$. The solution is then

$$\boxed{w = a_0 \sum_{n=0}^{\infty} z^n,}$$

which is the result we obtained by expanding the solution in Example 16.7.2.

Result 16.7.1 Consider the equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0.$$

If $p(z)$ is analytic at $z = z_0$ then z_0 is called an ordinary point of the differential equation. The Taylor series expansion of the solution can be found by substituting $w = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ into the equation and equating powers of $(z - z_0)$. The radius of convergence of the series is at least the distance to the nearest singularity of $p(z)$ in the complex plane.

16.7.2 Regular Singular Points

If the coefficient function $p(z)$ has a simple pole at $z = z_0$ then z_0 is a *regular singular point* of the first order differential equation.

Example 16.7.4 Consider the equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha}{z} w = 0, \qquad \alpha \neq 0.$$

This equation has a regular singular point at $z = 0$. The solution is $w = cz^{-\alpha}$. Depending on the value of α, the solution can have three different kinds of behavior.

α is a negative integer. The solution is analytic in the finite complex plane.

α is a positive integer The solution has a pole at the origin. w is analytic in the annulus, $0 < |z|$.

α is not an integer. w has a branch point at $z = 0$. The solution is analytic in the cut annulus $0 < |z| < \infty$, $\theta_0 < \arg z < \theta_0 + 2\pi$.

Consider the differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0,$$

where $p(z)$ has a simple pole at the origin and is analytic in the annulus, $0 < |z| < r$, for some positive r. Recall that the solution is

$$\begin{aligned} w &= c \exp\left(-\int p(z)\,\mathrm{d}z\right) \\ &= c \exp\left(-\int \frac{b_0}{z} + p(z) - \frac{b_0}{z}\,\mathrm{d}z\right) \\ &= c \exp\left(-b_0 \log z - \int \frac{zp(z) - b_0}{z}\,\mathrm{d}z\right) \\ &= cz^{-b_0} \exp\left(-\int \frac{zp(z) - b_0}{z}\,\mathrm{d}z\right) \end{aligned}$$

The exponential factor has a removable singularity at $z = 0$ and is analytic in $|z| < r$. We consider the following cases for the z^{-b_0} factor:

$\mathbf{b_0}$ is a negative integer. Since z^{-b_0} is analytic at the origin, the solution to the differential equation is analytic in the circle $|z| < r$.

$\mathbf{b_0}$ is a positive integer. The solution has a pole of order $-b_0$ at the origin and is analytic in the annulus $0 < |z| < r$.

$\mathbf{b_0}$ is not an integer. The solution has a branch point at the origin and thus is not single-valued. The solution is analytic in the cut annulus $0 < |z| < r$, $\theta_0 < \arg z < \theta_0 + 2\pi$.

Since the exponential factor has a convergent Taylor series in $|z| < r$, the solution can be expanded in a series of the form

$$w = z^{-b_0} \sum_{n=0}^{\infty} a_n z^n, \quad \text{where } a_0 \neq 0 \text{ and } b_0 = \lim_{z \to 0} z\, p(z).$$

In the case of a regular singular point at $z = z_0$, the series is

$$w = (z - z_0)^{-b_0} \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad \text{where } a_0 \neq 0 \text{ and } b_0 = \lim_{z \to z_0} (z - z_0)\, p(z).$$

Series of this form are known as *Frobenius series*. Since we can write the solution as

$$w = c(z - z_0)^{-b_0} \exp\left(- \int \left(p(z) - \frac{b_0}{z - z_0} \right) \mathrm{d}z \right),$$

we see that the Frobenius expansion of the solution will have a radius of convergence at least the distance to the nearest singularity of $p(z)$.

Result 16.7.2 Consider the equation,

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0,$$

where $p(z)$ has a simple pole at $z = z_0$, $p(z)$ is analytic in some annulus, $0 < |z - z_0| < r$, and $\lim_{z \to z_0}(z - z_0)p(z) = \beta$. The solution to the differential equation has a Frobenius series expansion of the form

$$w = (z - z_0)^{-\beta} \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_0 \neq 0.$$

The radius of convergence of the expansion will be at least the distance to the nearest singularity of $p(z)$.

Example 16.7.5 We will find the first two nonzero terms in the series solution about $z = 0$ of the differential equation,

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{\sin z} w = 0.$$

First we note that the coefficient function has a simple pole at $z = 0$ and

$$\lim_{z \to 0} \frac{z}{\sin z} = \lim_{z \to 0} \frac{1}{\cos z} = 1.$$

Thus we look for a series solution of the form

$$w = z^{-1} \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 \neq 0.$$

The nearest singularities of $1/\sin z$ in the complex plane are at $z = \pm\pi$. Thus the radius of convergence of the series will be at least π.

Substituting the first three terms of the expansion into the differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}z}(a_0 z^{-1} + a_1 + a_2 z) + \frac{1}{\sin z}(a_0 z^{-1} + a_1 + a_2 z) = O(z).$$

Recall that the Taylor expansion of $\sin z$ is $\sin z = z - \frac{1}{6}z^3 + O(z^5)$.

$$\left(z - \frac{z^3}{6} + O(z^5)\right)(-a_0 z^{-2} + a_2) + (a_0 z^{-1} + a_1 + a_2 z) = O(z^2)$$

$$-a_0 z^{-1} + \left(a_2 + \frac{a_0}{6}\right) z + a_0 z^{-1} + a_1 + a_2 z = O(z^2)$$

$$a_1 + \left(2a_2 + \frac{a_0}{6}\right) z = O(z^2)$$

a_0 is arbitrary. Equating powers of z,

$$z^0: \qquad a_1 = 0.$$

$$z^1: \qquad 2a_2 + \frac{a_0}{6} = 0.$$

Thus the solution has the expansion,

$$\boxed{w = a_0 \left(z^{-1} - \frac{z}{12} \right) + O(z^2).}$$

In Figure 16.5 the exact solution is plotted in a solid line and the two term approximation is plotted in a dashed line. The two term approximation is very good near the point $x = 0$.

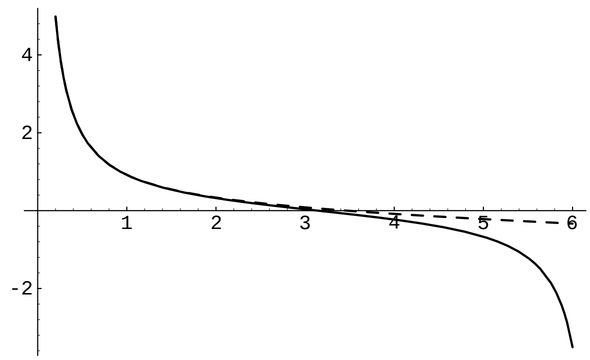


Figure 16.5: Plot of the Exact Solution and the Two Term Approximation.

Example 16.7.6 Find the first two nonzero terms in the series expansion about $z = 0$ of the solution to

$$w' - i \frac{\cos z}{z} w = 0.$$

Since $\frac{\cos z}{z}$ has a simple pole at $z = 0$ and $\lim_{z \to 0} -i \cos z = -i$ we see that the Frobenius series will have the form

$$w = z^i \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 \neq 0.$$

Recall that $\cos z$ has the Taylor expansion $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$. Substituting the Frobenius expansion into the differential equation yields

$$z \left(i z^{i-1} \sum_{n=0}^{\infty} a_n z^n + z^i \sum_{n=0}^{\infty} n a_n z^{n-1} \right) - i \left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \right) \left(z^i \sum_{n=0}^{\infty} a_n z^n \right) = 0$$
$$\sum_{n=0}^{\infty} (n+i) a_n z^n - i \left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \right) \left(\sum_{n=0}^{\infty} a_n z^n \right) = 0.$$

Equating powers of z,

$$\begin{aligned}
z^0 &: \quad i a_0 - i a_0 = 0 && \Rightarrow a_0 \text{ is arbitrary} \\
z^1 &: \quad (1+i) a_1 - i a_1 = 0 && \Rightarrow a_1 = 0 \\
z^2 &: \quad (2+i) a_2 - i a_2 + \frac{i}{2} a_0 = 0 && \Rightarrow a_2 = -\frac{i}{4} a_0.
\end{aligned}$$

Thus the solution is

$$\boxed{w = a_0 z^i \left(1 - \frac{i}{4} z^2 + O(z^3) \right).}$$

16.7.3 Irregular Singular Points

If a point is not an ordinary point or a regular singular point then it is called an *irregular singular point*. The following equations have irregular singular points at the origin.

- $w' + \sqrt{z}w = 0$

- $w' - z^{-2}w = 0$

- $w' + \exp(1/z)w = 0$

Example 16.7.7 Consider the differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \alpha z^\beta w = 0, \qquad \alpha \neq 0, \quad \beta \neq -1, 0, 1, 2, \ldots$$

This equation has an irregular singular point at the origin. Solving this equation,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\exp\left(\int \alpha z^\beta \, \mathrm{d}z\right) w\right) = 0$$

$$\boxed{w = c \exp\left(-\frac{\alpha}{\beta+1} z^{\beta+1}\right) = c \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\alpha}{\beta+1}\right)^n z^{(\beta+1)n}.}$$

If β is not an integer, then the solution has a branch point at the origin. If β is an integer, $\beta < -1$, then the solution has an essential singularity at the origin. The solution cannot be expanded in a Frobenius series, $w = z^\lambda \sum_{n=0}^{\infty} a_n z^n$.

Although we will not show it, this result holds for any irregular singular point of the differential equation. We cannot approximate the solution near an irregular singular point using a Frobenius expansion.

Now would be a good time to summarize what we have discovered about solutions of first order differential equations in the complex plane.

Result 16.7.3 Consider the first order differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0.$$

Ordinary Points If $p(z)$ is analytic at $z = z_0$ then z_0 is an ordinary point of the differential equation. The solution can be expanded in the Taylor series $w = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. The radius of convergence of the series is at least the distance to the nearest singularity of $p(z)$ in the complex plane.

Regular Singular Points If $p(z)$ has a simple pole at $z = z_0$ and is analytic in some annulus $0 < |z - z_0| < r$ then z_0 is a regular singular point of the differential equation. The solution at z_0 will either be analytic, have a pole, or have a branch point. The solution can be expanded in the Frobenius series $w = (z - z_0)^{-\beta} \sum_{n=0}^{\infty} a_n (z - z_0)^n$ where $a_0 \neq 0$ and $\beta = \lim_{z \to z_0} (z - z_0) p(z)$. The radius of convergence of the Frobenius series will be at least the distance to the nearest singularity of $p(z)$.

Irregular Singular Points If the point $z = z_0$ is not an ordinary point or a regular singular point, then it is an irregular singular point of the differential equation. The solution cannot be expanded in a Frobenius series about that point.

16.7.4 The Point at Infinity

Now we consider the behavior of first order linear differential equations at the point at infinity. Recall from complex variables that the complex plane together with the point at infinity is called the extended complex plane. To study the behavior of a function $f(z)$ at infinity, we make the transformation $z = \frac{1}{\zeta}$ and study the behavior of $f(1/\zeta)$ at $\zeta = 0$.

Example 16.7.8 Let's examine the behavior of $\sin z$ at infinity. We make the substitution $z = 1/\zeta$ and find the Laurent expansion about $\zeta = 0$.

$$\sin(1/\zeta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,\zeta^{(2n+1)}}$$

Since $\sin(1/\zeta)$ has an essential singularity at $\zeta = 0$, $\sin z$ has an essential singularity at infinity.

We use the same approach if we want to examine the behavior at infinity of a differential equation. Starting with the first order differential equation,

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0,$$

we make the substitution

$$z = \frac{1}{\zeta}, \qquad \frac{\mathrm{d}}{\mathrm{d}z} = -\zeta^2 \frac{\mathrm{d}}{\mathrm{d}\zeta}, \qquad w(z) = u(\zeta)$$

to obtain

$$-\zeta^2 \frac{\mathrm{d}u}{\mathrm{d}\zeta} + p(1/\zeta)u = 0$$
$$\frac{\mathrm{d}u}{\mathrm{d}\zeta} - \frac{p(1/\zeta)}{\zeta^2} u = 0.$$

Result 16.7.4 The behavior at infinity of

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0$$

is the same as the behavior at $\zeta = 0$ of

$$\frac{\mathrm{d}u}{\mathrm{d}\zeta} - \frac{p(1/\zeta)}{\zeta^2}u = 0.$$

Example 16.7.9 Classify the singular points of the equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{z^2 + 9}w = 0.$$

Rewriting this equation as

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{(z - 3i)(z + 3i)}w = 0,$$

we see that $z = 3i$ and $z = -3i$ are regular singular points. The transformation $z = 1/\zeta$ yields the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}\zeta} - \frac{1}{\zeta^2}\frac{1}{(1/\zeta)^2 + 9}u = 0$$
$$\frac{\mathrm{d}u}{\mathrm{d}\zeta} - \frac{1}{9\zeta^2 + 1}u = 0$$

Since the equation for u has a ordinary point at $\zeta = 0$, $z = \infty$ is a ordinary point of the equation for w.

16.8 Exercises

Exact Equations

Exercise 16.1 (mathematica/ode/first_order/exact.nb)
Find the general solution $y = y(x)$ of the equations

$$1.\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + xy + y^2}{x^2},$$

$$2.\ (4y - 3x)\,\mathrm{d}x + (y - 2x)\,\mathrm{d}y = 0.$$

Hint, Solution

Exercise 16.2 (mathematica/ode/first_order/exact.nb)
Determine whether or not the following equations can be made exact. If so find the corresponding general solution.

$$1.\ (3x^2 - 2xy + 2)\,\mathrm{d}x + (6y^2 - x^2 + 3)\,\mathrm{d}y = 0$$

$$2.\ \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ax + by}{bx + cy}$$

Hint, Solution

Exercise 16.3 (mathematica/ode/first_order/exact.nb)
Find the solutions of the following differential equations which satisfy the given initial condition. In each case determine the interval in which the solution is defined.

$$1.\ \frac{\mathrm{d}y}{\mathrm{d}x} = (1 - 2x)y^2, \quad y(0) = -1/6.$$

$$2.\ x\,\mathrm{d}x + y\,\mathrm{e}^{-x}\,\mathrm{d}y = 0, \quad y(0) = 1.$$

Hint, Solution

Exercise 16.4
Show that

$$\mu(x, y) = \frac{1}{xM(x, y) + yN(x, y)}$$

is an integrating factor for the homogeneous equation,

$$M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hint, Solution

Exercise 16.5
Are the following equations exact? If so, solve them.

1. $(4y - x)y' - (9x^2 + y - 1) = 0$
2. $(2x - 2y)y' + (2x + 4y) = 0.$

Hint, Solution

Exercise 16.6
Solve the following differential equations by inspection. That is, group terms into exact derivatives and then integrate. $f(x)$ and $g(x)$ are known functions.

1. $g(x)y'(x) + g'(x)y(x) = f(x)$
2. $\frac{y'(x)}{y(x)} = f(x)$
3. $y^\alpha(x)y'(x) = f(x)$
4. $\frac{y'}{\cos x} + y\frac{\tan x}{\cos x} = \cos x$

Hint, Solution

Exercise 16.7 (mathematica/ode/first_order/exact.nb)
Suppose we have a differential equation of the form $dy/dt = f(y/t)$. Differential equations of this form are called homogeneous equations. Since the right side only depends on the single variable y/t, it suggests itself to make the substitution $y/t = v$ or $y = tv$.

1. Show that this substitution replaces the equation $dy/dt = f(y/t)$ by the equivalent equation $tdv/dt + v = f(v)$, which is separable.

2. Find the general solution of the equation $dy/dt = 2(y/t) + (y/t)^2$.

Hint, Solution

Exercise 16.8 (mathematica/ode/first_order/exact.nb)
Find all functions $f(t)$ such that the differential equation

$$y^2 \sin t + y f(t) \frac{\mathrm{d}y}{\mathrm{d}t} = 0 \tag{16.5}$$

is exact. Solve the differential equation for these $f(t)$.
Hint, Solution

The First Order, Linear Differential Equation

Exercise 16.9 (mathematica/ode/first_order/linear.nb)
Solve the differential equation

$$y' + \frac{y}{\sin x} = 0.$$

Hint, Solution

Exercise 16.10 (mathematica/ode/first_order/linear.nb)
Solve the differential equation

$$y' - \frac{1}{x} y = x^\alpha.$$

Hint, Solution

Initial Conditions

Exercise 16.11 (mathematica/ode/first_order/exact.nb)
Find the solutions of the following differential equations which satisfy the given initial conditions:

1. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + xy = x^{2n+1}, \quad y(1) = 1, \quad n \in \mathbb{Z}$
2. $\dfrac{\mathrm{d}y}{\mathrm{d}x} - 2xy = 1, \quad y(0) = 1$

Hint, Solution

Exercise 16.12 (mathematica/ode/first_order/exact.nb)
Show that if $\alpha > 0$ and $\lambda > 0$, then for any real β, every solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \alpha y(x) = \beta \,\mathrm{e}^{-\lambda x}$$

satisfies $\lim_{x \to +\infty} y(x) = 0$. (The case $\alpha = \lambda$ requires special treatment.) Find the solution for $\beta = \lambda = 1$ which satisfies $y(0) = 1$. Sketch this solution for $0 \leq x < \infty$ for several values of α. In particular, show what happens when $\alpha \to 0$ and $\alpha \to \infty$.
Hint, Solution

Well-Posed Problems

Exercise 16.13
Find the solutions of

$$t\frac{\mathrm{d}y}{\mathrm{d}t} + Ay = 1 + t^2$$

which are bounded at $t = 0$. Consider all (real) values of A.
Hint, Solution

Equations in the Complex Plane

Exercise 16.14
Find the Taylor series expansion about the origin of the solution to

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{1-z}w = 0$$

with the substitution $w = \sum_{n=0}^{\infty} a_n z^n$. What is the radius of convergence of the series? What is the distance to the nearest singularity of $\frac{1}{1-z}$?
Hint, Solution

Exercise 16.15
Classify the singular points of the following first order differential equations, (include the point at infinity).

1. $w' + \frac{\sin z}{z} w = 0$
2. $w' + \frac{1}{z-3} w = 0$
3. $w' + z^{1/2} w = 0$

Hint, Solution

Exercise 16.16
Consider the equation

$$w' + z^{-2} w = 0.$$

The point $z = 0$ is an irregular singular point of the differential equation. Thus we know that we cannot expand the solution about $z = 0$ in a Frobenius series. Try substituting the series solution

$$w = z^\lambda \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 \neq 0$$

into the differential equation anyway. What happens?
Hint, Solution

16.9 Hints

Exact Equations

Hint 16.1

1.

2.

Hint 16.2

1. The equation is exact. Determine the primitive u by solving the equations $u_x = P$, $u_y = Q$.

2. The equation can be made exact.

Hint 16.3

1. This equation is separable. Integrate to get the general solution. Apply the initial condition to determine the constant of integration.

2. Ditto. You will have to numerically solve an equation to determine where the solution is defined.

Hint 16.4

Hint 16.5

Hint 16.6

1. $\frac{\mathrm{d}}{\mathrm{d}x}[uv] = u'v + uv'$
2. $\frac{\mathrm{d}}{\mathrm{d}x} \log u = \frac{1}{u}$
3. $\frac{\mathrm{d}}{\mathrm{d}x} u^c = u^{c-1} u'$

Hint 16.7

Hint 16.8

The First Order, Linear Differential Equation

Hint 16.9
Look in the appendix for the integral of $\csc x$.

Hint 16.10
Make sure you consider the case $\alpha = 0$.

Initial Conditions

Hint 16.11

Hint 16.12

Well-Posed Problems

Hint 16.13

Equations in the Complex Plane

Hint 16.14
The radius of convergence of the series and the distance to the nearest singularity of $\frac{1}{1-z}$ are not the same.

Hint 16.15

Hint 16.16
Try to find the value of λ by substituting the series into the differential equation and equating powers of z.

16.10 Solutions

Exact Equations

Solution 16.1

1.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + xy + y^2}{x^2}$$

Since the right side is a homogeneous function of order zero, this is a homogeneous differential equation. We make the change of variables $u = y/x$ and then solve the differential equation for u.

$$xu' + u = 1 + u + u^2$$
$$\frac{\mathrm{d}u}{1 + u^2} = \frac{\mathrm{d}x}{x}$$
$$\arctan(u) = \ln|x| + c$$
$$u = \tan(\ln(|cx|))$$
$$\boxed{y = x\tan(\ln(|cx|))}$$

2.

$$(4y - 3x)\,\mathrm{d}x + (y - 2x)\,\mathrm{d}y = 0$$

Since the coefficients are homogeneous functions of order one, this is a homogeneous differential equation.

We make the change of variables $u = y/x$ and then solve the differential equation for u.

$$\left(4\frac{y}{x} - 3\right) \mathrm{d}x + \left(\frac{y}{x} - 2\right) \mathrm{d}y = 0$$
$$(4u - 3)\,\mathrm{d}x + (u - 2)(u\,\mathrm{d}x + x\,\mathrm{d}u) = 0$$
$$(u^2 + 2u - 3)\,\mathrm{d}x + x(u - 2)\,\mathrm{d}u = 0$$
$$\frac{\mathrm{d}x}{x} + \frac{u-2}{(u+3)(u-1)}\,\mathrm{d}u = 0$$
$$\frac{\mathrm{d}x}{x} + \left(\frac{5/4}{u+3} - \frac{1/4}{u-1}\right) \mathrm{d}u = 0$$
$$\ln(x) + \frac{5}{4}\ln(u+3) - \frac{1}{4}\ln(u-1) = c$$
$$\frac{x^4(u+3)^5}{u-1} = c$$
$$\frac{x^4(y/x+3)^5}{y/x-1} = c$$
$$\boxed{\frac{(y+3x)^5}{y-x} = c}$$

Solution 16.2

1.

$$(3x^2 - 2xy + 2)\,\mathrm{d}x + (6y^2 - x^2 + 3)\,\mathrm{d}y = 0$$

We check if this form of the equation, $P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$, is exact.

$$P_y = -2x, \qquad Q_x = -2x$$

Since $P_y = Q_x$, the equation is exact. Now we find the primitive $u(x, y)$ which satisfies

$$\mathrm{d}u = (3x^2 - 2xy + 2)\,\mathrm{d}x + (6y^2 - x^2 + 3)\,\mathrm{d}y.$$

The primitive satisfies the partial differential equations

$$u_x = P, \quad u_y = Q. \tag{16.6}$$

We integrate the first equation of 16.6 to determine u up to a function of integration.

$$u_x = 3x^2 - 2xy + 2$$
$$u = x^3 - x^2 y + 2x + f(y)$$

We substitute this into the second equation of 16.6 to determine the function of integration up to an additive constant.

$$-x^2 + f'(y) = 6y^2 - x^2 + 3$$
$$f'(y) = 6y^2 + 3$$
$$f(y) = 2y^3 + 3y$$

The solution of the differential equation is determined by the implicit equation $u = c$.

$$\boxed{x^3 - x^2 y + 2x + 2y^3 + 3y = c}$$

2.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ax + by}{bx + cy}$$
$$(ax + by)\,\mathrm{d}x + (bx + cy)\,\mathrm{d}y = 0$$

We check if this form of the equation, $P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$, is exact.

$$P_y = b, \qquad Q_x = b$$

Since $P_y = Q_x$, the equation is exact. Now we find the primitive $u(x, y)$ which satisfies

$$\mathrm{d}u = (ax + by)\,\mathrm{d}x + (bx + cy)\,\mathrm{d}y$$

The primitive satisfies the partial differential equations

$$u_x = P, \quad u_y = Q. \tag{16.7}$$

We integrate the first equation of 16.7 to determine u up to a function of integration.

$$\begin{aligned} u_x &= ax + by \\ u &= \frac{1}{2}ax^2 + bxy + f(y) \end{aligned}$$

We substitute this into the second equation of 16.7 to determine the function of integration up to an additive constant.

$$\begin{aligned} bx + f'(y) &= bx + cy \\ f'(y) &= cy \\ f(y) &= \frac{1}{2}cy^2 \end{aligned}$$

The solution of the differential equation is determined by the implicit equation $u = d$.

$$\boxed{ax^2 + 2bxy + cy^2 = d}$$

Solution 16.3

Note that since these equations are nonlinear, we cannot predict where the solutions will be defined from the equation alone.

1. This equation is separable. We integrate to get the general solution.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= (1-2x)y^2 \\
\frac{\mathrm{d}y}{y^2} &= (1-2x)\,\mathrm{d}x \\
-\frac{1}{y} &= x - x^2 + c \\
y &= \frac{1}{x^2 - x - c}
\end{aligned}$$

Now we apply the initial condition.

$$\begin{aligned}
y(0) = \frac{1}{-c} &= -\frac{1}{6} \\
y &= \frac{1}{x^2 - x - 6} \\
\Aboxed{y &= \frac{1}{(x+2)(x-3)}}
\end{aligned}$$

The solution is defined on the interval $(-2 \ldots 3)$.

2. This equation is separable. We integrate to get the general solution.

$$\begin{aligned}
x\,\mathrm{d}x + y\,\mathrm{e}^{-x}\,\mathrm{d}y &= 0 \\
x\,\mathrm{e}^{x}\,\mathrm{d}x + y\,\mathrm{d}y &= 0 \\
(x-1)\,\mathrm{e}^{x} + \frac{1}{2}y^2 &= c \\
y &= \sqrt{2(c + (1-x)\,\mathrm{e}^{x})}
\end{aligned}$$

We apply the initial condition to determine the constant of integration.

$$y(0) = \sqrt{2(c+1)} = 1$$
$$c = -\frac{1}{2}$$
$$\boxed{y = \sqrt{2(1-x)\,\mathrm{e}^x - 1}}$$

The function $2(1-x)\,\mathrm{e}^x - 1$ is plotted in Figure 16.6. We see that the argument of the square root in the solution is non-negative only on an interval about the origin. Because $2(1-x)\,\mathrm{e}^x - 1 == 0$ is a mixed algebraic / transcendental equation, we cannot solve it analytically. The solution of the differential equation is defined on the interval $(-1.67835 \ldots 0.768039)$.

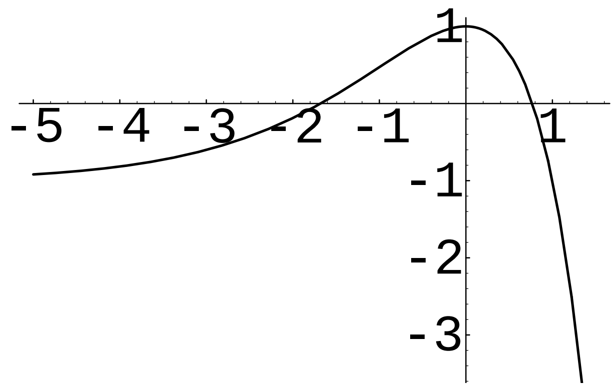


Figure 16.6: The function $2(1-x)\,\mathrm{e}^x - 1$.

Solution 16.4

We consider the homogeneous equation,

$$M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

That is, both M and N are homogeneous of degree n. Multiplying by

$$\mu(x, y) = \frac{1}{xM(x, y) + yN(x, y)}$$

will make the equation exact. To prove this we use the result that

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is exact if and only if $P_y = Q_x$.

$$\begin{aligned} P_y &= \frac{\partial}{\partial y}\left[\frac{M}{xM + yN}\right] \\ &= \frac{M_y(xM + yN) - M(xM_y + N + yN_y)}{(xM + yN)^2} \end{aligned}$$

$$\begin{aligned} Q_x &= \frac{\partial}{\partial x}\left[\frac{N}{xM + yN}\right] \\ &= \frac{N_x(xM + yN) - N(M + xM_x + yN_x)}{(xM + yN)^2} \end{aligned}$$

$$\begin{aligned} M_y(xM + yN) - M(xM_y + N + yN_y) &= N_x(xM + yN) - N(M + xM_x + yN_x) \\ yM_yN - yMN_y &= xMN_x - xM_xN \\ xM_xN + yM_yN &= xMN_x + yMN_y \end{aligned}$$

With Euler's theorem, this reduces to the identity,

$$nMN = nMN.$$

Thus the equation is exact. $\mu(x, y)$ is an integrating factor for the homogeneous equation.

Solution 16.5

1. We consider the differential equation,

$$(4y - x)y' - (9x^2 + y - 1) = 0.$$

$$P_y = \frac{\partial}{\partial y}\left(1 - y - 9x^2\right) = -1$$
$$Q_x = \frac{\partial}{\partial x}(4y - x) = -1$$

This equation is exact. It is simplest to solve the equation by rearranging terms to form exact derivatives.

$$4yy' - xy' - y + 1 - 9x^2 = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[2y^2 - xy\right] + 1 - 9x^2 = 0$$
$$2y^2 - xy + x - 3x^3 + c = 0$$
$$\boxed{y = \frac{1}{4}\left(x \pm \sqrt{x^2 - 8(c + x - 3x^3)}\right)}$$

2. We consider the differential equation,

$$(2x - 2y)y' + (2x + 4y) = 0.$$

$$P_y = \frac{\partial}{\partial y}(2x + 4y) = 4$$
$$Q_x = \frac{\partial}{\partial x}(2x - 2y) = 2$$

Since $P_y \neq Q_x$, this is not an exact equation.

Solution 16.6

1.

$$g(x)y'(x) + g'(x)y(x) = f(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[g(x)y(x)] = f(x)$$
$$\boxed{y(x) = \frac{1}{g(x)} \int f(x)\,\mathrm{d}x + \frac{c}{g(x)}}$$

2.

$$\frac{y'(x)}{y(x)} = f(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\log(y(x)) = f(x)$$
$$\log(y(x)) = \int f(x)\,\mathrm{d}x + c$$
$$y(x) = \mathrm{e}^{\int f(x)\,\mathrm{d}x + c}$$
$$\boxed{y(x) = a\,\mathrm{e}^{\int f(x)\,\mathrm{d}x}}$$

3.

$$y^\alpha(x)y'(x) = f(x)$$
$$\frac{y^{\alpha+1}(x)}{\alpha+1} = \int f(x)\,\mathrm{d}x + c$$
$$\boxed{y(x) = \left((\alpha+1)\int f(x)\,\mathrm{d}x + a\right)^{1/(\alpha+1)}}$$

4.

$$\frac{y'}{\cos x} + y\frac{\tan x}{\cos x} = \cos x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\cos x}\right) = \cos x$$
$$\frac{y}{\cos x} = \sin x + c$$
$$\boxed{y(x) = \sin x \cos x + c \cos x}$$

Solution 16.7

1. We substitute $y = tv$ into the differential equation and simplify.

$$y' = f\left(\frac{y}{t}\right)$$
$$tv' + v = f(v)$$
$$tv' = f(v) - v$$
$$\boxed{\frac{v'}{f(v) - v} = \frac{1}{t}} \tag{16.8}$$

The final equation is separable.

2. We start with the homogeneous differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2\left(\frac{y}{t}\right) + \left(\frac{y}{t}\right)^2.$$

We substitute $y = tv$ to obtain Equation 16.8, and solve the separable equation.

$$\begin{aligned}
\frac{v'}{v^2 + v} &= \frac{1}{t} \\
\frac{v'}{v(v+1)} &= \frac{1}{t} \\
\frac{v'}{v} - \frac{v'}{v+1} &= \frac{1}{t} \\
\log v - \log(v+1) &= \log t + c \\
\log\left(\frac{v}{v+1}\right) &= \log(ct) \\
\frac{v}{v+1} &= ct \\
v &= \frac{ct}{1 - ct} \\
v &= \frac{t}{c - t} \\
\Aboxed{y &= \frac{t^2}{c - t}}
\end{aligned}$$

Solution 16.8
Recall that the differential equation

$$P(x, y) + Q(x, y)y' = 0$$

is exact if and only if $P_y = Q_x$. For Equation 16.5, this criterion is

$$\begin{gathered}
2y \sin t = y f'(t) \\
f'(t) = 2 \sin t \\
\boxed{f(t) = 2(a - \cos t).}
\end{gathered}$$

In this case, the differential equation is

$$y^2 \sin t + 2yy'(a - \cos t) = 0.$$

We can integrate this exact equation by inspection.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(y^2(a - \cos t)\right) = 0$$
$$y^2(a - \cos t) = c$$
$$\boxed{y = \pm \frac{c}{\sqrt{a - \cos t}}}$$

The First Order, Linear Differential Equation

Solution 16.9
Consider the differential equation

$$y' + \frac{y}{\sin x} = 0.$$

The solution is

$$y = c\,\mathrm{e}^{\int -1/\sin x \,\mathrm{d}x}$$
$$= c\,\mathrm{e}^{-\log(\tan(x/2))}$$

$$\boxed{y = c \cot\left(\frac{x}{2}\right).}$$

Solution 16.10

$$y' - \frac{1}{x}y = x^\alpha$$

The integrating factor is

$$\exp\left(\int -\frac{1}{x}\,\mathrm{d}x\right) = \exp(-\log x) = \frac{1}{x}.$$

$$\begin{aligned}
\frac{1}{x}y' - \frac{1}{x^2}y &= x^{\alpha-1}\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}y\right) &= x^{\alpha-1}\\
\frac{1}{x}y &= \int x^{\alpha-1}\,\mathrm{d}x + c\\
y &= x\int x^{\alpha-1}\,\mathrm{d}x + cx
\end{aligned}$$

$$\boxed{y = \begin{cases} \frac{x^{\alpha+1}}{\alpha} + cx & \text{for } \alpha \neq 0,\\ x\log x + cx & \text{for } \alpha = 0.\end{cases}}$$

Initial Conditions

Solution 16.11

1.

$$y' + xy = x^{2n+1}, \quad y(1) = 1, \quad n \in \mathbb{Z}$$

The integrating factor is

$$I(x) = \mathrm{e}^{\int x\,\mathrm{d}x} = \mathrm{e}^{x^2/2}.$$

We multiply by the integrating factor and integrate. Since the initial condition is given at $x = 1$, we will take the lower bound of integration to be that point.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2/2} y\right) = x^{2n+1}\,\mathrm{e}^{x^2/2}$$
$$y = \mathrm{e}^{-x^2/2}\int_1^x \xi^{2n+1}\,\mathrm{e}^{\xi^2/2}\,\mathrm{d}\xi + c\,\mathrm{e}^{-x^2/2}$$

We choose the constant of integration to satisfy the initial condition.

$$\boxed{y = \mathrm{e}^{-x^2/2}\int_1^x \xi^{2n+1}\,\mathrm{e}^{\xi^2/2}\,\mathrm{d}\xi + \mathrm{e}^{(1-x^2)/2}}$$

If $n \geq 0$ then we can use integration by parts to write the integral as a sum of terms. If $n < 0$ we can write the integral in terms of the exponential integral function. However, the integral form above is as nice as any other and we leave the answer in that form.

2.

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy(x) = 1, \quad y(0) = 1.$$

The integrating factor is

$$I(x) = \mathrm{e}^{\int -2x\,\mathrm{d}x} = \mathrm{e}^{-x^2}.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x^2} y\right) = \mathrm{e}^{-x^2}$$
$$y = \mathrm{e}^{x^2}\int_0^x \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi + c\,\mathrm{e}^{x^2}$$

We choose the constant of integration to satisfy the initial condition.

$$\boxed{y = \mathrm{e}^{x^2} \left(1 + \int_0^x \mathrm{e}^{-\xi^2} \, \mathrm{d}\xi \right)}$$

We can write the answer in terms of the *Error function*,

$$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-\xi^2} \, \mathrm{d}\xi.$$

$$\boxed{y = \mathrm{e}^{x^2} \left(1 + \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) \right)}$$

Solution 16.12

The integrating factor is,

$$I(x) = \mathrm{e}^{\int \alpha \, \mathrm{d}x} = \mathrm{e}^{\alpha x}.$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\mathrm{e}^{\alpha x} y \right) = \beta \, \mathrm{e}^{(\alpha - \lambda) x}$$
$$y = \beta \, \mathrm{e}^{-\alpha x} \int \mathrm{e}^{(\alpha - \lambda) x} \, \mathrm{d}x + c \, \mathrm{e}^{-\alpha x}$$

For $\alpha \neq \lambda$, the solution is

$$y = \beta \, \mathrm{e}^{-\alpha x} \frac{\mathrm{e}^{(\alpha - \lambda) x}}{\alpha - \lambda} + c \, \mathrm{e}^{-\alpha x}$$
$$\boxed{y = \frac{\beta}{\alpha - \lambda} \mathrm{e}^{-\lambda x} + c \, \mathrm{e}^{-\alpha x}}$$

Clearly the solution vanishes as $x \to \infty$.

For $\alpha = \lambda$, the solution is

$$y = \beta\,\mathrm{e}^{-\alpha x} x + c\,\mathrm{e}^{-\alpha x}$$
$$\boxed{y = (c + \beta x)\,\mathrm{e}^{-\alpha x}}$$

We use L'Hospital's rule to show that the solution vanishes as $x \to \infty$.

$$\lim_{x\to\infty} \frac{c + \beta x}{\mathrm{e}^{\alpha x}} = \lim_{x\to\infty} \frac{\beta}{\alpha\,\mathrm{e}^{\alpha x}} = 0$$

For $\beta = \lambda = 1$, the solution is

$$y = \begin{cases} \frac{1}{\alpha-1}\,\mathrm{e}^{-x} + c\,\mathrm{e}^{-\alpha x} & \text{for } \alpha \neq 1, \\ (c + x)\,\mathrm{e}^{-x} & \text{for } \alpha = 1. \end{cases}$$

The solution which satisfies the initial condition is

$$\boxed{y = \begin{cases} \frac{1}{\alpha-1}\left(\mathrm{e}^{-x} + (\alpha - 2)\,\mathrm{e}^{-\alpha x}\right) & \text{for } \alpha \neq 1, \\ (1 + x)\,\mathrm{e}^{-x} & \text{for } \alpha = 1. \end{cases}}$$

In Figure 16.7 the solution is plotted for $\alpha = 1/16, 1/8, \dots, 16$.

Consider the solution in the limit as $\alpha \to 0$.

$$\begin{aligned} \lim_{\alpha\to 0} y(x) &= \lim_{\alpha\to 0} \frac{1}{\alpha - 1}\left(\mathrm{e}^{-x} + (\alpha - 2)\,\mathrm{e}^{-\alpha x}\right) \\ &= 2 - \mathrm{e}^{-x} \end{aligned}$$

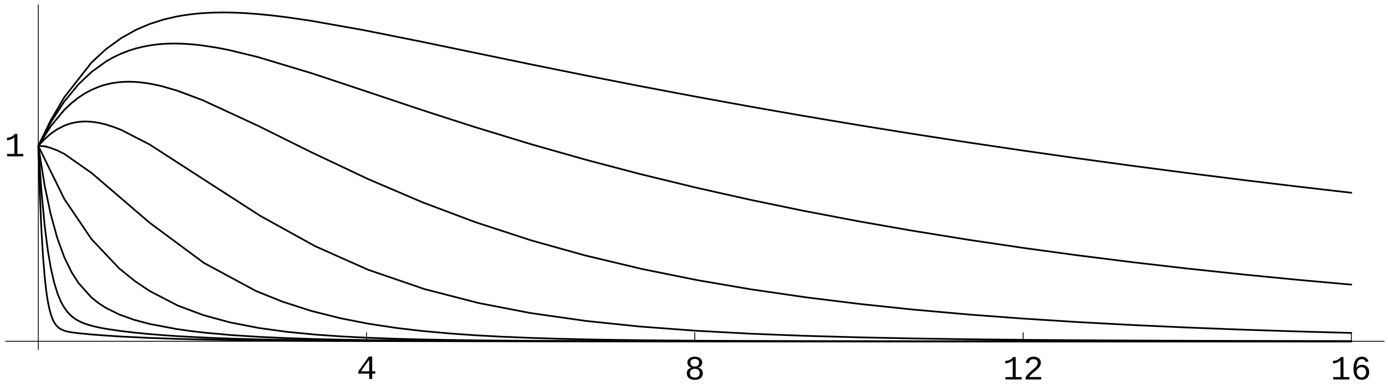


Figure 16.7: The Solution for a Range of α

In the limit as $\alpha \to \infty$ we have,

$$\begin{aligned}
\lim_{\alpha\to\infty} y(x) &= \lim_{\alpha\to\infty} \frac{1}{\alpha - 1}\left(\mathrm{e}^{-x} + (\alpha - 2)\,\mathrm{e}^{-\alpha x}\right) \\
&= \lim_{\alpha\to\infty} \frac{\alpha - 2}{\alpha - 1}\,\mathrm{e}^{-\alpha x} \\
&= \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } x > 0. \end{cases}
\end{aligned}$$

This behavior is shown in Figure 16.8. The first graph plots the solutions for $\alpha = 1/128, 1/64, \ldots, 1$. The second graph plots the solutions for $\alpha = 1, 2, \ldots, 128$.

Well-Posed Problems

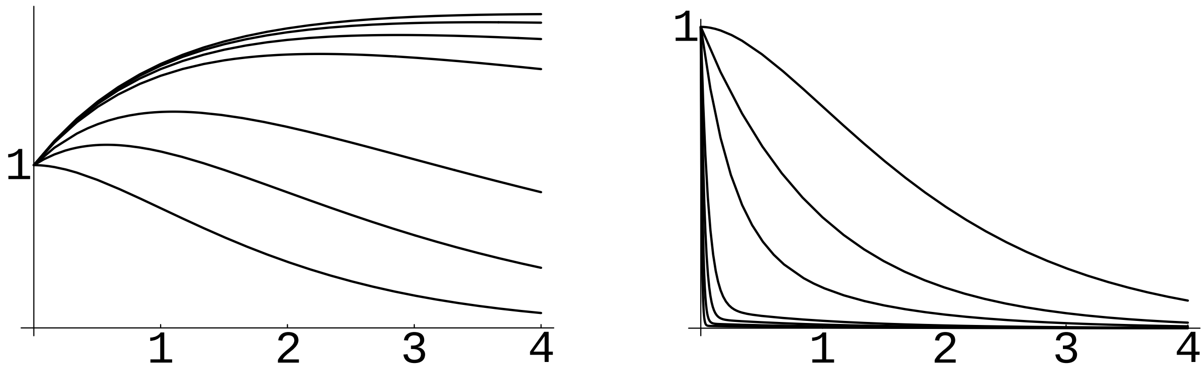


Figure 16.8: The Solution as $\alpha \to 0$ and $\alpha \to \infty$

Solution 16.13
First we write the differential equation in the standard form.

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{A}{t}y = \frac{1}{t} + t$$

The integrating factor is

$$I(t) = \mathrm{e}^{\int A/t \, \mathrm{d}t} = \mathrm{e}^{A \log t} = t^A$$

We multiply the differential equation by the integrating factor and integrate.

$$\begin{gathered}
\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{A}{t} y = \frac{1}{t} + t \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(t^A y\right) = t^{A-1} + t^{A+1} \\
t^A y = \begin{cases} \frac{t^A}{A} + \frac{t^{A+2}}{A+2} + c, & A \neq 0, -2 \\ \log t + \frac{1}{2} t^2 + c, & A = 0 \\ -\frac{1}{2} t^{-2} + \log t + c, & A = -2 \end{cases} \\
y = \begin{cases} \frac{1}{A} + \frac{t^2}{A+2} + c t^{-A}, & A \neq -2 \\ \log t + \frac{1}{2} t^2 + c, & A = 0 \\ -\frac{1}{2} + t^2 \log t + c t^2, & A = -2 \end{cases}
\end{gathered}$$

For positive A, the solution is bounded at the origin only for $c = 0$. For $A = 0$, there are no bounded solutions. For negative A, the solution is bounded there for any value of c and thus we have a one-parameter family of solutions.

In summary, the solutions which are bounded at the origin are:

$$\boxed{y = \begin{cases} \frac{1}{A} + \frac{t^2}{A+2}, & A > 0 \\ \frac{1}{A} + \frac{t^2}{A+2} + c t^{-A}, & A < 0,\ A \neq -2 \\ -\frac{1}{2} + t^2 \log t + c t^2, & A = -2 \end{cases}}$$

Equations in the Complex Plane

Solution 16.14

We substitute $w = \sum_{n=0}^{\infty} a_n z^n$ into the equation $\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{1-z} w = 0$.

$$\frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=0}^{\infty} a_n z^n + \frac{1}{1-z} \sum_{n=0}^{\infty} a_n z^n = 0$$

$$(1-z) \sum_{n=1}^{\infty} n a_n z^{n-1} + \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} z^n - \sum_{n=0}^{\infty} n a_n z^n + \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} \left((n+1) a_{n+1} - (n-1) a_n\right) z^n = 0$$

Equating powers of z to zero, we obtain the relation,

$$a_{n+1} = \frac{n-1}{n+1} a_n.$$

a_0 is arbitrary. We can compute the rest of the coefficients from the recurrence relation.

$$a_1 = \frac{-1}{1} a_0 = -a_0$$
$$a_2 = \frac{0}{2} a_1 = 0$$

We see that the coefficients are zero for $n \geq 2$. Thus the Taylor series expansion, (and the exact solution), is

$$\boxed{w = a_0 (1 - z).}$$

The radius of convergence of the series in infinite. The nearest singularity of $\frac{1}{1-z}$ is at $z = 1$. Thus we see the radius of convergence can be greater than the distance to the nearest singularity of the coefficient function, $p(z)$.

Solution 16.15

1. Consider the equation $w' + \frac{\sin z}{z} w = 0$. The point $z = 0$ is the only point we need to examine in the finite plane. Since $\frac{\sin z}{z}$ has a removable singularity at $z = 0$, there are no singular points in the finite plane. The substitution $z = \frac{1}{\zeta}$ yields the equation

$$u' - \frac{\sin(1/\zeta)}{\zeta} u = 0.$$

Since $\frac{\sin(1/\zeta)}{\zeta}$ has an essential singularity at $\zeta = 0$, the point at infinity is an irregular singular point of the original differential equation.

2. Consider the equation $w' + \frac{1}{z-3} w = 0$. Since $\frac{1}{z-3}$ has a simple pole at $z = 3$, the differential equation has a regular singular point there. Making the substitution $z = 1/\zeta$, $w(z) = u(\zeta)$

$$u' - \frac{1}{\zeta^2(1/\zeta - 3)} u = 0$$
$$u' - \frac{1}{\zeta(1 - 3\zeta)} u = 0.$$

Since this equation has a simple pole at $\zeta = 0$, the original equation has a regular singular point at infinity.

3. Consider the equation $w' + z^{1/2} w = 0$. There is an irregular singular point at $z = 0$. With the substitution $z = 1/\zeta$, $w(z) = u(\zeta)$,

$$u' - \frac{\zeta^{-1/2}}{\zeta^2} u = 0$$
$$u' - \zeta^{-5/2} u = 0.$$

We see that the point at infinity is also an irregular singular point of the original differential equation.

Solution 16.16
We start with the equation

$$w' + z^{-2}w = 0.$$

Substituting $w = z^\lambda \sum_{n=0}^\infty a_n z^n$, $a_0 \neq 0$ yields

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^\lambda \sum_{n=0}^\infty a_n z^n\right) + z^{-2} z^\lambda \sum_{n=0}^\infty a_n z^n = 0$$
$$\lambda z^{\lambda-1} \sum_{n=0}^\infty a_n z^n + z^\lambda \sum_{n=1}^\infty n a_n z^{n-1} + z^\lambda \sum_{n=0}^\infty a_n z^{n-2} = 0$$

The lowest power of z in the expansion is $z^{\lambda-2}$. The coefficient of this term is a_0. Equating powers of z demands that $a_0 = 0$ which contradicts our initial assumption that it was nonzero. Thus we cannot find a λ such that the solution can be expanded in the form,

$$w = z^\lambda \sum_{n=0}^\infty a_n z^n, \qquad a_0 \neq 0.$$

Chapter 17

First Order Systems of Differential Equations

We all agree that your theory is crazy, but is it crazy enough?

- Niels Bohr

17.1 Matrices and Jordan Canonical Form

Functions of Square Matrices. Consider a function $f(x)$ with a Taylor series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

We can define the function to take square matrices as arguments. The function of the square matrix $\mathbf{A}$ is defined in terms of the Taylor series.

$$f(\mathbf{A}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \mathbf{A}^n$$

(Note that this definition is usually not the most convenient method for computing a function of a matrix. Use the Jordan canonical form for that.)

Eigenvalues and Eigenvectors. Consider a square matrix $\mathbf{A}$. A nonzero vector $\mathbf{x}$ is an *eigenvector* of the matrix with *eigenvalue* λ if

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

Note that we can write this equation as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

This equation has solutions for nonzero $\mathbf{x}$ if and only if $\mathbf{A} - \lambda\mathbf{I}$ is singular, $(\det(\mathbf{A} - \lambda\mathbf{I}) = 0)$. We define the *characteristic polynomial* of the matrix $\chi(\lambda)$ as this determinant.

$$\chi(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$$

The roots of the characteristic polynomial are the eigenvalues of the matrix. The eigenvectors of distinct eigenvalues are linearly independent. Thus if a matrix has distinct eigenvalues, the eigenvectors form a basis.

If λ is a root of $\chi(\lambda)$ of multiplicity m then there are up to m linearly independent eigenvectors corresponding to that eigenvalue. That is, it has from 1 to m eigenvectors.

Diagonalizing Matrices. Consider an $n \times n$ matrix $\mathbf{A}$ that has a complete set of n linearly independent eigenvectors. $\mathbf{A}$ may or may not have distinct eigenvalues. Consider the matrix $\mathbf{S}$ with eigenvectors as columns.

$$\mathbf{S} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$$

$\mathbf{A}$ is diagonalized by the similarity transformation:

$$\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}.$$

$\mathbf{\Lambda}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ as the diagonal elements. Furthermore, the k^{th} diagonal element is λ_k, the eigenvalue corresponding to the the eigenvector, $\mathbf{x}_k$.

Generalized Eigenvectors. A vector $\mathbf{x}_k$ is a *generalized eigenvector of rank* k if

$$(\mathbf{A} - \lambda \mathbf{I})^k \mathbf{x}_k = \mathbf{0} \quad \text{but} \quad (\mathbf{A} - \lambda \mathbf{I})^{\mathbf{k-1}} \mathbf{x_k} \neq \mathbf{0}.$$

Eigenvectors are generalized eigenvectors of rank 1. An $n \times n$ matrix has n linearly independent generalized eigenvectors. A *chain* of generalized eigenvectors generated by the rank m generalized eigenvector $\mathbf{x}_m$ is the set: $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$, where

$$\mathbf{x}_k = (\mathbf{A} - \lambda \mathbf{I})\mathbf{x}_{k+1}, \quad \text{for} \quad k = m-1, \ldots, 1.$$

Computing Generalized Eigenvectors. Let λ be an eigenvalue of multiplicity m. Let n be the smallest integer such that

$$\text{rank}\,(\text{nullspace}\,((A - \lambda I)^n)) = m.$$

Let N_k denote the number of eigenvalues of rank k. These have the value:

$$N_k = \text{rank}\left(\text{nullspace}\left((A - \lambda I)^k\right)\right) - \text{rank}\left(\text{nullspace}\left((A - \lambda I)^{k-1}\right)\right).$$

One can compute the generalized eigenvectors of a matrix by looping through the following three steps until all the the N_k are zero:

1. Select the largest k for which N_k is positive. Find a generalized eigenvector $\mathbf{x}_k$ of rank k which is linearly independent of all the generalized eigenvectors found thus far.

2. From $\mathbf{x}_k$ generate the chain of eigenvectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$. Add this chain to the known generalized eigenvectors.

3. Decrement each positive N_k by one.

Example 17.1.1 Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}.$$

The characteristic polynomial of the matrix is

$$\begin{aligned} \chi(\lambda) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -3 & 2 & 4-\lambda \end{vmatrix} \\ &= (1-\lambda)^2(4-\lambda) + 3 + 4 + 3(1-\lambda) - 2(4-\lambda) + 2(1-\lambda) \\ &= -(\lambda-2)^3. \end{aligned}$$

Thus we see that $\lambda = 2$ is an eigenvalue of multiplicity 3. $\mathbf{A} - 2\mathbf{I}$ is

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}$$

The rank of the nullspace space of $\mathbf{A} - 2\mathbf{I}$ is less than 3.

$$(\mathbf{A} - 2\mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

The rank of nullspace $((\mathbf{A} - 2\mathbf{I})^2)$ is less than 3 as well, so we have to take one more step.

$$(\mathbf{A} - 2\mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of nullspace $((\mathbf{A}-2\mathbf{I})^3)$ is 3. Thus there are generalized eigenvectors of ranks 1, 2 and 3. The generalized eigenvector of rank 3 satisfies:

$$\begin{aligned}(\mathbf{A}-2\mathbf{I})^3\mathbf{x}_3 &= \mathbf{0}\\ \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\mathbf{x}_3 &= \mathbf{0}\end{aligned}$$

We choose the solution

$$\mathbf{x}_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

Now to compute the chain generated by $\mathbf{x}_3$.

$$\begin{aligned}\mathbf{x}_2 &= (\mathbf{A}-2\mathbf{I})\mathbf{x}_3 = \begin{pmatrix}-1\\2\\-3\end{pmatrix}\\ \mathbf{x}_1 &= (\mathbf{A}-2\mathbf{I})\mathbf{x}_2 = \begin{pmatrix}0\\-1\\1\end{pmatrix}\end{aligned}$$

Thus a set of generalized eigenvectors corresponding to the eigenvalue $\lambda = 2$ are

$$\boxed{\mathbf{x}_1 = \begin{pmatrix}0\\-1\\1\end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix}-1\\2\\-3\end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}.}$$

Jordan Block. A Jordan block is a square matrix which has the constant, λ, on the diagonal and ones on the first super-diagonal:

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

Jordan Canonical Form. A matrix $\mathbf{J}$ is in Jordan canonical form if all the elements are zero except for Jordan blocks $\mathbf{J}_k$ along the diagonal.

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{J}_{n-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{J}_n \end{pmatrix}$$

The Jordan canonical form of a matrix is obtained with the similarity transformation:

$$\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S},$$

where $\mathbf{S}$ is the matrix of the generalized eigenvectors of $\mathbf{A}$ and the generalized eigenvectors are grouped in chains.

Example 17.1.2 Again consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}.$$

Since $\lambda = 2$ is an eigenvalue of multiplicity 3, the Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

In Example 17.1.1 we found the generalized eigenvectors of $\mathbf{A}$. We define the matrix with generalized eigenvectors as columns:

$$\mathbf{S} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -3 & 0 \end{pmatrix}.$$

We can verify that $\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

$$\begin{aligned} \mathbf{J} &= \mathbf{S}^{-1}\mathbf{A}\mathbf{S} \\ &= \begin{pmatrix} 0 & -3 & -2 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

Functions of Matrices in Jordan Canonical Form. The function of an $n \times n$ Jordan block is the upper-triangular matrix:

$$f(\mathbf{J}_k) = \begin{pmatrix} f(\lambda) & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} & \cdots & \frac{f^{(n-2)}(\lambda)}{(n-2)!} & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ 0 & f(\lambda) & \frac{f'(\lambda)}{1!} & \cdots & \frac{f^{(n-3)}(\lambda)}{(n-3)!} & \frac{f^{(n-2)}(\lambda)}{(n-2)!} \\ 0 & 0 & f(\lambda) & \ddots & \frac{f^{(n-4)}(\lambda)}{(n-4)!} & \frac{f^{(n-3)}(\lambda)}{(n-3)!} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & f(\lambda) & \frac{f'(\lambda)}{1!} \\ 0 & 0 & 0 & \cdots & 0 & f(\lambda) \end{pmatrix}$$

The function of a matrix in Jordan canonical form is

$$f(\mathbf{J}) = \begin{pmatrix} f(\mathbf{J}_1) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & f(\mathbf{J}_2) & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ddots & f(\mathbf{J}_{n-1}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & f(\mathbf{J}_n) \end{pmatrix}$$

The Jordan canonical form of a matrix satisfies:

$$f(\mathbf{J}) = \mathbf{S}^{-1} f(\mathbf{A}) \mathbf{S},$$

where $\mathbf{S}$ is the matrix of the generalized eigenvectors of $\mathbf{A}$. This gives us a convenient method for computing functions of matrices.

Example 17.1.3 Consider the matrix exponential function $\mathrm{e}^{\mathbf{A}}$ for our old friend:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}.$$

In Example 17.1.2 we showed that the Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since all the derivatives of e^λ are just e^λ, it is especially easy to compute $\mathrm{e}^{\mathbf{J}}$.

$$\mathrm{e}^{\mathbf{J}} = \begin{pmatrix} \mathrm{e}^2 & \mathrm{e}^2 & \mathrm{e}^2/2 \\ 0 & \mathrm{e}^2 & \mathrm{e}^2 \\ 0 & 0 & \mathrm{e}^2 \end{pmatrix}$$

We find $\mathrm{e}^{\mathbf{A}}$ with a similarity transformation of $\mathrm{e}^{\mathbf{J}}$. We use the matrix of generalized eigenvectors found in Example 17.1.2.

$$\mathrm{e}^{\mathbf{A}} = \mathbf{S}\,\mathrm{e}^{\mathbf{J}}\mathbf{S}^{-1}$$

$$\mathrm{e}^{\mathbf{A}} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^2 & \mathrm{e}^2 & \mathrm{e}^2/2 \\ 0 & \mathrm{e}^2 & \mathrm{e}^2 \\ 0 & 0 & \mathrm{e}^2 \end{pmatrix} \begin{pmatrix} 0 & -3 & -2 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\boxed{\mathrm{e}^{\mathbf{A}} = \begin{pmatrix} 0 & 2 & 2 \\ 3 & 1 & -1 \\ -5 & 3 & 5 \end{pmatrix} \frac{\mathrm{e}^2}{2}}$$

17.2 Systems of Differential Equations

The homogeneous differential equation

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

has the solution

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c}$$

where $\mathbf{c}$ is a vector of constants. The solution subject to the initial condition, $\mathbf{x}(t_0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)}\mathbf{x}_0.$$

The homogeneous differential equation

$$\mathbf{x}'(t) = \frac{1}{t}\mathbf{A}\mathbf{x}(t)$$

has the solution

$$\mathbf{x}(t) = t^{\mathbf{A}}\mathbf{c} \equiv \mathrm{e}^{\mathbf{A}\,\mathrm{Log}\,t}\mathbf{c},$$

where $\mathbf{c}$ is a vector of constants. The solution subject to the initial condition, $\mathbf{x}(t_0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = \left(\frac{t}{t_0}\right)^{\mathbf{A}}\mathbf{x}_0 \equiv \mathrm{e}^{\mathbf{A}\,\mathrm{Log}\,(t/t_0)}\mathbf{x}_0.$$

The inhomogeneous problem

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

has the solution

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \mathrm{e}^{\mathbf{A}t}\int_{t_0}^{t} \mathrm{e}^{-\mathbf{A}\tau}\mathbf{f}(\tau)\,d\tau.$$

Example 17.2.1 Consider the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \mathbf{x}.$$

The general solution of the system of differential equations is

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c}.$$

In Example 17.1.3 we found $\mathrm{e}^{\mathbf{A}}$. $\mathbf{A}t$ is just a constant times $\mathbf{A}$. The eigenvalues of $\mathbf{A}t$ are $\{\lambda_k t\}$ where $\{\lambda_k\}$ are the eigenvalues of $\mathbf{A}$. The generalized eigenvectors of $\mathbf{A}t$ are the same as those of $\mathbf{A}$.

Consider $\mathrm{e}^{\mathbf{J}t}$. The derivatives of $f(\lambda) = \mathrm{e}^{\lambda t}$ are $f'(\lambda) = t\,\mathrm{e}^{\lambda t}$ and $f''(\lambda) = t^2\,\mathrm{e}^{\lambda t}$. Thus we have

$$\mathrm{e}^{\mathbf{J}t} = \begin{pmatrix} \mathrm{e}^{2t} & t\,\mathrm{e}^{2t} & t^2\,\mathrm{e}^{2t}/2 \\ 0 & \mathrm{e}^{2t} & t\,\mathrm{e}^{2t} \\ 0 & 0 & \mathrm{e}^{2t} \end{pmatrix}$$

$$\mathrm{e}^{\mathbf{J}t} = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \mathrm{e}^{2t}$$

We find e^{At} with a similarity transformation.

$$\mathrm{e}^{\mathbf{A}t} = \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\mathbf{S}^{-1}$$

$$\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \mathrm{e}^{2t} \begin{pmatrix} 0 & -3 & -2 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} 1-t & t & t \\ 2t - t^2/2 & 1 - t + t^2/2 & -t + t^2/2 \\ -3t + t^2/2 & 2t - t^2/2 & 1 + 2t - t^2/2 \end{pmatrix} \mathrm{e}^{2t}$$

The solution of the system of differential equations is

$$\boxed{\mathbf{x}(t) = \left(c_1 \begin{pmatrix} 1-t \\ 2t - t^2/2 \\ -3t + t^2/2 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 - t + t^2/2 \\ 2t - t^2/2 \end{pmatrix} + c_3 \begin{pmatrix} t \\ -t + t^2/2 \\ 1 + 2t - t^2/2 \end{pmatrix} \right) \mathrm{e}^{2t}}$$

Example 17.2.2 Consider the Euler equation system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{1}{t}\mathbf{A}\mathbf{x} \equiv \frac{1}{t}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{x}.$$

The solution is $\mathbf{x}(t) = t^{\mathbf{A}}\mathbf{c}$. Note that $\mathbf{A}$ is almost in Jordan canonical form. It has a one on the sub-diagonal instead of the super-diagonal. It is clear that a function of $\mathbf{A}$ is defined

$$f(\mathbf{A}) = \begin{pmatrix} f(1) & 0 \\ f'(1) & f(1) \end{pmatrix}.$$

The function $f(\lambda) = t^\lambda$ has the derivative $f'(\lambda) = t^\lambda \log t$. Thus the solution of the system is

$$\boxed{\mathbf{x}(t) = \begin{pmatrix} t & 0 \\ t\log t & t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 \begin{pmatrix} t \\ t\log t \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ t \end{pmatrix}}$$

Example 17.2.3 Consider an inhomogeneous system of differential equations.

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} + \mathbf{f}(t) \equiv \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \quad t > 0.$$

The general solution is

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c} + \mathrm{e}^{\mathbf{A}t} \int \mathrm{e}^{-\mathbf{A}t} f(t)\, dt.$$

First we find homogeneous solutions. The characteristic equation for the matrix is

$$\chi(\lambda) = \begin{vmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{vmatrix} = \lambda^2 = 0$$

$\lambda = 0$ is an eigenvalue of multiplicity 2. Thus the Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since rank (nullspace ($\mathbf{A} - 0\mathbf{I}$)) = 1 there is only one eigenvector. A generalized eigenvector of rank 2 satisfies

$$(\mathbf{A} - 0\mathbf{I})^2 \mathbf{x}_2 = \mathbf{0}$$
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x}_2 = \mathbf{0}$$

We choose

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Now we generate the chain from $\mathbf{x}_2$.

$$\mathbf{x}_1 = (\mathbf{A} - 0\mathbf{I})\mathbf{x}_2 = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

We define the matrix of generalized eigenvectors $\mathbf{S}$.

$$\mathbf{S} = \begin{pmatrix} 4 & 1 \\ 8 & 0 \end{pmatrix}$$

The derivative of $f(\lambda) = \mathrm{e}^{\lambda t}$ is $f'(\lambda) = t\,\mathrm{e}^{\lambda t}$. Thus

$$\mathrm{e}^{\mathbf{J}t} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

The homogeneous solution of the differential equation system is $\mathbf{x}_h = \mathrm{e}^{\mathbf{A}t}\mathbf{c}$ where

$$\mathrm{e}^{\mathbf{A}t} = \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\mathbf{S}^{-1}$$

$$\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} 4 & 1 \\ 8 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1/8 \\ 1 & -1/2 \end{pmatrix}$$

$$\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} 1+4t & -2t \\ 8t & 1-4t \end{pmatrix}$$

The general solution of the inhomogeneous system of equations is

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c} + \mathrm{e}^{\mathbf{A}t}\int \mathrm{e}^{-\mathbf{A}t} f(t)\,dt$$

$$\mathbf{x}(t) = \begin{pmatrix} 1+4t & -2t \\ 8t & 1-4t \end{pmatrix}\mathbf{c} + \begin{pmatrix} 1+4t & -2t \\ 8t & 1-4t \end{pmatrix}\int \begin{pmatrix} 1-4t & 2t \\ -8t & 1+4t \end{pmatrix}\begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix} dt$$

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1+4t \\ 8t \end{pmatrix} + c_2\begin{pmatrix} -2t \\ 1-4t \end{pmatrix} + \begin{pmatrix} 2 - 2\operatorname{Log} t + \frac{6}{t} - \frac{1}{2t^2} \\ 4 - 4\operatorname{Log} t + \frac{13}{t} \end{pmatrix}$$

We can tidy up the answer a little bit. First we take linear combinations of the homogeneous solutions to obtain a simpler form.

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2t \\ 4t-1 \end{pmatrix} + \begin{pmatrix} 2 - 2\operatorname{Log} t + \frac{6}{t} - \frac{1}{2t^2} \\ 4 - 4\operatorname{Log} t + \frac{13}{t} \end{pmatrix}$$

Then we subtract 2 times the first homogeneous solution from the particular solution.

$$\boxed{\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2t \\ 4t-1 \end{pmatrix} + \begin{pmatrix} -2\operatorname{Log} t + \frac{6}{t} - \frac{1}{2t^2} \\ -4\operatorname{Log} t + \frac{13}{t} \end{pmatrix}}$$

17.3 Exercises

Exercise 17.1 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Hint, Solution

Exercise 17.2 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Hint, Solution

Exercise 17.3 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hint, Solution

Exercise 17.4 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Hint, Solution

Exercise 17.5 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Hint, Solution

Exercise 17.6 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$$

Hint, Solution

Exercise 17.7

1. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \mathbf{x}. \tag{17.1}$$

   (a) Show that $\lambda = 2$ is an eigenvalue of multiplicity 3 of the coefficient matrix $\mathbf{A}$, and that there is only one corresponding eigenvector, namely

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(b) Using the information in part (i), write down one solution $\mathbf{x}^{(1)}(t)$ of the system (17.1). There is no other solution of a purely exponential form $\mathbf{x} = \boldsymbol{\xi}\, e^{\lambda t}$.

(c) To find a second solution use the form $\mathbf{x} = \boldsymbol{\xi} t\, e^{2t} + \boldsymbol{\eta}\, e^{2t}$, and find appropriate vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. This gives a solution of the system (17.1) which is independent of the one obtained in part (ii).

(d) To find a third linearly independent solution use the form $\mathbf{x} = \boldsymbol{\xi}(t^2/2)\, e^{2t} + \boldsymbol{\eta} t\, e^{2t} + \boldsymbol{\zeta}\, e^{2t}$. Show that $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}.$$

The first two equations can be taken to coincide with those obtained in part (iii). Solve the third equation, and write down a third independent solution of the system (17.1).

2. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \mathbf{x}. \tag{17.2}$$

(a) Show that $\lambda = 1$ is an eigenvalue of multiplicity 3 of the coefficient matrix $\mathbf{A}$, and that there are only two linearly independent eigenvectors, which we may take as

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$$

Find two independent solutions of equation (17.2).

(b) To find a third solution use the form $\mathbf{x} = \boldsymbol{\xi} t\, e^{t} + \boldsymbol{\eta} e^{t}$; then show that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ must satisfy

$$(\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \quad (\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}.$$

Show that the most general solution of the first of these equations is $\boldsymbol{\xi} = c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2$, where c_1 and c_2 are arbitrary constants. Show that, in order to solve the second of these equations it is necessary to take $c_1 = c_2$. Obtain such a vector $\boldsymbol{\eta}$, and use it to obtain a third independent solution of the system (17.2).

Hint, Solution

Exercise 17.8 (mathematica/ode/systems/systems.nb)
Consider the system of ODE's

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

where $\mathbf{A}$ is the constant 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$$

1. Find the eigenvalues and associated eigenvectors of $\mathbf{A}$. [HINT: notice that $\lambda = -1$ is a root of the characteristic polynomial of $\mathbf{A}$.]

2. Use the results from part (a) to construct $e^{\mathbf{A}t}$ and therefore the solution to the initial value problem above.

3. Use the results of part (a) to find the general solution to

$$\frac{d\mathbf{x}}{dt} = \frac{1}{t}\mathbf{A}\mathbf{x}.$$

Hint, Solution

Exercise 17.9 (mathematica/ode/systems/systems.nb)

1. Find the general solution to

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

where

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

2. Solve

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} + \mathbf{g}(t), \quad \mathbf{x}(0) = \mathbf{0}$$

using **A** from part (a).

Hint, Solution

Exercise 17.10

Let **A** be an $n \times n$ matrix of constants. The system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{1}{t}\mathbf{A}\mathbf{x}, \tag{17.3}$$

is analogous to the Euler equation.

1. Verify that when **A** is a 2×2 constant matrix, elimination of (17.3) yields a second order Euler differential equation.

2. Now assume that **A** is an $n \times n$ matrix of constants. Show that this system, in analogy with the Euler equation has solutions of the form $\mathbf{x} = \mathbf{a}t^{\lambda}$ where **a** is a constant vector provided **a** and λ satisfy certain conditions.

3. Based on your experience with the treatment of multiple roots in the solution of constant coefficient systems, what form will the general solution of (17.3) take if λ is a multiple eigenvalue in the eigenvalue problem derived in part (b)?

4. Verify your prediction by deriving the general solution for the system

$$\frac{d\mathbf{x}}{dt} = \frac{1}{t}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\mathbf{x}.$$

Hint, Solution

Exercise 17.11

Use the matrix form of the method of variation of parameters to find the general solution of

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \quad t > 0.$$

Hint, Solution

17.4 Hints

Hint 17.1

Hint 17.2

Hint 17.3

Hint 17.4

Hint 17.5

Hint 17.6

Hint 17.7

Hint 17.8

Hint 17.9

Hint 17.10

Hint 17.11

17.5 Solutions

Solution 17.1

We consider an initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Method 1. Find Homogeneous Solutions. The matrix has the distinct eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$. The corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

The general solution of the system of differential equations is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} \mathrm{e}^{3t}.$$

We apply the initial condition to determine the constants.

$$\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2}$$

The solution subject to the initial condition is

$$\boxed{\mathbf{x} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \mathrm{e}^{3t}}$$

For large t, the solution looks like

$$\mathbf{x} \approx \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \mathrm{e}^{3t}.$$

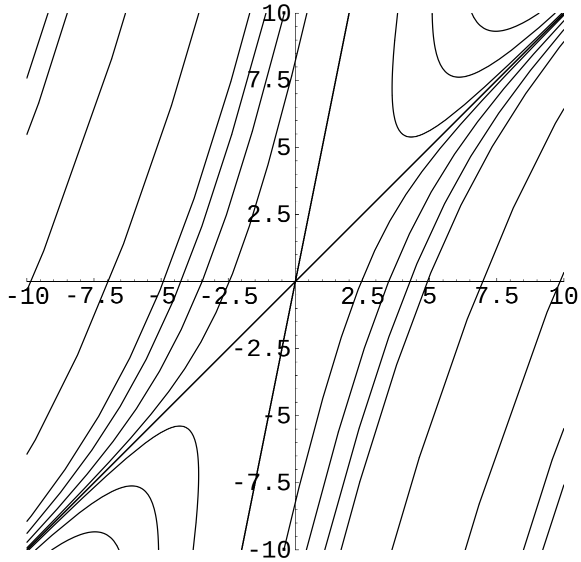


Figure 17.1: Homogeneous solutions in the phase plane.

Both coordinates tend to infinity.

Figure 17.1 show some homogeneous solutions in the phase plane.

Method 2. Use the Exponential Matrix. The Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

The solution of the initial value problem is $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0$.

$$
\begin{aligned}
\mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0 \\
&= \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\mathbf{S}^{-1}\mathbf{x}_0 \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{3t} \end{pmatrix} \frac{1}{4} \begin{pmatrix} 5 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} \mathrm{e}^{-t} + \mathrm{e}^{3t} \\ \mathrm{e}^{-t} + 5\,\mathrm{e}^{3t} \end{pmatrix}
\end{aligned}
$$

$$
\boxed{\mathbf{x} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \mathrm{e}^{3t}}
$$

Solution 17.2
We consider an initial value problem.

$$
\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}
$$

Method 1. Find Homogeneous Solutions. The matrix has the distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$. The corresponding eigenvectors are

$$
\mathbf{x}_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.
$$

The general solution of the system of differential equations is

$$
\mathbf{x} = c_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^{t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{2t} + c_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \mathrm{e}^{3t}.
$$

We apply the initial condition to determine the constants.

$$\begin{pmatrix} 0 & 1 & 2 \\ -2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$
$$c_1 = 1, \quad c_2 = 2, \quad c_3 = 0$$

The solution subject to the initial condition is

$$\boxed{\mathbf{x} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^t + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{2t}.}$$

As $t \to \infty$, all coordinates tend to infinity.

Method 2. Use the Exponential Matrix. The Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

The solution of the initial value problem is $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0$.

$$\begin{aligned}
\mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0 \\
&= \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\mathbf{S}^{-1}\mathbf{x}_0 \\
&= \begin{pmatrix} 0 & 1 & 2 \\ -2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^t & 0 & 0 \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & 0 & \mathrm{e}^{3t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ 4 & -2 & -4 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 2\,\mathrm{e}^{2t} \\ -2\,\mathrm{e}^t + 2\,\mathrm{e}^{2t} \\ \mathrm{e}^t \end{pmatrix}
\end{aligned}$$

$$\boxed{\mathbf{x} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^{t} + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \mathrm{e}^{2t}.}$$

Solution 17.3
We consider an initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Method 1. Find Homogeneous Solutions. The matrix has the distinct eigenvalues $\lambda_1 = -1 - \mathrm{i}$, $\lambda_2 = -1 + \mathrm{i}$. The corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} 2 - \mathrm{i} \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 + \mathrm{i} \\ 1 \end{pmatrix}.$$

The general solution of the system of differential equations is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 - \mathrm{i} \\ 1 \end{pmatrix} \mathrm{e}^{(-1-\mathrm{i})t} + c_2 \begin{pmatrix} 2 + \mathrm{i} \\ 1 \end{pmatrix} \mathrm{e}^{(-1+\mathrm{i})t}.$$

We can take the real and imaginary parts of either of these solution to obtain real-valued solutions.

$$\begin{pmatrix} 2 + \mathrm{i} \\ 1 \end{pmatrix} \mathrm{e}^{(-1+\mathrm{i})t} = \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} \mathrm{e}^{-t} + \mathrm{i} \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix} \mathrm{e}^{-t}$$

$$\mathbf{x} = c_1 \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} \mathrm{e}^{-t} + c_2 \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix} \mathrm{e}^{-t}$$

We apply the initial condition to determine the constants.

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$c_1 = 1, \quad c_2 = -1$$

The solution subject to the initial condition is

$$\boxed{\mathbf{x} = \begin{pmatrix} \cos(t) - 3\sin(t) \\ \cos(t) - \sin(t) \end{pmatrix} \mathrm{e}^{-t}.}$$

Plotted in the phase plane, the solution spirals in to the origin at t increases. Both coordinates tend to zero as $t \to \infty$.

Method 2. Use the Exponential Matrix. The Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} -1 - \mathrm{i} & 0 \\ 0 & -1 + \mathrm{i} \end{pmatrix}.$$

The solution of the initial value problem is $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0$.

$$\begin{aligned}
\mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0 \\
&= \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\mathbf{S}^{-1}\mathbf{x}_0 \\
&= \begin{pmatrix} 2 - \mathrm{i} & 2 + \mathrm{i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{(-1-\mathrm{i})t} & 0 \\ 0 & \mathrm{e}^{(-1+\mathrm{i})t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} \mathrm{i} & 1 - \mathrm{i}2 \\ -\mathrm{i} & 1 + \mathrm{i}2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (\cos(t) - 3\sin(t))\,\mathrm{e}^{-t} \\ (\cos(t) - \sin(t))\,\mathrm{e}^{-t} \end{pmatrix}
\end{aligned}$$

$$\boxed{\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-t}\cos(t) - \begin{pmatrix} 3 \\ 1 \end{pmatrix} \mathrm{e}^{-t}\sin(t)}$$

Solution 17.4

We consider an initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Method 1. Find Homogeneous Solutions. The matrix has the distinct eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1 - \mathrm{i}\sqrt{2}$, $\lambda_3 = -1 + \mathrm{i}\sqrt{2}$. The corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 + \mathrm{i}\sqrt{2} \\ -1 + \mathrm{i}\sqrt{2} \\ 3 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 2 - \mathrm{i}\sqrt{2} \\ -1 - \mathrm{i}\sqrt{2} \\ 3 \end{pmatrix}.$$

The general solution of the system of differential equations is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^{-2t} + c_2 \begin{pmatrix} 2 + \mathrm{i}\sqrt{2} \\ -1 + \mathrm{i}\sqrt{2} \\ 3 \end{pmatrix} \mathrm{e}^{(-1-\mathrm{i}\sqrt{2})t} + c_3 \begin{pmatrix} 2 - \mathrm{i}\sqrt{2} \\ -1 - \mathrm{i}\sqrt{2} \\ 3 \end{pmatrix} \mathrm{e}^{(-1+\mathrm{i}\sqrt{2})t}.$$

We can take the real and imaginary parts of the second or third solution to obtain two real-valued solutions.

$$\begin{pmatrix} 2 + \mathrm{i}\sqrt{2} \\ -1 + \mathrm{i}\sqrt{2} \\ 3 \end{pmatrix} \mathrm{e}^{(-1-\mathrm{i}\sqrt{2})t} = \begin{pmatrix} 2\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ -\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ 3\cos(\sqrt{2}t) \end{pmatrix} \mathrm{e}^{-t} + \mathrm{i} \begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) - 2\sin(\sqrt{2}t) \\ \sqrt{2}\cos(\sqrt{2}t) + \sin(\sqrt{2}t) \\ -3\sin(\sqrt{2}t) \end{pmatrix} \mathrm{e}^{-t}$$

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^{-2t} + c_2 \begin{pmatrix} 2\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ -\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ 3\cos(\sqrt{2}t) \end{pmatrix} \mathrm{e}^{-t} + c_3 \begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) - 2\sin(\sqrt{2}t) \\ \sqrt{2}\cos(\sqrt{2}t) + \sin(\sqrt{2}t) \\ -3\sin(\sqrt{2}t) \end{pmatrix} \mathrm{e}^{-t}$$

We apply the initial condition to determine the constants.

$$\begin{pmatrix} 2 & 2 & \sqrt{2} \\ -2 & -1 & \sqrt{2} \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$c_1 = \frac{1}{3}, \quad c_2 = -\frac{1}{9}, \quad c_3 = \frac{5}{9\sqrt{2}}$$

The solution subject to the initial condition is

$$\boxed{\mathbf{x} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^{-2t} + \frac{1}{6}\begin{pmatrix} 2\cos(\sqrt{2}t) - 4\sqrt{2}\sin(\sqrt{2}t) \\ 4\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ -2\cos(\sqrt{2}t) - 5\sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} \mathrm{e}^{-t}.}$$

As $t \to \infty$, all coordinates tend to infinity. Plotted in the phase plane, the solution would spiral in to the origin.

Method 2. Use the Exponential Matrix. The Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 - \mathrm{i}\sqrt{2} & 0 \\ 0 & 0 & -1 + \mathrm{i}\sqrt{2} \end{pmatrix}.$$

The solution of the initial value problem is $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0$.

$$\begin{aligned}
\mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0 \\
&= \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\mathbf{S}^{-1}\mathbf{x}_0 \\
&= \frac{1}{3}\begin{pmatrix} 6 & 2 + \mathrm{i}\sqrt{2} & 2 - \mathrm{i}\sqrt{2} \\ -6 & -1 + \mathrm{i}\sqrt{2} & -1 - \mathrm{i}\sqrt{2} \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-2t} & 0 & 0 \\ 0 & \mathrm{e}^{(-1-\mathrm{i}\sqrt{2})t} & 0 \\ 0 & 0 & \mathrm{e}^{(-1+\mathrm{i}\sqrt{2})t} \end{pmatrix} \\
&\quad \frac{1}{6}\begin{pmatrix} 2 & -2 & -2 \\ -1 - \mathrm{i}5\sqrt{2}/2 & 1 - \mathrm{i}2\sqrt{2} & 4 + \mathrm{i}\sqrt{2} \\ -1 + \mathrm{i}5\sqrt{2}/2 & 1 + \mathrm{i}2\sqrt{2} & 4 - \mathrm{i}\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\boxed{\mathbf{x} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^{-2t} + \frac{1}{6}\begin{pmatrix} 2\cos(\sqrt{2}t) - 4\sqrt{2}\sin(\sqrt{2}t) \\ 4\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ -2\cos(\sqrt{2}t) - 5\sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} \mathrm{e}^{-t}.}$$

Solution 17.5
We consider an initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Method 1. Find Homogeneous Solutions. The matrix has the double eigenvalue $\lambda_1 = \lambda_2 = -3$. There is only one corresponding eigenvector. We compute a chain of generalized eigenvectors.

$$(\mathbf{A} + 3\mathbf{I})^2 \mathbf{x}_2 = \mathbf{0}$$
$$\mathbf{0}\mathbf{x}_2 = \mathbf{0}$$
$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$(\mathbf{A} + 3\mathbf{I})\mathbf{x}_2 = \mathbf{x}_1$$
$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

The general solution of the system of differential equations is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-3t} + c_2 \left(\begin{pmatrix} 4 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \mathrm{e}^{-3t}.$$

We apply the initial condition to determine the constants.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
$$c_1 = 2, \quad c_2 = 1$$

The solution subject to the initial condition is

$$\boxed{\mathbf{x} = \begin{pmatrix} 3 + 4t \\ 2 + 4t \end{pmatrix} \mathrm{e}^{-3t}.}$$

Both coordinates tend to zero as $t \to \infty$.

Method 2. Use the Exponential Matrix. The Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}.$$

The solution of the initial value problem is $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0$.

$$\begin{aligned} \mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0 \\ &= \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\mathbf{S}^{-1}\mathbf{x}_0 \\ &= \begin{pmatrix} 1 & 1/4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-3t} & t\,\mathrm{e}^{-3t} \\ 0 & \mathrm{e}^{-3t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \end{aligned}$$

$$\boxed{\mathbf{x} = \begin{pmatrix} 3 + 4t \\ 2 + 4t \end{pmatrix} \mathrm{e}^{-3t}.}$$

Solution 17.6

We consider an initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$$

Method 1. Find Homogeneous Solutions. The matrix has the distinct eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$. The corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The general solution of the system of differential equations is

$$\mathbf{x} = c_1 \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} \mathrm{e}^{-t} + c_2 \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \mathrm{e}^{t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{2t}.$$

We apply the initial condition to determine the constants.

$$\begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 0 \\ 5 & 6 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$$

$$c_1 = 1, \quad c_2 = -4, \quad c_3 = -11$$

The solution subject to the initial condition is

$$\boxed{\mathbf{x} = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} \mathrm{e}^{-t} - 4 \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \mathrm{e}^{t} - 11 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{2t}.}$$

As $t \to \infty$, the first coordinate vanishes, the second coordinate tends to ∞ and the third coordinate tends to $-\infty$

Method 2. Use the Exponential Matrix. The Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The solution of the initial value problem is $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0$.

$$\begin{aligned} \mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0 \\ &= \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\mathbf{S}^{-1}\mathbf{x}_0 \\ &= \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 0 \\ 5 & 6 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-t} & 0 & 0 \\ 0 & \mathrm{e}^{t} & 0 \\ 0 & 0 & \mathrm{e}^{2t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ -7 & 6 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix} \end{aligned}$$

$$\boxed{\mathbf{x} = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} \mathrm{e}^{-t} - 4 \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \mathrm{e}^{t} - 11 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{2t}.}$$

Solution 17.7

1. (a) We compute the eigenvalues of the matrix.

$$\chi(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -3 & 2 & 4-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda-2)^3$$

$\lambda = 2$ is an eigenvalue of multiplicity 3. The rank of the null space of $\mathbf{A} - 2\mathbf{I}$ is 1. (The first two rows are linearly independent, but the third is a linear combination of the first two.)

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}$$

Thus there is only one eigenvector.

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \mathbf{0}$$

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(b) One solution of the system of differential equations is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathrm{e}^{2t}.$$

(c) We substitute the form $\mathbf{x} = \boldsymbol{\xi} t\, \mathrm{e}^{2t} + \boldsymbol{\eta}\, \mathrm{e}^{2t}$ into the differential equation.

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$
$$\boldsymbol{\xi}\, \mathrm{e}^{2t} + 2\boldsymbol{\xi} t\, \mathrm{e}^{2t} + 2\boldsymbol{\eta}\, \mathrm{e}^{2t} = \mathbf{A}\boldsymbol{\xi} t\, \mathrm{e}^{2t} + \mathbf{A}\boldsymbol{\eta}\, \mathrm{e}^{2t}$$
$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$$

We already have a solution of the first equation, we need the generalized eigenvector $\boldsymbol{\eta}$. Note that $\boldsymbol{\eta}$ is only determined up to a constant times $\boldsymbol{\xi}$. Thus we look for the solution whose second component vanishes to simplify the algebra.

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$$
$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ 0 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$
$$-\eta_1 + \eta_3 = 0, \quad 2\eta_1 - \eta_3 = 1, \quad -3\eta_1 + 2\eta_3 = -1$$
$$\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

A second linearly independent solution is

$$\boxed{\mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t\, \mathrm{e}^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{2t}.}$$

(d) To find a third solution we substutite the form $\mathbf{x} = \boldsymbol{\xi}(t^2/2)\, \mathrm{e}^{2t} + \boldsymbol{\eta} t\, \mathrm{e}^{2t} + \boldsymbol{\zeta}\, \mathrm{e}^{2t}$ into the differential equation.

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$
$$2\boldsymbol{\xi}(t^2/2)\, \mathrm{e}^{2t} + (\boldsymbol{\xi} + 2\boldsymbol{\eta}) t\, \mathrm{e}^{2t} + (\boldsymbol{\eta} + 2\boldsymbol{\zeta})\, \mathrm{e}^{2t} = \mathbf{A}\boldsymbol{\xi}(t^2/2)\, \mathrm{e}^{2t} + \mathbf{A}\boldsymbol{\eta} t\, \mathrm{e}^{2t} + \mathbf{A}\boldsymbol{\zeta}\, \mathrm{e}^{2t}$$
$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$$

We have already solved the first two equations, we need the generalized eigenvector $\boldsymbol{\zeta}$. Note that $\boldsymbol{\zeta}$ is only determined up to a constant times $\boldsymbol{\xi}$. Thus we look for the solution whose second component vanishes to simplify the algebra.

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ 0 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$-\zeta_1 + \zeta_3 = 1, \quad 2\zeta_1 - \zeta_3 = 0, \quad -3\zeta_1 + 2\zeta_3 = 1$$

$$\boldsymbol{\zeta} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

A third linearly independent solution is

$$\boxed{\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} (t^2/2)\, \mathrm{e}^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} t\, \mathrm{e}^{2t} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \mathrm{e}^{2t}}$$

2. (a) We compute the eigenvalues of the matrix.

$$\chi(\lambda) = \begin{vmatrix} 5-\lambda & -3 & -2 \\ 8 & -5-\lambda & -4 \\ -4 & 3 & 3-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3$$

$\lambda = 1$ is an eigenvalue of multiplicity 3. The rank of the null space of $\mathbf{A} - \mathbf{I}$ is 2. (The second and third rows are multiples of the first.)

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix}$$

Thus there are two eigenvectors.

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \mathbf{0}$$

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$$

Two linearly independent solutions of the differential equation are

$$\boxed{\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \mathrm{e}^t, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \mathrm{e}^t.}$$

(b) We substitute the form $\mathbf{x} = \boldsymbol{\xi} t \, \mathrm{e}^t + \boldsymbol{\eta} \, \mathrm{e}^t$ into the differential equation.

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$
$$\boldsymbol{\xi} \, \mathrm{e}^t + \boldsymbol{\xi} t \, \mathrm{e}^t + \boldsymbol{\eta} \, \mathrm{e}^t = \mathbf{A}\boldsymbol{\xi} t \, \mathrm{e}^t + \mathbf{A}\boldsymbol{\eta} \, \mathrm{e}^t$$
$$(\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \quad (\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$$

The general solution of the first equation is a linear combination of the two solutions we found in the previous part.

$$\boldsymbol{\xi} = c_1 \boldsymbol{\xi}_1 + c_2 \boldsymbol{\xi}_2$$

Now we find the generalized eigenvector, $\boldsymbol{\eta}$. Note that $\boldsymbol{\eta}$ is only determined up to a linear combination of $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. Thus we can take the first two components of $\boldsymbol{\eta}$ to be zero.

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \eta_3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$$

$$-2\eta_3 = c_1, \quad -4\eta_3 = 2c_2, \quad 2\eta_3 = 2c_1 - 3c_2$$

$$c_1 = c_2, \quad \eta_3 = -\frac{c_1}{2}$$

We see that we must take $c_1 = c_2$ in order to obtain a solution. We choose $c_1 = c_2 = 2$ A third linearly independent solution of the differential equation is

$$\boxed{\mathbf{x}^{(3)} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} t\, \mathrm{e}^t + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \mathrm{e}^t.}$$

Solution 17.8

1. The characteristic polynomial of the matrix is

$$\begin{aligned} \chi(\lambda) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -8 & -5 & -3-\lambda \end{vmatrix} \\ &= (1-\lambda)^2(-3-\lambda) + 8 - 10 - 5(1-\lambda) - 2(-3-\lambda) - 8(1-\lambda) \\ &= -\lambda^3 - \lambda^2 + 4\lambda + 4 \\ &= -(\lambda+2)(\lambda+1)(\lambda-2) \end{aligned}$$

Thus we see that the eigenvalues are $\lambda = -2, -1, 2$. The eigenvectors $\boldsymbol{\xi}$ satisfy

$$(\mathbf{A} - \lambda \mathbf{I})\boldsymbol{\xi} = \mathbf{0}.$$

For $\lambda = -2$, we have

$$(\mathbf{A} + 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}.$$

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ -8 & -5 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If we take $\xi_3 = 1$ then the first two rows give us the system,

$$\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

which has the solution $\xi_1 = -4/7$, $\xi_2 = 5/7$. For the first eigenvector we choose:

$$\boldsymbol{\xi} = \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}$$

For $\lambda = -1$, we have

$$(\mathbf{A} + \mathbf{I})\boldsymbol{\xi} = \mathbf{0}.$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If we take $\xi_3 = 1$ then the first two rows give us the system,

$$\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

which has the solution $\xi_1 = -3/2$, $\xi_2 = 2$. For the second eigenvector we choose:

$$\boldsymbol{\xi} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$$

For $\lambda = 2$, we have

$$(\mathbf{A} + \mathbf{I})\boldsymbol{\xi} = \mathbf{0}.$$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If we take $\xi_3 = 1$ then the first two rows give us the system,

$$\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

which has the solution $\xi_1 = 0$, $\xi_2 = -1$. For the third eigenvector we choose:

$$\boldsymbol{\xi} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

In summary, the eigenvalues and eigenvectors are

$$\boxed{\lambda = \{-2, -1, 2\}, \qquad \boldsymbol{\xi} = \left\{ \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}}$$

2. The matrix is diagonalized with the similarity transformation

$$\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S},$$

where $\mathbf{S}$ is the matrix with eigenvectors as columns:

$$\mathbf{S} = \begin{pmatrix} -4 & -3 & 0 \\ 5 & 4 & -1 \\ 7 & 2 & 1 \end{pmatrix}$$

The matrix exponential, $\mathrm{e}^{\mathbf{A}t}$ is given by

$$\mathrm{e}^{\mathbf{A}} = \mathbf{S}\,\mathrm{e}^{\mathbf{J}}\mathbf{S}^{-1}.$$

$$\mathrm{e}^{\mathbf{A}} = \begin{pmatrix} -4 & -3 & 0 \\ 5 & 4 & -1 \\ 7 & 2 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-2t} & 0 & 0 \\ 0 & \mathrm{e}^{-t} & 0 \\ 0 & 0 & \mathrm{e}^{2t} \end{pmatrix} \frac{1}{12} \begin{pmatrix} 6 & 3 & 3 \\ -12 & -4 & -4 \\ -18 & -13 & -1 \end{pmatrix}.$$

$$\boxed{\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} -2\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{-t} & -\,\mathrm{e}^{-2t} + \,\mathrm{e}^{-t} & -\,\mathrm{e}^{-2t} + \,\mathrm{e}^{-t} \\ \frac{5\,\mathrm{e}^{-2t} - 8\,\mathrm{e}^{-t} + 3\,\mathrm{e}^{t}}{2} & \frac{15\,\mathrm{e}^{-2t} - 16\,\mathrm{e}^{-t} + 13\,\mathrm{e}^{t}}{12} & \frac{15\,\mathrm{e}^{-2t} - 16\,\mathrm{e}^{-t} + \,\mathrm{e}^{t}}{12} \\ \frac{7\,\mathrm{e}^{-2t} - 4\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{t}}{2} & \frac{21\,\mathrm{e}^{-2t} - 8\,\mathrm{e}^{-t} - 13\,\mathrm{e}^{t}}{12} & \frac{21\,\mathrm{e}^{-2t} - 8\,\mathrm{e}^{-t} - \,\mathrm{e}^{t}}{12} \end{pmatrix}}$$

The solution of the initial value problem is $\mathrm{e}^{\mathbf{A}t}\mathbf{x}_0$.

3. The general solution of the Euler equation is

$$\boxed{c_1 \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix} t^{-2} + c_2 \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} t^{-1} + c_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} t^2.}$$

We could also write the solution as

$$\mathbf{x} = t^{\mathbf{A}}\mathbf{c} \equiv \mathrm{e}^{\mathbf{A}\log t}\mathbf{c},$$

Solution 17.9

1. The characteristic polynomial of the matrix is

$$\begin{aligned} \chi(\lambda) &= \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)^2(3-\lambda) \end{aligned}$$

Thus we see that the eigenvalues are $\lambda = 2, 2, 3$. Consider

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix}.$$

Since $\operatorname{rank}(\operatorname{nullspace}(\mathbf{A} - 2\mathbf{I})) = 1$ there is one eigenvector and one generalized eigenvector of rank two for $\lambda = 2$. The generalized eigenvector of rank two satisfies

$$(\mathbf{A} - 2\mathbf{I})^2 \boldsymbol{\xi}_2 = \mathbf{0}$$
$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$$

We choose the solution

$$\boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

The eigenvector for $\lambda = 2$ is

$$\boldsymbol{\xi}_1 = (\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvector for $\lambda = 3$ satisfies

$$(\mathbf{A} - 3\mathbf{I})^2 \boldsymbol{\xi} = \mathbf{0}$$
$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}$$

We choose the solution

$$\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvalues and generalized eigenvectors are

$$\lambda = \{2, 2, 3\}, \qquad \boldsymbol{\xi} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The matrix of eigenvectors and its inverse is

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \mathbf{S}^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The Jordan canonical form of the matrix, which satisfies $\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ is

$$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Recall that the function of a Jordan block is:

$$f\left(\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \right) = \begin{pmatrix} f(\lambda) & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} & \frac{f'''(\lambda)}{3!} \\ 0 & f(\lambda) & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} \\ 0 & 0 & f(\lambda) & \frac{f'(\lambda)}{1!} \\ 0 & 0 & 0 & f(\lambda) \end{pmatrix},$$

and that the function of a matrix in Jordan canonical form is

$$f\left(\begin{pmatrix} \mathbf{J}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J}_4 \end{pmatrix} \right) = \begin{pmatrix} f(\mathbf{J}_1) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & f(\mathbf{J}_2) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & f(\mathbf{J}_3) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & f(\mathbf{J}_4) \end{pmatrix}.$$

We want to compute $\mathrm{e}^{\mathbf{J}t}$ so we consider the function $f(\lambda) = \mathrm{e}^{\lambda t}$, which has the derivative $f'(\lambda) = t\,\mathrm{e}^{\lambda t}$. Thus we see that

$$\mathrm{e}^{\mathbf{J}t} = \begin{pmatrix} \mathrm{e}^{2t} & t\,\mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & 0 & \mathrm{e}^{3t} \end{pmatrix}$$

The exponential matrix is

$$\mathrm{e}^{\mathbf{A}t} = \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\mathbf{S}^{-1},$$

$$\boxed{\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} \mathrm{e}^{2t} & -(1+t)\,\mathrm{e}^{2t} + \mathrm{e}^{3t} & -\mathrm{e}^{2t} + \mathrm{e}^{3t} \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & -\mathrm{e}^{2t} + \mathrm{e}^{3t} & \mathrm{e}^{3t} \end{pmatrix}.}$$

The general solution of the homogeneous differential equation is

$$\boxed{\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\mathbf{C}.}$$

2. The solution of the inhomogeneous differential equation subject to the initial condition is

$$\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\mathbf{0} + \mathrm{e}^{\mathbf{At}} \int_{\mathbf{0}}^{\mathbf{t}} \mathrm{e}^{-\mathbf{A}\tau}\mathbf{g}(\tau)\,\mathbf{d}\tau$$

$$\boxed{\mathbf{x} = \mathrm{e}^{\mathbf{A}t} \int_0^t \mathrm{e}^{-\mathbf{A}\tau}\mathbf{g}(\tau)\,d\tau}$$

Solution 17.10

1.

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{1}{t}\mathbf{A}\mathbf{x}$$

$$t\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The first component of this equation is

$$tx_1' = ax_1 + bx_2.$$

We differentiate and multiply by t to obtain a second order coupled equation for x_1. We use (17.3) to eliminate the dependence on x_2.

$$\begin{aligned}
t^2x_1'' + tx_1' &= atx_1' + btx_2' \\
t^2x_1'' + (1-a)tx_1' &= b(cx_1 + dx_2) \\
t^2x_1'' + (1-a)tx_1' - bcx_1 &= d(tx_1' - ax_1) \\
t^2x_1'' + (1-a-d)tx_1' + (ad-bc)x_1 &= 0
\end{aligned}$$

Thus we see that x_1 satisfies a second order, Euler equation. By symmetry we see that x_2 satisfies,

$$t^2x_2'' + (1-b-c)tx_2' + (bc-ad)x_2 = 0.$$

2. We substitute $\mathbf{x} = \mathbf{a}t^\lambda$ into (17.3).

$$\begin{aligned}
\lambda\mathbf{a}t^{\lambda-1} &= \frac{1}{t}\mathbf{A}\mathbf{a}t^\lambda \\
\mathbf{A}\mathbf{a} &= \lambda\mathbf{a}
\end{aligned}$$

Thus we see that $\mathbf{x} = \mathbf{a}t^\lambda$ is a solution if λ is an eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{a}$.

3. Suppose that $\lambda = \alpha$ is an eigenvalue of multiplicity 2. If $\lambda = \alpha$ has two linearly independent eigenvectors, $\mathbf{a}$ and $\mathbf{b}$ then $\mathbf{a}t^\alpha$ and $\mathbf{b}t^\alpha$ are linearly independent solutions. If $\lambda = \alpha$ has only one linearly independent eigenvector, $\mathbf{a}$, then $\mathbf{a}t^\alpha$ is a solution. We look for a second solution of the form

$$\mathbf{x} = \boldsymbol{\xi}t^\alpha \log t + \boldsymbol{\eta}t^\alpha.$$

Substituting this into the differential equation yields

$$\alpha\boldsymbol{\xi}t^{\alpha-1}\log t + \boldsymbol{\xi}t^{\alpha-1} + \alpha\boldsymbol{\eta}t^{\alpha-1} = \mathbf{A}\boldsymbol{\xi}t^{\alpha-1}\log t + \mathbf{A}\boldsymbol{\eta}t^{\alpha-1}$$

We equate coefficients of $t^{\alpha-1}\log t$ and $t^{\alpha-1}$ to determine $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

$$(\mathbf{A} - \alpha\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \qquad (\mathbf{A} - \alpha\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$$

These equations have solutions because $\lambda = \alpha$ has generalized eigenvectors of first and second order.

Note that the change of independent variable $\tau = \log t$, $\mathbf{y}(\tau) = \mathbf{x}(t)$, will transform (17.3) into a constant coefficient system.

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\tau} = \mathbf{A}\mathbf{y}$$

Thus all the methods for solving constant coefficient systems carry over directly to solving (17.3). In the case of eigenvalues with multiplicity greater than one, we will have solutions of the form,

$$\boldsymbol{\xi}t^\alpha, \quad \boldsymbol{\xi}t^\alpha\log t + \boldsymbol{\eta}t^\alpha, \quad \boldsymbol{\xi}t^\alpha(\log t)^2 + \boldsymbol{\eta}t^\alpha\log t + \boldsymbol{\zeta}t^\alpha, \quad \ldots,$$

analogous to the form of the solutions for a constant coefficient system,

$$\boldsymbol{\xi}\,\mathrm{e}^{\alpha\tau}, \quad \boldsymbol{\xi}\tau\,\mathrm{e}^{\alpha\tau} + \boldsymbol{\eta}\,\mathrm{e}^{\alpha\tau}, \quad \boldsymbol{\xi}\tau^2\,\mathrm{e}^{\alpha\tau} + \boldsymbol{\eta}\tau\,\mathrm{e}^{\alpha\tau} + \boldsymbol{\zeta}\,\mathrm{e}^{\alpha\tau}, \quad \ldots.$$

4. **Method 1.** Now we consider

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{1}{t}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\mathbf{x}.$$

The characteristic polynomial of the matrix is

$$\chi(\lambda) = \begin{vmatrix}1-\lambda & 0\\ 1 & 1-\lambda\end{vmatrix} = (1-\lambda)^2.$$

$\lambda = 1$ is an eigenvalue of multiplicity 2. The equation for the associated eigenvectors is

$$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

There is only one linearly independent eigenvector, which we choose to be

$$\mathbf{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One solution of the differential equation is

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t.$$

We look for a second solution of the form

$$\mathbf{x}_2 = \mathbf{a} t \log t + \boldsymbol{\eta} t.$$

$\boldsymbol{\eta}$ satisfies the equation

$$(\mathbf{A} - I)\boldsymbol{\eta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solution is determined only up to an additive multiple of $\mathbf{a}$. We choose

$$\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus a second linearly independent solution is

$$\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t \log t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t.$$

The general solution of the differential equation is

$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} t + c_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} t \log t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \right).}$$

Method 2. Note that the matrix is lower triangular.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \frac{1}{t} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{17.4}$$

We have an uncoupled equation for x_1.

$$x_1' = \frac{1}{t} x_1$$
$$x_1 = c_1 t$$

By substituting the solution for x_1 into (17.4), we obtain an uncoupled equation for x_2.

$$x_2' = \frac{1}{t}(c_1 t + x_2)$$
$$x_2' - \frac{1}{t} x_2 = c_1$$
$$\left(\frac{1}{t} x_2 \right)' = \frac{c_1}{t}$$
$$\frac{1}{t} x_2 = c_1 \log t + c_2$$
$$x_2 = c_1 t \log t + c_2 t$$

Thus the solution of the system is

$$\mathbf{x} = \begin{pmatrix} c_1 t \\ c_1 t \log t + c_2 t \end{pmatrix},$$
$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} t \\ t \log t \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ t \end{pmatrix},}$$

which is equivalent to the solution we obtained previously.

Solution 17.11
Homogeneous Solution, Method 1. We designate the inhomogeneous system of differential equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t).$$

First we find homogeneous solutions. The characteristic equation for the matrix is

$$\chi(\lambda) = \begin{vmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{vmatrix} = \lambda^2 = 0$$

$\lambda = 0$ is an eigenvalue of multiplicity 2. The eigenvectors satisfy

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus we see that there is only one linearly independent eigenvector. We choose

$$\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

One homogeneous solution is then

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mathrm{e}^{0t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We look for a second homogeneous solution of the form

$$\mathbf{x}_2 = \boldsymbol{\xi} t + \boldsymbol{\eta}.$$

We substitute this into the homogeneous equation.

$$\mathbf{x}_2' = \mathbf{A}\mathbf{x}_2$$
$$\boldsymbol{\xi} = \mathbf{A}(\boldsymbol{\xi} t + \boldsymbol{\eta})$$

We see that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy

$$\mathbf{A}\boldsymbol{\xi} = 0, \qquad \mathbf{A}\boldsymbol{\eta} = \boldsymbol{\xi}.$$

We choose $\boldsymbol{\xi}$ to be the eigenvector that we found previously. The equation for $\boldsymbol{\eta}$ is then

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$\boldsymbol{\eta}$ is determined up to an additive multiple of $\boldsymbol{\xi}$. We choose

$$\boldsymbol{\eta} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}.$$

Thus a second homogeneous solution is

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}.$$

The general homogeneous solution of the system is

$$\mathbf{x}_h = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 2t - 1/2 \end{pmatrix}$$

We can write this in matrix notation using the fundamental matrix $\boldsymbol{\Psi}(t)$.

$$\mathbf{x}_h = \boldsymbol{\Psi}(t)\mathbf{c} = \begin{pmatrix} 1 & t \\ 2 & 2t - 1/2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Homogeneous Solution, Method 2. The similarity transform $\mathbf{C}^{-1}\mathbf{A}\mathbf{C}$ with

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 2 & -1/2 \end{pmatrix}$$

will convert the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}$$

to Jordan canonical form. We make the change of variables,

$$\mathbf{y} = \begin{pmatrix} 1 & 0 \\ 2 & -1/2 \end{pmatrix} \mathbf{x}.$$

The homogeneous system becomes

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \begin{pmatrix} 1 & 0 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1/2 \end{pmatrix} \mathbf{y}$$
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The equation for y_2 is

$$y_2' = 0.$$
$$y_2 = c_2$$

The equation for y_1 becomes

$$y_1' = c_2.$$
$$y_1 = c_1 + c_2 t$$

The solution for $\mathbf{y}$ is then

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

We multiply this by $\mathbf{C}$ to obtain the homogeneous solution for $\mathbf{x}$.

$$\mathbf{x}_h = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 2t - 1/2 \end{pmatrix}$$

Inhomogeneous Solution. By the method of variation of parameters, a particular solution is

$$\mathbf{x}_p = \mathbf{\Psi}(t) \int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\, dt.$$

$$\mathbf{x}_p = \begin{pmatrix} 1 & t \\ 2 & 2t - 1/2 \end{pmatrix} \int \begin{pmatrix} 1 - 4t & 2t \\ 4 & -2 \end{pmatrix} \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix} dt$$

$$\mathbf{x}_p = \begin{pmatrix} 1 & t \\ 2 & 2t - 1/2 \end{pmatrix} \int \begin{pmatrix} -2t^{-1} - 4t^{-2} + t^{-3} \\ 2t^{-2} + 4t^{-3} \end{pmatrix} dt$$

$$\mathbf{x}_p = \begin{pmatrix} 1 & t \\ 2 & 2t - 1/2 \end{pmatrix} \begin{pmatrix} -2\log t + 4t^{-1} - \frac{1}{2}t^{-2} \\ -2t^{-1} - 2t^{-2} \end{pmatrix}$$

$$\mathbf{x}_p = \begin{pmatrix} -2 - 2\log t + 2t^{-1} - \frac{1}{2}t^{-2} \\ -4 - 4\log t + 5t^{-1} \end{pmatrix}$$

By adding 2 times our first homogeneous solution, we obtain

$$\mathbf{x}_p = \begin{pmatrix} -2\log t + 2t^{-1} - \frac{1}{2}t^{-2} \\ -4\log t + 5t^{-1} \end{pmatrix}$$

The general solution of the system of differential equations is

$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 2t - 1/2 \end{pmatrix} + \begin{pmatrix} -2\log t + 2t^{-1} - \frac{1}{2}t^{-2} \\ -4\log t + 5t^{-1} \end{pmatrix}}$$

Chapter 18

Theory of Linear Ordinary Differential Equations

A little partyin' is good for the soul.

-Matt Metz

18.1 Nature of Solutions

Result 18.1.1 Consider the n^{th} order ordinary differential equation of the form

$$L[y] = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_0(x)y = f(x). \tag{18.1}$$

If the coefficient functions $p_{n-1}(x), \ldots, p_0(x)$ and the inhomogeneity $f(x)$ are continuous on some interval $a < x < b$ then the differential equation subject to the conditions,

$$y(x_0) = v_0, \quad y'(x_0) = v_1, \quad \ldots \quad y^{(n-1)}(x_0) = v_{n-1}, \qquad a < x_0 < b,$$

has a unique solution on the interval.

Linearity of the Operator. The differential operator L is linear. To verify this,

$$\begin{aligned}
L[cy] &= \frac{\mathrm{d}^n}{\mathrm{d}x^n}(cy) + p_{n-1}(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(cy) + \cdots + p_1(x)\frac{\mathrm{d}}{\mathrm{d}x}(cy) + p_0(x)(cy) \\
&= c\frac{\mathrm{d}^n}{\mathrm{d}x^n}y + cp_{n-1}(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}y + \cdots + cp_1(x)\frac{\mathrm{d}}{\mathrm{d}x}y + cp_0(x)y \\
&= cL[y] \\
L[y_1 + y_2] &= \frac{\mathrm{d}^n}{\mathrm{d}x^n}(y_1 + y_2) + p_{n-1}(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(y_1 + y_2) + \cdots + p_1(x)\frac{\mathrm{d}}{\mathrm{d}x}(y_1 + y_2) + p_0(x)(y_1 + y_2) \\
&= \frac{\mathrm{d}^n}{\mathrm{d}x^n}(y_1) + p_{n-1}(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(y_1) + \cdots + p_1(x)\frac{\mathrm{d}}{\mathrm{d}x}(y_1) + p_0(x)(y_1) \\
&\quad + \frac{\mathrm{d}^n}{\mathrm{d}x^n}(y_2) + p_{n-1}(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(y_2) + \cdots + p_1(x)\frac{\mathrm{d}}{\mathrm{d}x}(y_2) + p_0(x)(y_2) \\
&= L[y_1] + L[y_2].
\end{aligned}$$

Homogeneous Solutions. The general homogeneous equation has the form

$$L[y] = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_0(x)y = 0.$$

From the linearity of L, we see that if y_1 and y_2 are solutions to the homogeneous equation then $c_1 y_1 + c_2 y_2$ is also a solution, ($L[c_1 y_1 + c_2 y_2] = 0$).

On any interval where the coefficient functions are continuous, the n^{th} order linear homogeneous equation has n linearly independent solutions, $y_1, y_2, \ldots, y_n$. (We will study linear independence in Section 18.3.) The general solution to the homogeneous problem is then

$$y_h = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

Particular Solutions. Any function, y_p, that satisfies the inhomogeneous equation, $L[y_p] = f(x)$, is called a particular solution or particular integral of the equation. Note that for linear differential equations the particular solution is not unique. If y_p is a particular solution then $y_p + y_h$ is also a particular solution where y_h is any homogeneous solution.

The general solution to the problem $L[y] = f(x)$ is the sum of a particular solution and a linear combination of the homogeneous solutions

$$y = y_p + c_1 y_1 + \cdots + c_n y_n.$$

Example 18.1.1 Consider the differential equation

$$y'' - y' = 1.$$

You can verify that two homogeneous solutions are e^x and 1. A particular solution is $-x$. Thus the general solution is

$$y = -x + c_1 \mathrm{e}^x + c_2.$$

Real-Valued Solutions. If the coefficient function and the inhomogeneity in Equation 18.1 are real-valued, then the general solution can be written in terms of real-valued functions. Let y be any, homogeneous solution, (perhaps complex-valued). By taking the complex conjugate of the equation $L[y] = 0$ we show that $\bar{y}$ is a homogeneous solution as well.

$$\frac{\begin{aligned} L[y] &= 0 \\ \overline{L[y]} &= 0 \end{aligned}}{\begin{aligned} y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y &= 0 \\ \bar{y}^{(n)} + p_{n-1} \bar{y}^{(n-1)} + \cdots + p_0 \bar{y} &= 0 \\ L\left[\bar{y}\right] &= 0 \end{aligned}}$$

For the same reason, if y_p is a particular solution, then $\overline{y_p}$ is a particular solution as well.

Since the real and imaginary parts of a function y are linear combinations of y and $\bar{y}$,

$$\Re(y) = \frac{y + \bar{y}}{2}, \quad \Im(y) = \frac{y - \bar{y}}{\imath 2},$$

if y is a homogeneous solution then both $\Re y$ and $\Im(y)$ are homogeneous solutions. Likewise, if y_p is a particular solution then $\Re(y_p)$ is a particular solution.

$$L\left[\Re(y_p)\right] = L\left[\frac{y_p + \overline{y_p}}{2}\right] = \frac{f}{2} + \frac{f}{2} = f$$

Thus we see that the homogeneous solution, the particular solution and the general solution of a linear differential equation with real-valued coefficients and inhomogeneity can be written in terms of real-valued functions.

Result 18.1.2 The differential equation

$$L[y] = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_0(x) y = f(x)$$

with continuous coefficients and inhomogeneity has a general solution of the form

$$y = y_p + c_1 y_1 + \cdots + c_n y_n$$

where y_p is a particular solution, $L[y_p] = f$, and the y_k are linearly independent homogeneous solutions, $L[y_k] = 0$. If the coefficient functions and inhomogeneity are real-valued, then the general solution can be written in terms of real-valued functions.

18.2 Transformation to a First Order System

Any linear differential equation can be put in the form of a system of first order differential equations. Consider

$$y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y = f(x).$$

We introduce the functions,

$$y_1 = y, \quad y_2 = y', \quad , \ldots , \quad y_n = y^{(n-1)}.$$

The differential equation is equivalent to the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ \vdots &= \vdots \\ y_n' &= f(x) - p_{n-1} y_n - \cdots - p_0 y_1. \end{aligned}$$

The first order system is more useful when numerically solving the differential equation.

Example 18.2.1 Consider the differential equation

$$y'' + x^2 y' + \cos x \; y = \sin x.$$

The corresponding system of first order equations is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= \sin x - x^2 y_2 - \cos x \; y_1. \end{aligned}$$

18.3 The Wronskian

18.3.1 Derivative of a Determinant.

Before investigating the Wronskian, we will need a preliminary result from matrix theory. Consider an $n \times n$ matrix A whose elements $a_{ij}(x)$ are functions of x. We will denote the determinant by $\Delta[A(x)]$. We then have the following theorem.

Result 18.3.1 Let $a_{ij}(x)$, the elements of the matrix A, be differentiable functions of x. Then

$$\frac{\mathrm{d}}{\mathrm{d}x} \Delta[A(x)] = \sum_{k=1}^{n} \Delta_k[A(x)]$$

where $\Delta_k[A(x)]$ is the determinant of the matrix A with the k^{th} row replaced by the derivative of the k^{th} row.

Example 18.3.1 Consider the the matrix

$$A(x) = \begin{pmatrix} x & x^2 \\ x^2 & x^4 \end{pmatrix}$$

The determinant is $x^5 - x^4$ thus the derivative of the determinant is $5x^4 - 4x^3$. To check the theorem,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Delta[A(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\begin{vmatrix} x & x^2 \\ x^2 & x^4 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 2x \\ x^2 & x^4 \end{vmatrix} + \begin{vmatrix} x & x^2 \\ 2x & 4x^3 \end{vmatrix} \\
&= x^4 - 2x^3 + 4x^4 - 2x^3 \\
&= 5x^4 - 4x^3.
\end{aligned}$$

18.3.2 The Wronskian of a Set of Functions.

A set of functions $\{y_1, y_2, \ldots, y_n\}$ is linearly dependent on an interval if there are constants $c_1, \ldots, c_n$ not all zero such that

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0 \tag{18.2}$$

identically on the interval. The set is linearly independent if all of the constants must be zero to satisfy $c_1 y_1 + \cdots c_n y_n = 0$ on the interval.

Consider a set of functions $\{y_1, y_2, \ldots, y_n\}$ that are linearly dependent on a given interval and $n-1$ times differentiable. There are a set of constants, not all zero, that satisfy equation 18.2

Differentiating equation 18.2 $n-1$ times gives the equations,

$$\begin{aligned}
c_1 y_1' + c_2 y_2' + \cdots + c_n y_n' &= 0 \\
c_1 y_1'' + c_2 y_2'' + \cdots + c_n y_n'' &= 0 \\
&\cdots \\
c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \cdots + c_n y_n^{(n-1)} &= 0.
\end{aligned}$$

We could write the problem to find the constants as

$$\begin{pmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ y_1'' & y_2'' & \dots & y_n'' \\ \vdots & \vdots & \ddots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = 0$$

From linear algebra, we know that this equation has a solution for a nonzero constant vector only if the determinant of the matrix is zero. Here we define the **Wronskian** ,$W(x)$, of a set of functions.

$$W(x) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

Thus if a set of functions is linearly dependent on an interval, then the Wronskian is identically zero on that interval. Alternatively, if the Wronskian is identically zero, then the above matrix equation has a solution for a nonzero constant vector. This implies that the the set of functions is linearly dependent.

Result 18.3.2 The Wronskian of a set of functions vanishes identically over an interval if and only if the set of functions is linearly dependent on that interval. The Wronskian of a set of linearly independent functions does not vanish except possibly at isolated points.

Example 18.3.2 Consider the set, $\{x, x^2\}$. The Wronskian is

$$\begin{aligned} W(x) &= \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} \\ &= 2x^2 - x^2 \\ &= x^2. \end{aligned}$$

Thus the functions are independent.

Example 18.3.3 Consider the set $\{\sin x, \cos x, \mathrm{e}^{ix}\}$. The Wronskian is

$$W(x) = \begin{vmatrix} \sin x & \cos x & \mathrm{e}^{ix} \\ \cos x & -\sin x & i\,\mathrm{e}^{ix} \\ -\sin x & -\cos x & -\mathrm{e}^{ix} \end{vmatrix}.$$

Since the last row is a constant multiple of the first row, the determinant is zero. The functions are dependent. We could also see this with the identity $\mathrm{e}^{ix} = \cos x + i \sin x$.

18.3.3 The Wronskian of the Solutions to a Differential Equation

Consider the n^{th} order linear homogeneous differential equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0.$$

Let $\{y_1, y_2, \ldots, y_n\}$ be any set of n linearly independent solutions. Let $Y(x)$ be the matrix such that $W(x) = \Delta[Y(x)]$. Now let's differentiate $W(x)$.

$$\begin{aligned} W'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\Delta[Y(x)] \\ &= \sum_{k=1}^{n} \Delta_k[Y(x)] \end{aligned}$$

We note that the all but the last term in this sum is zero. To see this, let's take a look at the first term.

$$\Delta_1[Y(x)] = \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

The first two rows in the matrix are identical. Since the rows are dependent, the determinant is zero.

The last term in the sum is

$$\Delta_n[Y(x)] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}.$$

In the last row of this matrix we make the substitution $y_i^{(n)} = -p_{n-1}(x) y_i^{(n-1)} - \cdots - p_0(x) y_i$. Recalling that we can add a multiple of a row to another without changing the determinant, we add $p_0(x)$ times the first row, and $p_1(x)$ times the second row, etc., to the last row. Thus we have the determinant,

$$\begin{aligned} W'(x) &= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ -p_{n-1}(x) y_1^{(n-1)} & -p_{n-1}(x) y_2^{(n-1)} & \cdots & -p_{n-1}(x) y_n^{(n-1)} \end{vmatrix} \\ &= -p_{n-1}(x) \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \\ &= -p_{n-1}(x) W(x) \end{aligned}$$

Thus the Wronskian satisfies the first order differential equation,

$$W'(x) = -p_{n-1}(x) W(x).$$

Solving this equation we get a result known as **Abel's formula**.

$$W(x) = c \exp\left(-\int p_{n-1}(x) \, \mathrm{d}x \right)$$

Thus regardless of the particular set of solutions that we choose, we can compute their Wronskian up to a constant factor.

Result 18.3.3 The Wronskian of any linearly independent set of solutions to the equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0$$

is, (up to a multiplicative constant), given by

$$W(x) = \exp\left(-\int p_{n-1}(x)\,\mathrm{d}x\right).$$

Example 18.3.4 Consider the differential equation

$$y'' - 3y' + 2y = 0.$$

The Wronskian of the two independent solutions is

$$\begin{aligned} W(x) &= c\exp\left(-\int -3\,\mathrm{d}x\right) \\ &= c\,\mathrm{e}^{3x}. \end{aligned}$$

For the choice of solutions $\{\,\mathrm{e}^x, \mathrm{e}^{2x}\}$, the Wronskian is

$$W(x) = \begin{vmatrix} \mathrm{e}^x & \mathrm{e}^{2x} \\ \mathrm{e}^x & 2\,\mathrm{e}^{2x} \end{vmatrix} = 2\,\mathrm{e}^{3x} - \mathrm{e}^{3x} = \mathrm{e}^{3x}.$$

18.4 Well-Posed Problems

Consider the initial value problem for an n^{th} order linear differential equation.

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_0(x)y = f(x)$$
$$y(x_0) = v_1, \quad y'(x_0) = v_2, \quad \ldots, \quad y^{(n-1)}(x_0) = v_n$$

Since the general solution to the differential equation is a linear combination of the n homogeneous solutions plus the particular solution

$$y = y_p + c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

the problem to find the constants c_i can be written

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \ldots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} y_p(x_0) \\ y_p'(x_0) \\ \vdots \\ y_p^{(n-1)}(x_0) \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

From linear algebra we know that this system of equations has a unique solution only if the determinant of the matrix is nonzero. Note that the determinant of the matrix is just the Wronskian evaluated at x_0. Thus if the Wronskian vanishes at x_0, the initial value problem for the differential equation either has no solutions or infinitely many solutions. Such problems are said to be ill-posed. From Abel's formula for the Wronskian

$$W(x) = \exp\left(-\int p_{n-1}(x)\,\mathrm{d}x\right),$$

we see that the only way the Wronskian can vanish is if the value of the integral goes to ∞.

Example 18.4.1 Consider the initial value problem

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0, \qquad y(0) = y'(0) = 1.$$

The Wronskian

$$W(x) = \exp\left(-\int -\frac{2}{x}\,\mathrm{d}x\right) = \exp\left(2\log x\right) = x^2$$

vanishes at $x = 0$. Thus this problem is not well-posed.

The general solution of the differential equation is

$$y = c_1 x + c_2 x^2.$$

We see that the general solution cannot satisfy the initial conditions. If instead we had the initial conditions $y(0) = 0$, $y'(0) = 1$, then there would be an infinite number of solutions.

Example 18.4.2 Consider the initial value problem

$$y'' - \frac{2}{x^2} y = 0, \qquad y(0) = y'(0) = 1.$$

The Wronskian

$$W(x) = \exp\left(-\int 0\,\mathrm{d}x\right) = 1$$

does not vanish anywhere. However, this problem is not well-posed.

The general solution,

$$y = c_1 x^{-1} + c_2 x^2,$$

cannot satisfy the initial conditions. Thus we see that a non-vanishing Wronskian does not imply that the problem is well-posed.

Result 18.4.1 Consider the initial value problem

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_0(x)y = 0$$
$$y(x_0) = v_1, \quad y'(x_0) = v_2, \quad \ldots, \quad y^{(n-1)}(x_0) = v_n.$$

If the Wronskian

$$W(x) = \exp\left(-\int p_{n-1}(x)\,dx\right)$$

vanishes at $x = x_0$ then the problem is ill-posed. The problem may be ill-posed even if the Wronskian does not vanish.

18.5 The Fundamental Set of Solutions

Consider a set of linearly independent solutions $\{u_1, u_2, \ldots, u_n\}$ to an n^{th} order linear homogeneous differential equation. This is called the **fundamental set of solutions at $\mathbf{x_0}$** if they satisfy the relations

$$\begin{array}{cccc}
u_1(x_0) = 1 & u_2(x_0) = 0 & \ldots & u_n(x_0) = 0 \\
u_1'(x_0) = 0 & u_2'(x_0) = 1 & \ldots & u_n'(x_0) = 0 \\
\vdots & \vdots & \ddots & \vdots \\
u_1^{(n-1)}(x_0) = 0 & u_2^{(n-1)}(x_0) = 0 & \ldots & u_n^{(n-1)}(x_0) = 1
\end{array}$$

Knowing the fundamental set of solutions is handy because it makes the task of solving an initial value problem trivial. Say we are given the initial conditions,

$$y(x_0) = v_1, \quad y'(x_0) = v_2, \quad \ldots, \quad y^{(n-1)}(x_0) = v_n.$$

If the u_i's are a fundamental set then the solution that satisfies these constraints is just

$$y = v_1 u_1(x) + v_2 u_2(x) + \cdots + v_n u_n(x).$$

Of course in general, a set of solutions is not the fundamental set. If the Wronskian of the solutions is nonzero and finite we can generate a fundamental set of solutions that are linear combinations of our original set. Consider the case of a second order equation Let $\{y_1, y_2\}$ be two linearly independent solutions. We will generate the fundamental set of solutions, $\{u_1, u_2\}$.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

For $\{u_1, u_2\}$ to satisfy the relations that define a fundamental set, it must satisfy the matrix equation

$$\begin{pmatrix} u_1(x_0) & u_1'(x_0) \\ u_2(x_0) & u_2'(x_0) \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} y_1(x_0) & y_1'(x_0) \\ y_2(x_0) & y_2'(x_0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} y_1(x_0) & y_1'(x_0) \\ y_2(x_0) & y_2'(x_0) \end{pmatrix}^{-1}$$

If the Wronskian is non-zero and finite, we can solve for the constants, c_{ij}, and thus find the fundamental set of solutions. To generalize this result to an equation of order n, simply replace all the 2×2 matrices and vectors of length 2 with $n \times n$ matrices and vectors of length n. I presented the case of $n = 2$ simply to save having to write out all the ellipses involved in the general case. (It also makes for easier reading.)

Example 18.5.1 Two linearly independent solutions to the differential equation $y'' + y = 0$ are $y_1 = \mathrm{e}^{ix}$ and $y_2 = \mathrm{e}^{-ix}$.

$$\begin{pmatrix} y_1(0) & y_1'(0) \\ y_2(0) & y_2'(0) \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

To find the fundamental set of solutions, $\{u_1, u_2\}$, at $x = 0$ we solve the equation

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{-1}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}$$

The fundamental set is

$$u_1 = \frac{\mathrm{e}^{ix} + \mathrm{e}^{-ix}}{2}, \qquad u_2 = \frac{\mathrm{e}^{ix} - \mathrm{e}^{-ix}}{2i}.$$

Using trigonometric identities we can rewrite these as

$$u_1 = \cos x, \qquad u_2 = \sin x.$$

Result 18.5.1 The fundamental set of solutions at $x = x_0$, $\{u_1, u_2, \ldots, u_n\}$, to an n^{th} order linear differential equation, satisfy the relations

$$\begin{array}{cccc} u_1(x_0) = 1 & u_2(x_0) = 0 & \ldots & u_n(x_0) = 0 \\ u_1'(x_0) = 0 & u_2'(x_0) = 1 & \ldots & u_n'(x_0) = 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)}(x_0) = 0 & u_2^{(n-1)}(x_0) = 0 & \ldots & u_n^{(n-1)}(x_0) = 1. \end{array}$$

If the Wronskian of the solutions is nonzero and finite at the point x_0 then you can generate the fundamental set of solutions from any linearly independent set of solutions.

18.6 Adjoint Equations

For the n^{th} order linear differential operator

$$L[y] = p_n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_0 y$$

(where the p_j are complex-valued functions) we define the adjoint of L

$$L^*[y] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\overline{p_n} y) + (-1)^{n-1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(\overline{p_{n-1}} y) + \cdots + \overline{p_0} y.$$

Here $\overline{f}$ denotes the complex conjugate of f.

Example 18.6.1

$$L[y] = x y'' + \frac{1}{x} y' + y$$

has the adjoint

$$\begin{aligned} L^*[y] &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}[xy] - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x} y\right] + y \\ &= x y'' + 2y' - \frac{1}{x} y' + \frac{1}{x^2} y + y \\ &= x y'' + \left(2 - \frac{1}{x}\right) y' + \left(1 + \frac{1}{x^2}\right) y. \end{aligned}$$

Taking the adjoint of L^* yields

$$\begin{aligned} L^{**}[y] &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}[xy] - \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(2 - \frac{1}{x}\right) y\right] + \left(1 + \frac{1}{x^2}\right) y \\ &= x y'' + 2y' - \left(2 - \frac{1}{x}\right) y' - \left(\frac{1}{x^2}\right) y + \left(1 + \frac{1}{x^2}\right) y \\ &= x y'' + \frac{1}{x} y' + y. \end{aligned}$$

Thus by taking the adjoint of L^*, we obtain the original operator.

In general, $L^{**} = L$.

Consider $L[y] = p_n y^{(n)} + \cdots + p_0 y$. If each of the p_k is k times continuously differentiable and u and v are n times continuously differentiable on some interval, then on that interval

$$\overline{v}L[u] - u\overline{L^*[v]} = \frac{\mathrm{d}}{\mathrm{d}x}B[u, v]$$

where $B[u, v]$, the **bilinear concomitant**, is the bilinear form

$$B[u, v] = \sum_{m=1}^{n} \sum_{\substack{j+k=m-1 \\ j\geq 0, k\geq 0}} (-1)^j u^{(k)} (p_m \overline{v})^{(j)}.$$

This equation is known as **Lagrange's identity**. If L is a second order operator then

$$\begin{aligned} \overline{v}L[u] - u\overline{L^*[v]} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[u p_1 \overline{v} + u' p_2 \overline{v} - u(p_2 \overline{v})' \right] \\ &= u'' p_2 \overline{v} + u' p_1 \overline{v} + u\left[- p_2 \overline{v}'' + (-2p_2' + p_1)\overline{v}' + (-p_2'' + p_1')\overline{v} \right]. \end{aligned}$$

Example 18.6.2 Verify Lagrange's identity for the second order operator, $L[y] = p_2 y'' + p_1 y' + p_0 y$.

$$\begin{aligned} \overline{v}L[u] - u\overline{L^*[v]} &= \overline{v}(p_2 u'' + p_1 u' + p_0 u) - u\overline{\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\overline{p_2} v) - \frac{\mathrm{d}}{\mathrm{d}x}(\overline{p_1} v) + \overline{p_0} v \right)} \\ &= \overline{v}(p_2 u'' + p_1 u' + p_0 u) - u\overline{(\overline{p_2} v'' + (2\overline{p_2}' - \overline{p_1})v' + (\overline{p_2}'' - \overline{p_1}' + \overline{p_0})v)} \\ &= u'' p_2 \overline{v} + u' p_1 \overline{v} + u\left[- p_2 \overline{v}'' + (-2p_2' + p_1)\overline{v}' + (-p_2'' + p_1')\overline{v} \right]. \end{aligned}$$

We will not verify Lagrange's identity for the general case.

Integrating Lagrange's identity on its interval of validity gives us **Green's formula**.

$$\int_a^b \left(\overline{v}L[u] - u\overline{L^*[v]} \right) \, \mathrm{d}x = B[u,v]\big|_{x=b} - B[u,v]\big|_{x=a}$$

Result 18.6.1 The adjoint of the operator

$$L[y] = p_n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_0 y$$

is defined

$$L^*[y] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\overline{p_n} y) + (-1)^{n-1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(\overline{p_{n-1}} y) + \cdots + \overline{p_0} y.$$

If each of the p_k is k times continuously differentiable and u and v are n times continuously differentiable, then Lagrange's identity states

$$\overline{v}L[y] - u\overline{L^*[v]} = \frac{\mathrm{d}}{\mathrm{d}x} B[u,v] = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{m=1}^{n} \sum_{\substack{j+k=m-1 \\ j \geq 0, k \geq 0}} (-1)^j u^{(k)} (p_m \overline{v})^{(j)}.$$

Integrating Lagrange's identity on it's domain of validity yields Green's formula,

$$\int_a^b \left(\overline{v}L[u] - u\overline{L^*[v]} \right) \, \mathrm{d}x = B[u,v]\big|_{x=b} - B[u,v]\big|_{x=a}.$$

18.7 Exercises

Exercise 18.1
Determine a necessary condition for a second order linear differential equation to be exact.

Determine an equation for the integrating factor for a second order linear differential equation.
Hint, Solution

Exercise 18.2
Show that

$$y'' + xy' + y = 0$$

is exact. Find the solution.
Hint, Solution

Nature of Solutions

Exercise 18.3
On what intervals do the following problems have unique solutions?

1. $xy'' + 3y = x$
2. $x(x-1)y'' + 3xy' + 4y = 2$
3. $e^x y'' + x^2 y' + y = \tan x$

Hint, Solution

Exercise 18.4
Suppose you are able to find three linearly independent particular solutions $u_1(x)$, $u_2(x)$ and $u_3(x)$ of the second order linear differential equation $L[y] = f(x)$. What is the general solution?
Hint, Solution

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Exercise 18.5
Two solutions of $y'' - y = 0$ are e^x and e^{-x}. Show that the solutions are independent. Find the fundamental set of solutions at $x = 0$.
Hint, Solution

Adjoint Equations

Exercise 18.6
Find the adjoint of the Bessel equation of order ν,

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0,$$

and the Legendre equation of order α,

$$(1 - x^2) y'' - 2 x y' + \alpha(\alpha + 1) y = 0.$$

Hint, Solution

Exercise 18.7
Find the adjoint of

$$x^2 y'' - x y' + 3 y = 0.$$

Hint, Solution

18.8 Hints

Hint 18.1

Hint 18.2

Nature of Solutions

Hint 18.3

Hint 18.4
The difference of any two of the u_i's is a homogeneous solution.

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Hint 18.5

Adjoint Equations

Hint 18.6

Hint 18.7

18.9 Solutions

Solution 18.1

The second order, linear, homogeneous differential equation is

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{18.3}$$

The second order, linear, homogeneous, exact differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[P(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)y\right] = 0. \tag{18.4}$$

$$P(x)y'' + \left(P'(x) + f(x)\right)y' + f'(x)y = 0$$

Equating the coefficients of Equations 18.3 and 18.4 yields the set of equations,

$$P'(x) + f(x) = Q(x), \quad f'(x) = R(x).$$

We differentiate the first equation and substitute in the expression for $f'(x)$ from the second equation to determine a necessary condition for exactness.

$$\boxed{P''(x) - Q'(x) + R(x) = 0}$$

We multiply Equation 18.3 by the integrating factor $\mu(x)$ to obtain,

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0. \tag{18.5}$$

The corresponding exact equation is of the form,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)P(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)y\right] = 0. \tag{18.6}$$

$$\mu(x)P(x)y'' + (\mu'(x)P(x) + \mu(x)P'(x) + f(x))\,y' + f'(x)y = 0$$

Equating the coefficients of Equations 18.5 and 18.6 yields the set of equations,

$$\mu'P + \mu P' + f = \mu Q, \quad f' = \mu R.$$

We differentiate the first equation and substitute in the expression for f' from the second equation to find a differential equation for $\mu(x)$.

$$\mu''P + \mu'P' + \mu'P' + \mu P'' + \mu R = \mu'Q + \mu Q'$$
$$\boxed{P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0}$$

Solution 18.2
We consider the differential equation,

$$y'' + xy' + y = 0.$$

Since

$$(1)'' - (x)' + 1 = 0$$

we see that this is an exact equation. We rearrange terms to form exact derivatives and then integrate.

$$(y')' + (xy)' = 0$$
$$y' + xy = c$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{x^2/2}y\right] = c\,\mathrm{e}^{x^2/2}$$
$$\boxed{y = c\,\mathrm{e}^{-x^2/2}\int \mathrm{e}^{x^2/2}\,\mathrm{d}x + d\,\mathrm{e}^{-x^2/2}}$$

Nature of Solutions

Solution 18.3
Consider the initial value problem,

$$y'' + p(x)y' + q(x)y = f(x),$$
$$y(x_0) = y_0, \quad y'(x_0) = y_1.$$

If $p(x)$, $q(x)$ and $f(x)$ are continuous on an interval $(a \ldots b)$ with $x_0 \in (a \ldots b)$, then the problem has a unique solution on that interval.

1.

$$xy'' + 3y = x$$
$$y'' + \frac{3}{x}y = 1$$

Unique solutions exist on the intervals $(-\infty \ldots 0)$ and $(0 \ldots \infty)$.

2.

$$x(x-1)y'' + 3xy' + 4y = 2$$
$$y'' + \frac{3}{x-1}y' + \frac{4}{x(x-1)}y = \frac{2}{x(x-1)}$$

Unique solutions exist on the intervals $(-\infty \ldots 0)$, $(0 \ldots 1)$ and $(1 \ldots \infty)$.

3.

$$\mathrm{e}^x y'' + x^2 y' + y = \tan x$$
$$y'' + x^2\,\mathrm{e}^{-x} y' + \mathrm{e}^{-x} y = \mathrm{e}^{-x}\tan x$$

Unique solutions exist on the intervals $\left(\frac{(2n-1)\pi}{2} \ldots \frac{(2n+1)\pi}{2}\right)$ for $n \in \mathbb{Z}$.

Solution 18.4
We know that the general solution is

$$y = y_p + c_1 y_1 + c_2 y_2,$$

where y_p is a particular solution and y_1 and y_2 are linearly independent homogeneous solutions. Since y_p can be any particular solution, we choose $y_p = u_1$. Now we need to find two homogeneous solutions. Since $L[u_i] = f(x)$, $L[u_1 - u_2] = L[u_2 - u_3] = 0$. Finally, we note that since the u_i's are linearly independent, $y_1 = u_1 - u_2$ and $y_2 = u_2 - u_3$ are linearly independent. Thus the general solution is

$$\boxed{y = u_1 + c_1(u_1 - u_2) + c_2(u_2 - u_3).}$$

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Solution 18.5
The Wronskian of the solutions is

$$W(x) = \begin{vmatrix} \mathrm{e}^x & \mathrm{e}^{-x} \\ \mathrm{e}^x & -\mathrm{e}^{-x} \end{vmatrix} = -2.$$

Since the Wronskian is nonzero, the solutions are independent.

The fundamental set of solutions, $\{u_1, u_2\}$, is a linear combination of e^x and e^{-x}.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \mathrm{e}^x \\ \mathrm{e}^{-x} \end{pmatrix}$$

The coefficients are

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{0} & \mathrm{e}^{0} \\ \mathrm{e}^{-0} & -\mathrm{e}^{-0} \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$
$$= \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$u_1 = \frac{1}{2}(\mathrm{e}^{x} + \mathrm{e}^{-x}), \qquad u_2 = \frac{1}{2}(\mathrm{e}^{x} - \mathrm{e}^{-x}).$$

The fundamental set of solutions at $x = 0$ is

$$\boxed{\{\cosh x, \sinh x\}}$$

.

Adjoint Equations

Solution 18.6

1. The Bessel equation of order ν is

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0.$$

   The adjoint equation is

$$x^2 \mu'' + (4x - x)\mu' + (2 - 1 + x^2 - \nu^2)\mu = 0$$
$$\boxed{x^2 \mu'' + 3x\mu' + (1 + x^2 - \nu^2)\mu = 0.}$$

2. The Legendre equation of order α is

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

The adjoint equation is

$$(1 - x^2)\mu'' + (-4x + 2x)\mu' + (-2 + 2 + \alpha(\alpha + 1))\mu = 0$$
$$\boxed{(1 - x^2)\mu'' - 2x\mu' + \alpha(\alpha + 1)\mu = 0}$$

Solution 18.7
The adjoint of

$$x^2y'' - xy' + 3y = 0$$

is

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2y) + \frac{\mathrm{d}}{\mathrm{d}x}(xy) + 3y = 0$$
$$(x^2y'' + 4xy' + 2y) + (xy' + y) + 3y = 0$$
$$\boxed{x^2y'' + 5xy' + 6y = 0.}$$

Chapter 19

Techniques for Linear Differential Equations

My new goal in life is to take the meaningless drivel out of human interaction.

-Dave Ozenne

The n^{th} order linear homogeneous differential equation has the form

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0.$$

In general it is not possible to solve second order and higher linear differential equations. In this chapter we will examine equations that have special forms which allow us to either reduce the order of the equation or solve it.

19.1 Constant Coefficient Equations

The n^{th} order constant coefficient differential equation has the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

We will find that solving a constant coefficient differential equation is no more difficult than finding the roots of a polynomial.

19.1.1 Second Order Equations

Factoring the Differential Equation. Consider the second order constant coefficient differential equation

$$y'' + 2ay' + by = 0. \tag{19.1}$$

Just as we can factor the polynomial,

$$\lambda^2 + 2a\lambda + b = (\lambda - \alpha)(\lambda - \beta), \tag{19.2}$$

where

$$\alpha = -a + \sqrt{a^2 - b} \quad \text{and} \quad \beta = -a - \sqrt{a^2 - b},$$

we can factor the differential equation.

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2a\frac{\mathrm{d}}{\mathrm{d}x} + b \right) y = \left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha \right) \left(\frac{\mathrm{d}}{\mathrm{d}x} - \beta \right) y$$

Once we have factored the differential equation, we can solve it by solving a series of of two first order differential equations. We set $u = \left(\frac{\mathrm{d}}{\mathrm{d}x} - \beta \right) y$ to obtain a first order equation,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha \right) u = 0,$$

which has the solution

$$u = c_1 \, \mathrm{e}^{\alpha x}.$$

To find the solution of Equation 19.1, we solve

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - \beta \right) y = u = c_1 \, \mathrm{e}^{\alpha x}.$$

We multiply by the integrating factor and integrate.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\beta x} y\right) = c_1 \,\mathrm{e}^{(\alpha-\beta)x}$$
$$y = c_1 \,\mathrm{e}^{\beta x} \int \mathrm{e}^{(\alpha-\beta)x} \,\mathrm{d}x + c_2 \,\mathrm{e}^{\beta x}$$

We first consider the case when α and β are distinct.

$$y = c_1 \,\mathrm{e}^{\beta x} \frac{1}{\alpha-\beta} \,\mathrm{e}^{(\alpha-\beta)x} + c_2 \,\mathrm{e}^{\beta x}$$

We choose new constants to write the solution in a better form.

$$y = c_1 \,\mathrm{e}^{\alpha x} + c_2 \,\mathrm{e}^{\beta x}$$

Now we consider the case $\alpha = \beta$.

$$y = c_1 \,\mathrm{e}^{\alpha x} \int 1 \,\mathrm{d}x + c_2 \,\mathrm{e}^{\alpha x}$$
$$y = c_1 x \,\mathrm{e}^{\alpha x} + c_2 \,\mathrm{e}^{\alpha x}.$$

The solution of Equation 19.1 is

$$y = \begin{cases} c_1 \,\mathrm{e}^{\alpha x} + c_2 \,\mathrm{e}^{\beta x}, & \alpha \neq \beta, \\ c_1 \,\mathrm{e}^{\alpha x} + c_2 x \,\mathrm{e}^{\alpha x}, & \alpha = \beta. \end{cases}$$

Example 19.1.1 Consider the differential equation: $y'' + y = 0$. We factor the equation.

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}\right) y = 0$$

The general solution of the differential equation is

$$y = c_1 \,\mathrm{e}^{\mathrm{i}x} + c_2 \,\mathrm{e}^{-\mathrm{i}x}.$$

Example 19.1.2 Consider the differential equation: $y'' = 0$. We factor the equation.

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - 0\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - 0\right) y = 0$$

The general solution of the differential equation is

$$y = c_1 \mathrm{e}^{0x} + c_2 x \,\mathrm{e}^{0x}$$
$$y = c_1 + c_2 x.$$

Substituting the Form of the Solution into the Differential Equation. Note that if we substitute $y = \mathrm{e}^{\lambda x}$ into the differential equation 19.1, we will obtain the quadratic polynomial equation 19.2 for λ.

$$y'' + 2ay' + by = 0$$
$$\lambda^2 \mathrm{e}^{\lambda x} + 2a\lambda \,\mathrm{e}^{\lambda x} + b \,\mathrm{e}^{\lambda x} = 0$$
$$\lambda^2 + 2a\lambda + b = 0.$$

This gives us a superficially different method for solving constant coefficient equations. We substitute $y = \mathrm{e}^{\lambda x}$ into the differential equation. Let α and β be the roots of the quadratic in λ. If the roots are distinct, then the linearly independent solutions are $y_1 = \mathrm{e}^{\alpha x}$ and $y_2 = \mathrm{e}^{\beta x}$. If the quadratic has a double root at $\lambda = \alpha$, then the linearly independent solutions are $y_1 = \mathrm{e}^{\alpha x}$ and $y_2 = x \,\mathrm{e}^{\alpha x}$.

Example 19.1.3 Consider the equation

$$y'' - 3y' + 2y = 0.$$

The substitution $y = \mathrm{e}^{\lambda x}$ yields

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0.$$

Thus the solutions are e^{x} and e^{2x}.

Example 19.1.4 Consider the equation

$$y'' - 2y' + 4y = 0.$$

The substitution $y = \mathrm{e}^{\lambda x}$ yields

$$\lambda^2 - 2\lambda + 4 = (\lambda - 2)^2 = 0.$$

Thus the solutions are e^{2x} and $x\,\mathrm{e}^{2x}$.

Shift Invariance. Note that if $u(x)$ is a solution of a constant coefficient equation, then $u(x + c)$ is also a solution. This is useful in applying initial or boundary conditions.

Example 19.1.5 Consider the problem

$$y'' - 3y' + 2y = 0, \qquad y(0) = a, \quad y'(0) = b.$$

We know that the general solution is

$$y = c_1\,\mathrm{e}^{x} + c_2\,\mathrm{e}^{2x}.$$

Applying the initial conditions, we obtain the equations,

$$c_1 + c_2 = a, \quad c_1 + 2c_2 = b.$$

The solution is

$$y = (2a - b)\,\mathrm{e}^{x} + (b - a)\,\mathrm{e}^{2x}.$$

Now suppose we wish to solve the same differential equation with the boundary conditions $y(1) = a$ and $y'(1) = b$. All we have to do is shift the solution to the right.

$$y = (2a - b)\,\mathrm{e}^{x-1} + (b - a)\,\mathrm{e}^{2(x-1)}.$$

Result 19.1.1 . Consider the second order constant coefficient equation

$$y'' + 2ay' + by = 0.$$

The general solution of this differential equation is

$$y = \begin{cases} e^{-ax}\left(c_1 e^{\sqrt{a^2-b}\,x} + c_2 e^{-\sqrt{a^2-b}\,x}\right) & \text{if } a^2 > b, \\ e^{-ax}\left(c_1 \cos(\sqrt{b-a^2}\,x) + c_2 \sin(\sqrt{b-a^2}\,x)\right) & \text{if } a^2 < b, \\ e^{-ax}(c_1 + c_2 x) & \text{if } a^2 = b. \end{cases}$$

The fundamental set of solutions at $x = 0$ is

$$\begin{cases} \left\{ e^{-ax}\left(\cosh(\sqrt{a^2-b}\,x) + \frac{a}{\sqrt{a^2-b}}\sinh(\sqrt{a^2-b}\,x)\right), e^{-ax}\frac{1}{\sqrt{a^2-b}}\sinh(\sqrt{a^2-b}\,x)\right\} & \text{if } a^2 > b, \\ \left\{ e^{-ax}\left(\cos(\sqrt{b-a^2}\,x) + \frac{a}{\sqrt{b-a^2}}\sin(\sqrt{b-a^2}\,x)\right), e^{-ax}\frac{1}{\sqrt{b-a^2}}\sin(\sqrt{b-a^2}\,x)\right\} & \text{if } a^2 < b, \\ \{(1+ax)\,e^{-ax}, x\,e^{-ax}\} & \text{if } a^2 = b. \end{cases}$$

To obtain the fundamental set of solutions at the point $x = \xi$, substitute $(x - \xi)$ for x in the above solutions.

19.1.2 Higher Order Equations

The constant coefficient equation of order n has the form

$$L[y] = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{19.3}$$

The substitution $y = e^{\lambda x}$ will transform this differential equation into an algebraic equation.

$$L[e^{\lambda x}] = \lambda^n e^{\lambda x} + a_{n-1}\lambda^{n-1} e^{\lambda x} + \cdots + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0$$
$$\left(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0\right) e^{\lambda x} = 0$$
$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

Assume that the roots of this equation, $\lambda_1, \ldots, \lambda_n$, are distinct. Then the n linearly independent solutions of Equation 19.3 are

$$e^{\lambda_1 x}, \ldots, e^{\lambda_n x}.$$

If the roots of the algebraic equation are not distinct then we will not obtain all the solutions of the differential equation. Suppose that $\lambda_1 = \alpha$ is a double root. We substitute $y = e^{\lambda x}$ into the differential equation.

$$L[e^{\lambda x}] = [(\lambda - \alpha)^2 (\lambda - \lambda_3) \cdots (\lambda - \lambda_n)] e^{\lambda x} = 0$$

Setting $\lambda = \alpha$ will make the left side of the equation zero. Thus $y = e^{\alpha x}$ is a solution. Now we differentiate both sides of the equation with respect to λ and interchange the order of differentiation.

$$\frac{d}{d\lambda} L[e^{\lambda x}] = L\left[\frac{d}{d\lambda} e^{\lambda x}\right] = L\left[x e^{\lambda x}\right]$$

Let $p(\lambda) = (\lambda - \lambda_3) \cdots (\lambda - \lambda_n)$. We calculate $L\left[x e^{\lambda x}\right]$ by applying L and then differentiating with respect to λ.

$$\begin{aligned}
L\left[x e^{\lambda x}\right] &= \frac{d}{d\lambda} L[e^{\lambda x}] \\
&= \frac{d}{d\lambda}[(\lambda - \alpha)^2 (\lambda - \lambda_3) \cdots (\lambda - \lambda_n)] e^{\lambda x} \\
&= \frac{d}{d\lambda}[(\lambda - \alpha)^2 p(\lambda)] e^{\lambda x} \\
&= \left[2(\lambda - \alpha) p(\lambda) + (\lambda - \alpha)^2 p'(\lambda) + (\lambda - \alpha)^2 p(\lambda) x\right] e^{\lambda x} \\
&= (\lambda - \alpha) \left[2p(\lambda) + (\lambda - \alpha) p'(\lambda) + (\lambda - \alpha) p(\lambda) x\right] e^{\lambda x}
\end{aligned}$$

Since setting $\lambda = \alpha$ will make this expression zero, $L[x\,\mathrm{e}^{\alpha x}] = 0$, $x\,\mathrm{e}^{\alpha x}$ is a solution of Equation 19.3. You can verify that $\mathrm{e}^{\alpha x}$ and $x\,\mathrm{e}^{\alpha x}$ are linearly independent. Now we have generated all of the solutions for the differential equation.

If $\lambda = \alpha$ is a root of multiplicity m then by repeatedly differentiating with respect to λ you can show that the corresponding solutions are

$$\mathrm{e}^{\alpha x}, x\,\mathrm{e}^{\alpha x}, x^2\,\mathrm{e}^{\alpha x}, \ldots, x^{m-1}\,\mathrm{e}^{\alpha x}.$$

Example 19.1.6 Consider the equation

$$y''' - 3y' + 2y = 0.$$

The substitution $y = \mathrm{e}^{\lambda x}$ yields

$$\lambda^3 - 3\lambda + 2 = (\lambda - 1)^2(\lambda + 2) = 0.$$

Thus the general solution is

$$\boxed{y = c_1\,\mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x} + c_3\,\mathrm{e}^{-2x}.}$$

19.1.3 Real-Valued Solutions

If the coefficients of the differential equation are real, then the solution can be written in terms of real-valued functions (Result 18.1.2). For a real root $\lambda = \alpha$ of the polynomial in λ, the corresponding solution, $y = \mathrm{e}^{\alpha x}$, is real-valued.

Now recall that the complex roots of a polynomial with real coefficients occur in complex conjugate pairs. Assume that $\alpha \pm \mathrm{i}\beta$ are roots of

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0.$$

The corresponding solutions of the differential equation are $\mathrm{e}^{(\alpha+\mathrm{i}\beta)x}$ and $\mathrm{e}^{(\alpha-\mathrm{i}\beta)x}$. Note that the linear combinations

$$\frac{\mathrm{e}^{(\alpha+\mathrm{i}\beta)x} + \mathrm{e}^{(\alpha-\mathrm{i}\beta)x}}{2} = \mathrm{e}^{\alpha x}\cos(\beta x), \qquad \frac{\mathrm{e}^{(\alpha+\mathrm{i}\beta)x} - \mathrm{e}^{(\alpha-\mathrm{i}\beta)x}}{\mathrm{i}2} = \mathrm{e}^{\alpha x}\sin(\beta x),$$

are real-valued solutions of the differential equation. We could also obtain real-valued solution by taking the real and imaginary parts of either $\mathrm{e}^{(\alpha+\mathrm{i}\beta)x}$ or $\mathrm{e}^{(\alpha-\mathrm{i}\beta)x}$.

$$\Re\left(\mathrm{e}^{(\alpha+\mathrm{i}\beta)x}\right) = \mathrm{e}^{\alpha x}\cos(\beta x), \qquad \Im\left(\mathrm{e}^{(\alpha+\mathrm{i}\beta)x}\right) = \mathrm{e}^{\alpha x}\sin(\beta x)$$

Result 19.1.2 Consider the n^{th} order constant coefficient equation

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0.$$

Let the factorization of the algebraic equation obtained with the substitution $y = \mathrm{e}^{\lambda x}$ be

$$(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_p)^{m_p} = 0.$$

A set of linearly independent solutions is given by

$$\{\mathrm{e}^{\lambda_1 x}, x\,\mathrm{e}^{\lambda_1 x}, \ldots, x^{m_1-1}\,\mathrm{e}^{\lambda_1 x}, \ldots, \mathrm{e}^{\lambda_p x}, x\,\mathrm{e}^{\lambda_p x}, \ldots, x^{m_p-1}\,\mathrm{e}^{\lambda_p x}\}.$$

If the coefficients of the differential equation are real, then we can find a real-valued set of solutions.

Example 19.1.7 Consider the equation

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + 2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0.$$

The substitution $y = \mathrm{e}^{\lambda x}$ yields

$$\lambda^4 + 2\lambda^2 + 1 = (\lambda - i)^2(\lambda + i)^2 = 0.$$

Thus the linearly independent solutions are

$$\mathrm{e}^{\mathrm{i}x}, x\,\mathrm{e}^{\mathrm{i}x}, \ \mathrm{e}^{-\mathrm{i}x} \text{ and } x\,\mathrm{e}^{-\mathrm{i}x}.$$

Noting that

$$\mathrm{e}^{ix} = \cos(x) + \mathrm{i}\sin(x),$$

we can write the general solution in terms of sines and cosines.

$$\boxed{y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x}$$

Example 19.1.8 Consider the equation

$$y'' - 2y' + 2y = 0.$$

The substitution $y = \mathrm{e}^{\lambda x}$ yields

$$\lambda^2 - 2\lambda + 2 = (\lambda - 1 - \mathrm{i})(\lambda - 1 + \mathrm{i}) = 0.$$

The linearly independent solutions are

$$\mathrm{e}^{(1+\mathrm{i})x}, \quad \text{and} \quad \mathrm{e}^{(1-\mathrm{i})x}.$$

We can write the general solution in terms of real functions.

$$\boxed{y = c_1\,\mathrm{e}^{x} \cos x + c_2\,\mathrm{e}^{x} \sin x}$$

19.2 Euler Equations

Consider the equation

$$L[y] = x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + ax \frac{\mathrm{d}y}{\mathrm{d}x} + by = 0, \quad x > 0.$$

Let's say, for example, that y has units of distance and x has units of time. Note that each term in the differential equation has the same dimension.

$$(\text{time})^2 \frac{(\text{distance})}{(\text{time})^2} = (\text{time}) \frac{(\text{distance})}{(\text{time})} = (\text{distance})$$

Thus this is a second order Euler, or equidimensional equation. We know that the first order Euler equation, $xy' + ay = 0$, has the solution $y = cx^a$. Thus for the second order equation we will try a solution of the form $y = x^\lambda$. The substitution $y = x^\lambda$ will transform the differential equation into an algebraic equation.

$$\begin{gathered} L[x^\lambda] = x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}[x^\lambda] + ax \frac{\mathrm{d}}{\mathrm{d}x}[x^\lambda] + bx^\lambda = 0 \\ \lambda(\lambda - 1)x^\lambda + a\lambda x^\lambda + bx^\lambda = 0 \\ \lambda(\lambda - 1) + a\lambda + b = 0 \end{gathered}$$

Factoring yields

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0.$$

If the two roots, λ_1 and λ_2, are distinct then the general solution is

$$y = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}.$$

If the roots are not distinct, $\lambda_1 = \lambda_2 = \lambda$, then we only have the one solution, $y = x^\lambda$. To generate the other solution we use the same approach as for the constant coefficient equation. We substitute $y = x^\lambda$ into the differential equation and differentiate with respect to λ.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda} L[x^\lambda] &= L[\frac{\mathrm{d}}{\mathrm{d}\lambda} x^\lambda] \\ &= L[\ln x \; x^\lambda] \end{aligned}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} x^\lambda = \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathrm{e}^{\lambda \ln x} = \ln x \ \mathrm{e}^{\lambda \ln x} = \ln x \ x^\lambda.$$

Now we apply L and then differentiate with respect to λ.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda} L[x^\lambda] &= \frac{\mathrm{d}}{\mathrm{d}\lambda} (\lambda - \alpha)^2 x^\lambda \\ &= 2(\lambda - \alpha) x^\lambda + (\lambda - \alpha)^2 \ln x \ x^\lambda \end{aligned}$$

Equating these two results,

$$L[\ln x \ x^\lambda] = 2(\lambda - \alpha) x^\lambda + (\lambda - \alpha)^2 \ln x \ x^\lambda.$$

Setting $\lambda = \alpha$ will make the right hand side zero. Thus $y = \ln x \ x^\alpha$ is a solution.

If you are in the mood for a little algebra you can show by repeatedly differentiating with respect to λ that if $\lambda = \alpha$ is a root of multiplicity m in an n^{th} order Euler equation then the associated solutions are

$$x^\alpha, \ln x \ x^\alpha, (\ln x)^2 x^\alpha, \ldots, (\ln x)^{m-1} x^\alpha.$$

Example 19.2.1 Consider the Euler equation

$$x y'' - y' + \frac{y}{x} = 0.$$

The substitution $y = x^\lambda$ yields the algebraic equation

$$\lambda(\lambda - 1) - \lambda + 1 = (\lambda - 1)^2 = 0.$$

Thus the general solution is

$$\boxed{y = c_1 x + c_2 x \ln x.}$$

19.2.1 Real-Valued Solutions

If the coefficients of the Euler equation are real, then the solution can be written in terms of functions that are real-valued when x is real and positive, (Result 18.1.2). If $\alpha \pm i\beta$ are the roots of

$$\lambda(\lambda - 1) + a\lambda + b = 0$$

then the corresponding solutions of the Euler equation are

$$x^{\alpha + i\beta} \quad \text{and} \quad x^{\alpha - i\beta}.$$

We can rewrite these as

$$x^{\alpha} e^{i\beta \ln x} \quad \text{and} \quad x^{\alpha} e^{-i\beta \ln x}.$$

Note that the linear combinations

$$\frac{x^{\alpha} e^{i\beta \ln x} + x^{\alpha} e^{-i\beta \ln x}}{2} = x^{\alpha} \cos(\beta \ln x), \quad \text{and} \quad \frac{x^{\alpha} e^{i\beta \ln x} - x^{\alpha} e^{-i\beta \ln x}}{2i} = x^{\alpha} \sin(\beta \ln x),$$

are real-valued solutions when x is real and positive. Equivalently, we could take the real and imaginary parts of either $x^{\alpha + i\beta}$ or $x^{\alpha - i\beta}$.

$$\Re\left(x^{\alpha} e^{i\beta \ln x}\right) = x^{\alpha} \cos(\beta \ln x), \qquad \Im\left(x^{\alpha} e^{i\beta \ln x}\right) = x^{\alpha} \sin(\beta \ln x)$$

Result 19.2.1 Consider the second order Euler equation

$$x^2y'' + (2a+1)xy' + by = 0.$$

The general solution of this differential equation is

$$y = \begin{cases} x^{-a}\left(c_1 x^{\sqrt{a^2-b}} + c_2 x^{-\sqrt{a^2-b}}\right) & \text{if } a^2 > b, \\ x^{-a}\left(c_1 \cos\left(\sqrt{b-a^2}\,\ln x\right) + c_2 \sin\left(\sqrt{b-a^2}\,\ln x\right)\right) & \text{if } a^2 < b, \\ x^{-a}\left(c_1 + c_2 \ln x\right) & \text{if } a^2 = b. \end{cases}$$

The fundamental set of solutions at $x = \xi$ is

$$y = \begin{cases} \left\{ \left(\frac{x}{\xi}\right)^{-a}\left(\cosh\left(\sqrt{a^2-b}\,\ln\frac{x}{\xi}\right) + \frac{a}{\sqrt{a^2-b}}\sinh\left(\sqrt{a^2-b}\,\ln\frac{x}{\xi}\right)\right), \right. & \\ \qquad \left. \left(\frac{x}{\xi}\right)^{-a}\frac{\xi}{\sqrt{a^2-b}}\sinh\left(\sqrt{a^2-b}\,\ln\frac{x}{\xi}\right)\right\} & \text{if } a^2 > b, \\ \left\{ \left(\frac{x}{\xi}\right)^{-a}\left(\cos\left(\sqrt{b-a^2}\,\ln\frac{x}{\xi}\right) + \frac{a}{\sqrt{b-a^2}}\sin\left(\sqrt{b-a^2}\,\ln\frac{x}{\xi}\right)\right), \right. & \\ \qquad \left. \left(\frac{x}{\xi}\right)^{-a}\frac{\xi}{\sqrt{b-a^2}}\sin\left(\sqrt{b-a^2}\,\ln\frac{x}{\xi}\right)\right\} & \text{if } a^2 < b, \\ \left\{ \left(\frac{x}{\xi}\right)^{-a}\left(1 + a\ln\frac{x}{\xi}\right), \left(\frac{x}{\xi}\right)^{-a}\xi\ln\frac{x}{\xi}\right\} & \text{if } a^2 = b. \end{cases}$$

Example 19.2.2 Consider the Euler equation

$$x^2y'' - 3xy' + 13y = 0.$$

The substitution $y = x^\lambda$ yields

$$\lambda(\lambda-1) - 3\lambda + 13 = (\lambda - 2 - i3)(\lambda - 2 + i3) = 0.$$

The linearly independent solutions are

$$\left\{x^{2+i3}, x^{2-i3}\right\}.$$

We can put this in a more understandable form.

$$x^{2+i3} = x^2 e^{i3 \ln x} = x^2 \cos(3 \ln x) + x^2 \sin(3 \ln x)$$

We can write the general solution in terms of real-valued functions.

$$\boxed{y = c_1 x^2 \cos(3 \ln x) + c_2 x^2 \sin(3 \ln x)}$$

Result 19.2.2 Consider the n^{th} order Euler equation

$$x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0.$$

Let the factorization of the algebraic equation obtained with the substitution $y = x^\lambda$ be

$$(\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p} = 0.$$

A set of linearly independent solutions is given by

$$\{x^{\lambda_1}, \ln x\ x^{\lambda_1}, \ldots, (\ln x)^{m_1 - 1} x^{\lambda_1}, \ldots, x^{\lambda_p}, \ln x\ x^{\lambda_p}, \ldots, (\ln x)^{m_p - 1} x^{\lambda_p}\}.$$

If the coefficients of the differential equation are real, then we can find a set of solutions that are real valued when x is real and positive.

19.3 Exact Equations

Exact equations have the form

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x, y, y', y'', \ldots) = f(x).$$

If you can write an equation in the form of an exact equation, you can integrate to reduce the order by one, (or solve the equation for first order). We will consider a few examples to illustrate the method.

Example 19.3.1 Consider the equation

$$y'' + x^2 y' + 2xy = 0.$$

We can rewrite this as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y' + x^2 y\right] = 0.$$

Integrating yields a first order inhomogeneous equation.

$$y' + x^2 y = c_1$$

We multiply by the integrating factor $I(x) = \exp(\int x^2 \, \mathrm{d}x)$ to make this an exact equation.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^3/3} y\right) = c_1 \, \mathrm{e}^{x^3/3}$$

$$\mathrm{e}^{x^3/3} y = c_1 \int \mathrm{e}^{x^3/3} \, \mathrm{d}x + c_2$$

$$\boxed{y = c_1 \, \mathrm{e}^{-x^3/3} \int \mathrm{e}^{x^3/3} \, \mathrm{d}x + c_2 \, \mathrm{e}^{-x^3/3}}$$

Result 19.3.1 If you can write a differential equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x, y, y', y'', \ldots) = f(x),$$

then you can integrate to reduce the order of the equation.

$$F(x, y, y', y'', \ldots) = \int f(x)\,\mathrm{d}x + c$$

19.4 Equations Without Explicit Dependence on y

Example 19.4.1 Consider the equation

$$y'' + \sqrt{x}y' = 0.$$

This is a second order equation for y, but note that it is a first order equation for y'. We can solve directly for y'.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left(\frac{2}{3}x^{3/2}\right)y'\right) = 0$$

$$y' = c_1 \exp\left(-\frac{2}{3}x^{3/2}\right)$$

Now we just integrate to get the solution for y.

$$\boxed{y = c_1 \int \exp\left(-\frac{2}{3}x^{3/2}\right)\mathrm{d}x + c_2}$$

Result 19.4.1 If an n^{th} order equation does not explicitly depend on y then you can consider it as an equation of order $n - 1$ for y'.

19.5 Reduction of Order

Consider the second order linear equation

$$L[y] \equiv y'' + p(x)y' + q(x)y = f(x).$$

Suppose that we know one homogeneous solution y_1. We make the substitution $y = uy_1$ and use that $L[y_1] = 0$.

$$\begin{gathered}
L[uy_1] = 0u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0 \\
u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0 \\
u''y_1 + u'(2y_1' + py_1) = 0
\end{gathered}$$

Thus we have reduced the problem to a first order equation for u'. An analogous result holds for higher order equations.

Result 19.5.1 Consider the n^{th} order linear differential equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x).$$

Let y_1 be a solution of the homogeneous equation. The substitution $y = uy_1$ will transform the problem into an $(n-1)^{th}$ order equation for u'. For the second order problem

$$y'' + p(x)y' + q(x)y = f(x)$$

this reduced equation is

$$u''y_1 + u'(2y_1' + py_1) = f(x).$$

Example 19.5.1 Consider the equation

$$y'' + xy' - y = 0.$$

By inspection we see that $y_1 = x$ is a solution. We would like to find another linearly independent solution. The substitution $y = xu$ yields

$$xu'' + (2 + x^2)u' = 0$$
$$u'' + \left(\frac{2}{x} + x\right) u' = 0$$

The integrating factor is $I(x) = \exp(2 \ln x + x^2/2) = x^2 \exp(x^2/2)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\,\mathrm{e}^{x^2/2} u'\right) = 0$$
$$u' = c_1 x^{-2}\,\mathrm{e}^{-x^2/2}$$
$$u = c_1 \int x^{-2}\,\mathrm{e}^{-x^2/2}\,\mathrm{d}x + c_2$$
$$y = c_1 x \int x^{-2}\,\mathrm{e}^{-x^2/2}\,\mathrm{d}x + c_2 x$$

Thus we see that a second solution is

$$\boxed{y_2 = x \int x^{-2}\,\mathrm{e}^{-x^2/2}\,\mathrm{d}x.}$$

19.6 *Reduction of Order and the Adjoint Equation

Let L be the linear differential operator

$$L[y] = p_n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_0 y,$$

where each p_j is a j times continuously differentiable complex valued function. Recall that the adjoint of L is

$$L^*[y] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\overline{p_n} y) + (-1)^{n-1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(\overline{p_{n-1}} y) + \cdots + \overline{p_0} y.$$

If u and v are n times continuously differentiable, then Lagrange's identity states

$$\overline{v}L[u] - u\overline{L^*[v]} = \frac{\mathrm{d}}{\mathrm{d}x}B[u, v],$$

where

$$B[u, v] = \sum_{m=1}^{n} \sum_{\substack{j+k=m-1 \\ j\geq 0, k\geq 0}} (-1)^j u^{(k)} (p_m \overline{v})^{(j)}.$$

For second order equations,

$$B[u, v] = u p_1 \overline{v} + u' p_2 \overline{v} - u(p_2 \overline{v})'.$$

(See Section 18.6.)

If we can find a solution to the homogeneous adjoint equation, $L^*[y] = 0$, then we can reduce the order of the equation $L[y] = f(x)$. Let ψ satisfy $L^*[\psi] = 0$. Substituting $u = y$, $v = \psi$ into Lagrange's identity yields

$$\overline{\psi}L[y] - y\overline{L^*[\psi]} = \frac{\mathrm{d}}{\mathrm{d}x}B[y, \psi]$$

$$\overline{\psi}L[y] = \frac{\mathrm{d}}{\mathrm{d}x}B[y, \psi].$$

The equation $L[y] = f(x)$ is equivalent to the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}B[y, \psi] = \overline{\psi} f$$

$$B[y, \psi] = \int \overline{\psi(x)} f(x) \, \mathrm{d}x,$$

which is a linear equation in y of order $n - 1$.

Example 19.6.1 Consider the equation

$$L[y] = y'' - x^2 y' - 2xy = 0.$$

Method 1. Note that this is an exact equation.

$$\frac{\mathrm{d}}{\mathrm{d}x}(y' - x^2 y) = 0$$
$$y' - x^2 y = c_1$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x^3/3} y\right) = c_1\,\mathrm{e}^{-x^3/3}$$
$$\boxed{y = c_1\,\mathrm{e}^{x^3/3}\int \mathrm{e}^{-x^3/3}\,\mathrm{d}x + c_2\,\mathrm{e}^{x^3/3}}$$

Method 2. The adjoint equation is

$$L^*[y] = y'' + x^2 y' = 0.$$

By inspection we see that $\psi = \text{(constant)}$ is a solution of the adjoint equation. To simplify the algebra we will choose $\psi = 1$. Thus the equation $L[y] = 0$ is equivalent to

$$B[y, 1] = c_1$$
$$y(-x^2) + \frac{\mathrm{d}}{\mathrm{d}x}[y](1) - y\frac{\mathrm{d}}{\mathrm{d}x}[1] = c_1$$
$$y' - x^2 y = c_1.$$

By using the adjoint equation to reduce the order we obtain the same solution as with Method 1.

19.7 Exercises

Constant Coefficient Equations

Exercise 19.1 (mathematica/ode/techniques_linear/constant.nb)
Find the solution of each one of the following initial value problems. Sketch the graph of the solution and describe its behavior as t increases.

1. $6y'' - 5y' + y = 0$, $y(0) = 4$, $y'(0) = 0$
2. $y'' - 2y' + 5y = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = 2$
3. $y'' + 4y' + 4y = 0$, $y(-1) = 2$, $y'(-1) = 1$

Hint, Solution

Exercise 19.2 (mathematica/ode/techniques_linear/constant.nb)
Substitute $y = \mathrm{e}^{\lambda x}$ to find two linearly independent solutions to

$$y'' - 4y' + 13y = 0.$$

that are real-valued when x is real-valued.
Hint, Solution

Exercise 19.3 (mathematica/ode/techniques_linear/constant.nb)
Find the general solution to

$$y''' - y'' + y' - y = 0.$$

Write the solution in terms of functions that are real-valued when x is real-valued.
Hint, Solution

Exercise 19.4
Substitute $y = \mathrm{e}^{\lambda x}$ to find the fundamental set of solutions at $x = 0$ for the equations:

1. $y'' + y = 0$,
2. $y'' - y = 0$,
3. $y'' = 0$.

What are the fundamental sets of solutions at $x = 1$ for these equations.
Hint, Solution

Exercise 19.5
Find the general solution of

$$y'' + 2ay' + by = 0$$

for $a, b \in \mathbb{R}$. There are three distinct forms of the solution depending on the sign of $a^2 - b$.
Hint, Solution

Exercise 19.6
Find the fundamental set of solutions of

$$y'' + 2ay' + by = 0$$

at the point $x = 0$, for $a, b \in \mathbb{R}$. Use the general solutions obtained in Exercise 19.5.
Hint, Solution

Exercise 19.7
Consider a ball of mass m hanging by an ideal spring of spring constant k. The ball is suspended in a fluid which damps the motion. This resistance has a coefficient of friction, μ. Find the differential equation for the

displacement of the mass from its equilibrium position by balancing forces. Denote this displacement by $y(t)$. If the damping force is weak, the mass will have a decaying, oscillatory motion. If the damping force is strong, the mass will not oscillate. The displacement will decay to zero. The value of the damping which separates these two behaviors is called critical damping.

Find the solution which satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$. Use the solutions obtained in Exercise 19.6 or refer to Result 19.1.1.

Consider the case $m = k = 1$. Find the coefficient of friction for which the displacement of the mass decays most rapidly. Plot the displacement for strong, weak and critical damping.
Hint, Solution

Exercise 19.8
Show that $y = c\cos(x - \phi)$ is the general solution of $y'' + y = 0$ where c and ϕ are constants of integration. (It is not sufficient to show that $y = c\cos(x - \phi)$ satisfies the differential equation. $y = 0$ satisfies the differential equation, but is is certainly not the general solution.) Find constants c and ϕ such that $y = \sin(x)$.

Is $y = c\cosh(x - \phi)$ the general solution of $y'' - y = 0$? Are there constants c and ϕ such that $y = \sinh(x)$?
Hint, Solution

Exercise 19.9 (mathematica/ode/techniques_linear/constant.nb)
Let $y(t)$ be the solution of the initial-value problem

$$y'' + 5y' + 6y = 0; \qquad y(0) = 1, \quad y'(0) = V.$$

For what values of V does $y(t)$ remain nonnegative for all $t > 0$?
Hint, Solution

Exercise 19.10 (mathematica/ode/techniques_linear/constant.nb)
Find two linearly independent solutions of

$$y'' + \operatorname{sign}(x) y = 0, \quad -\infty < x < \infty.$$

where $\operatorname{sign}(x) = \pm 1$ according as x is positive or negative. (The solution should be continuous and have a continuous first derivative.)
Hint, Solution

Euler Equations

Exercise 19.11
Find the general solution of

$$x^2 y'' + x y' + y = 0, \qquad x > 0.$$

Hint, Solution

Exercise 19.12
Substitute $y = x^\lambda$ to find the general solution of

$$x^2 y'' - 2xy + 2y = 0.$$

Hint, Solution

Exercise 19.13 (mathematica/ode/techniques_linear/constant.nb)
Substitute $y = x^\lambda$ to find the general solution of

$$x y''' + y'' + \frac{1}{x} y' = 0.$$

Write the solution in terms of functions that are real-valued when x is real-valued and positive.
Hint, Solution

Exercise 19.14
Find the general solution of

$$x^2 y'' + (2a + 1)xy' + by = 0.$$

Hint, Solution

Exercise 19.15
Show that

$$y_1 = \mathrm{e}^{ax}, \quad y_2 = \lim_{\alpha \to a} \frac{\mathrm{e}^{\alpha x} - \mathrm{e}^{-\alpha x}}{\alpha}$$

are linearly indepedent solutions of

$$y'' - a^2 y = 0$$

for all values of a. It is common to abuse notation and write the second solution as

$$y_2 = \frac{\mathrm{e}^{ax} - \mathrm{e}^{-ax}}{a}$$

where the limit is taken if $a = 0$. Likewise show that

$$y_1 = x^a, \quad y_2 = \frac{x^a - x^{-a}}{a}$$

are linearly indepedent solutions of

$$x^2 y'' + xy' - a^2 y = 0$$

for all values of a.
Hint, Solution

Exercise 19.16 (mathematica/ode/techniques_linear/constant.nb)
Find two linearly independent solutions (i.e., the general solution) of

$$\text{(a) } x^2y'' - 2xy' + 2y = 0, \quad \text{(b) } x^2y'' - 2y = 0, \quad \text{(c) } x^2y'' - xy' + y = 0.$$

Hint, Solution

Exact Equations

Exercise 19.17
Solve the differential equation

$$y'' + y' \sin x + y \cos x = 0.$$

Hint, Solution

Equations Without Explicit Dependence on y
Reduction of Order

Exercise 19.18
Consider

$$(1 - x^2)y'' - 2xy' + 2y = 0, \quad -1 < x < 1.$$

Verify that $y = x$ is a solution. Find the general solution.
Hint, Solution

Exercise 19.19
Consider the differential equation

$$y'' - \frac{x+1}{x} y' + \frac{1}{x} y = 0.$$

Since the coefficients sum to zero, $(1 - \frac{x+1}{x} + \frac{1}{x} = 0)$, $y = \mathrm{e}^x$ is a solution. Find another linearly independent solution.
Hint, Solution

Exercise 19.20
One solution of

$$(1 - 2x)y'' + 4xy' - 4y = 0$$

is $y = x$. Find the general solution.
Hint, Solution

Exercise 19.21
Find the general solution of

$$(x - 1)y'' - xy' + y = 0,$$

given that one solution is $y = \mathrm{e}^x$. (you may assume $x > 1$)
Hint, Solution

*Reduction of Order and the Adjoint Equation

19.8 Hints

Constant Coefficient Equations

Hint 19.1

Hint 19.2

Hint 19.3
It is a constant coefficient equation.

Hint 19.4
Use the fact that if $u(x)$ is a solution of a constant coefficient equation, then $u(x + c)$ is also a solution.

Hint 19.5
Substitute $y = e^{\lambda x}$ into the differential equation.

Hint 19.6
The fundamental set of solutions is a linear combination of the homogeneous solutions.

Hint 19.7
The force on the mass due to the spring is $-ky(t)$. The frictional force is $-\mu y'(t)$.

Note that the initial conditions describe the second fundamental solution at $t = 0$.

Note that for large t, $t\,e^{\alpha t}$ is much small than $e^{\beta t}$ if $\alpha < \beta$. (Prove this.)

Hint 19.8
By definition, the general solution of a second order differential equation is a two parameter family of functions that satisfies the differential equation. The trigonometric identities in Appendix Q may be useful.

Hint 19.9

Hint 19.10

Euler Equations

Hint 19.11

Hint 19.12

Hint 19.13

Hint 19.14
Substitute $y = x^\lambda$ into the differential equation. Consider the three cases: $a^2 > b$, $a^2 < b$ and $a^2 = b$.

Hint 19.15

Hint 19.16

Exact Equations

Hint 19.17
It is an exact equation.

Equations Without Explicit Dependence on y
Reduction of Order

Hint 19.18

Hint 19.19
Use reduction of order to find the other solution.

Hint 19.20
Use reduction of order to find the other solution.

Hint 19.21

*Reduction of Order and the Adjoint Equation

19.9 Solutions

Constant Coefficient Equations

Solution 19.1

1. We consider the problem

$$6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0.$$

We make the substitution $y = \mathrm{e}^{\lambda x}$ in the differential equation.

$$6\lambda^2 - 5\lambda + 1 = 0$$
$$(2\lambda - 1)(3\lambda - 1) = 0$$
$$\lambda = \left\{ \frac{1}{3}, \frac{1}{2} \right\}$$

The general solution of the differential equation is

$$y = c_1 \mathrm{e}^{t/3} + c_2 \mathrm{e}^{t/2}.$$

We apply the initial conditions to determine the constants.

$$c_1 + c_2 = 4, \quad \frac{c_1}{3} + \frac{c_2}{2} = 0$$
$$c_1 = 12, \quad c_2 = -8$$

The solution subject to the initial conditions is

$$\boxed{y = 12 \mathrm{e}^{t/3} - 8 \mathrm{e}^{t/2}.}$$

The solution is plotted in Figure 19.1. The solution tends to $-\infty$ as $t \to \infty$.

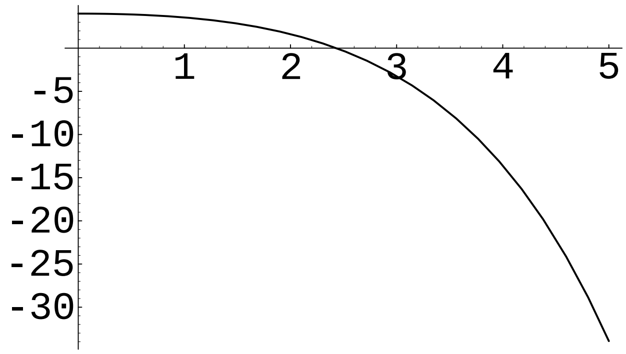

Figure 19.1: The solution of $6y'' - 5y' + y = 0$, $y(0) = 4$, $y'(0) = 0$.

2. We consider the problem

$$y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2.$$

We make the substitution $y = \mathrm{e}^{\lambda x}$ in the differential equation.

$$\lambda^2 - 2\lambda + 5 = 0$$
$$\lambda = 1 \pm \sqrt{1 - 5}$$
$$\lambda = \{1 + i2, 1 - i2\}$$

The general solution of the differential equation is

$$y = c_1 \mathrm{e}^t \cos(2t) + c_2 \mathrm{e}^t \sin(2t).$$

We apply the initial conditions to determine the constants.

$$y(\pi/2) = 0 \quad \Rightarrow \quad -c_1 \mathrm{e}^{\pi/2} = 0 \quad \Rightarrow \quad c_1 = 0$$
$$y'(\pi/2) = 2 \quad \Rightarrow \quad -2c_2 \mathrm{e}^{\pi/2} = 2 \quad \Rightarrow \quad c_2 = -\mathrm{e}^{-\pi/2}$$

The solution subject to the initial conditions is

$$\boxed{y = -\,e^{t-\pi/2}\sin(2t).}$$

The solution is plotted in Figure 19.2. The solution oscillates with an amplitude that tends to ∞ as $t \to \infty$.

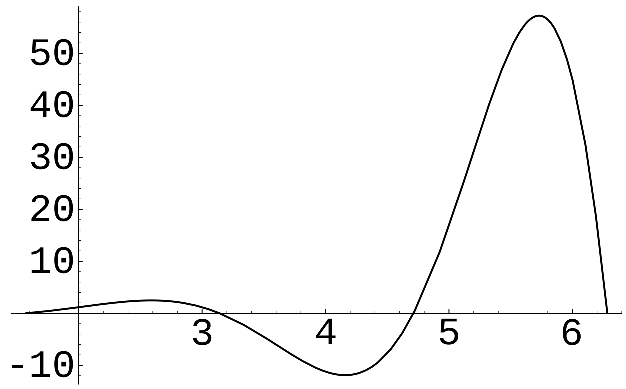

Figure 19.2: The solution of $y'' - 2y' + 5y = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = 2$.

3. We consider the problem

$$y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1.$$

We make the substitution $y = e^{\lambda x}$ in the differential equation.

$$\lambda^2 + 4\lambda + 4 = 0$$
$$(\lambda + 2)^2 = 0$$
$$\lambda = -2$$

The general solution of the differential equation is

$$y = c_1 \mathrm{e}^{-2t} + c_2 t \mathrm{e}^{-2t}.$$

We apply the initial conditions to determine the constants.

$$c_1 \mathrm{e}^2 - c_2 \mathrm{e}^2 = 2, \quad -2c_1 \mathrm{e}^2 + 3c_2 \mathrm{e}^2 = 1$$
$$c_1 = 7 \mathrm{e}^{-2}, \quad c_2 = 5 \mathrm{e}^{-2}$$

The solution subject to the initial conditions is

$$\boxed{y = (7 + 5t) \mathrm{e}^{-2(t+1)}}$$

The solution is plotted in Figure 19.3. The solution vanishes as $t \to \infty$.

$$\lim_{t\to\infty} (7 + 5t) \mathrm{e}^{-2(t+1)} = \lim_{t\to\infty} \frac{7 + 5t}{\mathrm{e}^{2(t+1)}} = \lim_{t\to\infty} \frac{5}{2 \mathrm{e}^{2(t+1)}} = 0$$

Solution 19.2

$$y'' - 4y' + 13y = 0.$$

With the substitution $y = \mathrm{e}^{\lambda x}$ we obtain

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda \mathrm{e}^{\lambda x} + 13 \mathrm{e}^{\lambda x} = 0$$
$$\lambda^2 - 4\lambda + 13 = 0$$
$$\lambda = 2 \pm 3i.$$

Thus two linearly independent solutions are

$$\mathrm{e}^{(2+3i)x}, \quad \text{and} \quad \mathrm{e}^{(2-3i)x}.$$

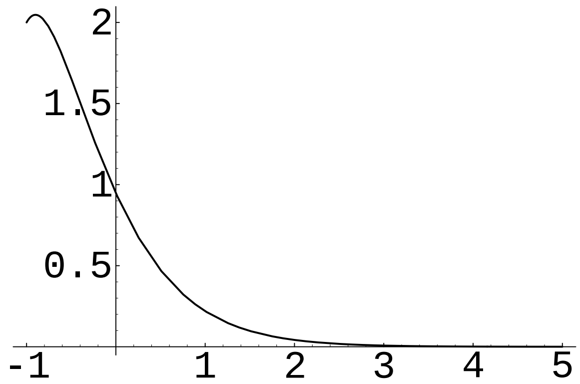

Figure 19.3: The solution of $y'' + 4y' + 4y = 0$, $y(-1) = 2$, $y'(-1) = 1$.

Noting that

$$
\begin{aligned}
e^{(2+3i)x} &= e^{2x}[\cos(3x) + i\sin(3x)] \\
e^{(2-3i)x} &= e^{2x}[\cos(3x) - i\sin(3x)],
\end{aligned}
$$

we can write the two linearly independent solutions

$$\boxed{y_1 = e^{2x}\cos(3x), \qquad y_2 = e^{2x}\sin(3x).}$$

Solution 19.3
We note that

$$y''' - y'' + y' - y = 0$$

is a constant coefficient equation. The substitution, $y = e^{\lambda x}$, yields

$$
\begin{gathered}
\lambda^3 - \lambda^2 + \lambda - 1 = 0 \\
(\lambda - 1)(\lambda - i)(\lambda + i) = 0.
\end{gathered}
$$

The corresponding solutions are e^x, e^{ix}, and e^{-ix}. We can write the general solution as

$$\boxed{y = c_1 e^x + c_2 \cos x + c_3 \sin x.}$$

Solution 19.4
We start with the equation $y'' + y = 0$. We substitute $y = e^{\lambda x}$ into the differential equation to obtain

$$\lambda^2 + 1 = 0, \qquad \lambda = \pm i.$$

A linearly independent set of solutions is

$$\{ e^{ix}, e^{-ix} \}.$$

The fundamental set of solutions has the form

$$\begin{aligned} y_1 &= c_1 e^{ix} + c_2 e^{-ix}, \\ y_2 &= c_3 e^{ix} + c_4 e^{-ix}. \end{aligned}$$

By applying the constraints

$$\begin{aligned} y_1(0) = 1, \quad y_1'(0) = 0, \\ y_2(0) = 0, \quad y_2'(0) = 1, \end{aligned}$$

we obtain

$$\begin{aligned} y_1 &= \frac{e^{ix} + e^{-ix}}{2} = \cos x, \\ y_2 &= \frac{e^{ix} + e^{-ix}}{2i} = \sin x. \end{aligned}$$

Now consider the equation $y'' - y = 0$. By substituting $y = e^{\lambda x}$ we find that a set of solutions is

$$\{ e^x, e^{-x} \}.$$

By taking linear combinations of these we see that another set of solutions is

$$\{\cosh x, \sinh x\}.$$

Note that this is the fundamental set of solutions.

Next consider $y'' = 0$. We can find the solutions by substituting $y = e^{\lambda x}$ or by integrating the equation twice. The fundamental set of solutions as $x = 0$ is

$$\{1, x\}.$$

Note that if $u(x)$ is a solution of a constant coefficient differential equation, then $u(x + c)$ is also a solution. Also note that if $u(x)$ satisfies $y(0) = a$, $y'(0) = b$, then $u(x - x_0)$ satisfies $y(x_0) = a$, $y'(x_0) = b$. Thus the fundamental sets of solutions at $x = 1$ are

1. $\{\cos(x - 1), \sin(x - 1)\}$,
2. $\{\cosh(x - 1), \sinh(x - 1)\}$,
3. $\{1, x - 1\}$.

Solution 19.5

We substitute $y = e^{\lambda x}$ into the differential equation.

$$y'' + 2ay' + by = 0$$
$$\lambda^2 + 2a\lambda + b = 0$$
$$\lambda = -a \pm \sqrt{a^2 - b}$$

If $a^2 > b$ then the two roots are distinct and real. The general solution is

$$y = c_1 e^{\left(-a+\sqrt{a^2-b}\right)x} + c_2 e^{\left(-a-\sqrt{a^2-b}\right)x}.$$

If $a^2 < b$ then the two roots are distinct and complex-valued. We can write them as

$$\lambda = -a \pm i\sqrt{b - a^2}.$$

The general solution is

$$y = c_1 \,\mathrm{e}^{\left(-a+i\sqrt{b-a^2}\right)x} + c_2 \,\mathrm{e}^{\left(-a-i\sqrt{b-a^2}\right)x}.$$

By taking the sum and difference of the two linearly independent solutions above, we can write the general solution as

$$y = c_1 \,\mathrm{e}^{-ax} \cos\left(\sqrt{b-a^2}\, x\right) + c_2 \,\mathrm{e}^{-ax} \sin\left(\sqrt{b-a^2}\, x\right).$$

If $a^2 = b$ then the only root is $\lambda = -a$. The general solution in this case is then

$$y = c_1 \,\mathrm{e}^{-ax} + c_2 x \,\mathrm{e}^{-ax}.$$

In summary, the general solution is

$$\boxed{y = \begin{cases} \mathrm{e}^{-ax}\left(c_1 \,\mathrm{e}^{\sqrt{a^2-b}\,x} + c_2 \,\mathrm{e}^{-\sqrt{a^2-b}\,x}\right) & \text{if } a^2 > b, \\ \mathrm{e}^{-ax}\left(c_1 \cos\left(\sqrt{b-a^2}\,x\right) + c_2 \sin\left(\sqrt{b-a^2}\,x\right)\right) & \text{if } a^2 < b, \\ \mathrm{e}^{-ax}(c_1 + c_2 x) & \text{if } a^2 = b. \end{cases}}$$

Solution 19.6
First we note that the general solution can be written,

$$y = \begin{cases} \mathrm{e}^{-ax}\left(c_1 \cosh\left(\sqrt{a^2-b}\,x\right) + c_2 \sinh\left(\sqrt{a^2-b}\,x\right)\right) & \text{if } a^2 > b, \\ \mathrm{e}^{-ax}\left(c_1 \cos\left(\sqrt{b-a^2}\,x\right) + c_2 \sin\left(\sqrt{b-a^2}\,x\right)\right) & \text{if } a^2 < b, \\ \mathrm{e}^{-ax}(c_1 + c_2 x) & \text{if } a^2 = b. \end{cases}$$

We first consider the case $a^2 > b$. The derivative is

$$y' = \mathrm{e}^{-ax}\left(\left(-ac_1 + \sqrt{a^2-b}\,c_2\right)\cosh\left(\sqrt{a^2-b}\,x\right) + \left(-ac_2 + \sqrt{a^2-b}\,c_1\right)\sinh\left(\sqrt{a^2-b}\,x\right)\right).$$

The conditions, $y_1(0) = 1$ and $y_1'(0) = 0$, for the first solution become,

$$c_1 = 1, \qquad -ac_1 + \sqrt{a^2-b}\,c_2 = 0,$$
$$c_1 = 1, \qquad c_2 = \frac{a}{\sqrt{a^2-b}}.$$

The conditions, $y_2(0) = 0$ and $y_2'(0) = 1$, for the second solution become,

$$c_1 = 0, \qquad -ac_1 + \sqrt{a^2-b}\,c_2 = 1,$$
$$c_1 = 0, \qquad c_2 = \frac{1}{\sqrt{a^2-b}}.$$

The fundamental set of solutions is

$$\left\{\mathrm{e}^{-ax}\left(\cosh\left(\sqrt{a^2-b}\,x\right) + \frac{a}{\sqrt{a^2-b}}\sinh\left(\sqrt{a^2-b}\,x\right)\right), \mathrm{e}^{-ax}\frac{1}{\sqrt{a^2-b}}\sinh\left(\sqrt{a^2-b}\,x\right)\right\}.$$

Now consider the case $a^2 < b$. The derivative is

$$y' = \mathrm{e}^{-ax}\left(\left(-ac_1 + \sqrt{b-a^2}\,c_2\right)\cos\left(\sqrt{b-a^2}\,x\right) + \left(-ac_2 - \sqrt{b-a^2}\,c_1\right)\sin\left(\sqrt{b-a^2}\,x\right)\right).$$

Clearly, the fundamental set of solutions is

$$\left\{\mathrm{e}^{-ax}\left(\cos\left(\sqrt{b-a^2}\,x\right) + \frac{a}{\sqrt{b-a^2}}\sin\left(\sqrt{b-a^2}\,x\right)\right), \mathrm{e}^{-ax}\frac{1}{\sqrt{b-a^2}}\sin\left(\sqrt{b-a^2}\,x\right)\right\}.$$

Finally we consider the case $a^2 = b$. The derivative is

$$y' = \mathrm{e}^{-ax}(-ac_1 + c_2 + -ac_2 x).$$

The conditions, $y_1(0) = 1$ and $y_1'(0) = 0$, for the first solution become,

$$c_1 = 1, \qquad -ac_1 + c_2 = 0,$$
$$c_1 = 1, \qquad c_2 = a.$$

The conditions, $y_2(0) = 0$ and $y_2'(0) = 1$, for the second solution become,

$$c_1 = 0, \qquad -ac_1 + c_2 = 1,$$
$$c_1 = 0, \qquad c_2 = 1.$$

The fundamental set of solutions is

$$\left\{(1 + ax)\,\mathrm{e}^{-ax}, x\,\mathrm{e}^{-ax}\right\}.$$

In summary, the fundamental set of solutions at $x = 0$ is

$$\boxed{\begin{cases} \left\{\mathrm{e}^{-ax}\left(\cosh\left(\sqrt{a^2-b}\,x\right) + \frac{a}{\sqrt{a^2-b}}\sinh\left(\sqrt{a^2-b}\,x\right)\right), \mathrm{e}^{-ax}\frac{1}{\sqrt{a^2-b}}\sinh\left(\sqrt{a^2-b}\,x\right)\right\} & \text{if } a^2 > b, \\ \left\{\mathrm{e}^{-ax}\left(\cos\left(\sqrt{b-a^2}\,x\right) + \frac{a}{\sqrt{b-a^2}}\sin\left(\sqrt{b-a^2}\,x\right)\right), \mathrm{e}^{-ax}\frac{1}{\sqrt{b-a^2}}\sin\left(\sqrt{b-a^2}\,x\right)\right\} & \text{if } a^2 < b, \\ \left\{(1+ax)\,\mathrm{e}^{-ax}, x\,\mathrm{e}^{-ax}\right\} & \text{if } a^2 = b. \end{cases}}$$

Solution 19.7

Let $y(t)$ denote the displacement of the mass from equilibrium. The forces on the mass are $-ky(t)$ due to the spring and $-\mu y'(t)$ due to friction. We equate the external forces to $my''(t)$ to find the differential equation of the motion.

$$my'' = -ky - \mu y'$$
$$\boxed{y'' + \frac{\mu}{m}y' + \frac{k}{m}y = 0}$$

The solution which satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$ is

$$\boxed{y(t) = \begin{cases} \mathrm{e}^{-\mu t/(2m)} \frac{2m}{\sqrt{\mu^2-4km}} \sinh\left(\sqrt{\mu^2 - 4km}\, t/(2m)\right) & \text{if } \mu^2 > km, \\ \mathrm{e}^{-\mu t/(2m)} \frac{2m}{\sqrt{4km-\mu^2}} \sin\left(\sqrt{4km - \mu^2}\, t/(2m)\right) & \text{if } \mu^2 < km, \\ t\,\mathrm{e}^{-\mu t/(2m)} & \text{if } \mu^2 = km. \end{cases}}$$

We respectively call these cases: strongly damped, weakly damped and critically damped. In the case that $m = k = 1$ the solution is

$$y(t) = \begin{cases} \mathrm{e}^{-\mu t/2} \frac{2}{\sqrt{\mu^2-4}} \sinh\left(\sqrt{\mu^2 - 4}\, t/2\right) & \text{if } \mu > 2, \\ \mathrm{e}^{-\mu t/2} \frac{2}{\sqrt{4-\mu^2}} \sin\left(\sqrt{4 - \mu^2}\, t/2\right) & \text{if } \mu < 2, \\ t\,\mathrm{e}^{-t} & \text{if } \mu = 2. \end{cases}$$

Note that when t is large, $t\,\mathrm{e}^{-t}$ is much smaller than $\mathrm{e}^{-\mu t/2}$ for $\mu < 2$. To prove this we examine the ratio of these functions as $t \to \infty$.

$$\begin{aligned} \lim_{t\to\infty} \frac{t\,\mathrm{e}^{-t}}{\mathrm{e}^{-\mu t/2}} &= \lim_{t\to\infty} \frac{t}{\mathrm{e}^{(1-\mu/2)t}} \\ &= \lim_{t\to\infty} \frac{1}{(1-\mu/2)\,\mathrm{e}^{(1-\mu)t}} \\ &= 0 \end{aligned}$$

Using this result, we see that the critically damped solution decays faster than the weakly damped solution.

We can write the strongly damped solution as

$$\mathrm{e}^{-\mu t/2} \frac{2}{\sqrt{\mu^2 - 4}} \left(\mathrm{e}^{\sqrt{\mu^2-4}\,t/2} - \mathrm{e}^{-\sqrt{\mu^2-4}\,t/2} \right).$$

For large t, the dominant factor is $\mathrm{e}^{\left(\sqrt{\mu^2-4}-\mu\right)t/2}$. Note that for $\mu > 2$,

$$\sqrt{\mu^2 - 4} = \sqrt{(\mu+2)(\mu-2)} > \mu - 2.$$

Therefore we have the bounds

$$-2 < \sqrt{\mu^2 - 4} - \mu < 0.$$

This shows that the critically damped solution decays faster than the strongly damped solution. $\mu = 2$ gives the fastest decaying solution. Figure 19.4 shows the solution for $\mu = 4$, $\mu = 1$ and $\mu = 2$.

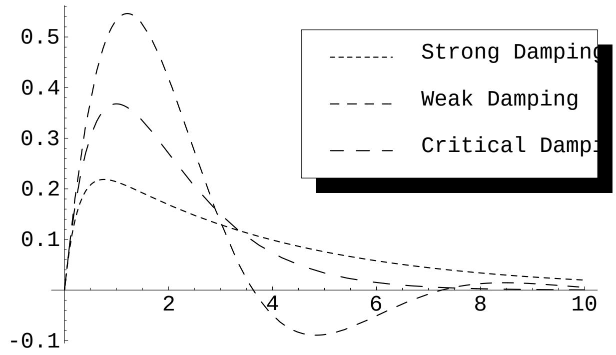

Figure 19.4: Strongly, weakly and critically damped solutions.

Solution 19.8

Clearly $y = c\cos(x - \phi)$ satisfies the differential equation $y'' + y = 0$. Since it is a two-parameter family of functions, it must be the general solution.

Using a trigonometric identity we can rewrite the solution as

$$y = c\cos\phi\cos x + c\sin\phi\sin x.$$

Setting this equal to $\sin x$ gives us the two equations

$$c\cos\phi = 0,$$
$$c\sin\phi = 1,$$

which has the solutions $c = 1$, $\phi = (2n + 1/2)\pi$, and $c = -1$, $\phi = (2n - 1/2)\pi$, for $n \in \mathbb{Z}$.

Clearly $y = c\cosh(x - \phi)$ satisfies the differential equation $y'' - y = 0$. Since it is a two-parameter family of functions, it must be the general solution.

Using a trigonometric identity we can rewrite the solution as

$$y = c\cosh\phi\cosh x + c\sinh\phi\sinh x.$$

Setting this equal to $\sinh x$ gives us the two equations

$$c\cosh\phi = 0,$$
$$c\sinh\phi = 1,$$

which has the solutions $c = -i$, $\phi = i(2n + 1/2)\pi$, and $c = i$, $\phi = i(2n - 1/2)\pi$, for $n \in \mathbb{Z}$.

Solution 19.9

We substitute $y = \mathrm{e}^{\lambda t}$ into the differential equation.

$$\lambda^2 \mathrm{e}^{\lambda t} + 5\lambda \mathrm{e}^{\lambda t} + 6 \mathrm{e}^{\lambda t} = 0$$
$$\lambda^2 + 5\lambda + 6 = 0$$
$$(\lambda + 2)(\lambda + 3) = 0$$

The general solution of the differential equation is

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-3t}.$$

The initial conditions give us the constraints:

$$c_1 + c_2 = 1,$$
$$-2c_1 - 3c_2 = V.$$

The solution subject to the initial conditions is

$$\boxed{y = (3 + V)\,\mathrm{e}^{-2t} - (2 + V)\,\mathrm{e}^{-3t}.}$$

This solution will be non-negative for $t > 0$ if $V \geq -3$.

Solution 19.10

For negative x, the differential equation is

$$y'' - y = 0.$$

We substitute $y = \mathrm{e}^{\lambda x}$ into the differential equation to find the solutions.

$$\lambda^2 - 1 = 0$$
$$\lambda = \pm 1$$
$$y = \left\{ \mathrm{e}^{x}, \mathrm{e}^{-x} \right\}$$

We can take linear combinations to write the solutions in terms of the hyperbolic sine and cosine.

$$y = \{\cosh(x), \sinh(x)\}$$

For positive x, the differential equation is

$$y'' + y = 0.$$

We substitute $y = \mathrm{e}^{\lambda x}$ into the differential equation to find the solutions.

$$\lambda^2 + 1 = 0$$
$$\lambda = \pm i$$
$$y = \left\{ \mathrm{e}^{ix}, \mathrm{e}^{-ix} \right\}$$

We can take linear combinations to write the solutions in terms of the sine and cosine.

$$y = \{\cos(x), \sin(x)\}$$

We will find the fundamental set of solutions at $x = 0$. That is, we will find a set of solutions, $\{y_1, y_2\}$ that satisfy the conditions:

$$y_1(0) = 1 \quad y_1'(0) = 0$$
$$y_2(0) = 0 \quad y_2'(0) = 1$$

Clearly these solutions are

$$\boxed{y_1 = \begin{cases} \cosh(x) & x < 0 \\ \cos(x) & x \geq 0 \end{cases} \qquad y_2 = \begin{cases} \sinh(x) & x < 0 \\ \sin(x) & x \geq 0 \end{cases}}$$

Euler Equations

Solution 19.11
We consider an Euler equation,

$$x^2 y'' + xy' + y = 0, \quad x > 0.$$

We make the change of independent variable $\xi = \ln x$, $u(\xi) = y(x)$ to obtain

$$u'' + u = 0.$$

We make the substitution $u(\xi) = \mathrm{e}^{\lambda\xi}$.

$$\lambda^2 + 1 = 0$$
$$\lambda = \pm i$$

A set of linearly independent solutions for $u(\xi)$ is

$$\{\mathrm{e}^{i\xi}, \mathrm{e}^{-i\xi}\}.$$

Since

$$\cos\xi = \frac{\mathrm{e}^{i\xi} + \mathrm{e}^{-i\xi}}{2} \quad \text{and} \quad \sin\xi = \frac{\mathrm{e}^{i\xi} - \mathrm{e}^{-i\xi}}{2i},$$

another linearly independent set of solutions is

$$\{\cos\xi, \sin\xi\}.$$

The general solution for $y(x)$ is

$$\boxed{y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x).}$$

Solution 19.12

Consider the differential equation

$$x^2 y'' - 2xy + 2y = 0.$$

With the substitution $y = x^\lambda$ this equation becomes

$$\begin{gathered}\lambda(\lambda - 1) - 2\lambda + 2 = 0 \\ \lambda^2 - 3\lambda + 2 = 0 \\ \lambda = 1, 2.\end{gathered}$$

The general solution is then

$$\boxed{y = c_1 x + c_2 x^2.}$$

Solution 19.13

We note that

$$x y''' + y'' + \frac{1}{x} y' = 0$$

is an Euler equation. The substitution $y = x^\lambda$ yields

$$\begin{gathered}\lambda^3 - 3\lambda^2 + 2\lambda + \lambda^2 - \lambda + \lambda = 0 \\ \lambda^3 - 2\lambda^2 + 2\lambda = 0.\end{gathered}$$

The three roots of this algebraic equation are

$$\lambda = 0, \quad \lambda = 1 + i, \quad \lambda = 1 - i$$

The corresponding solutions to the differential equation are

$$\begin{array}{lll} y = x^0 & y = x^{1+i} & y = x^{1-i} \\ y = 1 & y = x\,\mathrm{e}^{i \ln x} & y = x\,\mathrm{e}^{-i \ln x}. \end{array}$$

We can write the general solution as

$$\boxed{y = c_1 + c_2 x \cos(\ln x) + c_3 \sin(\ln x).}$$

Solution 19.14

We substitute $y = x^\lambda$ into the differential equation.

$$\begin{gathered}
x^2 y'' + (2a+1)xy' + by = 0 \\
\lambda(\lambda - 1) + (2a+1)\lambda + b = 0 \\
\lambda^2 + 2a\lambda + b = 0 \\
\lambda = -a \pm \sqrt{a^2 - b}
\end{gathered}$$

For $a^2 > b$ then the general solution is

$$y = c_1 x^{-a+\sqrt{a^2-b}} + c_2 x^{-a-\sqrt{a^2-b}}.$$

For $a^2 < b$, then the general solution is

$$y = c_1 x^{-a+i\sqrt{b-a^2}} + c_2 x^{-a-i\sqrt{b-a^2}}.$$

By taking the sum and difference of these solutions, we can write the general solution as

$$y = c_1 x^{-a} \cos\left(\sqrt{b-a^2}\,\ln x\right) + c_2 x^{-a} \sin\left(\sqrt{b-a^2}\,\ln x\right).$$

For $a^2 = b$, the quadratic in lambda has a double root at $\lambda = a$. The general solution of the differential equation is

$$y = c_1 x^{-a} + c_2 x^{-a} \ln x.$$

In summary, the general solution is:

$$\boxed{y = \begin{cases} x^{-a}\left(c_1 x^{\sqrt{a^2-b}} + c_2 x^{-\sqrt{a^2-b}}\right) & \text{if } a^2 > b, \\ x^{-a}\left(c_1 \cos\left(\sqrt{b-a^2}\,\ln x\right) + c_2 \sin\left(\sqrt{b-a^2}\,\ln x\right)\right) & \text{if } a^2 < b, \\ x^{-a}\left(c_1 + c_2 \ln x\right) & \text{if } a^2 = b. \end{cases}}$$

Solution 19.15

For $a \neq 0$, two linearly independent solutions of

$$y'' - a^2 y = 0$$

are

$$y_1 = \mathrm{e}^{ax}, \quad y_2 = \mathrm{e}^{-ax}.$$

For $a = 0$, we have

$$y_1 = \mathrm{e}^{0x} = 1, \quad y_2 = x\,\mathrm{e}^{0x} = x.$$

In this case the solution are defined by

$$y_1 = \left[\mathrm{e}^{ax}\right]_{a=0}, \quad y_2 = \left[\frac{\mathrm{d}}{\mathrm{d}a}\,\mathrm{e}^{ax}\right]_{a=0}.$$

By the definition of differentiation, $f'(0)$ is

$$f'(0) = \lim_{a \to 0} \frac{f(a) - f(-a)}{2a}.$$

Thus the second solution in the case $a = 0$ is

$$y_2 = \lim_{a \to 0} \frac{\mathrm{e}^{ax} - \mathrm{e}^{-ax}}{a}$$

Consider the solutions

$$y_1 = \mathrm{e}^{ax}, \quad y_2 = \lim_{\alpha \to a} \frac{\mathrm{e}^{\alpha x} - \mathrm{e}^{-\alpha x}}{\alpha}.$$

Clearly y_1 is a solution for all a. For $a \neq 0$, y_2 is a linear combination of e^{ax} and e^{-ax} and is thus a solution. Since the coefficient of e^{-ax} in this linear combination is non-zero, it is linearly independent to y_1. For $a = 0$, y_2 is one half the derivative of e^{ax} evaluated at $a = 0$. Thus it is a solution.

For $a \neq 0$, two linearly independent solutions of

$$x^2 y'' + x y' - a^2 y = 0$$

are

$$y_1 = x^a, \quad y_2 = x^{-a}.$$

For $a = 0$, we have

$$y_1 = [x^a]_{a=0} = 1, \quad y_2 = \left[\frac{\mathrm{d}}{\mathrm{d}a} x^a\right]_{a=0} = \ln x.$$

Consider the solutions

$$y_1 = x^a, \quad y_2 = \frac{x^a - x^{-a}}{a}$$

Clearly y_1 is a solution for all a. For $a \neq 0$, y_2 is a linear combination of x^a and x^{-a} and is thus a solution. For $a = 0$, y_2 is one half the derivative of x^a evaluated at $a = 0$. Thus it is a solution.

Solution 19.16

1.

$$x^2 y'' - 2 x y' + 2 y = 0$$

We substitute $y = x^\lambda$ into the differential equation.

$$\lambda(\lambda - 1) - 2\lambda + 2 = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$
$$(\lambda - 1)(\lambda - 2) = 0$$
$$\boxed{y = c_1 x + c_2 x^2}$$

2.

$$x^2 y'' - 2y = 0$$

We substitute $y = x^\lambda$ into the differential equation.

$$\lambda(\lambda - 1) - 2 = 0$$
$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda + 1)(\lambda - 2) = 0$$
$$\boxed{y = \frac{c_1}{x} + c_2 x^2}$$

3.

$$x^2 y'' - xy' + y = 0$$

We substitute $y = x^\lambda$ into the differential equation.

$$\lambda(\lambda - 1) - \lambda + 1 = 0$$
$$\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)^2 = 0$$

Since there is a double root, the solution is:

$$\boxed{y = c_1 x + c_2 x \ln x.}$$

Exact Equations

Solution 19.17
We note that

$$y'' + y' \sin x + y \cos x = 0$$

is an exact equation.

$$\frac{\mathrm{d}}{\mathrm{d}x}[y' + y\sin x] = 0$$
$$y' + y\sin x = c_1$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y\,\mathrm{e}^{-\cos x}\right] = c_1\,\mathrm{e}^{-\cos x}$$
$$\boxed{y = c_1\,\mathrm{e}^{\cos x}\int \mathrm{e}^{-\cos x}\,\mathrm{d}x + c_2\,\mathrm{e}^{\cos x}}$$

Equations Without Explicit Dependence on y

Reduction of Order

Solution 19.18

$$(1 - x^2)y'' - 2xy' + 2y = 0, \quad -1 < x < 1$$

We substitute $y = x$ into the differential equation to check that it is a solution.

$$(1 - x^2)(0) - 2x(1) + 2x = 0$$

We look for a second solution of the form $y = xu$. We substitute this into the differential equation and use the fact that x is a solution.

$$
\begin{gathered}
(1-x^2)(xu''+2u')-2x(xu'+u)+2xu=0 \\
(1-x^2)(xu''+2u')-2x(xu')=0 \\
(1-x^2)xu''+(2-4x^2)u'=0 \\
\frac{u''}{u'}=\frac{2-4x^2}{x(x^2-1)} \\
\frac{u''}{u'}=-\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x} \\
\ln(u')=-2\ln(x)-\ln(1-x)-\ln(1+x)+\text{const} \\
\ln(u')=\ln\left(\frac{c}{x^2(1-x)(1+x)}\right) \\
u'=\frac{c}{x^2(1-x)(1+x)} \\
u'=c\left(\frac{1}{x^2}+\frac{1}{2(1-x)}+\frac{1}{2(1+x)}\right) \\
u=c\left(-\frac{1}{x}-\frac{1}{2}\ln(1-x)+\frac{1}{2}\ln(1+x)\right)+\text{const} \\
u=c\left(-\frac{1}{x}+\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right)+\text{const}
\end{gathered}
$$

A second linearly independent solution is

$$
\boxed{y=-1+\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right).}
$$

Solution 19.19

We are given that $y = \mathrm{e}^x$ is a solution of

$$y'' - \frac{x+1}{x} y' + \frac{1}{x} y = 0.$$

To find another linearly independent solution, we will use reduction of order. Substituting

$$\begin{aligned} y &= u\,\mathrm{e}^x \\ y' &= (u' + u)\,\mathrm{e}^x \\ y'' &= (u'' + 2u' + u)\,\mathrm{e}^x \end{aligned}$$

into the differential equation yields

$$\begin{gathered} u'' + 2u' + u - \frac{x+1}{x}(u' + u) + \frac{1}{x} u = 0. \\ u'' + \frac{x-1}{x} u' = 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[u' \exp\left(\int \left(1 - \frac{1}{x}\right) \mathrm{d}x \right) \right] = 0 \\ u'\,\mathrm{e}^{x - \ln x} = c_1 \\ u' = c_1 x\,\mathrm{e}^{-x} \\ u = c_1 \int x\,\mathrm{e}^{-x}\,\mathrm{d}x + c_2 \\ u = c_1 (x\,\mathrm{e}^{-x} + \mathrm{e}^{-x}) + c_2 \\ y = c_1 (x+1) + c_2\,\mathrm{e}^x \end{gathered}$$

Thus a second linearly independent solution is

$$\boxed{y = x + 1.}$$

Solution 19.20
We are given that $y = x$ is a solution of

$$(1-2x)y'' + 4xy' - 4y = 0.$$

To find another linearly independent solution, we will use reduction of order. Substituting

$$\begin{aligned} y &= xu \\ y' &= xu' + u \\ y'' &= xu'' + 2u' \end{aligned}$$

into the differential equation yields

$$\begin{gathered} (1-2x)(xu'' + 2u') + 4x(xu' + u) - 4xu = 0, \\ (1-2x)xu'' + (4x^2 - 4x + 2)u' = 0, \\ \frac{u''}{u'} = \frac{4x^2 - 4x + 2}{x(2x-1)}, \\ \frac{u''}{u'} = 2 - \frac{2}{x} + \frac{2}{2x-1}, \\ \ln(u') = 2x - 2\ln x + \ln(2x-1) + \text{const}, \\ u' = c_1 \left(\frac{2}{x} - \frac{1}{x^2} \right) e^{2x}, \\ u = c_1 \frac{1}{x} e^{2x} + c_2, \\ \boxed{y = c_1 e^{2x} + c_2 x.} \end{gathered}$$

Solution 19.21
One solution of

$$(x-1)y'' - xy' + y = 0,$$

is $y_1 = e^x$. We find a second solution with reduction of order. We make the substitution $y_2 = u\,e^x$ in the differential equation. We determine u up to an additive constant.

$$
\begin{gathered}
(x-1)(u''+2u'+u)\,e^x - x(u'+u)\,e^x + u\,e^x = 0 \\
(x-1)u'' + (x-2)u' = 0 \\
\frac{u''}{u'} = -\frac{x-2}{x-1} = -1 + \frac{1}{x-1} \\
\ln|u'| = -x + \ln|x-1| + c \\
u' = c(x-1)\,e^{-x} \\
u = -cx\,e^{-x}
\end{gathered}
$$

The second solution of the differential equation is $y_2 = x$.

*Reduction of Order and the Adjoint Equation

Chapter 20

Techniques for Nonlinear Differential Equations

In mathematics you don't understand things. You just get used to them.

- Johann von Neumann

20.1 Bernoulli Equations

Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. One of the most important such equations is the *Bernoulli equation*

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)y^{\alpha}, \quad \alpha \neq 1.$$

The change of dependent variable $u = y^{1-\alpha}$ will yield a first order linear equation for u which when solved will give us an implicit solution for y. (See Exercise **??**.)

Result 20.1.1 The Bernoulli equation $y' + p(t)y = q(t)y^\alpha,\ \alpha \neq 1$ can be transformed to the first order linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + (1-\alpha)p(t)u = (1-\alpha)q(t)$$

with the change of variables $u = y^{1-\alpha}$.

Example 20.1.1 Consider the Bernoulli equation

$$y' = \frac{2}{x}y + y^2.$$

First we divide by y^2.

$$y^{-2}y' = \frac{2}{x}y^{-1} + 1$$

We make the change of variable $u = y^{-1}$.

$$-u' = \frac{2}{x}u + 1$$
$$u' + \frac{2}{x}u = -1$$

The integrating factor is $I(x) = \exp(\int \frac{2}{x}\,dx) = x^2$.

$$\begin{aligned}
\frac{d}{dx}(x^2 u) &= -x^2 \\
x^2 u &= -\frac{1}{3}x^3 + c \\
u &= -\frac{1}{3}x + \frac{c}{x^2} \\
y &= \left(-\frac{1}{3}x + \frac{c}{x^2}\right)^{-1}
\end{aligned}$$

Thus the solution for y is

$$\boxed{y = \frac{3x^2}{c - x^2}.}$$

20.2 Riccati Equations

Factoring Second Order Operators. Consider the second order linear equation

$$L[y] = \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)\right] y = y'' + p(x)y' + q(x)y = f(x).$$

If we were able to factor the linear operator L into the form

$$L = \left[\frac{\mathrm{d}}{\mathrm{d}x} + a(x)\right]\left[\frac{\mathrm{d}}{\mathrm{d}x} + b(x)\right], \tag{20.1}$$

then we would be able to solve the differential equation. Factoring reduces the problem to a system of first order equations. We start with the factored equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + a(x)\right]\left[\frac{\mathrm{d}}{\mathrm{d}x} + b(x)\right] y = f(x).$$

We set $u = \left[\frac{\mathrm{d}}{\mathrm{d}x} + b(x)\right] y$ and solve the problem

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + a(x)\right] u = f(x).$$

Then to obtain the solution we solve

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + b(x)\right] y = u.$$

Example 20.2.1 Consider the equation

$$y'' + \left(x - \frac{1}{x}\right) y' + \left(\frac{1}{x^2} - 1\right) y = 0.$$

Let's say by some insight or just random luck we are able to see that this equation can be factored into

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + x\right] \left[\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x}\right] y = 0.$$

We first solve the equation

$$\begin{aligned} \left[\frac{\mathrm{d}}{\mathrm{d}x} + x\right] u &= 0. \\ u' + xu &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2/2} u\right) &= 0 \\ u &= c_1\, \mathrm{e}^{-x^2/2} \end{aligned}$$

Then we solve for y with the equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x}\right] y = u = c_1 \,\mathrm{e}^{-x^2/2}.$$

$$y' - \frac{1}{x} y = c_1 \,\mathrm{e}^{-x^2/2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1} y\right) = c_1 x^{-1} \,\mathrm{e}^{-x^2/2}$$

$$\boxed{y = c_1 x \int x^{-1} \,\mathrm{e}^{-x^2/2}\, dx + c_2 x}$$

If we were able to solve for a and b in Equation 20.1 in terms of p and q then we would be able to solve any second order differential equation. Equating the two operators,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p\frac{\mathrm{d}}{\mathrm{d}x} + q &= \left[\frac{\mathrm{d}}{\mathrm{d}x} + a\right]\left[\frac{\mathrm{d}}{\mathrm{d}x} + b\right] \\ &= \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (a+b)\frac{\mathrm{d}}{\mathrm{d}x} + (b' + ab).\end{aligned}$$

Thus we have the two equations

$$a + b = p, \quad \text{and} \quad b' + ab = q.$$

Eliminating a,

$$\begin{aligned} b' + (p - b)b &= q \\ b' &= b^2 - pb + q \end{aligned}$$

Now we have a nonlinear equation for b that is no easier to solve than the original second order linear equation.

Riccati Equations. Equations of the form

$$y' = a(x)y^2 + b(x)y + c(x)$$

are called Riccati equations. From the above derivation we see that for every second order differential equation there is a corresponding Riccati equation. Now we will show that the converse is true.

We make the substitution

$$y = -\frac{u'}{au}, \qquad y' = -\frac{u''}{au} + \frac{(u')^2}{au^2} + \frac{a'u'}{a^2u},$$

in the Riccati equation.

$$\begin{gathered}
y' = ay^2 + by + c \\
-\frac{u''}{au} + \frac{(u')^2}{au^2} + \frac{a'u'}{a^2u} = a\frac{(u')^2}{a^2u^2} - b\frac{u'}{au} + c \\
-\frac{u''}{au} + \frac{a'u'}{a^2u} + b\frac{u'}{au} - c = 0 \\
u'' - \left(\frac{a'}{a} + b\right)u' + acu = 0
\end{gathered}$$

Now we have a second order linear equation for u.

Result 20.2.1 The substitution $y = -\frac{u'}{au}$ transforms the Riccati equation

$$y' = a(x)y^2 + b(x)y + c(x)$$

into the second order linear equation

$$u'' - \left(\frac{a'}{a} + b\right)u' + acu = 0.$$

Example 20.2.2 Consider the Riccati equation

$$y' = y^2 + \frac{1}{x}y + \frac{1}{x^2}.$$

With the substitution $y = -\frac{u'}{u}$ we obtain

$$u'' - \frac{1}{x}u' + \frac{1}{x^2}u = 0.$$

This is an Euler equation. The substitution $u = x^\lambda$ yields

$$\lambda(\lambda - 1) - \lambda + 1 = (\lambda - 1)^2 = 0.$$

Thus the general solution for u is

$$u = c_1 x + c_2 x \log x.$$

Since $y = -\frac{u'}{u}$,

$$y = -\frac{c_1 + c_2(1 + \log x)}{c_1 x + c_2 x \log x}$$

$$\boxed{y = -\frac{1 + c(1 + \log x)}{x + cx \log x}}$$

20.3 Exchanging the Dependent and Independent Variables

Some differential equations can be put in a more elementary form by exchanging the dependent and independent variables. If the new equation can be solved, you will have an implicit solution for the initial equation. We will consider a few examples to illustrate the method.

Example 20.3.1 Consider the equation

$$y' = \frac{1}{y^3 - xy^2}.$$

Instead of considering y to be a function of x, consider x to be a function of y. That is, $x = x(y)$, $x' = \frac{\mathrm{d}x}{\mathrm{d}y}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y^3 - xy^2}$$
$$\frac{\mathrm{d}x}{\mathrm{d}y} = y^3 - xy^2$$
$$x' + y^2 x = y^3$$

Now we have a first order equation for x.

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\mathrm{e}^{y^3/3} x\right) = y^3\,\mathrm{e}^{y^3/3}$$
$$\boxed{x = \mathrm{e}^{-y^3/3} \int y^3\,\mathrm{e}^{y^3/3}\,dy + c\,\mathrm{e}^{-y^3/3}}$$

Example 20.3.2 Consider the equation

$$y' = \frac{y}{y^2 + 2x}.$$

Interchanging the dependent and independent variables yields

$$\frac{1}{x'} = \frac{y}{y^2 + 2x}$$
$$x' = y + 2\frac{x}{y}$$
$$x' - 2\frac{x}{y} = y$$
$$\frac{\mathrm{d}}{\mathrm{d}y}(y^{-2}x) = y^{-1}$$
$$y^{-2}x = \log y + c$$
$$\boxed{x = y^2 \log y + cy^2}$$

Result 20.3.1 Some differential equations can be put in a simpler form by exchanging the dependent and independent variables. Thus a differential equation for $y(x)$ can be written as an equation for $x(y)$. Solving the equation for $x(y)$ will give an implicit solution for $y(x)$.

20.4 Autonomous Equations

Autonomous equations have no explicit dependence on x. The following are examples.

- $y'' + 3y' - 2y = 0$
- $y'' = y + (y')^2$
- $y''' + y''y = 0$

The change of variables $u(y) = y'$ reduces an n^{th} order autonomous equation in y to a non-autonomous equation of order $n-1$ in $u(y)$. Writing the derivatives of y in terms of u,

$$\begin{aligned}
y' &= u(y) \\
y'' &= \frac{\mathrm{d}}{\mathrm{d}x} u(y) \\
&= \frac{\mathrm{d}y}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}y} u(y) \\
&= y' u' \\
&= u' u \\
y''' &= (u'' u + (u')^2) u.
\end{aligned}$$

Thus we see that the equation for $u(y)$ will have an order of one less than the original equation.

> **Result 20.4.1** Consider an autonomous differential equation for $y(x)$, (autonomous equations have no explicit dependence on x.) The change of variables $u(y) = y'$ reduces an n^{th} order autonomous equation in y to a non-autonomous equation of order $n-1$ in $u(y)$.

Example 20.4.1 Consider the equation

$$y'' = y + (y')^2.$$

With the substitution $u(y) = y'$, the equation becomes

$$\begin{aligned}
u' u &= y + u^2 \\
u' &= u + y u^{-1}.
\end{aligned}$$

We recognize this as a Bernoulli equation. The substitution $v = u^2$ yields

$$
\begin{gathered}
\frac{1}{2} v' = v + y \\
v' - 2v = 2y \\
\frac{\mathrm{d}}{\mathrm{d}y}\left(\mathrm{e}^{-2y} v\right) = 2y\,\mathrm{e}^{-2y} \\
v(y) = c_1\,\mathrm{e}^{2y} + \mathrm{e}^{2y} \int 2y\,\mathrm{e}^{-2y}\,dy \\
v(y) = c_1\,\mathrm{e}^{2y} + \mathrm{e}^{2y}\left(-y\,\mathrm{e}^{-2y} + \int \mathrm{e}^{-2y}\,dy\right) \\
v(y) = c_1\,\mathrm{e}^{2y} + \mathrm{e}^{2y}\left(-y\,\mathrm{e}^{-2y} - \frac{1}{2}\,\mathrm{e}^{-2y}\right) \\
v(y) = c_1\,\mathrm{e}^{2y} - y - \frac{1}{2}.
\end{gathered}
$$

Now we solve for u.

$$
\begin{gathered}
u(y) = \left(c_1\,\mathrm{e}^{2y} - y - \frac{1}{2}\right)^{1/2}. \\
\frac{\mathrm{d}y}{\mathrm{d}x} = \left(c_1\,\mathrm{e}^{2y} - y - \frac{1}{2}\right)^{1/2}
\end{gathered}
$$

This equation is separable.

$$
\begin{gathered}
dx = \frac{dy}{\left(c_1\,\mathrm{e}^{2y} - y - \frac{1}{2}\right)^{1/2}} \\
\boxed{x + c_2 = \int \frac{1}{\left(c_1\,\mathrm{e}^{2y} - y - \frac{1}{2}\right)^{1/2}}\,dy}
\end{gathered}
$$

Thus we finally have arrived at an implicit solution for $y(x)$.

Example 20.4.2 Consider the equation

$$y'' + y^3 = 0.$$

With the change of variables, $u(y) = y'$, the equation becomes

$$u'u + y^3 = 0.$$

This equation is separable.

$$\begin{aligned}
u\,du &= -y^3\,dy \\
\frac{1}{2}u^2 &= -\frac{1}{4}y^4 + c_1 \\
u &= \left(2c_1 - \frac{1}{2}y^4\right)^{1/2} \\
y' &= \left(2c_1 - \frac{1}{2}y^4\right)^{1/2} \\
\frac{dy}{(2c_1 - \frac{1}{2}y^4)^{1/2}} &= dx
\end{aligned}$$

Integrating gives us the implicit solution

$$\boxed{\int \frac{1}{(2c_1 - \frac{1}{2}y^4)^{1/2}}\,dy = x + c_2.}$$

20.5 *Equidimensional-in-x Equations

Differential equations that are invariant under the change of variables $x = c\,\xi$ are said to be equidimensional-in-x. For a familiar example from linear equations, we note that the Euler equation is equidimensional-in-x. Writing the new derivatives under the change of variables,

$$x = c\,\xi, \qquad \frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}\xi}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{1}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}, \qquad \dots.$$

Example 20.5.1 Consider the Euler equation

$$y'' + \frac{2}{x}y' + \frac{3}{x^2}y = 0.$$

Under the change of variables, $x = c\,\xi$, $y(x) = u(\xi)$, this equation becomes

$$\frac{1}{c^2}u'' + \frac{2}{c\,\xi}\frac{1}{c}u' + \frac{3}{c^2\,\xi^2}u = 0$$
$$u'' + \frac{2}{\xi}u' + \frac{3}{\xi^2}u = 0.$$

Thus this equation is invariant under the change of variables $x = c\,\xi$.

Example 20.5.2 For a nonlinear example, consider the equation

$$y''\,y' + \frac{y''}{x\,y} + \frac{y'}{x^2} = 0.$$

With the change of variables $x = c\,\xi$, $y(x) = u(\xi)$ the equation becomes

$$\frac{u''}{c^2}\frac{u'}{c} + \frac{u''}{c^3\,\xi\,u} + \frac{u'}{c^3\,\xi^2} = 0$$
$$u''\,u' + \frac{u''}{\xi\,u} + \frac{u'}{\xi^2} = 0.$$

We see that this equation is also equidimensional-in-x.

You may recall that the change of variables $x = \mathrm{e}^t$ reduces an Euler equation to a constant coefficient equation. To generalize this result to nonlinear equations we will see that the same change of variables reduces an equidimensional-in-x equation to an autonomous equation.

Writing the derivatives with respect to x in terms of t,

$$x = \mathrm{e}^t, \qquad \frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{e}^{-t}\frac{\mathrm{d}}{\mathrm{d}t}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}t}$$

$$x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} = x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) - x\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{\mathrm{d}}{\mathrm{d}t}.$$

Example 20.5.3 Consider the equation in Example 20.5.2

$$y''\,y' + \frac{y''}{x\,y} + \frac{y'}{x^2} = 0.$$

Applying the change of variables $x = \mathrm{e}^t$, $y(x) = u(t)$ yields an autonomous equation for $u(t)$.

$$x^2\,y''\,x\,y' + \frac{x^2\,y''}{y} + x\,y' = 0$$

$$\boxed{(u'' - u')u' + \frac{u'' - u'}{u} + u' = 0}$$

Result 20.5.1 A differential equation that is invariant under the change of variables $x = c\,\xi$ is equidimensional-in-x. Such an equation can be reduced to autonomous equation of the same order with the change of variables, $x = \mathrm{e}^t$.

20.6 *Equidimensional-in-y Equations

A differential equation is said to be equidimensional-in-y if it is invariant under the change of variables $y(x) = c\,v(x)$. Note that all linear homogeneous equations are equidimensional-in-y.

Example 20.6.1 Consider the linear equation

$$y'' + p(x)y' + q(x)y = 0.$$

With the change of variables $y(x) = cv(x)$ the equation becomes

$$\begin{gathered} cv'' + p(x)cv' + q(x)cv = 0 \\ v'' + p(x)v' + q(x)v = 0 \end{gathered}$$

Thus we see that the equation is invariant under the change of variables.

Example 20.6.2 For a nonlinear example, consider the equation

$$y''y + (y')^2 - y^2 = 0.$$

Under the change of variables $y(x) = cv(x)$ the equation becomes.

$$\begin{gathered} cv''cv + (cv')^2 - (cv)^2 = 0 \\ v''v + (v')^2 - v^2 = 0. \end{gathered}$$

Thus we see that this equation is also equidimensional-in-y.

The change of variables $y(x) = \mathrm{e}^{u(x)}$ reduces an n^{th} order equidimensional-in-y equation to an equation of order $n-1$ for u'. Writing the derivatives of $\mathrm{e}^{u(x)}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{e}^u &= u'\,\mathrm{e}^u\\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}\,\mathrm{e}^u &= (u'' + (u')^2)\,\mathrm{e}^u\\ \frac{d^3}{dx^3}\,\mathrm{e}^u &= (u''' + 3u''u'' + (u')^3)\,\mathrm{e}^u.\end{aligned}$$

Example 20.6.3 Consider the linear equation in Example 20.6.1

$$y'' + p(x)y' + q(x)y = 0.$$

Under the change of variables $y(x) = \mathrm{e}^{u(x)}$ the equation becomes

$$(u'' + (u')^2)\,\mathrm{e}^u + p(x)u'\,\mathrm{e}^u + q(x)\,\mathrm{e}^u = 0$$
$$\boxed{u'' + (u')^2 + p(x)u' + q(x) = 0.}$$

Thus we have a Riccati equation for u'. This transformation might seem rather useless since linear equations are usually easier to work with than nonlinear equations, but it is often useful in determining the asymptotic behavior of the equation.

Example 20.6.4 From Example 20.6.2 we have the equation

$$y''y + (y')^2 - y^2 = 0.$$

The change of variables $y(x) = \mathrm{e}^{u(x)}$ yields

$$\begin{aligned}(u'' + (u')^2)\,\mathrm{e}^u\,\mathrm{e}^u + (u'\,\mathrm{e}^u)^2 - (\,\mathrm{e}^u)^2 &= 0\\ u'' + 2(u')^2 - 1 &= 0\\ u'' &= -2(u')^2 + 1\end{aligned}$$

Now we have a Riccati equation for u'. We make the substitution $u' = \frac{v'}{2v}$.

$$\frac{v''}{2v} - \frac{(v')^2}{2v^2} = -2\frac{(v')^2}{4v^2} + 1$$
$$v'' - 2v = 0$$
$$v = c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x}$$
$$u' = 2\sqrt{2}\frac{c_1 \mathrm{e}^{\sqrt{2}x} - c_2 \mathrm{e}^{-\sqrt{2}x}}{c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x}}$$
$$u = 2\int \frac{c_1\sqrt{2}\,\mathrm{e}^{\sqrt{2}x} - c_2\sqrt{2}\,\mathrm{e}^{-\sqrt{2}x}}{c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x}}\,dx + c_3$$
$$u = 2\log\left(c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x}\right) + c_3$$
$$y = \left(c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x}\right)^2 \mathrm{e}^{c_3}$$

The constants are redundant, the general solution is

$$\boxed{y = \left(c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x}\right)^2}$$

Result 20.6.1 A differential equation is equidimensional-in-y if it is invariant under the change of variables $y(x) = cv(x)$. An n^{th} order equidimensional-in-y equation can be reduced to an equation of order $n-1$ in u' with the change of variables $y(x) = \mathrm{e}^{u(x)}$.

20.7 *Scale-Invariant Equations

Result 20.7.1 An equation is scale invariant if it is invariant under the change of variables, $x = c\xi$, $y(x) = c^\alpha v(\xi)$, for some value of α. A scale-invariant equation can be transformed to an equidimensional-in-x equation with the change of variables, $y(x) = x^\alpha u(x)$.

Example 20.7.1 Consider the equation

$$y'' + x^2 y^2 = 0.$$

Under the change of variables $x = c\xi$, $y(x) = c^\alpha v(\xi)$ this equation becomes

$$\frac{c^\alpha}{c^2} v''(\xi) + c^2 x^2 c^{2\alpha} v^2(\xi) = 0.$$

Equating powers of c in the two terms yields $\alpha = -4$.

Introducing the change of variables $y(x) = x^{-4} u(x)$ yields

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[x^{-4} u(x)\right] + x^2 (x^{-4} u(x))^2 = 0$$
$$x^{-4} u'' - 8x^{-5} u' + 20 x^{-6} u + x^{-6} u^2 = 0$$
$$\boxed{x^2 u'' - 8xu' + 20u + u^2 = 0.}$$

We see that the equation for u is equidimensional-in-x.

20.8 Exercises

Exercise 20.1

1. Find the general solution and the singular solution of the Clairaut equation,

$$y = xp + p^2.$$

2. Show that the singular solution is the envelope of the general solution.

Hint, Solution

Bernoulli Equations

Exercise 20.2 (mathematica/ode/techniques_nonlinear/bernoulli.nb)
Consider the Bernoulli equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)y^\alpha.$$

1. Solve the Bernoulli equation for $\alpha = 1$.

2. Show that for $\alpha \neq 1$ the substitution $u = y^{1-\alpha}$ reduces Bernoulli's equation to a linear equation.

3. Find the general solution to the following equations.

   (a)
$$t^2 \frac{\mathrm{d}y}{\mathrm{d}t} + 2ty - y^3 = 0, \ t > 0$$

   (b)
$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy + y^2 = 0$$

Hint, Solution

Exercise 20.3
Consider a population, y. Let the birth rate of the population be proportional to y with constant of proportionality 1. Let the death rate of the population be proportional to y^2 with constant of proportionality $1/1000$. Assume that the population is large enough so that you can consider y to be continuous. What is the population as a function of time if the initial population is y_0?
Hint, Solution

Exercise 20.4
Show that the transformation $u = y^{1-n}$ reduces the equation to a linear first order equation. Solve the equations

1. $t^2 \frac{\mathrm{d}y}{\mathrm{d}t} + 2ty - y^3 = 0 \quad t > 0$
2. $\frac{\mathrm{d}y}{\mathrm{d}t} = (\Gamma \cos t + T)\, y - y^3$, Γ and T are real constants. (From a fluid flow stability problem.)

Hint, Solution

Riccati Equations

Exercise 20.5

1. Consider the Ricatti equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)y^2 + b(x)y + c(x).$$

Substitute

$$y = y_p(x) + \frac{1}{u(x)}$$

into the Ricatti equation, where y_p is some particular solution to obtain a first order linear differential equation for u.

2. Consider a Ricatti equation,

$$y' = 1 + x^2 - 2xy + y^2.$$

Verify that $y_p(x) = x$ is a particular solution. Make the substitution $y = y_p + 1/u$ to find the general solution.

What would happen if you continued this method, taking the general solution for y_p? Would you be able to find a more general solution?

3. The substitution

$$y = -\frac{u'}{au}$$

gives us the second order, linear, homogeneous differential equation,

$$u'' - \left(\frac{a'}{a} + b\right) u' + acu = 0.$$

The general solution for u has two constants of integration. However, the solution for y should only have one constant of integration as it satisfies a first order equation. Write y in terms of the solution for u and verify tha y has only one constant of integration.

Hint, Solution

Exchanging the Dependent and Independent Variables

Exercise 20.6
Solve the differential equation

$$y' = \frac{\sqrt{y}}{xy + y}.$$

Hint, Solution

Autonomous Equations
***Equidimensional-in-x Equations**
***Equidimensional-in-y Equations**
***Scale-Invariant Equations**

20.9 Hints

Hint 20.1

Bernoulli Equations

Hint 20.2

Hint 20.3
The differential equation governing the population is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y - \frac{y^2}{1000}, \quad y(0) = y_0.$$

This is a Bernoulli equation.

Hint 20.4

Riccati Equations

Hint 20.5

Exchanging the Dependent and Independent Variables

Hint 20.6
Exchange the dependent and independent variables.

Autonomous Equations

***Equidimensional-in-x Equations**
***Equidimensional-in-y Equations**
***Scale-Invariant Equations**

20.10 Solutions

Solution 20.1

We consider the Clairaut equation,

$$y = xp + p^2. \tag{20.2}$$

1. We differentiate Equation 20.2 with respect to x to obtain a second order differential equation.

$$y' = y' + xy'' + 2y'y''$$
$$y''(2y' + x) = 0$$

Equating the first or second factor to zero will lead us to two distinct solutions.

$$y'' = 0 \quad \text{or} \quad y' = -\frac{x}{2}$$

If $y'' = 0$ then $y' \equiv p$ is a constant, (say $y' = c$). From Equation 20.2 we see that the general solution is,

$$\boxed{y(x) = cx + c^2.} \tag{20.3}$$

Recall that the general solution of a first order differential equation has one constant of integration.

If $y' = -x/2$ then $y = -x^2/4 + \text{const}$. We determine the constant by substituting the expression into Equation 20.2.

$$-\frac{x^2}{4} + c = x\left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2$$

Thus we see that a singular solution of the Clairaut equation is

$$\boxed{y(x) = -\frac{1}{4}x^2.} \tag{20.4}$$

Recall that a singular solution of a first order nonlinear differential equation has no constant of integration.

2. Equating the general and singular solutions, $y(x)$, and their derivatives, $y'(x)$, gives us the system of equations,

$$cx + c^2 = -\frac{1}{4}x^2, \qquad c = -\frac{1}{2}x.$$

Since the first equation is satisfied for $c = -x/2$, we see that the solution $y = cx + c^2$ is tangent to the solution $y = -x^2/4$ at the point $(-2c, -|c|)$. The solution $y = cx+c^2$ is plotted for $c = \dots, -1/4, 0, 1/4, \dots$ in Figure 20.1.

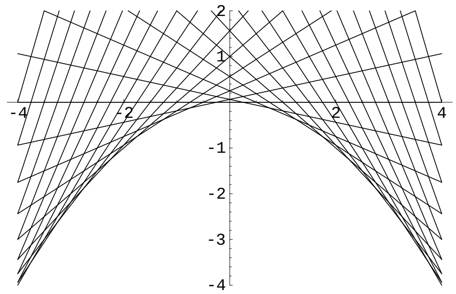

Figure 20.1: The Envelope of $y = cx + c^2$.

The envelope of a one-parameter family $F(x, y, c) = 0$ is given by the system of equations,

$$F(x, y, c) = 0, \qquad F_c(x, y, c) = 0.$$

For the family of solutions $y = cx + c^2$ these equations are

$$y = cx + c^2, \qquad 0 = x + 2c.$$

Substituting the solution of the second equation, $c = -x/2$, into the first equation gives the envelope,

$$y = \left(-\frac{1}{2}x\right)x + \left(-\frac{1}{2}x\right)^2 = -\frac{1}{4}x^2.$$

Thus we see that the singular solution is the envelope of the general solution.

Bernoulli Equations

Solution 20.2

1.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y &= q(t)y \\
\frac{\mathrm{d}y}{y} &= (q-p)\,\mathrm{d}t \\
\ln y &= \int (q-p)\,\mathrm{d}t + c \\
&\boxed{y = c\,\mathrm{e}^{\int (q-p)\,\mathrm{d}t}}
\end{aligned}$$

2. We consider the Bernoulli equation,

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)y^{\alpha}, \quad \alpha \neq 1.$$

We divide by y^{α}.

$$y^{-\alpha}y' + p(t)y^{1-\alpha} = q(t)$$

This suggests the change of dependent variable $u = y^{1-\alpha}$, $u' = (1-\alpha)y^{-\alpha}y'$.

$$\begin{aligned}
\frac{1}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}y^{1-\alpha} + p(t)y^{1-\alpha} &= q(t) \\
\frac{\mathrm{d}u}{\mathrm{d}t} + (1-\alpha)p(t)u &= (1-\alpha)q(t)
\end{aligned}$$

Thus we obtain a linear equation for u which when solved will give us an implicit solution for y.

3. (a)

$$t^2 \frac{\mathrm{d}y}{\mathrm{d}t} + 2ty - y^3 = 0, \quad t > 0$$

$$t^2 \frac{y'}{y^3} + 2t \frac{1}{y^2} = 1$$

We make the change of variables $u = y^{-2}$.

$$-\frac{1}{2} t^2 u' + 2tu = 1$$

$$u' - \frac{4}{t} u = -\frac{2}{t^2}$$

The integrating factor is

$$\mu = \mathrm{e}^{\int (-4/t)\,\mathrm{d}t} = \mathrm{e}^{-4 \ln t} = t^{-4}.$$

We multiply by the integrating factor and integrate to obtain the solution.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{-4} u\right) = -2t^{-6}$$

$$u = \frac{2}{5} t^{-1} + ct^4$$

$$y^{-2} = \frac{2}{5} t^{-1} + ct^4$$

$$y = \pm \frac{1}{\sqrt{\frac{2}{5} t^{-1} + ct^4}} \quad \boxed{y = \pm \frac{\sqrt{5t}}{\sqrt{2 + ct^5}}}$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy + y^2 = 0$$

$$\frac{y'}{y^2} + \frac{2x}{y} = -1$$

We make the change of variables $u = y^{-1}$.

$$u' - 2xu = 1$$

The integrating factor is

$$\mu = \mathrm{e}^{\int (-2x)\,\mathrm{d}x} = \mathrm{e}^{-x^2}.$$

We multiply by the integrating factor and integrate to obtain the solution.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x^2} u\right) = \mathrm{e}^{-x^2}$$

$$u = \mathrm{e}^{x^2} \int \mathrm{e}^{-x^2}\,\mathrm{d}x + c\,\mathrm{e}^{x^2}$$

$$\boxed{y = \frac{\mathrm{e}^{-x^2}}{\int \mathrm{e}^{-x^2}\,\mathrm{d}x + c}}$$

Solution 20.3

The differential equation governing the population is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y - \frac{y^2}{1000}, \qquad y(0) = y_0.$$

We recognize this as a Bernoulli equation. The substitution $u(t) = 1/y(t)$ yields

$$-\frac{\mathrm{d}u}{\mathrm{d}t} = u - \frac{1}{1000}, \quad u(0) = \frac{1}{y_0}.$$

$$u' + u = \frac{1}{1000}$$

$$u = \frac{1}{y_0}\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{-t}}{1000}\int_0^t \mathrm{e}^{\tau}\,d\tau$$

$$u = \frac{1}{1000} + \left(\frac{1}{y_0} - \frac{1}{1000}\right)\mathrm{e}^{-t}$$

Solving for $y(t)$,

$$\boxed{y(t) = \left(\frac{1}{1000} + \left(\frac{1}{y_0} - \frac{1}{1000}\right) e^{-t}\right)^{-1}.}$$

As a check, we see that as $t \to \infty$, $y(t) \to 1000$, which is an equilibrium solution of the differential equation.

$$\frac{dy}{dt} = 0 = y - \frac{y^2}{1000} \quad \Rightarrow \quad y = 1000.$$

Solution 20.4

1.

$$t^2 \frac{dy}{dt} + 2ty - y^3 = 0$$
$$\frac{dy}{dt} + 2t^{-1}y = t^{-2}y^3$$

We make the change of variables $u(t) = y^{-2}(t)$.

$$u' - 4t^{-1}u = -2t^{-2}$$

This gives us a first order, linear equation. The integrating factor is

$$I(t) = e^{\int -4t^{-1}\, dt} = e^{-4\log t} = t^{-4}.$$

We multiply by the integrating factor and integrate.

$$\frac{d}{dt}\left(t^{-4}u\right) = -2t^{-6}$$
$$t^{-4}u = \frac{2}{5}t^{-5} + c$$
$$u = \frac{2}{5}t^{-1} + ct^4$$

Finally we write the solution in terms of $y(t)$.

$$y(t) = \pm \frac{1}{\sqrt{\frac{2}{5} t^{-1} + c t^4}}$$

$$\boxed{y(t) = \pm \frac{\sqrt{5t}}{\sqrt{2 + c t^5}}}$$

2.

$$\frac{\mathrm{d}y}{\mathrm{d}t} - (\Gamma \cos t + T)\, y = -y^3$$

We make the change of variables $u(t) = y^{-2}(t)$.

$$u' + 2\,(\Gamma \cos t + T)\, u = 2$$

This gives us a first order, linear equation. The integrating factor is

$$I(t) = \mathrm{e}^{\int 2(\Gamma \cos t + T)\, dt} = \mathrm{e}^{2(\Gamma \sin t + T t)}$$

We multiply by the integrating factor and integrate.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2(\Gamma \sin t + T t)} u\right) = 2\,\mathrm{e}^{2(\Gamma \sin t + T t)}$$

$$u = 2\,\mathrm{e}^{-2(\Gamma \sin t + T t)} \left(\int \mathrm{e}^{2(\Gamma \sin t + T t)}\, dt + c\right)$$

Finally we write the solution in terms of $y(t)$.

$$\boxed{y = \pm \frac{\mathrm{e}^{\Gamma \sin t + T t}}{\sqrt{2\left(\int \mathrm{e}^{2(\Gamma \sin t + T t)}\, dt + c\right)}}}$$

Riccati Equations

Solution 20.5
We consider the Ricatti equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)y^2 + b(x)y + c(x). \tag{20.5}$$

1. We substitute

$$y = y_p(x) + \frac{1}{u(x)}$$

into the Ricatti equation, where y_p is some particular solution.

$$y_p' - \frac{u'}{u^2} = +a(x)\left(y_p^2 + 2\frac{y_p}{u} + \frac{1}{u^2}\right) + b(x)\left(y_p + \frac{1}{u}\right) + c(x)$$
$$-\frac{u'}{u^2} = b(x)\frac{1}{u} + a(x)\left(2\frac{y_p}{u} + \frac{1}{u^2}\right)$$
$$\boxed{u' = -\left(b + 2ay_p\right)u - a}$$

We obtain a first order linear differential equation for u whose solution will contain one constant of integration.

2. We consider a Ricatti equation,

$$y' = 1 + x^2 - 2xy + y^2. \tag{20.6}$$

We verify that $y_p(x) = x$ is a solution.

$$1 = 1 + x^2 - 2xx + x^2$$

Substituting $y = y_p + 1/u$ into Equation 20.6 yields,

$$u' = -\left(-2x + 2x\right)u - 1$$
$$u = -x + c$$
$$\boxed{y = x + \frac{1}{c - x}}$$

What would happen if we continued this method? Since $y = x + \frac{1}{c-x}$ is a solution of the Ricatti equation we can make the substitution,

$$y = x + \frac{1}{c - x} + \frac{1}{u(x)}, \tag{20.7}$$

which will lead to a solution for y which has two constants of integration. Then we could repeat the process, substituting the sum of that solution and $1/u(x)$ into the Ricatti equation to find a solution with three constants of integration. We know that the general solution of a first order, ordinary differential equation has only one constant of integration. Does this method for Ricatti equations violate this theorem? There's only one way to find out. We substitute Equation 20.7 into the Ricatti equation.

$$u' = -\left(-2x + 2\left(x + \frac{1}{c - x}\right)\right)u - 1$$
$$u' = -\frac{2}{c - x}u - 1$$
$$u' + \frac{2}{c - x}u = -1$$

The integrating factor is

$$I(x) = \mathrm{e}^{2/(c-x)} = \mathrm{e}^{-2\log(c-x)} = \frac{1}{(c - x)^2}.$$

Upon multiplying by the integrating factor, the equation becomes exact.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{(c-x)^2}u\right) = -\frac{1}{(c-x)^2}$$
$$u = (c-x)^2\frac{-1}{c-x} + b(c-x)^2$$
$$u = x - c + b(c-x)^2$$

Thus the Ricatti equation has the solution,

$$y = x + \frac{1}{c-x} + \frac{1}{x-c+b(c-x)^2}.$$

It appears that we we have found a solution that has two constants of integration, but appearances can be deceptive. We do a little algebraic simplification of the solution.

$$y = x + \frac{1}{c-x} + \frac{1}{(b(c-x)-1)(c-x)}$$
$$y = x + \frac{(b(c-x)-1)+1}{(b(c-x)-1)(c-x)}$$
$$y = x + \frac{b}{b(c-x)-1}$$
$$y = x + \frac{1}{(c-1/b)-x}$$

This is actually a solution, (namely the solution we had before), with one constant of integration, (namely $c - 1/b$). Thus we see that repeated applications of the procedure will not produce more general solutions.

3. The substitution

$$y = -\frac{u'}{au}$$

gives us the second order, linear, homogeneous differential equation,

$$u'' - \left(\frac{a'}{a} + b \right) u' + acu = 0.$$

The solution to this linear equation is a linear combination of two homogeneous solutions, u_1 and u_2.

$$u = c_1 u_1(x) + c_2 u_2(x)$$

The solution of the Ricatti equation is then

$$y = -\frac{c_1 u_1'(x) + c_2 u_2'(x)}{a(x)(c_1 u_1(x) + c_2 u_2(x))}.$$

Since we can divide the numerator and denominator by either c_1 or c_2, this answer has only one constant of integration, (namely c_1/c_2 or c_2/c_1).

Exchanging the Dependent and Independent Variables

Solution 20.6
Exchanging the dependent and independent variables in the differential equation,

$$y' = \frac{\sqrt{y}}{xy + y},$$

yields

$$x'(y) = y^{1/2} x + y^{1/2}.$$

This is a first order differential equation for $x(y)$.

$$x' - y^{1/2}x = y^{1/2}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[x \exp\left(-\frac{2y^{3/2}}{3}\right)\right] = y^{1/2}\exp\left(-\frac{2y^{3/2}}{3}\right)$$

$$x \exp\left(-\frac{2y^{3/2}}{3}\right) = -\exp\left(-\frac{2y^{3/2}}{3}\right) + c_1$$

$$x = -1 + c_1 \exp\left(\frac{2y^{3/2}}{3}\right)$$

$$\frac{x+1}{c_1} = \exp\left(\frac{2y^{3/2}}{3}\right)$$

$$\log\left(\frac{x+1}{c_1}\right) = \frac{2}{3}y^{3/2}$$

$$y = \left(\frac{3}{2}\log\left(\frac{x+1}{c_1}\right)\right)^{2/3}$$

$$\boxed{y = \left(c + \frac{3}{2}\log(x+1)\right)^{2/3}}$$

Autonomous Equations
*Equidimensional-in-x Equations
*Equidimensional-in-y Equations
*Scale-Invariant Equations

Chapter 21

Transformations and Canonical Forms

Prize intensity more than extent. Excellence resides in quality not in quantity. The best is always few and rare - abundance lowers value. Even among men, the giants are usually really dwarfs. Some reckon books by the thickness, as if they were written to exercise the brawn more than the brain. Extent alone never rises above mediocrity; it is the misfortune of universal geniuses that in attempting to be at home everywhere are so nowhere. Intensity gives eminence and rises to the heroic in matters sublime.

-Balthasar Gracian

21.1 The Constant Coefficient Equation

The solution of any second order linear homogeneous differential equation can be written in terms of the solutions to either

$$y'' = 0, \quad \text{or} \quad y'' - y = 0$$

Consider the general equation

$$y'' + ay' + by = 0.$$

We can solve this differential equation by making the substitution $y = \mathrm{e}^{\lambda x}$. This yields the algebraic equation

$$\lambda^2 + a\lambda + b = 0.$$

$$\lambda = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$$

There are two cases to consider. If $a^2 \neq 4b$ then the solutions are

$$y_1 = \mathrm{e}^{(-a+\sqrt{a^2-4b})x/2}, \qquad y_2 = \mathrm{e}^{(-a-\sqrt{a^2-4b})x/2}$$

If $a^2 = 4b$ then we have

$$y_1 = \mathrm{e}^{-ax/2}, \qquad y_2 = x\,\mathrm{e}^{-ax/2}$$

Note that regardless of the values of a and b the solutions are of the form

$$y = \mathrm{e}^{-ax/2} u(x)$$

We would like to write the solutions to the general differential equation in terms of the solutions to simpler differential equations. We make the substitution

$$y = \mathrm{e}^{\lambda x} u$$

The derivatives of y are

$$y' = \mathrm{e}^{\lambda x}(u' + \lambda u)$$
$$y'' = \mathrm{e}^{\lambda x}(u'' + 2\lambda u' + \lambda^2 u)$$

Substituting these into the differential equation yields

$$u'' + (2\lambda + a)u' + (\lambda^2 + a\lambda + b)u = 0$$

In order to get rid of the u' term we choose

$$\lambda = -\frac{a}{2}.$$

The equation is then

$$u'' + \left(b - \frac{a^2}{4}\right) u = 0.$$

There are now two cases to consider.

Case 1. If $b = a^2/4$ then the differential equation is

$$u'' = 0$$

which has solutions 1 and x. The general solution for y is then

$$y = \mathrm{e}^{-ax/2}(c_1 + c_2 x).$$

Case 2. If $b \neq a^2/4$ then the differential equation is

$$u'' - \left(\frac{a^2}{4} - b\right) u = 0.$$

We make the change variables

$$u(x) = v(\xi), \qquad x = \mu\xi.$$

The derivatives in terms of ξ are

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} = \frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}\xi}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}\xi} = \frac{1}{\mu^2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}.$$

The differential equation for v is

$$\frac{1}{\mu^2}v'' - \left(\frac{a^2}{4} - b\right) v = 0$$

$$v'' - \mu^2 \left(\frac{a^2}{4} - b\right) v = 0$$

We choose

$$\mu = \left(\frac{a^2}{4} - b\right)^{-1/2}$$

to obtain

$$v'' - v = 0$$

which has solutions $\mathrm{e}^{\pm\xi}$. The solution for y is

$$y = \mathrm{e}^{\lambda x} \left(c_1 \, \mathrm{e}^{x/\mu} + c_2 \, \mathrm{e}^{-x/\mu}\right)$$

$$y = \mathrm{e}^{-ax/2} \left(c_1 \, \mathrm{e}^{\sqrt{a^2/4-b} \; x} + c_2 \, \mathrm{e}^{-\sqrt{a^2/4-b} \; x}\right)$$

21.2 Normal Form

21.2.1 Second Order Equations

Consider the second order equation

$$y'' + p(x)y' + q(x)y = 0. \tag{21.1}$$

Through a change of dependent variable, this equation can be transformed to

$$u'' + I(x)y = 0.$$

This is known as the **normal form** of (21.1). The function $I(x)$ is known as the **invariant** of the equation.

Now to find the change of variables that will accomplish this transformation. We make the substitution $y(x) = a(x)u(x)$ in (21.1).

$$au'' + 2a'u' + a''u + p(au' + a'u) + qau = 0$$

$$u'' + \left(2\frac{a'}{a} + p\right)u' + \left(\frac{a''}{a} + \frac{pa'}{a} + q\right)u = 0$$

To eliminate the u' term, $a(x)$ must satisfy

$$2\frac{a'}{a} + p = 0$$

$$a' + \frac{1}{2}pa = 0$$

$$a = c\exp\left(-\frac{1}{2}\int p(x)\,dx\right).$$

For this choice of a, our differential equation for u becomes

$$u'' + \left(q - \frac{p^2}{4} - \frac{p'}{2}\right)u = 0.$$

Two differential equations having the same normal form are called **equivalent**.

Result 21.2.1 The change of variables

$$y(x) = \exp\left(-\frac{1}{2}\int p(x)\,dx\right)u(x)$$

transforms the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

into its normal form

$$u'' + I(x)u = 0$$

where the invariant of the equation, $I(x)$, is

$$I(x) = q - \frac{p^2}{4} - \frac{p'}{2}.$$

21.2.2 Higher Order Differential Equations

Consider the third order differential equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0.$$

We can eliminate the y'' term. Making the change of dependent variable

$$\begin{aligned}
y &= u \exp\left(-\frac{1}{3}\int p(x)\,dx\right) \\
y' &= \left[u' - \frac{1}{3}pu\right] \exp\left(-\frac{1}{3}\int p(x)\,dx\right) \\
y'' &= \left[u'' - \frac{2}{3}pu' + \frac{1}{9}(p^2 - 3p')u\right] \exp\left(-\frac{1}{3}\int p(x)\,dx\right) \\
y'' &= \left[u''' - pu'' + \frac{1}{3}(p^2 - 3p')u' + \frac{1}{27}(9p' - 9p'' - p^3)u\right] \exp\left(-\frac{1}{3}\int p(x)\,dx\right)
\end{aligned}$$

yields the differential equation

$$u''' + \frac{1}{3}(3q - 3p' - p^2)u' + \frac{1}{27}(27r - 9pq - 9p'' + 2p^3)u = 0.$$

Result 21.2.2 The change of variables

$$y(x) = \exp\left(-\frac{1}{n}\int p_{n-1}(x)\,dx\right) u(x)$$

transforms the differential equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_0(x)y = 0$$

into the form

$$u^{(n)} + a_{n-2}(x)u^{(n-2)} + a_{n-3}(x)u^{(n-3)} + \cdots + a_0(x)u = 0.$$

21.3 Transformations of the Independent Variable

21.3.1 Transformation to the form u" + a(x) u = 0

Consider the second order linear differential equation

$$y'' + p(x)y' + q(x)y = 0.$$

We make the change of independent variable

$$\xi = f(x), \qquad u(\xi) = y(x).$$

The derivatives in terms of ξ are

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} = f'\frac{\mathrm{d}}{\mathrm{d}\xi}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} = f'\frac{\mathrm{d}}{\mathrm{d}\xi}f'\frac{\mathrm{d}}{\mathrm{d}\xi} = (f')^2\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + f''\frac{\mathrm{d}}{\mathrm{d}\xi}$$

The differential equation becomes

$$(f')^2u'' + f''u' + pf'u' + qu = 0.$$

In order to eliminate the u' term, f must satisfy

$$f'' + pf' = 0$$

$$f' = \exp\left(-\int p(x)\,dx\right)$$

$$f = \int \exp\left(-\int p(x)\,dx\right)\,dx.$$

The differential equation for u is then

$$u'' + \frac{q}{(f')^2} u = 0$$

$$u''(\xi) + q(x) \exp\left(2 \int p(x)\, dx\right) u(\xi) = 0.$$

Result 21.3.1 The change of variables

$$\xi = \int \exp\left(-\int p(x)\, dx\right) dx, \qquad u(\xi) = y(x)$$

transforms the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

into

$$u''(\xi) + q(x) \exp\left(2 \int p(x)\, dx\right) u(\xi) = 0.$$

21.3.2 Transformation to a Constant Coefficient Equation

Consider the second order linear differential equation

$$y'' + p(x)y' + q(x)y = 0.$$

With the change of independent variable

$$\xi = f(x), \qquad u(\xi) = y(x),$$

the differential equation becomes

$$(f')^2 u'' + (f'' + pf')u' + qu = 0.$$

For this to be a constant coefficient equation we must have

$$(f')^2 = c_1 q, \qquad \text{and} \qquad f'' + pf' = c_2 q,$$

for some constants c_1 and c_2. Solving the first condition,

$$f' = c\sqrt{q},$$

$$f = c \int \sqrt{q(x)}\, dx.$$

The second constraint becomes

$$\frac{f'' + pf'}{q} = \text{const}$$
$$\frac{\frac{1}{2} c q^{-1/2} q' + pcq^{1/2}}{q} = \text{const}$$
$$\frac{q' + 2pq}{q^{3/2}} = \text{const}.$$

Result 21.3.2 Consider the differential equation

$$y'' + p(x)y' + q(x)y = 0.$$

If the expression

$$\frac{q' + 2pq}{q^{3/2}}$$

is a constant then the change of variables

$$\xi = c \int \sqrt{q(x)}\, dx, \quad u(\xi) = y(x),$$

will yield a constant coefficient differential equation. (Here c is an arbitrary constant.)

21.4 Integral Equations

Volterra's Equations. Volterra's integral equation of the first kind has the form

$$\int_a^x N(x, \xi) f(\xi)\, d\xi = f(x).$$

The Volterra equation of the second kind is

$$y(x) = f(x) + \lambda \int_a^x N(x, \xi) y(\xi)\, d\xi.$$

$N(x, \xi)$ is known as the kernel of the equation.

Fredholm's Equations. Fredholm's integral equations of the first and second kinds are

$$\int_a^b N(x,\xi) f(\xi)\, d\xi = f(x),$$

$$y(x) = f(x) + \lambda \int_a^b N(x,\xi) y(\xi)\, d\xi.$$

21.4.1 Initial Value Problems

Consider the initial value problem

$$y'' + p(x)y' + q(x)y = f(x), \qquad y(a) = \alpha, \quad y'(a) = \beta.$$

Integrating this equation twice yields

$$\int_a^x \int_a^\eta y''(\xi) + p(\xi)y'(\xi) + q(\xi)y(\xi)\, d\xi\, d\eta = \int_a^x \int_a^\eta f(\xi)\, d\xi\, d\eta$$

$$\int_a^x (x-\xi)[y''(\xi) + p(\xi)y'(\xi) + q(\xi)y(\xi)]\, d\xi = \int_a^x (x-\xi) f(\xi)\, d\xi.$$

Now we use integration by parts.

$$\Big[(x-\xi)y'(\xi)\Big]_a^x - \int_a^x -y'(\xi)\, d\xi + \Big[(x-\xi)p(\xi)y(\xi)\Big]_a^x - \int_a^x [(x-\xi)p'(\xi) - p(\xi)]y(\xi)\, d\xi$$
$$+ \int_a^x (x-\xi)q(\xi)y(\xi)\, d\xi = \int_a^x (x-\xi)f(\xi)\, d\xi.$$

$$-(x-a)y'(a) + y(x) - y(a) - (x-a)p(a)y(a) - \int_a^x [(x-\xi)p'(\xi) - p(\xi)]y(\xi)\, d\xi$$
$$+ \int_a^x (x-\xi)q(\xi)y(\xi)\, d\xi = \int_a^x (x-\xi)f(\xi)\, d\xi.$$

We obtain a Volterra integral equation of the second kind for $y(x)$.

$$\boxed{y(x) = \int_a^x (x-\xi) f(\xi)\, d\xi + (x-a)(\alpha p(a) + \beta) + \alpha + \int_a^x \left\{ (x-\xi)[p'(\xi) - q(\xi)] - p(\xi) \right\} y(\xi)\, d\xi.}$$

Note that the initial conditions for the differential equation are "built into" the Volterra equation. Setting $x = a$ in the Volterra equation yields $y(a) = \alpha$. Differentiating the Volterra equation,

$$y'(x) = \int_a^x f(\xi)\, d\xi + (\alpha p(a) + \beta) - p(x) y(x) + \int_a^x [p'(\xi) - q(\xi)] - p(\xi) y(\xi)\, d\xi$$

and setting $x = a$ yields

$$y'(a) = \alpha p(a) + \beta - p(a)\alpha = \beta.$$

(Recall from calculus that

$$\frac{\mathrm{d}}{\mathrm{d}x} \int^x g(x,\xi)\, d\xi = g(x,x) + \int^x \frac{\partial}{\partial x}[g(x,\xi)]\, d\xi.)$$

Result 21.4.1 The initial value problem

$$y'' + p(x) y' + q(x) y = f(x), \qquad y(a) = \alpha, \quad y'(a) = \beta.$$

is equivalent to the Volterra equation of the second kind

$$y(x) = F(x) + \int_a^x N(x,\xi) y(\xi)\, d\xi$$

where

$$F(x) = \int_a^x (x-\xi) f(\xi)\, d\xi + (x-a)(\alpha p(a) + \beta) + \alpha$$
$$N(x,\xi) = (x-\xi)[p'(\xi) - q(\xi)] - p(\xi).$$

21.4.2 Boundary Value Problems

Consider the boundary value problem

$$y'' = f(x), \qquad y(a) = \alpha, \quad y(b) = \beta. \tag{21.2}$$

To obtain a problem with homogeneous boundary conditions, we make the change of variable

$$y(x) = u(x) + \alpha + \frac{\beta - \alpha}{b - a}(x - a)$$

to obtain the problem

$$u'' = f(x), \qquad u(a) = u(b) = 0.$$

Now we will use Green's functions to write the solution as an integral. First we solve the problem

$$G'' = \delta(x - \xi), \qquad G(a|\xi) = G(b|\xi) = 0.$$

The homogeneous solutions of the differential equation that satisfy the left and right boundary conditions are

$$c_1(x - a) \quad \text{and} \quad c_2(x - b).$$

Thus the Green's function has the form

$$G(x|\xi) = \begin{cases} c_1(x - a), & \text{for } x \le \xi \\ c_2(x - b), & \text{for } x \ge \xi \end{cases}$$

Imposing continuity of $G(x|\xi)$ at $x = \xi$ and a unit jump of $G(x|\xi)$ at $x = \xi$, we obtain

$$G(x|\xi) = \begin{cases} \frac{(x-a)(\xi-b)}{b-a}, & \text{for } x \le \xi \\ \frac{(x-b)(\xi-a)}{b-a}, & \text{for } x \ge \xi \end{cases}$$

Thus the solution of the (21.2) is

$$y(x) = \alpha + \frac{\beta - \alpha}{b - a}(x - a) + \int_a^b G(x|\xi) f(\xi)\, d\xi.$$

Now consider the boundary value problem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(a) = \alpha, \quad y(b) = \beta.$$

From the above result we can see that the solution satisfies

$$y(x) = \alpha + \frac{\beta - \alpha}{b - a}(x - a) + \int_a^b G(x|\xi)[f(\xi) - p(\xi)y'(\xi) - q(\xi)y(\xi)]\, d\xi.$$

Using integration by parts, we can write

$$\begin{aligned} -\int_a^b G(x|\xi)p(\xi)y'(\xi)\, d\xi &= -\big[G(x|\xi)p(\xi)y(\xi)\big]_a^b + \int_a^b \left[\frac{\partial G(x|\xi)}{\partial \xi}p(\xi) + G(x|\xi)p'(\xi)\right] y(\xi)\, d\xi \\ &= \int_a^b \left[\frac{\partial G(x|\xi)}{\partial \xi}p(\xi) + G(x|\xi)p'(\xi)\right] y(\xi)\, d\xi. \end{aligned}$$

Substituting this into our expression for $y(x)$,

$$y(x) = \alpha + \frac{\beta - \alpha}{b - a}(x - a) + \int_a^b G(x|\xi)f(\xi)\, d\xi + \int_a^b \left[\frac{\partial G(x|\xi)}{\partial \xi}p(\xi) + G(x|\xi)[p'(\xi) - q(\xi)]\right] y(\xi)\, d\xi,$$

we obtain a Fredholm integral equation of the second kind.

Result 21.4.2 The boundary value problem

$$y'' + p(x)y' + q(x)y = f(x), \qquad y(a) = \alpha, \quad y(b) = \beta.$$

is equivalent to the Fredholm equation of the second kind

$$y(x) = F(x) + \int_a^b N(x, \xi) y(\xi) \, d\xi$$

where

$$F(x) = \alpha + \frac{\beta - \alpha}{b - a}(x - a) + \int_a^b G(x|\xi) f(\xi) \, d\xi,$$

$$N(x, \xi) = \int_a^b H(x|\xi) y(\xi) \, d\xi,$$

$$G(x|\xi) = \begin{cases} \frac{(x-a)(\xi-b)}{b-a}, & \text{for } x \leq \xi \\ \frac{(x-b)(\xi-a)}{b-a}, & \text{for } x \geq \xi, \end{cases}$$

$$H(x|\xi) = \begin{cases} \frac{(x-a)}{b-a} p(\xi) + \frac{(x-a)(\xi-b)}{b-a} [p'(\xi) - q(\xi)] & \text{for } x \leq \xi \\ \frac{(x-b)}{b-a} p(\xi) + \frac{(x-b)(\xi-a)}{b-a} [p'(\xi) - q(\xi)] & \text{for } x \geq \xi. \end{cases}$$

21.5 Exercises

The Constant Coefficient Equation
Normal Form

Exercise 21.1
Solve the differential equation

$$y'' + \left(2 + \frac{4}{3}x\right) y' + \frac{1}{9}\left(24 + 12x + 4x^2\right) y = 0.$$

Hint, Solution

Transformations of the Independent Variable
Integral Equations

Exercise 21.2
Show that the solution of the differential equation

$$y'' + 2(a + bx)y' + (c + dx + ex^2)y = 0$$

can be written in terms of one of the following canonical forms:

$$\begin{aligned} &v'' + (\xi^2 + A)v = 0 \\ &v'' = \xi v \\ &v'' + v = 0 \\ &v'' = 0. \end{aligned}$$

Hint, Solution

Exercise 21.3
Show that the solution of the differential equation

$$y'' + 2\left(a + \frac{b}{x}\right) y' + \left(c + \frac{d}{x} + \frac{e}{x^2}\right) y = 0$$

can be written in terms of one of the following canonical forms:

$$v'' + \left(1 + \frac{A}{\xi} + \frac{B}{\xi^2}\right) v = 0$$
$$v'' + \left(\frac{1}{\xi} + \frac{A}{\xi^2}\right) v = 0$$
$$v'' + \frac{A}{\xi^2} v = 0$$

Hint, Solution

Exercise 21.4
Show that the second order Euler equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 x \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0$$

can be transformed to a constant coefficient equation.
Hint, Solution

Exercise 21.5
Solve Bessel's equation of order 1/2,

$$y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right) y = 0.$$

Hint, Solution

21.6 Hints

The Constant Coefficient Equation
Normal Form

Hint 21.1
Transform the equation to normal form.

Transformations of the Independent Variable
Integral Equations

Hint 21.2
Transform the equation to normal form and then apply the scale transformation $x = \lambda\xi + \mu$.

Hint 21.3
Transform the equation to normal form and then apply the scale transformation $x = \lambda\xi$.

Hint 21.4
Make the change of variables $x = \mathrm{e}^t$, $y(x) = u(t)$. Write the derivatives with respect to x in terms of t.

$$\begin{aligned} x &= \mathrm{e}^t \\ dx &= \mathrm{e}^t dt \\ \frac{\mathrm{d}}{\mathrm{d}x} &= \mathrm{e}^{-t}\frac{\mathrm{d}}{\mathrm{d}t} \\ x\frac{\mathrm{d}}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}t} \end{aligned}$$

Hint 21.5
Transform the equation to normal form.

21.7 Solutions

The Constant Coefficient Equation
Normal Form

Solution 21.1

$$y'' + \left(2 + \frac{4}{3}x\right) y' + \frac{1}{9}\left(24 + 12x + 4x^2\right) y = 0$$

To transform the equation to normal form we make the substitution

$$\begin{aligned} y &= \exp\left(-\frac{1}{2}\int\left(2 + \frac{4}{3}x\right) dx\right) u \\ &= \mathrm{e}^{-x-x^2/3} u \end{aligned}$$

The invariant of the equation is

$$\begin{aligned} I(x) &= \frac{1}{9}\left(24 + 12x + 4x^2\right) - \frac{1}{4}\left(2 + \frac{4}{3}x\right)^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(2 + \frac{4}{3}x\right) \\ &= 1. \end{aligned}$$

The normal form of the differential equation is then

$$u'' + u = 0$$

which has the general solution

$$u = c_1 \cos x + c_2 \sin x$$

Thus the equation for y has the general solution

$$\boxed{y = c_1\,\mathrm{e}^{-x-x^2/3}\cos x + c_2\,\mathrm{e}^{-x-x^2/3}\sin x.}$$

Solution 21.2

The substitution that will transform the equation to normal form is

$$\begin{aligned} y &= \exp\left(-\frac{1}{2}\int 2(a+bx)\,dx\right)u \\ &= \mathrm{e}^{-ax-bx^2/2}u. \end{aligned}$$

The invariant of the equation is

$$\begin{aligned} I(x) &= c + dx + ex^2 - \frac{1}{4}(2(a+bx))^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(2(a+bx)) \\ &= c - b - a^2 + (d - 2ab)x + (e - b^2)x^2 \\ &\equiv \alpha + \beta x + \gamma x^2 \end{aligned}$$

The normal form of the differential equation is

$$u'' + (\alpha + \beta x + \gamma x^2)u = 0$$

We consider the following cases:

$\gamma = 0.$

$\beta = 0.$

$\alpha = 0.$ We immediately have the equation

$$u'' = 0.$$

$\alpha \neq 0.$ With the change of variables

$$v(\xi) = u(x), \qquad x = \alpha^{-1/2}\xi,$$

we obtain

$$v'' + v = 0.$$

$\beta \neq 0$. We have the equation

$$y'' + (\alpha + \beta x)y = 0.$$

The scale transformation $x = \lambda\xi + \mu$ yields

$$v'' + \lambda^2(\alpha + \beta(\lambda\xi + \mu))y = 0$$

$$v'' = [\beta\lambda^3\xi + \lambda^2(\beta\mu + \alpha)]v.$$

Choosing

$$\lambda = (-\beta)^{-1/3}, \qquad \mu = -\frac{\alpha}{\beta}$$

yields the differential equation

$$v'' = \xi v.$$

$\gamma \neq 0$. The scale transformation $x = \lambda\xi + \mu$ yields

$$v'' + \lambda^2[\alpha + \beta(\lambda\xi + \mu) + \gamma(\lambda\xi + \mu)^2]v = 0$$

$$v'' + \lambda^2[\alpha + \beta\mu + \gamma\mu^2 + \lambda(\beta + 2\gamma\mu)\xi + \lambda^2\gamma\xi^2]v = 0.$$

Choosing

$$\lambda = \gamma^{-1/4}, \qquad \mu = -\frac{\beta}{2\gamma}$$

yields the differential equation

$$v'' + (\xi^2 + A)v = 0$$

where

$$A = \gamma^{-1/2} - \frac{1}{4}\beta\gamma^{-3/2}.$$

Solution 21.3
The substitution that will transform the equation to normal form is

$$\begin{aligned} y &= \exp\left(-\frac{1}{2}\int 2\left(a + \frac{b}{x}\right)\,dx\right) u \\ &= x^{-b}\,\mathrm{e}^{-ax} u. \end{aligned}$$

The invariant of the equation is

$$\begin{aligned} I(x) &= c + \frac{d}{x} + \frac{e}{x^2} - \frac{1}{4}\left(2\left(a + \frac{b}{x}\right)\right)^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(2\left(a + \frac{b}{x}\right)\right) \\ &= c - a^x + \frac{d - 2ab}{x} + \frac{e + b - b^2}{x^2} \\ &\equiv \alpha + \frac{\beta}{x} + \frac{\gamma}{x^2}. \end{aligned}$$

The invariant form of the differential equation is

$$u'' + \left(\alpha + \frac{\beta}{x} + \frac{\gamma}{x^2}\right) u = 0.$$

We consider the following cases:

$\alpha = 0.$

$\beta = 0.$ We immediately have the equation

$$u'' + \frac{\gamma}{x^2} u = 0.$$

$\beta \neq 0.$ We have the equation

$$u'' + \left(\frac{\beta}{x} + \frac{\gamma}{x^2} \right) u = 0.$$

The scale transformation $u(x) = v(\xi)$, $x = \lambda \xi$ yields

$$v'' + \left(\frac{\beta\lambda}{\xi} + \frac{\gamma}{\xi^2} \right) u = 0.$$

Choosing $\lambda = \beta^{-1}$, we obtain

$$v'' + \left(\frac{1}{\xi} + \frac{\gamma}{\xi^2} \right) u = 0.$$

$\alpha \neq 0.$ The scale transformation $x = \lambda \xi$ yields

$$v'' + \left(\alpha\lambda^2 + \frac{\beta\lambda}{\xi} + \frac{\gamma}{\xi^2} \right) v = 0.$$

Choosing $\lambda = \alpha^{-1/2}$, we obtain

$$v'' + \left(1 + \frac{\alpha^{-1/2}\beta}{\xi} + \frac{\gamma}{\xi^2} \right) v = 0.$$

Solution 21.4
We write the derivatives with respect to x in terms of t.

$$\begin{aligned} x &= \mathrm{e}^t \\ dx &= \mathrm{e}^t dt \\ \frac{\mathrm{d}}{\mathrm{d}x} &= \mathrm{e}^{-t}\frac{\mathrm{d}}{\mathrm{d}t} \\ x\frac{\mathrm{d}}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}t} \end{aligned}$$

Now we express $x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ in terms of t.

$$x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} = x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) - x\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{\mathrm{d}}{\mathrm{d}t}$$

Thus under the change of variables, $x = \mathrm{e}^t$, $y(x) = u(t)$, the Euler equation becomes

$$u'' - u' + a_1 u' + a_0 u = 0$$
$$\boxed{u'' + (a_1 - 1)u' + a_0 u = 0.}$$

Solution 21.5
The transformation

$$y = \exp\left(-\frac{1}{2}\int \frac{1}{x}\, dx\right) = x^{-1/2}u$$

will put the equation in normal form. The invariant is

$$I(x) = \left(1 - \frac{1}{4x^2}\right) - \frac{1}{4}\left(\frac{1}{x^2}\right) - \frac{1}{2}\frac{-1}{x^2} = 1.$$

Thus we have the differential equation

$$u'' + u = 0,$$

with the solution

$$u = c_1 \cos x + c_2 \sin x.$$

The solution of Bessel's equation of order 1/2 is

$$\boxed{y = c_1 x^{-1/2} \cos x + c_2 x^{-1/2} \sin x.}$$

Chapter 22

The Dirac Delta Function

I do not know what I appear to the world; but to myself I seem to have been only like a boy playing on a seashore, and diverting myself now and then by finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

- Sir Issac Newton

22.1 Derivative of the Heaviside Function

The Heaviside function $H(x)$ is defined

$$H(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases}$$

The derivative of the Heaviside function is zero for $x \neq 0$. At $x = 0$ the derivative is undefined. We will represent the derivative of the Heaviside function by the Dirac delta function, $\delta(x)$. The delta function is zero for $x \neq 0$ and infinite at the point $x = 0$. Since the derivative of $H(x)$ is undefined, $\delta(x)$ is not a function in the conventional sense of the word. One can derive the properties of the delta function rigorously, but the treatment in this text will be almost entirely heuristic.

The Dirac delta function is defined by the properties

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0, \\ \infty & \text{for } x = 0, \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

The second property comes from the fact that $\delta(x)$ represents the derivative of $H(x)$. The Dirac delta function is conceptually pictured in Figure 22.1.

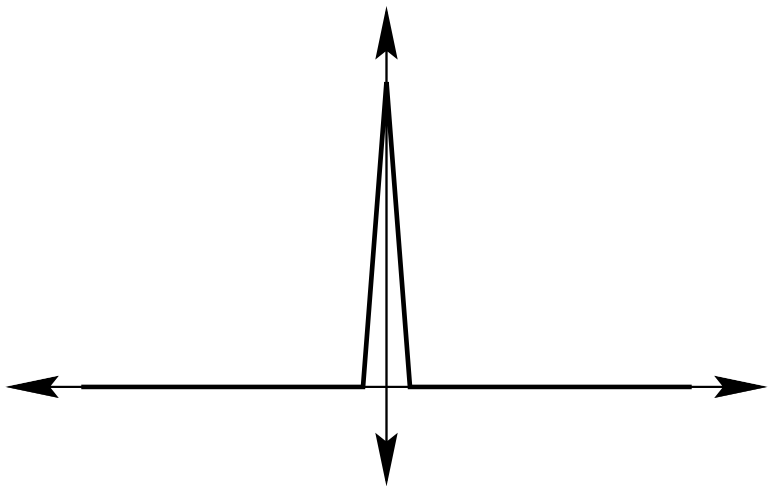

Figure 22.1: The Dirac Delta Function.

Let $f(x)$ be a continuous function that vanishes at infinity. Consider the integral

$$\int_{-\infty}^{\infty} f(x)\delta(x)\, dx.$$

Using integration by parts,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\delta(x)\,dx &= \left[f(x)H(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)H(x)\,dx \\ &= -\int_{0}^{\infty} f'(x)\,dx \\ &= [-f(x)]_0^{\infty} \\ &= f(0).\end{aligned}$$

We assumed that $f(x)$ vanishes at infinity in order to use integration by parts to evaluate the integral. However, since the delta function is zero for $x \neq 0$, the integrand is nonzero only at $x = 0$. Thus the behavior of the function at infinity should not affect the value of the integral. Thus it is reasonable that $f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)\,dx$ holds for all continuous functions. Changing variables and noting that $\delta(x)$ is symmetric we have

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)\,d\xi.$$

This formula is very important in solving inhomogeneous differential equations.

22.2 The Delta Function as a Limit

Consider a function $b(x, \epsilon)$ defined by

$$b(x,\epsilon) = \begin{cases} 0 & \text{for } |x| > \epsilon/2 \\ \frac{1}{\epsilon} & \text{for } |x| < \epsilon/2. \end{cases}$$

The graph of $b(x, 1/10)$ is shown in Figure 22.2.

The Dirac delta function $\delta(x)$ can be thought of as $b(x,\epsilon)$ in the limit as $\epsilon \to 0$. Note that the delta function so defined satisfies the properties,

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0 \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

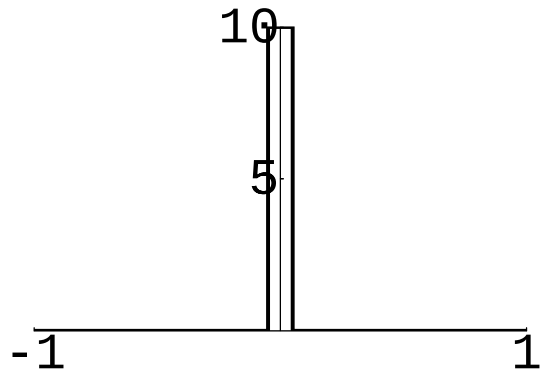

Figure 22.2: Graph of $b(x, 1/10)$.

Delayed Limiting Process. When the Dirac delta function appears inside an integral, we can think of the delta function as a delayed limiting process. That is,

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx \equiv \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(x)b(x,\epsilon)\,dx.$$

Let $f(x)$ be a continuous function and let $F'(x) = f(x)$. The integral of $f(x)\delta(x)$ is then

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\delta(x)\,dx &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-\epsilon/2}^{\epsilon/2} f(x)\,dx \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}[F(x)]_{-\epsilon/2}^{\epsilon/2} \\ &= \lim_{\epsilon\to 0}\frac{F(\epsilon/2)-F(-\epsilon/2)}{\epsilon} \\ &= F'(0) \\ &= f(0).\end{aligned}$$

22.3 Higher Dimensions

We can define a Dirac delta function in n-dimensional Cartesian space, $\delta_n(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$. It is defined by the following two properties.

$$\delta_n(\mathbf{x}) = 0 \quad \text{for} \quad \mathbf{x} \neq \mathbf{0}$$
$$\int_{\mathbb{R}^n} \delta_n(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 1$$

It is easy to verify, that the n-dimensional Dirac delta function can be written as a product of 1-dimensional Dirac delta functions.

$$\delta_n(\mathbf{x}) = \prod_{k=1}^{n} \delta(x_k)$$

22.4 Non-Rectangular Coordinate Systems

We can derive Dirac delta functions in non-rectangular coordinate systems by making a change of variables in the relation,

$$\int_{\mathbb{R}^n} \delta_n(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 1$$

Where the transformation is non-singular, one merely divides the Dirac delta function by the Jacobian of the transformation to the coordinate system.

Example 22.4.1 Consider the Dirac delta function in cylindrical coordinates, (r, θ, z). The Jacobian is $J = r$.

$$\int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\infty} \delta_3 \left(\mathbf{x} - \mathbf{x}_0\right) r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z = 1$$

For $r_0 \neq 0$, the Dirac Delta function is

$$\delta_3(\mathbf{x} - \mathbf{x}_0) = \frac{1}{r}\delta(r - r_0)\,\delta(\theta - \theta_0)\,\delta(z - z_0)$$

since it satisfies the two defining properties.

$$\frac{1}{r}\delta(r - r_0)\,\delta(\theta - \theta_0)\,\delta(z - z_0) = 0 \quad \text{for} \quad (r, \theta, z) \neq (r_0, \theta_0, z_0)$$

$$\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \frac{1}{r}\delta(r - r_0)\,\delta(\theta - \theta_0)\,\delta(z - z_0)\, r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$
$$= \int_0^{\infty} \delta(r - r_0)\,\mathrm{d}r \int_0^{2\pi} \delta(\theta - \theta_0)\,\mathrm{d}\theta \int_{-\infty}^{\infty} \delta(z - z_0)\,\mathrm{d}z = 1$$

For $r_0 = 0$, we have

$$\delta_3(\mathbf{x} - \mathbf{x}_0) = \frac{1}{2\pi r}\delta(r)\,\delta(z - z_0)$$

since this again satisfies the two defining properties.

$$\frac{1}{2\pi r}\delta(r)\,\delta(z - z_0) = 0 \quad \text{for} \quad (r, z) \neq (0, z_0)$$
$$\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi r}\delta(r)\,\delta(z - z_0)\, r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \frac{1}{2\pi}\int_0^{\infty} \delta(r)\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_{-\infty}^{\infty} \delta(z - z_0)\,\mathrm{d}z = 1$$

22.5 Exercises

Exercise 22.1

Let $f(x)$ be a function that is continuous except for a jump discontinuity at $x = 0$. Using a delayed limiting process, show that

$$\frac{f(0^-) + f(0^+)}{2} = \int_{-\infty}^{\infty} f(x)\delta(x)\,dx.$$

Hint, Solution

Exercise 22.2

Let $y = y(x)$ be defined on some interval Assume $y(x)$ is continuously differentiable and that $y'(x) \neq 0$. Show that

$$\delta(x - x_0) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1} \delta(y - y_0)$$

where $y_0 = y(x_0)$.

Hint, Solution

Exercise 22.3

Determine the Dirac delta function in spherical coordinates, (r, θ, ϕ).

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi$$

Hint, Solution

22.6 Hints

Hint 22.1

Hint 22.2
Make a change of variables in the integral

$$\int \delta(x - x_0)\, \mathrm{d}x.$$

Hint 22.3
Consider the special cases $\phi_0 = 0, \pi$ and $r_0 = 0$.

22.7 Solutions

Solution 22.1
Let $F'(x) = f(x)$.

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)\delta(x)\,dx &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{-\infty}^{\infty} f(x) b(x,\epsilon)\,dx \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(\int_{-\epsilon/2}^{0} f(x) b(x,\epsilon)\,dx + \int_{0}^{\epsilon/2} f(x) b(x,\epsilon)\,dx \right) \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left([F(0) - F(-\epsilon/2)] + [F(\epsilon/2) - F(0)] \right) \\
&= \lim_{\epsilon\to 0} \frac{1}{2} \left(\frac{F(0) - F(-\epsilon/2)}{\epsilon/2} + \frac{F(\epsilon/2) - F(0)}{\epsilon/2} \right) \\
&= \frac{F'(0^-) + F'(0^+)}{2} \\
&= \frac{f(0^-) + f(0^+)}{2}
\end{aligned}
$$

Solution 22.2
We prove the identity by making a change of variables in the integral of $\delta(x - x_0)$.

$$
\int_a^b \delta(x - x_0)\,\mathrm{d}x = \int_{y(a)}^{y(b)} \delta(y - y_0) \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)^{-1} \mathrm{d}y
$$

$$
\delta(x - x_0) = \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)^{-1} \delta(y - y_0)
$$

Solution 22.3

We consider the Dirac delta function in spherical coordinates, (r, θ, ϕ). The Jacobian is $J = r^2 \sin(\phi)$.

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} \delta_3(\mathbf{x} - \mathbf{x}_0)\, r^2 \sin(\phi)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi = 1$$

For $r_0 \neq 0$, and $\phi_0 \neq 0, \pi$, the Dirac Delta function is

$$\delta_3(\mathbf{x} - \mathbf{x}_0) = \frac{1}{r^2 \sin(\phi)} \delta(r - r_0)\, \delta(\theta - \theta_0)\, \delta(\phi - \phi_0)$$

since it satisfies the two defining properties.

$$\frac{1}{r^2 \sin(\phi)} \delta(r - r_0)\, \delta(\theta - \theta_0)\, \delta(\phi - \phi_0) = 0 \quad \text{for} \quad (r, \theta, \phi) \neq (r_0, \theta_0, \phi_0)$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} \frac{1}{r^2 \sin(\phi)} \delta(r - r_0)\, \delta(\theta - \theta_0)\, \delta(\phi - \phi_0)\, r^2 \sin(\phi)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$$
$$= \int_0^{\infty} \delta(r - r_0)\, \mathrm{d}r \int_0^{2\pi} \delta(\theta - \theta_0)\, \mathrm{d}\theta \int_0^{\pi} \delta(\phi - \phi_0)\, \mathrm{d}\phi = 1$$

For $\phi_0 = 0$ or $\phi_0 = \pi$, the Dirac delta function is

$$\delta_3(\mathbf{x} - \mathbf{x}_0) = \frac{1}{2\pi r^2 \sin(\phi)} \delta(r - r_0)\, \delta(\phi - \phi_0).$$

We check that the value of the integral is unity.

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi r^2 \sin(\phi)} \delta(r - r_0)\, \delta(\phi - \phi_0)\, r^2 \sin(\phi)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$$
$$= \frac{1}{2\pi} \int_0^{\infty} \delta(r - r_0)\, \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \delta(\phi - \phi_0)\, \mathrm{d}\phi = 1$$

For $r_0 = 0$ the Dirac delta function is

$$\delta_3(\mathbf{x}) = \frac{1}{4\pi r^2}\delta(r)$$

We verify that the value of the integral is unity.

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\frac{1}{4\pi r^2}\delta(r-r_0)\,r^2\sin(\phi)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{1}{4\pi}\int_0^{\infty}\delta(r)\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin(\phi)\,\mathrm{d}\phi = 1$$

Chapter 23

Inhomogeneous Differential Equations

Feelin' stupid? I know I am!

-Homer Simpson

23.1 Particular Solutions

Consider the n^{th} order linear homogeneous equation

$$L[y] \equiv y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0.$$

Let $\{y_1, y_2, \ldots, y_n\}$ be a set of linearly independent homogeneous solutions, $L[y_k] = 0$. We know that the general solution of the homogeneous equation is a linear combination of the homogeneous solutions.

$$y_h = \sum_{k=1}^{n} c_k y_k(x)$$

Now consider the n^{th} order linear *inhomogeneous* equation

$$L[y] \equiv y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x).$$

Any function y_p which satisfies this equation is called a *particular solution* of the differential equation. We want to know the general solution of the inhomogeneous equation. Later in this chapter we will cover methods of constructing this solution; now we consider the form of the solution.

Let y_p be a particular solution. Note that y_p+h is a particular solution if h satisfies the homogeneous equation.

$$L[y_p + h] = L[y_p] + L[h] = f + 0 = f$$

Therefore $y_p + y_h$ satisfies the homogeneous equation. We show that this is the general solution of the inhomogeneous equation. Let y_p and η_p both be solutions of the inhomogeneous equation $L[y] = f$. The difference of y_p and η_p is a homogeneous solution.

$$L[y_p - \eta_p] = L[y_p] - L[\eta_p] = f - f = 0$$

y_p and η_p differ by a linear combination of the homogeneous solutions $\{y_k\}$. Therefore the general solution of $L[y] = f$ is the sum of any particular solution y_p and the general homogeneous solution y_h.

$$y_p + y_h = y_p(x) + \sum_{k=1}^{n} c_k y_k(x)$$

Result 23.1.1 The general solution of the n^{th} order linear inhomogeneous equation $L[y] = f(x)$ is

$$y = y_p + c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

where y_p is a particular solution, $\{y_1, \ldots, y_n\}$ is a set of linearly independent homogeneous solutions, and the c_k's are arbitrary constants.

Example 23.1.1 The differential equation

$$y'' + y = \sin(2x)$$

has the two homogeneous solutions

$$y_1 = \cos x, \qquad y_2 = \sin x,$$

and a particular solution

$$y_p = -\frac{1}{3}\sin(2x).$$

We can add any combination of the homogeneous solutions to y_p and it will still be a particular solution. For example,

$$\begin{aligned}\eta_p &= -\frac{1}{3}\sin(2x) - \frac{1}{3}\sin x \\ &= -\frac{2}{3}\sin\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)\end{aligned}$$

is a particular solution.

23.2 Method of Undetermined Coefficients

The first method we present for computing particular solutions is the *method of undetermined coefficients*. For some simple differential equations, (primarily constant coefficient equations), and some simple inhomogeneities we are able to guess the form of a particular solution. This form will contain some unknown parameters. We substitute this form into the differential equation to determine the parameters and thus determine a particular solution.

Later in this chapter we will present general methods which work for any linear differential equation and any inhogeneity. Thus one might wonder why I would present a method that works only for some simple problems. (And why it is called a "method" if it amounts to no more than guessing.) The answer is that guessing an answer is less grungy than computing it with the formulas we will develop later. Also, the process of this guessing is not random, there is rhyme and reason to it.

Consider an n^{th} order constant coefficient, inhomogeneous equation.

$$L[y] \equiv y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

If $f(x)$ is one of a few simple forms, then we can guess the form of a particular solution. Below we enumerate some cases.

$\mathbf{f = p(x).}$ If f is an m^{th} order polynomial, $f(x) = p_m x^m + \cdots + p_1 x + p_0$, then guess

$$y_p = c_m x^m + \cdots c_1 x + c_0.$$

$\mathbf{f = p(x)\,e^{ax}.}$ If f is a polynomial times an exponential then guess

$$y_p = (c_m x^m + \cdots c_1 x + c_0)\,\mathrm{e}^{ax}.$$

$\mathbf{f = p(x)\,e^{ax}\cos(bx).}$ If f is a cosine or sine times a polynomial and perhaps an exponential, $f(x) = p(x)\,\mathrm{e}^{ax}\cos(bx)$ or $f(x) = p(x)\,\mathrm{e}^{ax}\sin(bx)$ then guess

$$y_p = (c_m x^m + \cdots c_1 x + c_0)\,\mathrm{e}^{ax}\cos(bx) + (d_m x^m + \cdots d_1 x + d_0)\,\mathrm{e}^{ax}\sin(bx).$$

Likewise for hyperbolic sines and hyperbolic cosines.

Example 23.2.1 Consider

$$y'' - 2y' + y = t^2.$$

The homogeneous solutions are $y_1 = \mathrm{e}^t$ and $y_2 = t\,\mathrm{e}^t$. We guess a particular solution of the form

$$y_p = at^2 + bt + c.$$

We substitute the expression into the differential equation and equate coefficients of powers of t to determine the parameters.

$$
\begin{gathered}
y_p'' - 2y_p' + y_p = t^2 \\
(2a) - 2(2at + b) + (at^2 + bt + c) = t^2 \\
(a-1)t^2 + (b - 4a)t + (2a - 2b + c) = 0 \\
a - 1 = 0, \quad b - 4a = 0, \quad 2a - 2b + c = 0 \\
a = 1, \quad b = 4, \quad c = 6
\end{gathered}
$$

A particular solution is

$$
y_p = t^2 + 4t + 6.
$$

If the inhomogeneity is a sum of terms, $L[y] = f \equiv f_1 + \cdots + f_k$, you can solve the problems $L[y] = f_1, \ldots, L[y] = f_k$ independently and then take the sum of the solutions as a particular solution of $L[y] = f$.

Example 23.2.2 Consider

$$
L[y] \equiv y'' - 2y' + y = t^2 + \mathrm{e}^{2t}. \tag{23.1}
$$

The homogeneous solutions are $y_1 = \mathrm{e}^t$ and $y_2 = t\,\mathrm{e}^t$. We already know a particular solution to $L[y] = t^2$. We seek a particular solution to $L[y] = \mathrm{e}^{2t}$. We guess a particular solution of the form

$$
y_p = a\,\mathrm{e}^{2t}.
$$

We substitute the expression into the differential equation to determine the parameter.

$$
\begin{gathered}
y_p'' - 2y_p' + y_p = \mathrm{e}^{2t} \\
4ae^{2t} - 4a\,\mathrm{e}^{2t} + a\,\mathrm{e}^{2t} = \mathrm{e}^{2t} \\
a = 1
\end{gathered}
$$

A particular solution of $L[y] = \mathrm{e}^{2t}$ is $y_p = \mathrm{e}^{2t}$. Thus a particular solution of Equation 23.1 is

$$
y_p = t^2 + 4t + 6 + \mathrm{e}^{2t}.
$$

The above guesses will not work if the inhomogeneity is a homogeneous solution. In this case, multiply the guess by the lowest power of x such that the guess does not contain homogeneous solutions.

Example 23.2.3 Consider

$$L[y] \equiv y'' - 2y' + y = \mathrm{e}^t.$$

The homogeneous solutions are $y_1 = \mathrm{e}^t$ and $y_2 = t\,\mathrm{e}^t$. Guessing a particular solution of the form $y_p = a\,\mathrm{e}^t$ would not work because $L[\mathrm{e}^t] = 0$. We guess a particular solution of the form

$$y_p = at^2\,\mathrm{e}^t$$

We substitute the expression into the differential equation and equate coefficients of like terms to determine the parameters.

$$\begin{aligned} y_p'' - 2y_p' + y_p &= \mathrm{e}^t \\ (at^2 + 4at + 2a)\,\mathrm{e}^t - 2(at^2 + 2at)\,\mathrm{e}^t + at^2\,\mathrm{e}^t &= \mathrm{e}^t \\ 2a\,\mathrm{e}^t &= \mathrm{e}^t \\ a &= \frac{1}{2} \end{aligned}$$

A particular solution is

$$y_p = \frac{t^2}{2}\,\mathrm{e}^t.$$

Example 23.2.4 Consider

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = x, \quad x > 0.$$

The homogeneous solutions are $y_1 = \cos(\ln x)$ and $y_2 = \sin(\ln x)$. We guess a particular solution of the form

$$y_p = ax^3$$

We substitute the expression into the differential equation and equate coefficients of like terms to determine the parameter.

$$y_p'' + \frac{1}{x} y_p' + \frac{1}{x^2} y_p = x$$
$$6ax + 3ax + ax = x$$
$$a = \frac{1}{10}$$

A particular solution is

$$y_p = \frac{x^3}{10}.$$

23.3 Variation of Parameters

In this section we present a method for computing a particular solution of an inhomogeneous equation given that we know the homogeneous solutions. We will first consider second order equations and then generalize the result for n^{th} order equations.

23.3.1 Second Order Differential Equations

Consider the second order inhomogeneous equation,

$$L[y] \equiv y'' + p(x) y' + q(x) y = f(x), \quad \text{on } a < x < b.$$

We assume that the coefficient functions in the differential equation are continuous on $[a \ldots b]$. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions to the homogeneous equation. Since the Wronskian,

$$W(x) = \exp\left(- \int p(x) \, \mathrm{d}x \right),$$

is non-vanishing, we know that these solutions exist. We seek a particular solution of the form,

$$y_p = u_1(x)y_1 + u_2(x)y_2.$$

We compute the derivatives of y_p.

$$\begin{aligned} y_p' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \\ y_p'' &= u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2'' \end{aligned}$$

We substitute the expression for y_p and its derivatives into the inhomogeneous equation and use the fact that y_1 and y_2 are homogeneous solutions to simplify the equation.

$$u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2'' + p(u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2') + q(u_1y_1 + u_2y_2) = f$$
$$u_1''y_1 + 2u_1'y_1' + u_2''y_2 + 2u_2'y_2' + p(u_1'y_1 + u_2'y_2) = f$$

This is an ugly equation for u_1 and u_2, however, we have an ace up our sleeve. Since u_1 and u_2 are undetermined functions of x, we are free to impose a constraint. We choose this constraint to simplify the algebra.

$$u_1'y_1 + u_2'y_2 = 0$$

This constraint simplifies the derivatives of y_p,

$$\begin{aligned} y_p' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \\ &= u_1y_1' + u_2y_2' \\ y_p'' &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''. \end{aligned}$$

We substitute the new expressions for y_p and its derivatives into the inhomogeneous differential equation to obtain a much simpler equation than before.

$$u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) = f(x)$$
$$u_1'y_1' + u_2'y_2' + u_1L[y_1] + u_2L[y_2] = f(x)$$
$$u_1'y_1' + u_2'y_2' = f(x).$$

With the constraint, we have a system of linear equations for u_1' and u_2'.

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(x). \end{aligned}$$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

We solve this system using Kramer's rule. (See Appendix S.)

$$u_1' = -\frac{f(x) y_2}{W(x)} \qquad u_2' = \frac{f(x) y_1}{W(x)}$$

Here $W(x)$ is the Wronskian.

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

We integrate to get u_1 and u_2. This gives us a particular solution.

$$y_p = -y_1 \int \frac{f(x) y_2(x)}{W(x)} \, \mathrm{d}x + y_2 \int \frac{f(x) y_1(x)}{W(x)} \, \mathrm{d}x.$$

Result 23.3.1 Let y_1 and y_2 be linearly independent homogeneous solutions of

$$L[y] = y'' + p(x) y' + q(x) y = f(x).$$

A particular solution is

$$y_p = -y_1(x) \int \frac{f(x) y_2(x)}{W(x)} \, \mathrm{d}x + y_2(x) \int \frac{f(x) y_1(x)}{W(x)} \, \mathrm{d}x,$$

where $W(x)$ is the Wronskian of y_1 and y_2.

Example 23.3.1 Consider the equation,

$$y'' + y = \cos(2x).$$

The homogeneous solutions are $y_1 = \cos x$ and $y_2 = \sin x$. We compute the Wronskian.

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

We use variation of parameters to find a particular solution.

$$\begin{aligned}
y_p &= -\cos(x) \int \cos(2x)\sin(x)\,\mathrm{d}x + \sin(x) \int \cos(2x)\cos(x)\,\mathrm{d}x \\
&= -\frac{1}{2}\cos(x) \int \big(\sin(3x) - \sin(x)\big)\,\mathrm{d}x + \frac{1}{2}\sin(x) \int \big(\cos(3x) + \cos(x)\big)\,\mathrm{d}x \\
&= -\frac{1}{2}\cos(x)\left(-\frac{1}{3}\cos(3x) + \cos(x)\right) + \frac{1}{2}\sin(x)\left(\frac{1}{3}\sin(3x) + \sin(x)\right) \\
&= \frac{1}{2}\big(\sin^2(x) - \cos^2(x)\big) + \frac{1}{6}\big(\cos(3x)\cos(x) + \sin(3x)\sin(x)\big) \\
&= -\frac{1}{2}\cos(2x) + \frac{1}{6}\cos(2x) \\
&= -\frac{1}{3}\cos(2x)
\end{aligned}$$

The general solution of the inhomogeneous equation is

$$\boxed{y = -\frac{1}{3}\cos(2x) + c_1\cos(x) + c_2\sin(x).}$$

23.3.2 Higher Order Differential Equations

Consider the n^{th} order inhomogeneous equation,

$$L[y] = y(n) + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x), \quad \text{on } a < x < b.$$

We assume that the coefficient functions in the differential equation are continuous on $[a \dots b]$. Let $\{y_1, \dots, y_n\}$ be a set of linearly independent solutions to the homogeneous equation. Since the Wronskian,

$$W(x) = \exp\left(-\int p_{n-1}(x)\,\mathrm{d}x\right),$$

is non-vanishing, we know that these solutions exist. We seek a particular solution of the form

$$y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n.$$

Since $\{u_1, \dots, u_n\}$ are undetermined functions of x, we are free to impose $n-1$ constraints. We choose these constraints to simplify the algebra.

$$\begin{aligned}
u_1' y_1 &+ u_2' y_2 &+ \cdots &+ u_n' y_n &= 0 \\
u_1' y_1' &+ u_2' y_2' &+ \cdots &+ u_n' y_n' &= 0 \\
\vdots &+ \vdots &+ \vdots &+ \vdots &= 0 \\
u_1' y_1^{(n-2)} &+ u_2' y_2^{(n-2)} &+ \cdots &+ u_n' y_n^{(n-2)} &= 0
\end{aligned}$$

We differentiate the expression for y_p, utilizing our constraints.

$$\begin{aligned}
y_p &= u_1 y_1 &+ u_2 y_2 &+ \cdots + u_n y_n \\
y_p' &= u_1 y_1' &+ u_2 y_2' &+ \cdots + u_n y_n' \\
y_p'' &= u_1 y_1'' &+ u_2 y_2'' &+ \cdots + u_n y_n'' \\
\vdots &= \vdots &+ \vdots &+ \vdots + \vdots \\
y_p^{(n)} &= u_1 y_1^{(n)} &+ u_2 y_2^{(n)} &+ \cdots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)}
\end{aligned}$$

We substitute y_p and its derivatives into the inhomogeneous differential equation and use the fact that the y_k are homogeneous solutions.

$$u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} + p_{n-1}(u_1 y_1^{(n-1)} + \cdots + u_n y_n^{(n-1)}) + \cdots + p_0(u_1 y_1 + \cdots u_n y_n) = f$$
$$u_1 L[y_1] + u_2 L[y_2] + \cdots + u_n L[y_n] + u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = f$$
$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = f.$$

With the constraints, we have a system of linear equations for $\{u_1, \ldots, u_n\}$.

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix}.$$

We solve this system using Kramer's rule. (See Appendix S.)

$$u_k' = (-1)^{n+k+1} \frac{W[y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n]}{W[y_1, y_2, \ldots, y_n]} f, \quad \text{for } k = 1, \ldots, n,$$

Here W is the Wronskian.

We integrating to obtain the u_k's.

$$u_k = (-1)^{n+k+1} \int \frac{W[y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n](x)}{W[y_1, y_2, \ldots, y_n](x)} f(x) \, \mathrm{d}x, \quad \text{for } k = 1, \ldots, n$$

Result 23.3.2 Let $\{y_1, \dots, y_n\}$ be linearly independent homogeneous solutions of

$$L[y] = y(n) + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x), \quad \text{on } a < x < b.$$

A particular solution is

$$y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n.$$

where

$$u_k = (-1)^{n+k+1} \int \frac{W[y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n](x)}{W[y_1, y_2, \dots, y_n](x)} f(x)\,\mathrm{d}x, \quad \text{for } k = 1, \dots, n,$$

and $W[y_1, y_2, \dots, y_n](x)$ is the Wronskian of $\{y_1(x), \dots, y_n(x)\}$.

23.4 Piecewise Continuous Coefficients and Inhomogeneities

Example 23.4.1 Consider the problem

$$y'' - y = \mathrm{e}^{-\alpha|x|}, \qquad y(\pm\infty) = 0, \qquad \alpha > 0, \alpha \neq 1.$$

The homogeneous solutions of the differential equation are e^x and e^{-x}. We use variation of parameters to find a particular solution for $x > 0$.

$$\begin{aligned} y_p &= -e^x \int^x \frac{e^{-\xi} e^{-\alpha\xi}}{-2} \, d\xi + e^{-x} \int^x \frac{e^{\xi} e^{-\alpha\xi}}{-2} \, d\xi \\ &= \frac{1}{2} e^x \int^x e^{-(\alpha+1)\xi} \, d\xi - \frac{1}{2} e^{-x} \int^x e^{(1-\alpha)\xi} \, d\xi \\ &= -\frac{1}{2(\alpha+1)} e^{-\alpha x} + \frac{1}{2(\alpha-1)} e^{-\alpha x} \\ &= \frac{e^{-\alpha x}}{\alpha^2 - 1}, \quad \text{for } x > 0 \end{aligned}$$

A particular solution for $x < 0$ is

$$y_p = \frac{e^{\alpha x}}{\alpha^2 - 1}, \quad \text{for } x < 0.$$

Thus a particular solution is

$$y_p = \frac{e^{-\alpha|x|}}{\alpha^2 - 1}.$$

The general solution is

$$y = \frac{1}{\alpha^2 - 1} e^{-\alpha|x|} + c_1 e^x + c_2 e^{-x}.$$

Applying the boundary conditions, we see that $c_1 = c_2 = 0$. Apparently the solution is

$$y = \frac{e^{-\alpha|x|}}{\alpha^2 - 1}.$$

This function is plotted in Figure 23.1. This function satisfies the differential equation for positive and negative x. It also satisfies the boundary conditions. However, this is NOT a solution to the differential equation. Since the differential equation has no singular points and the inhomogeneous term is continuous, the solution must be twice continuously differentiable. Since the derivative of $\mathrm{e}^{-\alpha|x|}/(\alpha^2-1)$ has a jump discontinuity at $x=0$, the second derivative does not exist. Thus this function could not possibly be a solution to the differential equation. In the next example we examine the right way to solve this problem.

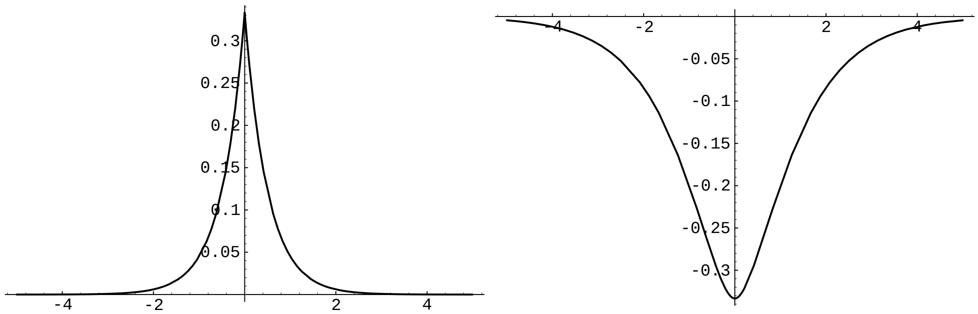

Figure 23.1: The Incorrect and Correct Solution to the Differential Equation.

Example 23.4.2 Again consider

$$y'' - y = \mathrm{e}^{-\alpha|x|}, \qquad y(\pm\infty) = 0, \qquad \alpha > 0, \alpha \neq 1.$$

Separating this into two problems for positive and negative x,

$$\begin{aligned} y_-'' - y_- &= \mathrm{e}^{\alpha x}, \quad y_-(-\infty) = 0, \quad \text{on } -\infty < x \leq 0, \\ y_+'' - y_+ &= \mathrm{e}^{-\alpha x}, \quad y_+(\infty) = 0, \quad \text{on } 0 \leq x < \infty. \end{aligned}$$

In order for the solution over the whole domain to be twice differentiable, the solution and it's first derivative must be continuous. Thus we impose the additional boundary conditions

$$y_-(0) = y_+(0), \qquad y_-'(0) = y_+'(0).$$

The solutions that satisfy the two differential equations and the boundary conditions at infinity are

$$y_- = \frac{\mathrm{e}^{\alpha x}}{\alpha^2 - 1} + c_- \mathrm{e}^x, \qquad y_+ = \frac{\mathrm{e}^{-\alpha x}}{\alpha^2 - 1} + c_+ \mathrm{e}^{-x}.$$

The two additional boundary conditions give us the equations

$$\begin{aligned} y_-(0) = y_+(0) \quad &\rightarrow \quad c_- = c_+ \\ y_-'(0) = y_+'(0) \quad &\rightarrow \quad \frac{\alpha}{\alpha^2 - 1} + c_- = -\frac{\alpha}{\alpha^2 - 1} - c_+. \end{aligned}$$

We solve these two equations to determine c_- and c_+.

$$c_- = c_+ = -\frac{\alpha}{\alpha^2 - 1}$$

Thus the solution over the whole domain is

$$y = \begin{cases} \frac{\mathrm{e}^{\alpha x} - \alpha \mathrm{e}^x}{\alpha^2 - 1} & \text{for } x < 0, \\ \frac{\mathrm{e}^{-\alpha x} - \alpha \mathrm{e}^{-x}}{\alpha^2 - 1} & \text{for } x > 0 \end{cases}$$

$$\boxed{y = \frac{\mathrm{e}^{-\alpha |x|} - \alpha \mathrm{e}^{-|x|}}{\alpha^2 - 1}.}$$

This function is plotted in Figure 23.1. You can verify that this solution is twice continuously differentiable.

23.5 Inhomogeneous Boundary Conditions

23.5.1 Eliminating Inhomogeneous Boundary Conditions

Consider the n^{th} order equation

$$L[y] = f(x), \quad \text{for } a < x < b,$$

subject to the linear inhomogeneous boundary conditions

$$B_j[y] = \gamma_j, \quad \text{for } j = 1, \dots, n,$$

where the boundary conditions are of the form

$$B[y] \equiv \alpha_0 y(a) + \alpha_1 y'(a) + \cdots + y_{n-1} y^{(n-1)}(a) + \beta_0 y(b) + \beta_1 y'(b) + \cdots + \beta_{n-1} y^{(n-1)}$$

Let $g(x)$ be an n-times continuously differentiable function that satisfies the boundary conditions. Substituting $y = u + g$ into the differential equation and boundary conditions yields

$$L[u] = f(x) - L[g], \qquad B_j[u] = b_j - B_j[g] = 0 \quad \text{for } j = 1, \dots, n.$$

Note that the problem for u has homogeneous boundary conditions. Thus a problem with inhomogeneous boundary conditions can be reduced to one with homogeneous boundary conditions. This technique is of limited usefulness for ordinary differential equations but is important for solving some partial differential equation problems.

Example 23.5.1 Consider the problem

$$y'' + y = \cos 2x, \qquad y(0) = 1, \quad y(\pi) = 2.$$

$g(x) = \frac{x}{\pi} + 1$ satisfies the boundary conditions. Substituting $y = u + g$ yields

$$u'' + u = \cos 2x - \frac{x}{\pi} - 1, \qquad y(0) = y(\pi) = 0.$$

Example 23.5.2 Consider

$$y'' + y = \cos 2x, \qquad y'(0) = y(\pi) = 1.$$

$g(x) = \sin x - \cos x$ satisfies the inhomogeneous boundary conditions. Substituting $y = u + \sin x - \cos x$ yields

$$u'' + u = \cos 2x, \qquad u'(0) = u(\pi) = 0.$$

Note that since $g(x)$ satisfies the homogeneous equation, the inhomogeneous term in the equation for u is the same as that in the equation for y.

Example 23.5.3 Consider

$$y'' + y = \cos 2x, \qquad y(0) = \frac{2}{3}, \quad y(\pi) = -\frac{4}{3}.$$

$g(x) = \cos x - \frac{1}{3}$ satisfies the boundary conditions. Substituting $y = u + \cos x - \frac{1}{3}$ yields

$$u'' + u = \cos 2x + \frac{1}{3}, \qquad u(0) = u(\pi) = 0.$$

Result 23.5.1 The n^{th} order differential equation with boundary conditions

$$L[y] = f(x), \qquad B_j[y] = b_j, \quad \text{for } j = 1, \ldots, n$$

has the solution $y = u + g$ where u satisfies

$$L[u] = f(x) - L[g], \qquad B_j[u] = 0, \quad \text{for } j = 1, \ldots, n$$

and g is any n-times continuously differentiable function that satisfies the inhomogeneous boundary conditions.

23.5.2 Separating Inhomogeneous Equations and Inhomogeneous Boundary Conditions

Now consider a problem with inhomogeneous boundary conditions

$$L[y] = f(x), \quad B_1[y] = \gamma_1, \quad B_2[y] = \gamma_2.$$

In order to solve this problem, we solve the two problems

$$L[u] = f(x), \quad B_1[u] = B_2[u] = 0, \quad \text{and}$$

$$L[v] = 0, \quad B_1[v] = \gamma_1, \quad B_2[v] = \gamma_2.$$

The solution for the problem with an inhomogeneous equation and inhomogeneous boundary conditions will be the sum of u and v. To verify this,

$$L[u+v] = L[u] + L[v] = f(x) + 0 = f(x),$$
$$B_i[u+v] = B_i[u] + B_i[v] = 0 + \gamma_i = \gamma_i.$$

This will be a useful technique when we develop Green functions.

Result 23.5.2 The solution to

$$L[y] = f(x), \quad B_1[y] = \gamma_1, \quad B_2[y] = \gamma_2,$$

is $y = u + v$ where

$$\begin{aligned} L[u] &= f(x), \quad & B_1[u] &= 0, \quad & B_2[u] &= 0, \quad \text{and} \\ L[v] &= 0, \quad & B_1[v] &= \gamma_1, \quad & B_2[v] &= \gamma_2. \end{aligned}$$

23.5.3 Existence of Solutions of Problems with Inhomogeneous Boundary Conditions

Consider the n^{th} order homogeneous differential equation

$$L[y] = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = f(x), \quad \text{for } a < x < b,$$

subject to the n inhomogeneous boundary conditions

$$B_j[y] = \gamma_j, \quad \text{for } j = 1, \ldots, n$$

where each boundary condition is of the form

$$B[y] \equiv \alpha_0 y(a) + \alpha_1 y'(a) + \cdots + \alpha_{n-1} y^{(n-1)}(a) + \beta_0 y(b) + \beta_1 y'(b) + \cdots + \beta_{n-1} y^{(n-1)}(b).$$

We assume that the coefficients in the differential equation are continuous on $[a, b]$. Since the Wronskian of the solutions of the differential equation,

$$W(x) = \exp\left(-\int p_{n-1}(x)\,\mathrm{d}x\right),$$

is non-vanishing on $[a, b]$, there are n linearly independent solution on that range. Let $\{y_1, \ldots, y_n\}$ be a set of linearly independent solutions of the homogeneous equation. From Result 23.3.2 we know that a particular solution y_p exists. The general solution of the differential equation is

$$y = y_p + c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

The n boundary conditions impose the matrix equation,

$$\begin{pmatrix} B_1[y_1] & B_1[y_2] & \cdots & B_1[y_n] \\ B_2[y_1] & B_2[y_2] & \cdots & B_2[y_n] \\ \vdots & \vdots & \ddots & \vdots \\ B_n[y_1] & B_n[y_2] & \cdots & B_n[y_n] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \gamma_1 - B_1[y_p] \\ \gamma_2 - B_2[y_p] \\ \vdots \\ \gamma_n - B_n[y_p] \end{pmatrix}$$

This equation has a unique solution if and only if the equation

$$\begin{pmatrix} B_1[y_1] & B_1[y_2] & \cdots & B_1[y_n] \\ B_2[y_1] & B_2[y_2] & \cdots & B_2[y_n] \\ \vdots & \vdots & \ddots & \vdots \\ B_n[y_1] & B_n[y_2] & \cdots & B_n[y_n] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

has only the trivial solution. (This is the case if and only if the determinant of the matrix is nonzero.) Thus the problem

$$L[y] = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = f(x), \quad \text{for } a < x < b,$$

subject to the n inhomogeneous boundary conditions

$$B_j[y] = \gamma_j, \quad \text{for } j = 1, \ldots, n,$$

has a unique solution if and only if the problem

$$L[y] = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0, \quad \text{for } a < x < b,$$

subject to the n homogeneous boundary conditions

$$B_j[y] = 0, \quad \text{for } j = 1, \ldots, n,$$

has only the trivial solution.

Result 23.5.3 The problem

$$L[y] = y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = f(x), \quad \text{for } a < x < b,$$

subject to the n inhomogeneous boundary conditions

$$B_j[y] = \gamma_j, \quad \text{for } j = 1, \ldots, n,$$

has a unique solution if and only if the problem

$$L[y] = y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0, \quad \text{for } a < x < b,$$

subject to

$$B_j[y] = 0, \quad \text{for } j = 1, \ldots, n,$$

has only the trivial solution.

23.6 Green Functions for First Order Equations

Consider the first order inhomogeneous equation

$$L[y] \equiv y' + p(x) y = f(x), \quad \text{for } x > a, \tag{23.2}$$

subject to a homogeneous initial condition, $B[y] \equiv y(a) = 0$.

The Green function $G(x|\xi)$ is defined as the solution to

$$L[G(x|\xi)] = \delta(x - \xi) \quad \text{subject to } G(a|\xi) = 0.$$

We can represent the solution to the inhomogeneous problem in Equation 23.2 as an integral involving the Green function. To show that

$$y(x) = \int_a^\infty G(x|\xi) f(\xi) \, \mathrm{d}\xi$$

is the solution, we apply the linear operator L to the integral. (Assume that the integral is uniformly convergent.)

$$\begin{aligned} L\left[\int_a^\infty G(x|\xi) f(\xi) \, \mathrm{d}\xi\right] &= \int_a^\infty L[G(x|\xi)] f(\xi) \, \mathrm{d}\xi \\ &= \int_a^\infty \delta(x-\xi) f(\xi) \, \mathrm{d}\xi \\ &= f(x) \end{aligned}$$

The integral also satisfies the initial condition.

$$\begin{aligned} B\left[\int_a^\infty G(x|\xi) f(\xi) \, \mathrm{d}\xi\right] &= \int_a^\infty B[G(x|\xi)] f(\xi) \, \mathrm{d}\xi \\ &= \int_a^\infty (0) f(\xi) \, \mathrm{d}\xi \\ &= 0 \end{aligned}$$

Now we consider the qualitiative behavior of the Green function. For $x \neq \xi$, the Green function is simply a homogeneous solution of the differential equation, however at $x = \xi$ we expect some singular behavior. $G'(x|\xi)$ will have a Dirac delta function type singularity. This means that $G(x|\xi)$ will have a jump discontinuity at $x = \xi$. We integrate the differential equation on the vanishing interval $(\xi^- \ldots \xi^+)$ to determine this jump.

$$\begin{gathered} G' + p(x)G = \delta(x-\xi) \\ G(\xi^+|\xi) - G(\xi^-|\xi) + \int_{\xi^-}^{\xi^+} p(x) G(x|\xi) \, \mathrm{d}x = 1 \\ G(\xi^+|\xi) - G(\xi^-|\xi) = 1 \end{gathered} \tag{23.3}$$

The homogeneous solution of the differential equation is

$$y_h = \mathrm{e}^{-\int p(x)\,\mathrm{d}x}$$

Since the Green function satisfies the homogeneous equation for $x \neq \xi$, it will be a constant times this homogeneous solution for $x < \xi$ and $x > \xi$.

$$G(x|\xi) = \begin{cases} c_1\,\mathrm{e}^{-\int p(x)\,\mathrm{d}x} & a < x < \xi \\ c_2\,\mathrm{e}^{-\int p(x)\,\mathrm{d}x} & \xi < x \end{cases}$$

In order to satisfy the homogeneous initial condition $G(a|\xi) = 0$, the Green function must vanish on the interval $(a \ldots \xi)$.

$$G(x|\xi) = \begin{cases} 0 & a < x < \xi \\ c\,\mathrm{e}^{-\int p(x)\,\mathrm{d}x} & \xi < x \end{cases}$$

The jump condition, (Equation 23.3), gives us the constraint $G(\xi^+|\xi) = 1$. This determines the constant in the homogeneous solution for $x > \xi$.

$$G(x|\xi) = \begin{cases} 0 & a < x < \xi \\ \mathrm{e}^{-\int_\xi^x p(t)\,\mathrm{d}t} & \xi < x \end{cases}$$

We can use the Heaviside function to write the Green function without using a case statement.

$$G(x|\xi) = \mathrm{e}^{-\int_\xi^x p(t)\,\mathrm{d}t} H(x - \xi)$$

Clearly the Green function is of little value in solving the inhomogeneous differential equation in Equation 23.2, as we can solve that problem directly. However, we will encounter first order Green function problems in solving some partial differential equations.

Result 23.6.1 The first order inhomogeneous differential equation with homogeneous initial condition

$$L[y] \equiv y' + p(x)y = f(x), \quad \text{for } a < x, \qquad y(a) = 0,$$

has the solution

$$y = \int_a^\infty G(x|\xi) f(\xi) \, \mathrm{d}\xi,$$

where $G(x|\xi)$ satisfies the equation

$$L[G(x|\xi)] = \delta(x - \xi), \quad \text{for } a < x, \qquad G(a|\xi) = 0.$$

The Green function is

$$G(x|\xi) = \mathrm{e}^{-\int_\xi^x p(t) \, \mathrm{d}t} H(x - \xi)$$

23.7 Green Functions for Second Order Equations

Consider the second order inhomogeneous equation

$$L[y] = y'' + p(x)y' + q(x)y = f(x), \quad \text{for } a < x < b, \tag{23.4}$$

subject to the homogeneous boundary conditions

$$B_1[y] = B_2[y] = 0.$$

The Green function $G(x|\xi)$ is defined as the solution to

$$L[G(x|\xi)] = \delta(x - \xi) \quad \text{subject to } B_1[G] = B_2[G] = 0.$$

The Green function is useful because you can represent the solution to the inhomogeneous problem in Equation 23.4 as an integral involving the Green function. To show that

$$y(x) = \int_a^b G(x|\xi) f(\xi) \,\mathrm{d}\xi$$

is the solution, we apply the linear operator L to the integral. (Assume that the integral is uniformly convergent.)

$$\begin{aligned} L\left[\int_a^b G(x|\xi) f(\xi) \,\mathrm{d}\xi\right] &= \int_a^b L[G(x|\xi)] f(\xi) \,\mathrm{d}\xi \\ &= \int_a^b \delta(x-\xi) f(\xi) \,\mathrm{d}\xi \\ &= f(x) \end{aligned}$$

The integral also satisfies the boundary conditions.

$$\begin{aligned} B_i\left[\int_a^b G(x|\xi) f(\xi) \,\mathrm{d}\xi\right] &= \int_a^b B_i[G(x|\xi)] f(\xi) \,\mathrm{d}\xi \\ &= \int_a^b [0] f(\xi) \,\mathrm{d}\xi \\ &= 0 \end{aligned}$$

One of the advantages of using Green functions is that once you find the Green function for a linear operator and certain homogeneous boundary conditions,

$$L[G] = \delta(x-\xi), \quad B_1[G] = B_2[G] = 0,$$

you can write the solution for any inhomogeneity, $f(x)$.

$$L[f] = f(x), \quad B_1[y] = B_2[y] = 0$$

You do not need to do any extra work to obtain the solution for a different inhomogeneous term.

Qualitatively, what kind of behavior will the Green function for a second order differential equation have? Will it have a delta function singularity; will it be continuous? To answer these questions we will first look at the behavior of integrals and derivatives of $\delta(x)$.

The integral of $\delta(x)$ is the Heaviside function, $H(x)$.

$$H(x) = \int_{-\infty}^{x} \delta(t)\,\mathrm{d}t = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

The integral of the Heaviside function is the ramp function, $r(x)$.

$$r(x) = \int_{-\infty}^{x} H(t)\,\mathrm{d}t = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x > 0 \end{cases}$$

The derivative of the delta function is zero for $x \neq 0$. At $x = 0$ it goes from 0 up to $+\infty$, down to $-\infty$ and then back up to 0.

In Figure 23.2 we see conceptually the behavior of the ramp function, the Heaviside function, the delta function, and the derivative of the delta function.

Figure 23.2: $r(x)$, $H(x)$, $\delta(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\delta(x)$

We write the differential equation for the Green function.

$$G''(x|\xi) + p(x)G'(x|\xi) + q(x)G(x|\xi) = \delta(x - \xi)$$

we see that only the $G''(x|\xi)$ term can have a delta function type singularity. If one of the other terms had a delta function type singularity then $G''(x|\xi)$ would be more singular than a delta function and there would be nothing in the right hand side of the equation to match this kind of singularity. Analogous to the progression from a delta function to a Heaviside function to a ramp function, we see that $G'(x|\xi)$ will have a jump discontinuity and $G(x|\xi)$ will be continuous.

Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation, $L[y] = 0$. Since the Green function satisfies the homogeneous equation for $x \neq \xi$, it will be a linear combination of the homogeneous solutions.

$$G(x|\xi) = \begin{cases} c_1 y_1 + c_2 y_2 & \text{for } x < \xi \\ d_1 y_1 + d_2 y_2 & \text{for } x > \xi \end{cases}$$

We require that $G(x|\xi)$ be continuous.

$$G(x|\xi)\big|_{x\to\xi^-} = G(x|\xi)\big|_{x\to\xi^+}$$

We can write this in terms of the homogeneous solutions.

$$c_1 y_1(\xi) + c_2 y_2(\xi) = d_1 y_1(\xi) + d_2 y_2(\xi)$$

We integrate $L[G(x|\xi)] = \delta(x - \xi)$ from ξ^- to $\xi+$.

$$\int_{\xi^-}^{\xi^+} \left[G''(x|\xi) + p(x)G'(x|\xi) + q(x)G(x|\xi)\right] \mathrm{d}x = \int_{\xi^-}^{\xi^+} \delta(x - \xi)\, \mathrm{d}x.$$

Since $G(x|\xi)$ is continuous and $G'(x|\xi)$ has only a jump discontinuity two of the terms vanish.

$$\int_{\xi^-}^{\xi^+} p(x)G'(x|\xi)\,\mathrm{d}x = 0 \qquad \text{and} \qquad \int_{\xi^-}^{\xi^+} q(x)G(x|\xi)\,\mathrm{d}x = 0$$

$$\int_{\xi^-}^{\xi^+} G''(x|\xi)\,\mathrm{d}x = \int_{\xi^-}^{\xi^+} \delta(x-\xi)\,\mathrm{d}x$$

$$\left[G'(x|\xi)\right]_{\xi^-}^{\xi^+} = \left[H(x-\xi)\right]_{\xi^-}^{\xi^+}$$

$$G'(\xi^+|\xi) - G'(\xi^-|\xi) = 1$$

We write this jump condition in terms of the homogeneous solutions.

$$d_1 y_1'(\xi) + d_2 y_2'(\xi) - c_1 y_1'(\xi) - c_2 y_2'(\xi) = 1$$

Combined with the two boundary conditions, this gives us a total of four equations to determine our four constants, c_1, c_2, d_1, and d_2.

Result 23.7.1 The second order inhomogeneous differential equation with homogeneous boundary conditions

$$L[y] = y'' + p(x)y' + q(x)y = f(x), \quad \text{for } a < x < b, \qquad B_1[y] = B_2[y] = 0,$$

has the solution

$$y = \int_a^b G(x|\xi) f(\xi) \, \mathrm{d}\xi,$$

where $G(x|\xi)$ satisfies the equation

$$L[G(x|\xi)] = \delta(x - \xi), \quad \text{for } a < x < b, \qquad B_1[G(x|\xi)] = B_2[G(x|\xi)] = 0.$$

$G(x|\xi)$ is continuous and $G'(x|\xi)$ has a jump discontinuity of height 1 at $x = \xi$.

Example 23.7.1 Solve the boundary value problem

$$y'' = f(x), \qquad y(0) = y(1) = 0,$$

using a Green function.

A pair of solutions to the homogeneous equation are $y_1 = 1$ and $y_2 = x$. First note that only the trivial solution to the homogeneous equation satisfies the homogeneous boundary conditions. Thus there is a unique solution to this problem.

The Green function satisfies

$$G''(x|\xi) = \delta(x - \xi), \qquad G(0|\xi) = G(1|\xi) = 0.$$

The Green function has the form

$$G(x|\xi) = \begin{cases} c_1 + c_2 x & \text{for } x < \xi \\ d_1 + d_2 x & \text{for } x > \xi. \end{cases}$$

Applying the two boundary conditions, we see that $c_1 = 0$ and $d_1 = -d_2$. The Green function now has the form

$$G(x|\xi) = \begin{cases} cx & \text{for } x < \xi \\ d(x-1) & \text{for } x > \xi. \end{cases}$$

Since the Green function must be continuous,

$$c\xi = d(\xi - 1) \quad \Rightarrow \quad d = c\frac{\xi}{\xi - 1}.$$

From the jump condition,

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}x} c\frac{\xi}{\xi-1}(x-1)\bigg|_{x=\xi} - \frac{\mathrm{d}}{\mathrm{d}x} cx\bigg|_{x=\xi} = 1 \\ c\frac{\xi}{\xi-1} - c = 1 \\ c = \xi - 1. \end{gathered}$$

Thus the Green function is

$$\boxed{G(x|\xi) = \begin{cases} (\xi - 1)x & \text{for } x < \xi \\ \xi(x-1) & \text{for } x > \xi. \end{cases}}$$

The Green function is plotted in Figure 23.3 for various values of ξ. The solution to $y'' = f(x)$ is

$$y(x) = \int_0^1 G(x|\xi) f(\xi)\,\mathrm{d}\xi$$

$$\boxed{y(x) = (x-1)\int_0^x \xi f(\xi)\,\mathrm{d}\xi + x\int_x^1 (\xi - 1) f(\xi)\,\mathrm{d}\xi.}$$

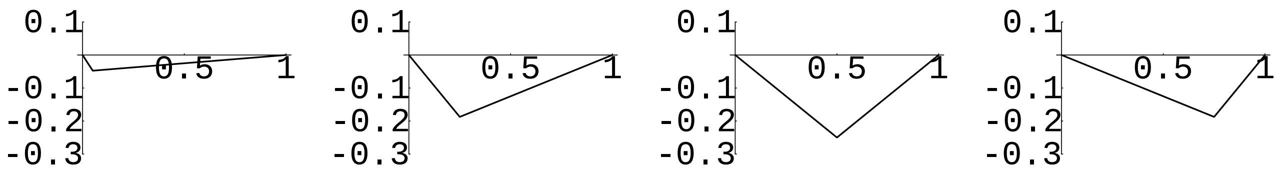

Figure 23.3: Plot of $G(x|0.05)$,$G(x|0.25)$,$G(x|0.5)$ and $G(x|0.75)$.

Example 23.7.2 Solve the boundary value problem

$$y'' = f(x), \qquad y(0) = 1, \quad y(1) = 2.$$

In Example 23.7.1 we saw that the solution to

$$u'' = f(x), \qquad u(0) = u(1) = 0$$

is

$$u(x) = (x-1)\int_0^x \xi f(\xi)\,\mathrm{d}\xi + x\int_x^1 (\xi - 1) f(\xi)\,\mathrm{d}\xi.$$

Now we have to find the solution to

$$v'' = 0, \qquad v(0) = 1, \quad u(1) = 2.$$

The general solution is

$$v = c_1 + c_2 x.$$

Applying the boundary conditions yields

$$v = 1 + x.$$

Thus the solution for y is

$$\boxed{y = 1 + x + (x-1)\int_0^x \xi f(\xi)\,\mathrm{d}\xi + x\int_x^1 (\xi - 1) f(\ xi)\,\mathrm{d}\xi.}$$

Example 23.7.3 Consider

$$y'' = x, \qquad y(0) = y(1) = 0.$$

Method 1. Integrating the differential equation twice yields

$$y = \frac{1}{6}x^3 + c_1 x + c_2.$$

Applying the boundary conditions, we find that the solution is

$$\boxed{y = \frac{1}{6}(x^3 - x).}$$

Method 2. Using the Green function to find the solution,

$$\begin{aligned} y &= (x-1)\int_0^x \xi^2\,\mathrm{d}\xi + x\int_x^1 (\xi-1)\xi\,\mathrm{d}\xi \\ &= (x-1)\frac{1}{3}x^3 + x\left(\frac{1}{3} - \frac{1}{2} - \frac{1}{3}x^3 + \frac{1}{2}x^2\right) \end{aligned}$$

$$\boxed{y = \frac{1}{6}(x^3 - x).}$$

Example 23.7.4 Find the solution to the differential equation

$$y'' - y = \sin x,$$

that is bounded for all x.

The Green function for this problem satisfies

$$G''(x|\xi) - G(x|\xi) = \delta(x - \xi).$$

The homogeneous solutions are $y_1 = \mathrm{e}^x$, and $y_2 = \mathrm{e}^{-x}$. The Green function has the form

$$G(x|\xi) = \begin{cases} c_1 \mathrm{e}^x + c_2 \mathrm{e}^{-x} & \text{for } x < \xi \\ d_1 \mathrm{e}^x + d_2 \mathrm{e}^{-x} & \text{for } x > \xi. \end{cases}$$

Since the solution must be bounded for all x, the Green function must also be bounded. Thus $c_2 = d_1 = 0$. The Green function now has the form

$$G(x|\xi) = \begin{cases} c\,\mathrm{e}^x & \text{for } x < \xi \\ d\,\mathrm{e}^{-x} & \text{for } x > \xi. \end{cases}$$

Requiring that $G(x|\xi)$ be continuous gives us the condition

$$c\,\mathrm{e}^{\xi} = d\,\mathrm{e}^{-\xi} \quad \Rightarrow \quad d = c\,\mathrm{e}^{2\xi}.$$

$G(x|\xi)$ has a jump discontinuity of height 1 at $x = \xi$.

$$\frac{\mathrm{d}}{\mathrm{d}x} c\,\mathrm{e}^{2\xi}\,\mathrm{e}^{-x}\bigg|_{x=\xi} - \frac{\mathrm{d}}{\mathrm{d}x} c\,\mathrm{e}^{x}\bigg|_{x=\xi} = 1$$
$$-c\,\mathrm{e}^{2\xi}\,\mathrm{e}^{-\xi} - c\,\mathrm{e}^{\xi} = 1$$
$$c = -\frac{1}{2}\,\mathrm{e}^{-\xi}$$

The Green function is then

$$G(x|\xi) = \begin{cases} -\frac{1}{2}\,\mathrm{e}^{x-\xi} & \text{for } x < \xi \\ -\frac{1}{2}\,\mathrm{e}^{-x+\xi} & \text{for } x > \xi \end{cases}$$

$$\boxed{G(x|\xi) = -\frac{1}{2}\,\mathrm{e}^{-|x-\xi|}.}$$

A plot of $G(x|0)$ is given in Figure 23.4. The solution to $y'' - y = \sin x$ is

$$\begin{aligned} y(x) &= \int_{-\infty}^{\infty} -\frac{1}{2}\,\mathrm{e}^{-|x-\xi|} \sin\xi \,\mathrm{d}\xi \\ &= -\frac{1}{2}\left(\int_{-\infty}^{x} \sin\xi\,\mathrm{e}^{x-\xi}\,\mathrm{d}\xi + \int_{x}^{\infty} \sin\xi\,\mathrm{e}^{-x+\xi}\,\mathrm{d}\xi \right) \\ &= -\frac{1}{2}\Big(-\frac{\sin x + \cos x}{2} + \frac{-\sin x + \cos x}{2}\Big) \end{aligned}$$

$$\boxed{y = \frac{1}{2}\sin x.}$$

23.7.1 Green Functions for Sturm-Liouville Problems

Consider the problem

$$\begin{gathered} L[y] = (p(x)y')' + q(x)y = f(x), \quad \text{subject to} \\ B_1[y] = \alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad B_2[y] = \beta_1 y(b) + \beta_2 y'(b) = 0. \end{gathered}$$

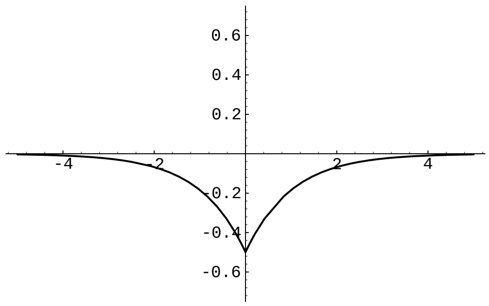

Figure 23.4: Plot of $G(x|0)$.

This is known as a Sturm-Liouville problem. Equations of this type often occur when solving partial differential equations. The Green function associated with this problem satisfies

$$L[G(x|\xi)] = \delta(x - \xi), \quad B_1[G(x|\xi)] = B_2[G(x|\xi)] = 0.$$

Let y_1 and y_2 be two non-trivial homogeneous solutions that satisfy the left and right boundary conditions, respectively.

$$L[y_1] = 0, \quad B_1[y_1] = 0, \qquad L[y_2] = 0, \quad B_2[y_2] = 0.$$

The Green function satisfies the homogeneous equation for $x \neq \xi$ and satisfies the homogeneous boundary conditions. Thus it must have the following form.

$$G(x|\xi) = \begin{cases} c_1(\xi) y_1(x) & \text{for } a \leq x \leq \xi, \\ c_2(\xi) y_2(x) & \text{for } \xi \leq x \leq b, \end{cases}$$

Here c_1 and c_2 are unknown functions of ξ.

The first constraint on c_1 and c_2 comes from the continuity condition.

$$G(\xi^-|\xi) = G(\xi^+|\xi)$$
$$c_1(\xi)y_1(\xi) = c_2(\xi)y_2(\xi)$$

We write the inhomogeneous equation in the standard form.

$$G''(x|\xi) + \frac{p'}{p}G'(x|\xi) + \frac{q}{p}G(x|\xi) = \frac{\delta(x-\xi)}{p}$$

The second constraint on c_1 and c_2 comes from the jump condition.

$$G'(\xi^+|\xi) - G'(\xi^-|\xi) = \frac{1}{p(\xi)}$$
$$c_2(\xi)y_2'(\xi) - c_1(\xi)y_1'(\xi) = \frac{1}{p(\xi)}$$

Now we have a system of equations to determine c_1 and c_2.

$$c_1(\xi)y_1(\xi) - c_2(\xi)y_2(\xi) = 0$$
$$c_1(\xi)y_1'(\xi) - c_2(\xi)y_2'(\xi) = -\frac{1}{p(\xi)}$$

We solve this system with Kramer's rule.

$$c_1(\xi) = -\frac{y_2(\xi)}{p(\xi)(-W(\xi))}, \quad c_2(\xi) = -\frac{y_1(\xi)}{p(\xi)(-W(\xi))}$$

Here $W(x)$ is the Wronskian of $y_1(x)$ and $y_2(x)$. The Green function is

$$\boxed{G(x|\xi) = \begin{cases} \frac{y_1(x)y_2(\xi)}{p(\xi)W(\xi)} & \text{for } a \leq x \leq \xi, \\ \frac{y_2(x)y_1(\xi)}{p(\xi)W(\xi)} & \text{for } \xi \leq x \leq b. \end{cases}}$$

The solution of the Sturm-Liouville problem is

$$y = \int_a^b G(x|\xi) f(\xi) \, \mathrm{d}\xi.$$

Result 23.7.2 The problem

$$L[y] = (p(x)y')' + q(x)y = f(x), \quad \text{subject to}$$
$$B_1[y] = \alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad B_2[y] = \beta_1 y(b) + \beta_2 y'(b) = 0.$$

has the Green function

$$G(x|\xi) = \begin{cases} \frac{y_1(x) y_2(\xi)}{p(\xi) W(\xi)} & \text{for } a \leq x \leq \xi, \\ \frac{y_2(x) y_1(\xi)}{p(\xi) W(\xi)} & \text{for } \xi \leq x \leq b, \end{cases}$$

where y_1 and y_2 are non-trivial homogeneous solutions that satisfy $B_1[y_1] = B_2[y_2] = 0$, and $W(x)$ is the Wronskian of y_1 and y_2.

Example 23.7.5 Consider the equation

$$y'' - y = f(x), \qquad y(0) = y(1) = 0.$$

A set of solutions to the homogeneous equation is $\{ \mathrm{e}^x, \mathrm{e}^{-x} \}$. Equivalently, one could use the set $\{\cosh x, \sinh x\}$. Note that $\sinh x$ satisfies the left boundary condition and $\sinh(x-1)$ satisfies the right boundary condition. The

Wronskian of these two homogeneous solutions is

$$\begin{aligned}
W(x) &= \begin{vmatrix} \sinh x & \sinh(x-1) \\ \cosh x & \cosh(x-1) \end{vmatrix} \\
&= \sinh x \cosh(x-1) - \cosh x \sinh(x-1) \\
&= \frac{1}{2}[\sinh(2x-1) + \sinh(1)] - \frac{1}{2}[\sinh(2x-1) - \sinh(1)] \\
&= \sinh(1).
\end{aligned}$$

The Green function for the problem is then

$$\boxed{G(x|\xi) = \begin{cases} \frac{\sinh x \sinh(\xi-1)}{\sinh(1)} & \text{for } 0 \leq x \leq \xi \\ \frac{\sinh(x-1)\sinh \xi}{\sinh(1)} & \text{for } \xi \leq x \leq 1. \end{cases}}$$

The solution to the problem is

$$\boxed{y = \frac{\sinh(x-1)}{\sinh(1)} \int_0^x \sinh(\xi) f(\xi)\, \mathrm{d}\xi + \frac{\sinh(x)}{\sinh(1)} \int_x^1 \sinh(\xi - 1) f(\xi)\, \mathrm{d}\xi.}$$

23.7.2 Initial Value Problems

Consider

$$L[y] = y'' + p(x)y' + q(x)y = f(x), \quad \text{for } a < x < b,$$

subject the the initial conditions

$$y(a) = \gamma_1, \qquad y'(a) = \gamma_2.$$

The solution is $y = u + v$ where

$$u'' + p(x)u' + q(x)u = f(x), \qquad u(a) = 0, \quad u'(a) = 0,$$

and

$$v'' + p(x)v' + q(x)v = 0, \qquad v(a) = \gamma_1, \quad v'(a) = \gamma_2.$$

Since the Wronskian

$$W(x) = c \exp\left(-\int p(x)\,\mathrm{d}x\right)$$

is non-vanishing, the solutions of the differential equation for v are linearly independent. Thus there is a unique solution for v that satisfies the initial conditions.

The Green function for u satisfies

$$G''(x|\xi) + p(x)G'(x|\xi) + q(x)G(x|\xi) = \delta(x-\xi), \qquad G(a|\xi) = 0, \quad G'(a|\xi) = 0.$$

The continuity and jump conditions are

$$G(\xi^-|\xi) = G(\xi^+|\xi), \qquad G'(\xi^-|\xi) + 1 = G'(\xi^+|\xi).$$

Let u_1 and u_2 be two linearly independent solutions of the differential equation. For $x < \xi$, $G(x|\xi)$ is a linear combination of these solutions. Since the Wronskian is non-vanishing, only the trivial solution satisfies the homogeneous initial conditions. The Green function must be

$$G(x|\xi) = \begin{cases} 0 & \text{for } x < \xi \\ u_\xi(x) & \text{for } x > \xi, \end{cases}$$

where $u_\xi(x)$ is the linear combination of u_1 and u_2 that satisfies

$$u_\xi(\xi) = 0, \qquad u'_\xi(\xi) = 1.$$

Note that the non-vanishing Wronskian ensures a unique solution for u_ξ. We can write the Green function in the form

$$G(x|\xi) = H(x-\xi)u_\xi(x).$$

This is known as the **causal solution**. The solution for u is

$$
\begin{aligned}
u &= \int_a^b G(x|\xi) f(\xi)\,\mathrm{d}\xi \\
&= \int_a^b H(x-\xi) u_\xi(x) f(\xi)\,\mathrm{d}\xi \\
&= \int_a^x u_\xi(x) f(\xi)\,\mathrm{d}\xi
\end{aligned}
$$

Now we have the solution for y,

$$y = v + \int_a^x u_\xi(x) f(\xi)\,\mathrm{d}\xi.$$

Result 23.7.3 The solution of the problem

$$y'' + p(x)y' + q(x)y = f(x), \qquad y(a) = \gamma_1, \quad y'(a) = \gamma_2,$$

is

$$y = y_h + \int_a^x y_\xi(x) f(\xi)\,\mathrm{d}\xi$$

where y_h is the combination of the homogeneous solutions of the equation that satisfy the initial conditions and $y_\xi(x)$ is the linear combination of homogeneous solutions that satisfy $y_\xi(\xi) = 0$, $y'_\xi(\xi) = 1$.

23.7.3 Problems with Unmixed Boundary Conditions

Consider

$$L[y] = y'' + p(x)y' + q(x)y = f(x), \quad \text{for } a < x < b,$$

subject the the unmixed boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = \gamma_1, \qquad \beta_1 y(b) + \beta_2 y'(b) = \gamma_2.$$

The solution is $y = u + v$ where

$$u'' + p(x)u' + q(x)u = f(x), \qquad \alpha_1 u(a) + \alpha_2 u'(a) = 0, \quad \beta_1 u(b) + \beta_2 u'(b) = 0,$$

and

$$v'' + p(x)v' + q(x)v = 0, \qquad \alpha_1 v(a) + \alpha_2 v'(a) = \gamma_1, \quad \beta_1 v(b) + \beta_2 v'(b) = \gamma_2.$$

The problem for v may have no solution, a unique solution or an infinite number of solutions. We consider only the case that there is a unique solution for v. In this case the homogeneous equation subject to homogeneous boundary conditions has only the trivial solution.

The Green function for u satisfies

$$G''(x|\xi) + p(x)G'(x|\xi) + q(x)G(x|\xi) = \delta(x - \xi),$$

$$\alpha_1 G(a|\xi) + \alpha_2 G'(a|\xi) = 0, \quad \beta_1 G(b|\xi) + \beta_2 G'(b|\xi) = 0.$$

The continuity and jump conditions are

$$G(\xi^-|\xi) = G(\xi^+|\xi), \qquad G'(\xi^-|\xi) + 1 = G'(\xi^+|\xi).$$

Let u_1 and u_2 be two solutions of the homogeneous equation that satisfy the left and right boundary conditions, respectively. The non-vanishing of the Wronskian ensures that these solutions exist. Let $W(x)$ denote the Wronskian of u_1 and u_2. Since the homogeneous equation with homogeneous boundary conditions has only the trivial solution, $W(x)$ is nonzero on $[a, b]$. The Green function has the form

$$G(x|\xi) = \begin{cases} c_1 u_1 & \text{for } x < \xi, \\ c_2 u_2 & \text{for } x > \xi. \end{cases}$$

The continuity and jump conditions for Green function gives us the equations

$$\begin{aligned} c_1 u_1(\xi) - c_2 u_2(\xi) &= 0 \\ c_1 u_1'(\xi) - c_2 u_2'(\xi) &= -1. \end{aligned}$$

Using Kramer's rule, the solution is

$$c_1 = \frac{u_2(\xi)}{W(\xi)}, \qquad c_2 = \frac{u_1(\xi)}{W(\xi)}.$$

Thus the Green function is

$$G(x|\xi) = \begin{cases} \frac{u_1(x) u_2(\xi)}{W(\xi)} & \text{for } x < \xi, \\ \frac{u_1(\xi) u_2(x)}{W(\xi)} & \text{for } x > \xi. \end{cases}$$

The solution for u is

$$u = \int_a^b G(x|\xi) f(\xi) \, \mathrm{d}\xi.$$

Thus if there is a unique solution for v, the solution for y is

$$y = v + \int_a^b G(x|\xi) f(\xi) \, \mathrm{d}\xi.$$

Result 23.7.4 Consider the problem

$$y'' + p(x)y' + q(x)y = f(x),$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = \gamma_1, \qquad \beta_1 y(b) + \beta_2 y'(b) = \gamma_2.$$

If the homogeneous differential equation subject to the inhomogeneous boundary conditions has the unique solution y_h, then the problem has the unique solution

$$y = y_h + \int_a^b G(x|\xi) f(\xi)\, \mathrm{d}\xi$$

where

$$G(x|\xi) = \begin{cases} \frac{u_1(x)u_2(\xi)}{W(\xi)} & \text{for } x < \xi, \\ \frac{u_1(\xi)u_2(x)}{W(\xi)} & \text{for } x > \xi, \end{cases}$$

u_1 and u_2 are solutions of the homogeneous differential equation that satisfy the left and right boundary conditions, respectively, and $W(x)$ is the Wronskian of u_1 and u_2.

23.7.4 Problems with Mixed Boundary Conditions

Consider

$$L[y] = y'' + p(x)y' + q(x)y = f(x), \quad \text{for } a < x < b,$$

subject the the mixed boundary conditions

$$B_1[y] = \alpha_{11} y(a) + \alpha_{12} y'(a) + \beta_{11} y(b) + \beta_{12} y'(b) = \gamma_1,$$

$$B_2[y] = \alpha_{21} y(a) + \alpha_{22} y'(a) + \beta_{21} y(b) + \beta_{22} y'(b) = \gamma_2.$$

The solution is $y = u + v$ where

$$u'' + p(x)u' + q(x)u = f(x), \qquad B_1[u] = 0, \quad B_2[u] = 0,$$

and

$$v'' + p(x)v' + q(x)v = 0, \qquad B_1[v] = \gamma_1, \quad B_2[v] = \gamma_2.$$

The problem for v may have no solution, a unique solution or an infinite number of solutions. Again we consider only the case that there is a unique solution for v. In this case the homogeneous equation subject to homogeneous boundary conditions has only the trivial solution.

Let y_1 and y_2 be two solutions of the homogeneous equation that satisfy the boundary conditions $B_1[y_1] = 0$ and $B_2[y_2] = 0$. Since the completely homogeneous problem has no solutions, we know that $B_1[y_2]$ and $B_2[y_1]$ are nonzero. The solution for v has the form

$$v = c_1 y_1 + c_2 y_2.$$

Applying the two boundary conditions yields

$$v = \frac{\gamma_2}{B_2[y_1]} y_1 + \frac{\gamma_1}{B_1[y_2]} y_2.$$

The Green function for u satisfies

$$G''(x|\xi) + p(x)G'(x|\xi) + q(x)G(x|\xi) = \delta(x - \xi), \qquad B_1[G] = 0, \quad B_2[G] = 0.$$

The continuity and jump conditions are

$$G(\xi^-|\xi) = G(\xi^+|\xi), \qquad G'(\xi^-|\xi) + 1 = G'(\xi^+|\xi).$$

We write the Green function as the sum of the causal solution and the two homogeneous solutions

$$G(x|\xi) = H(x-\xi)y_\xi(x) + c_1 y_1(x) + c_2 y_2(x)$$

With this form, the continuity and jump conditions are automatically satisfied. Applying the boundary conditions yields

$$B_1[G] = B_1[H(x-\xi)y_\xi] + c_2 B_1[y_2] = 0,$$
$$B_2[G] = B_2[H(x-\xi)y_\xi] + c_1 B_2[y_1] = 0,$$

$$B_1[G] = \beta_{11} y_\xi(b) + \beta_{12} y'_\xi(b) + c_2 B_1[y_2] = 0,$$
$$B_2[G] = \beta_{21} y_\xi(b) + \beta_{22} y'_\xi(b) + c_1 B_2[y_1] = 0,$$

$$G(x|\xi) = H(x-\xi)y_\xi(x) - \frac{\beta_{21} y_\xi(b) + \beta_{22} y'_\xi(b)}{B_2[y_1]} y_1(x) - \frac{\beta_{11} y_\xi(b) + \beta_{12} y'_\xi(b)}{B_1[y_2]} y_2(x).$$

Note that the Green function is well defined since $B_2[y_1]$ and $B_1[y_2]$ are nonzero. The solution for u is

$$u = \int_a^b G(x|\xi) f(\xi) \, \mathrm{d}\xi.$$

Thus if there is a unique solution for v, the solution for y is

$$y = \int_a^b G(x|\xi) f(\xi) \, \mathrm{d}\xi + \frac{\gamma_2}{B_2[y_1]} y_1 + \frac{\gamma_1}{B_1[y_2]} y_2.$$

Result 23.7.5 Consider the problem

$$y'' + p(x)y' + q(x)y = f(x),$$

$$B_1[y] = \alpha_{11}y(a) + \alpha_{12}y'(a) + \beta_{11}y(b) + \beta_{12}y'(b) = \gamma_1,$$

$$B_2[y] = \alpha_{21}y(a) + \alpha_{22}y'(a) + \beta_{21}y(b) + \beta_{22}y'(b) = \gamma_2.$$

If the homogeneous differential equation subject to the homogeneous boundary conditions has no solution, then the problem has the unique solution

$$y = \int_a^b G(x|\xi) f(\xi)\,\mathrm{d}\xi + \frac{\gamma_2}{B_2[y_1]} y_1 + \frac{\gamma_1}{B_1[y_2]} y_2,$$

where

$$G(x|\xi) = H(x-\xi)y_\xi(x) - \frac{\beta_{21}y_\xi(b) + \beta_{22}y_\xi'(b)}{B_2[y_1]} y_1(x) - \frac{\beta_{11}y_\xi(b) + \beta_{12}y_\xi'(b)}{B_1[y_2]} y_2(x),$$

y_1 and y_2 are solutions of the homogeneous differential equation that satisfy the first and second boundary conditions, respectively, and $y_\xi(x)$ is the solution of the homogeneous equation that satisfies $y_\xi(\xi) = 0$, $y_\xi'(\xi) = 1$.

23.8 Green Functions for Higher Order Problems

Consider the n_{th} order differential equation

$$L[y] = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0y = f(x) \quad \text{on } a < x < b,$$

subject to the n independent boundary conditions

$$B_j[y] = \gamma_j$$

where the boundary conditions are of the form

$$B[y] \equiv \sum_{k=0}^{n-1} \alpha_k y^{(k)}(a) + \sum_{k=0}^{n-1} \beta_k y^{(k)}(b).$$

We assume that the coefficient functions in the differential equation are continuous on $[a, b]$. The solution is $y = u + v$ where u and v satisfy

$$L[u] = f(x), \quad \text{with} \quad B_j[u] = 0,$$

and

$$L[v] = 0, \quad \text{with} \quad B_j[v] = \gamma_j$$

From Result 23.5.3, we know that if the completely homogeneous problem

$$L[w] = 0, \quad \text{with} \quad B_j[w] = 0,$$

has only the trivial solution, then the solution for y exists and is unique. We will construct this solution using Green functions.

First we consider the problem for v. Let $\{y_1, \dots, y_n\}$ be a set of linearly independent solutions. The solution for v has the form

$$v = c_1 y_1 + \cdots + c_n y_n$$

where the constants are determined by the matrix equation

$$\begin{pmatrix} B_1[y_1] & B_1[y_2] & \cdots & B_1[y_n] \\ B_2[y_1] & B_2[y_2] & \cdots & B_2[y_n] \\ \vdots & \vdots & \ddots & \vdots \\ B_n[y_1] & B_n[y_2] & \cdots & B_n[y_n] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}.$$

To solve the problem for u we consider the Green function satisfying

$$L[G(x|\xi)] = \delta(x - \xi), \quad \text{with} \quad B_j[G] = 0.$$

Let $y_\xi(x)$ be the linear combination of the homogeneous solutions that satisfy the conditions

$$\begin{aligned} y_\xi(\xi) &= 0 \\ y'_\xi(\xi) &= 0 \\ \vdots &= \vdots \\ y_\xi^{(n-2)}(\xi) &= 0 \\ y_\xi^{(n-1)}(\xi) &= 1. \end{aligned}$$

The causal solution is then

$$y_c(x) = H(x - \xi) y_\xi(x).$$

The Green function has the form

$$G(x|\xi) = H(x - \xi) y_\xi(x) + d_1 y_1(x) + \cdots + d_n y_n(x)$$

The constants are determined by the matrix equation

$$\begin{pmatrix} B_1[y_1] & B_1[y_2] & \cdots & B_1[y_n] \\ B_2[y_1] & B_2[y_2] & \cdots & B_2[y_n] \\ \vdots & \vdots & \ddots & \vdots \\ B_n[y_1] & B_n[y_2] & \cdots & B_n[y_n] \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} -B_1[H(x-\xi)y_\xi(x)] \\ -B_2[H(x-\xi)y_\xi(x)] \\ \vdots \\ -B_n[H(x-\xi)y_\xi(x)] \end{pmatrix}.$$

The solution for u then is

$$u = \int_a^b G(x|\xi) f(\xi) \, \mathrm{d}\xi.$$

Result 23.8.1 Consider the n_{th} order differential equation

$$L[y] = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0 y = f(x) \quad \text{on } a < x < b,$$

subject to the n independent boundary conditions

$$B_j[y] = \gamma_j$$

If the homogeneous differential equation subject to the homogeneous boundary conditions has only the trivial solution, then the problem has the unique solution

$$y = \int_a^b G(x|\xi) f(\xi)\, \mathrm{d}\xi + c_1 y_1 + \cdots c_n y_n$$

where

$$G(x|\xi) = H(x - \xi) y_\xi(x) + d_1 y_1(x) + \cdots + d_n y_n(x),$$

$\{y_1, \ldots, y_n\}$ is a set of solutions of the homogeneous differential equation, and the constants c_j and d_j can be determined by solving sets of linear equations.

Example 23.8.1 Consider the problem

$$y''' - y'' + y' - y = f(x),$$

$$y(0) = 1, \quad y'(0) = 2, \quad y(1) = 3.$$

The completely homogeneous associated problem is

$$w''' - w'' + w' - w = 0, \quad w(0) = w'(0) = w(1) = 0.$$

The solution of the differential equation is

$$w = c_1 \cos x + c_2 \sin x + c_2 \,\mathrm{e}^x.$$

The boundary conditions give us the equation

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \cos 1 & \sin 1 & e \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The determinant of the matrix is $e - \cos 1 - \sin 1 \neq 0$. Thus the homogeneous problem has only the trivial solution and the inhomogeneous problem has a unique solution.

We separate the inhomogeneous problem into the two problems

$$u''' - u'' + u' - u = f(x), \quad u(0) = u'(0) = u(1) = 0,$$

$$v''' - v'' + v' - v = 0, \quad v(0) = 1, \quad v'(0) = 2, \quad v(1) = 3,$$

First we solve the problem for v. The solution of the differential equation is

$$v = c_1 \cos x + c_2 \sin x + c_2 \,\mathrm{e}^x.$$

The boundary conditions yields the equation

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \cos 1 & \sin 1 & e \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

The solution for v is

$$v = \frac{1}{e - \cos 1 - \sin 1}\left[(e + \sin 1 - 3)\cos x + (2e - \cos 1 - 3)\sin x + (3 - \cos 1 - 2\sin 1)\,\mathrm{e}^x\right].$$

Now we find the Green function for the problem in u. The causal solution is

$$H(x-\xi)u_\xi(x) = H(x-\xi)\frac{1}{2}\big[(\sin\xi - \cos\xi)\cos x - (\sin\xi + \cos\xi)\sin\xi + \mathrm{e}^{-\xi}\,\mathrm{e}^{x}\big],$$

$$H(x-\xi)u_\xi(x) = \frac{1}{2}H(x-\xi)\big[\,\mathrm{e}^{x-\xi} - \cos(x-\xi) - \sin(x-\xi)\big].$$

The Green function has the form

$$G(x|\xi) = H(x-\xi)u_\xi(x) + c_1\cos x + c_2\sin x + c_3\,\mathrm{e}^{x}.$$

The constants are determined by the three conditions

$$\begin{aligned}
&\big[c_1\cos x + c_2\sin x + c_3\,\mathrm{e}^{x}\big]_{x=0} = 0,\\
&\left[\frac{\partial}{\partial x}\left(c_1\cos x + c_2\sin x + c_3\,\mathrm{e}^{x}\right)\right]_{x=0} = 0,\\
&\big[u_\xi(x) + c_1\cos x + c_2\sin x + c_3\,\mathrm{e}^{x}\big]_{x=1} = 0.
\end{aligned}$$

The Green function is

$$G(x|\xi) = \frac{1}{2}H(x-\xi)\big[\,\mathrm{e}^{x-\xi} - \cos(x-\xi) - \sin(x-\xi)\big] + \frac{\cos(1-\xi) + \sin(1-\xi) - \mathrm{e}^{1-\xi}}{2(\cos 1 + \sin 1 - e)}\big[\cos x + \sin x - \mathrm{e}^{x}\big]$$

The solution for v is

$$v = \int_0^1 G(x|\xi)f(\xi)\,\mathrm{d}\xi.$$

Thus the solution for y is

$$\begin{aligned}
y = \int_0^1 G(x|\xi)f(\xi)\,\mathrm{d}\xi + \frac{1}{e - \cos 1 - \sin 1}&\big[(e + \sin 1 - 3)\cos x\\
&+ (2e - \cos 1 - 3)\sin x + (3 - \cos 1 - 2\sin 1)\,\mathrm{e}^{x}\big].
\end{aligned}$$

23.9 Fredholm Alternative Theorem

Orthogonality. Two real vectors, u and v are orthogonal if $u \cdot v = 0$. Consider two functions, $u(x)$ and $v(x)$, defined in $[a, b]$. The dot product in vector space is analogous to the integral

$$\int_a^b u(x)v(x)\,\mathrm{d}x$$

in function space. Thus two real functions are orthogonal if

$$\int_a^b u(x)v(x)\,\mathrm{d}x = 0.$$

Consider the n^{th} order linear inhomogeneous differential equation

$$L[y] = f(x) \quad \text{on } [a, b],$$

subject to the linear inhomogeneous boundary conditions

$$B_j[y] = 0, \quad \text{for } j = 1, 2, \ldots n.$$

The Fredholm alternative theorem tells us if the problem has a unique solution, an infinite number of solutions, or no solution. Before presenting the theorem, we will consider a few motivating examples.

No Nontrivial Homogeneous Solutions. In the section on Green functions we showed that if the completely homogeneous problem has only the trivial solution then the inhomogeneous problem has a unique solution.

Nontrivial Homogeneous Solutions Exist. If there are nonzero solutions to the homogeneous problem $L[y] = 0$ that satisfy the homogeneous boundary conditions $B_j[y] = 0$ then the inhomogeneous problem $L[y] = f(x)$ subject to the same boundary conditions either has no solution or an infinite number of solutions.

Suppose there is a particular solution y_p that satisfies the boundary conditions. If there is a solution y_h to the homogeneous equation that satisfies the boundary conditions then there will be an infinite number of solutions since $y_p + cy_h$ is also a particular solution.

The question now remains: Given that there are homogeneous solutions that satisfy the boundary conditions, how do we know if a particular solution that satisfies the boundary conditions exists? Before we address this question we will consider a few examples.

Example 23.9.1 Consider the problem

$$y'' + y = \cos x, \qquad y(0) = y(\pi) = 0.$$

The two homogeneous solutions of the differential equation are

$$y_1 = \cos x, \quad \text{and} \quad y_2 = \sin x.$$

$y_2 = \sin x$ satisfies the boundary conditions. Thus we know that there are either no solutions or an infinite number of solutions. A particular solution is

$$\begin{aligned}
y_p &= -\cos x \int \frac{\cos x \sin x}{1}\, \mathrm{d}x + \sin x \int \frac{\cos^2 x}{1}\, \mathrm{d}x \\
&= -\cos x \int \frac{1}{2}\sin(2x)\, \mathrm{d}x + \sin x \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x) \right) \mathrm{d}x \\
&= \frac{1}{4}\cos x \cos(2x) + \sin x \left(\frac{1}{2}x + \frac{1}{4}\sin(2x) \right) \\
&= \frac{1}{2}x \sin x + \frac{1}{4}\left[\cos x \cos(2x) + \sin x \sin(2x) \right] \\
&= \frac{1}{2}x \sin x + \frac{1}{4}\cos x
\end{aligned}$$

The general solution is

$$y = \frac{1}{2}x \sin x + c_1 \cos x + c_2 \sin x.$$

Applying the two boundary conditions yields

$$y = \frac{1}{2}x \sin x + c \sin x.$$

Thus there are an infinite number of solutions.

Example 23.9.2 Consider the differential equation

$$y'' + y = \sin x, \qquad y(0) = y(\pi) = 0.$$

The general solution is

$$y = -\frac{1}{2} x \cos x + c_1 \cos x + c_2 \sin x.$$

Applying the boundary conditions,

$$\begin{aligned}
y(0) = 0 \quad &\Rightarrow \quad c_1 = 0 \\
y(\pi) = 0 \quad &\Rightarrow \quad -\frac{1}{2}\pi \cos(\pi) + c_2 \sin(\pi) = 0 \\
&\Rightarrow \quad \frac{\pi}{2} = 0.
\end{aligned}$$

Since this equation has no solution, there are no solutions to the inhomogeneous problem.

In both of the above examples there is a homogeneous solution $y = \sin x$ that satisfies the boundary conditions. In Example 23.9.1, the inhomogeneous term is $\cos x$ and there are an infinite number of solutions. In Example 23.9.2, the inhomogeneity is $\sin x$ and there are no solutions. In general, if the inhomogeneous term is orthogonal to all the homogeneous solutions that satisfy the boundary conditions then there are an infinite number of solutions. If not, there are no inhomogeneous solutions.

Result 23.9.1 Fredholm Alternative Theorem. Consider the n^{th} order inhomogeneous problem

$$L[y] = f(x) \quad \text{on} \quad [a, b] \quad \text{subject to} \quad B_j[y] = 0 \quad \text{for} \quad j = 1, 2, \ldots, n,$$

and the associated homogeneous problem

$$L[y] = 0 \quad \text{on} \quad [a, b] \quad \text{subject to} \quad B_j[y] = 0 \quad \text{for} \quad j = 1, 2, \ldots, n.$$

If the homogeneous problem has only the trivial solution then the inhomogeneous problem has a unique solution. If the homogeneous problem has m independent solutions, $\{y_1, y_2, \ldots, y_m\}$, then there are two possibilities:

- If $f(x)$ is orthogonal to each of the homogeneous solutions then there are an infinite number of solutions of the form

$$y = y_p + \sum_{j=1}^{m} c_j y_j.$$

- If $f(x)$ is not orthogonal to each of the homogeneous solutions then there are no inhomogeneous solutions.

Example 23.9.3 Consider the problem

$$y'' + y = \cos 2x, \qquad y(0) = 1, \quad y(\pi) = 2.$$

$\cos x$ and $\sin x$ are two linearly independent solutions to the homogeneous equation. $\sin x$ satisfies the homogeneous boundary conditions. Thus there are either an infinite number of solutions, or no solution.

To transform this problem to one with homogeneous boundary conditions, we note that $g(x) = \frac{x}{\pi} + 1$ and make the change of variables $y = u + g$ to obtain

$$u'' + u = \cos 2x - \frac{x}{\pi} - 1, \qquad y(0) = 0, \quad y(\pi) = 0.$$

Since $\cos 2x - \frac{x}{\pi} - 1$ is not orthogonal to $\sin x$, there is no solution to the inhomogeneous problem.

To check this, the general solution is

$$y = -\frac{1}{3}\cos 2x + c_1 \cos x + c_2 \sin x.$$

Applying the boundary conditions,

$$\begin{aligned} y(0) = 1 \qquad &\Rightarrow \qquad c_1 = \frac{4}{3} \\ y(\pi) = 2 \qquad &\Rightarrow \qquad -\frac{1}{3} - \frac{4}{3} = 2. \end{aligned}$$

Thus we see that the right boundary condition cannot be satisfied.

Example 23.9.4 Consider

$$y'' + y = \cos 2x, \qquad y'(0) = y(\pi) = 1.$$

There are no solutions to the homogeneous equation that satisfy the homogeneous boundary conditions. To check this, note that all solutions of the homogeneous equation have the form $u_h = c_1 \cos x + c_2 \sin x$.

$$\begin{aligned} u_h'(0) = 0 \qquad &\Rightarrow \qquad c_2 = 0 \\ u_h(\pi) = 0 \qquad &\Rightarrow \qquad c_1 = 0. \end{aligned}$$

From the Fredholm Alternative Theorem we see that the inhomogeneous problem has a unique solution.

To find the solution, start with

$$y = -\frac{1}{3}\cos 2x + c_1 \cos x + c_2 \sin x.$$

$$\begin{aligned} y'(0) = 1 \quad &\Rightarrow \quad c_2 = 1 \\ y(\pi) = 1 \quad &\Rightarrow \quad -\frac{1}{3} - c_1 = 1 \end{aligned}$$

Thus the solution is

$$\boxed{y = -\frac{1}{3}\cos 2x - \frac{4}{3}\cos x + \sin x.}$$

Example 23.9.5 Consider

$$y'' + y = \cos 2x, \qquad y(0) = \frac{2}{3}, \quad y(\pi) = -\frac{4}{3}.$$

$\cos x$ and $\sin x$ satisfy the homogeneous differential equation. $\sin x$ satisfies the homogeneous boundary conditions. Since $g(x) = \cos x - 1/3$ satisfies the boundary conditions, the substitution $y = u + g$ yields

$$u'' + u = \cos 2x + \frac{1}{3}, \qquad y(0) = 0, \quad y(\pi) = 0.$$

Now we check if $\sin x$ is orthogonal to $\cos 2x + \frac{1}{3}$.

$$\begin{aligned} \int_0^\pi \sin x \left(\cos 2x + \frac{1}{3}\right) \mathrm{d}x &= \int_0^\pi \frac{1}{2}\sin 3x - \frac{1}{2}\sin x + \frac{1}{3}\sin x \,\mathrm{d}x \\ &= \left[-\frac{1}{6}\cos 3x + \frac{1}{6}\cos x\right]_0^\pi \\ &= 0 \end{aligned}$$

Since $\sin x$ is orthogonal to the inhomogeneity, there are an infinite number of solutions to the problem for u, (and hence the problem for y).

As a check, then general solution for y is

$$y = -\frac{1}{3}\cos 2x + c_1 \cos x + c_2 \sin x.$$

Applying the boundary conditions,

$$y(0) = \frac{2}{3} \qquad \Rightarrow \qquad c_1 = 1$$
$$y(\pi) = -\frac{4}{3} \qquad \Rightarrow \qquad -\frac{4}{3} = -\frac{4}{3}.$$

Thus we see that c_2 is arbitrary. There are an infinite number of solutions of the form

$$\boxed{y = -\frac{1}{3}\cos 2x + \cos x + c \sin x.}$$

23.10 Exercises

Undetermined Coefficients

Exercise 23.1 (mathematica/ode/inhomogeneous/undetermined.nb)
Find the general solution of the following equations.

1. $y'' + 2y' + 5y = 3\sin(2t)$
2. $2y'' + 3y' + y = t^2 + 3\sin(t)$

Hint, Solution

Exercise 23.2 (mathematica/ode/inhomogeneous/undetermined.nb)
Find the solution of each one of the following initial value problems.

1. $y'' - 2y' + y = t\,\mathrm{e}^t + 4$, $y(0) = 1$, $y'(0) = 1$
2. $y'' + 2y' + 5y = 4\,\mathrm{e}^{-t}\cos(2t)$, $y(0) = 1$, $y'(0) = 0$

Hint, Solution

Variation of Parameters

Exercise 23.3 (mathematica/ode/inhomogeneous/variation.nb)
Use the method of variation of parameters to find a particular solution of the given differential equation.

1. $y'' - 5y' + 6y = 2\,\mathrm{e}^t$
2. $y'' + y = \tan(t)$, $0 < t < \pi/2$
3. $y'' - 5y' + 6y = g(t)$, for a given function g.

Hint, Solution

Exercise 23.4 (mathematica/ode/inhomogeneous/variation.nb)
Solve

$$y''(x) + y(x) = x, \quad y(0) = 1, \ y'(0) = 0.$$

Hint, Solution

Exercise 23.5 (mathematica/ode/inhomogeneous/variation.nb)
Solve

$$x^2 y''(x) - x y'(x) + y(x) = x.$$

Hint, Solution

Exercise 23.6 (mathematica/ode/inhomogeneous/variation.nb)

1. Find the general solution of $y'' + y = \mathrm{e}^x$.
2. Solve $y'' + \lambda^2 y = \sin x$, $y(0) = y'(0) = 0$. λ is an arbitrary real constant. Is there anything special about $\lambda = 1$?

Hint, Solution

Exercise 23.7 (mathematica/ode/inhomogeneous/variation.nb)
Consider the problem of solving the initial value problem

$$y'' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

1. Show that the general solution of $y'' + y = g(t)$ is

$$y(t) = \left(c_1 - \int_a^t g(\tau) \sin \tau \,\mathrm{d}\tau \right) \cos t + \left(c_2 + \int_b^t g(\tau) \cos \tau \,\mathrm{d}\tau \right) \sin t,$$

   where c_1 and c_2 are arbitrary constants and a and b are any conveniently chosen points.

2. Using the result of part (a) show that the solution satisfying the initial conditions $y(0) = 0$ and $y'(0) = 0$ is given by

$$y(t) = \int_0^t g(\tau) \sin(t - \tau) \, \mathrm{d}\tau.$$

   Notice that this equation gives a formula for computing the solution of the original initial value problem for any given inhomogeneous term $g(t)$. The integral is referred to as the *convolution* of $g(t)$ with $\sin t$.

3. Use the result of part (b) to solve the initial value problem,

$$y'' + y = \sin(\lambda t), \qquad y(0) = 0, \quad y'(0) = 0,$$

   where λ is a real constant. How does the solution for $\lambda = 1$ differ from that for $\lambda \neq 1$? The $\lambda = 1$ case provides an example of *resonant forcing*. Plot the solution for resonant and non-resonant forcing.

Hint, Solution

Exercise 23.8
Find the variation of parameters solution for the third order differential equation

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x).$$

Hint, Solution

Green Functions

Exercise 23.9
Use a Green function to solve

$$y'' = f(x), \quad y(-\infty) = y'(-\infty) = 0.$$

Verify the the solution satisfies the differential equation.
Hint, Solution

Exercise 23.10
Solve the initial value problem

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = x^2, \qquad y(0) = 0, \quad y'(0) = 1.$$

First use variation of parameters, and then solve the problem with a Green function.
Hint, Solution

Exercise 23.11
What are the continuity conditions at $x = \xi$ for the Green function for the problem

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x).$$

Hint, Solution

Exercise 23.12
Use variation of parameters and Green functions to solve

$$x^2y'' - 2xy' + 2y = \mathrm{e}^{-x}, \quad y(1) = 0, \quad y'(1) = 1.$$

Hint, Solution

Exercise 23.13
Find the Green function for

$$y'' - y = f(x), \qquad y'(0) = y(1) = 0.$$

Hint, Solution

Exercise 23.14
Find the Green function for

$$y'' - y = f(x), \qquad y(0) = y(\infty) = 0.$$

Hint, Solution

Exercise 23.15
Find the Green function for each of the following:

a) $xu'' + u' = f(x)$, $u(0^+)$ bounded, $u(1) = 0$.

b) $u'' - u = f(x)$, $u(-a) = u(a) = 0$.

c) $u'' - u = f(x)$, $u(x)$ bounded as $|x| \to \infty$.

d) Show that the Green function for (b) approaches that for (c) as $a \to \infty$.
Hint, Solution

Exercise 23.16

1. For what values of λ does the problem

$$y'' + \lambda y = f(x), \quad y(0) = y(\pi) = 0, \tag{23.5}$$

   have a unique solution? Find the Green functions for these cases.

2. For what values of α does the problem

$$y'' + 9y = 1 + \alpha x, \quad y(0) = y(\pi) = 0,$$

   have a solution? Find the solution.

3. For $\lambda = n^2$, $n \in \mathbb{Z}^+$ state in general the conditions on f in Equation 23.5 so that a solution will exist. What is the appropriate modified Green function (in terms of eigenfunctions)?

Hint, Solution

Exercise 23.17

Show that the inhomogeneous boundary value problem:

$$Lu \equiv (pu')' + qu = f(x), \quad a < x < b, \quad u(a) = \alpha, \quad u(b) = \beta$$

has the solution:

$$u(x) = \int_a^b g(x;\xi) f(\xi)\, \mathrm{d}\xi - \alpha p(a) g_\xi(x;a) + \beta p(b) g_\xi(x;b).$$

Hint, Solution

Exercise 23.18

The Green function for

$$u'' - k^2 u = f(x), \quad -\infty < x < \infty$$

subject to $|u(\pm\infty)| < \infty$ is

$$G(x;\xi) = -\frac{1}{2k}\,\mathrm{e}^{-k|x-\xi|}.$$

(We assume that $k > 0$.) Use the image method to find the Green function for the same equation on the semi-infinite interval $0 < x < \infty$ satisfying the boundary conditions,

$$\begin{array}{ll} \text{i)} & u(0) = 0 \quad |u(\infty)| < \infty, \\ \text{ii)} & u'(0) = 0 \quad |u(\infty)| < \infty. \end{array}$$

Express these results in simplified forms without absolute values.

Hint, Solution

Exercise 23.19

1. Determine the Green function for solving:

$$y'' - a^2 y = f(x), \quad y(0) = y'(L) = 0.$$

2. Take the limit as $L \to \infty$ to find the Green function on $(0, \infty)$ for the boundary conditions: $y(0) = 0$, $y'(\infty) = 0$. We assume here that $a > 0$. Use the limiting Green function to solve:

$$y'' - a^2 y = \mathrm{e}^{-x}, \quad y(0) = 0, \quad y'(\infty) = 0.$$

   Check that your solution satisfies all the conditions of the problem.

Hint, Solution

23.11 Hints

Undetermined Coefficients

Hint 23.1

Hint 23.2

Variation of Parameters

Hint 23.3

Hint 23.4

Hint 23.5

Hint 23.6

Hint 23.7

Hint 23.8
Look for a particular solution of the form

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3,$$

where the y_j's are homogeneous solutions. Impose the constraints

$$u_1' y_1 + u_2' y_2 + u_3' y_3 = 0$$
$$u_1' y_1' + u_2' y_2' + u_3' y_3' = 0.$$

To avoid some messy algebra when solving for u_j', use Kramer's rule.

Green Functions

Hint 23.9

Hint 23.10

Hint 23.11

Hint 23.12

Hint 23.13
$\cosh(x)$ and $\sinh(x-1)$ are homogeneous solutions that satisfy the left and right boundary conditions, respectively.

Hint 23.14
$\sinh(x)$ and e^{-x} are homogeneous solutions that satisfy the left and right boundary conditions, respectively.

Hint 23.15
The Green function for the differential equation

$$L[y] \equiv \frac{d}{dx}(p(x)y') + q(x)y = f(x),$$

subject to unmixed, homogeneous boundary conditions is

$$G(x|\xi) = \frac{y_1(x_<)y_2(x_>)}{p(\xi)W(\xi)},$$

$$G(x|\xi) = \begin{cases} \frac{y_1(x)y_2(\xi)}{p(\xi)W(\xi)} & \text{for } a \leq x \leq \xi, \\ \frac{y_1(\xi)y_2(x)}{p(\xi)W(\xi)} & \text{for } \xi \leq x \leq b, \end{cases}$$

where y_1 and y_2 are homogeneous solutions that satisfy the left and right boundary conditions, respectively.

Recall that if $y(x)$ is a solution of a homogeneous, constant coefficient differential equation then $y(x+c)$ is also a solution.

Hint 23.16
The problem has a Green function if and only if the inhomogeneous problem has a unique solution. The inhomogeneous problem has a unique solution if and only if the homogeneous problem has only the trivial solution.

Hint 23.17
Show that $g_\xi(x; a)$ and $g_\xi(x; b)$ are solutions of the homogeneous differential equation. Determine the value of these solutions at the boundary.

Hint 23.18

Hint 23.19

23.12 Solutions

Undetermined Coefficients

Solution 23.1

1. We consider

$$y'' + 2y' + 5y = 3\sin(2t).$$

We first find the homogeneous solution with the substitition $y = \mathrm{e}^{\lambda t}$.

$$\lambda^2 + 2\lambda + 5 = 0$$
$$\lambda = -1 \pm 2i$$

The homogeneous solution is

$$y_h = c_1 \,\mathrm{e}^{-t} \cos(2t) + c_2 \,\mathrm{e}^{-t} \sin(2t).$$

We guess a particular solution of the form

$$y_p = a\cos(2t) + b\sin(2t).$$

We substitute this into the differential equation to determine the coefficients.

$$y_p'' + 2y_p' + 5y_p = 3\sin(2t)$$
$$-4a\cos(2t) - 4b\sin(2t) - 4a\sin(2t) + 4b\sin(2t) + 5a\cos(2t) + 5b\sin(2t) = -3\sin(2t)$$
$$(a + 4b)\cos(2t) + (-3 - 4a + b)\sin(2t) = 0$$
$$a + 4b = 0, \quad -4a + b = 3$$
$$a = -\frac{12}{17}, \quad b = \frac{3}{17}$$

A particular solution is

$$y_p = \frac{3}{17}(\sin(2t) - 4\cos(2t)).$$

The general solution of the differential equation is

$$\boxed{y = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t) + \frac{3}{17}(\sin(2t) - 4\cos(2t)).}$$

2. We consider

$$2y'' + 3y' + y = t^2 + 3\sin(t)$$

We first find the homogeneous solution with the substitition $y = \mathrm{e}^{\lambda t}$.

$$2\lambda^2 + 3\lambda + 1 = 0$$
$$\lambda = \{-1, -1/2\}$$

The homogeneous solution is

$$y_h = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-t/2}.$$

We guess a particular solution of the form

$$y_p = at^2 + bt + c + d\cos(t) + e\sin(t).$$

We substitute this into the differential equation to determine the coefficients.

$$2y_p'' + 3y_p' + y_p = t^2 + 3\sin(t)$$

$$2(2a - d\cos(t) - e\sin(t)) + 3(2at + b - d\sin(t) + e\cos(t)) \\ + at^2 + bt + c + d\cos(t) + e\sin(t) = t^2 + 3\sin(t)$$

$$(a-1)t^2 + (6a+b)t + (4a+3b+c) + (-d+3e)\cos(t) - (3+3d+e)\sin(t) = 0$$
$$a - 1 = 0, \quad 6a + b = 0, \quad 4a + 3b + c = 0, \quad -d + 3e = 0, \quad 3 + 3d + e = 0$$
$$a = 1, \quad b = -6, \quad c = 14, \quad d = -\frac{9}{10}, \quad e = -\frac{3}{10}$$

A particular solution is

$$y_p = t^2 - 6t + 14 - \frac{3}{10}(3\cos(t) + \sin(t)).$$

The general solution of the differential equation is

$$\boxed{y = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-t/2} + t^2 - 6t + 14 - \frac{3}{10}(3\cos(t) + \sin(t)).}$$

Solution 23.2

1. We consider the problem

$$y'' - 2y' + y = t\,\mathrm{e}^t + 4, \quad y(0) = 1, \quad y'(0) = 1.$$

First we solve the homogeneous equation with the substitution $y = \mathrm{e}^{\lambda t}$.

$$\begin{gathered} \lambda^2 - 2\lambda + 1 = 0 \\ (\lambda - 1)^2 = 0 \\ \lambda = 1 \end{gathered}$$

The homogeneous solution is

$$y_h = c_1 \mathrm{e}^t + c_2 t\,\mathrm{e}^t.$$

We guess a particular solution of the form

$$y_p = at^3 \mathrm{e}^t + bt^2 \mathrm{e}^t + 4.$$

We substitute this into the inhomogeneous differential equation to determine the coefficients.

$$y_p'' - 2y_p' + y_p = t\,e^t + 4$$
$$(a(t^3 + 6t^2 + 6t) + b(t^2 + 4t + 2))\,e^t - 2(a(t^2 + 3t) + b(t + 2))\,e^t at^3\,e^t + bt^2\,e^t + 4 = t\,e^t + 4$$
$$(6a - 1)t + 2b = 0$$
$$6a - 1 = 0, \quad 2b = 0$$
$$a = \frac{1}{6}, \quad b = 0$$

A particular solution is

$$y_p = \frac{t^3}{6}\,e^t + 4.$$

The general solution of the differential equation is

$$y = c_1\,e^t + c_2 t\,e^t + \frac{t^3}{6}\,e^t + 4.$$

We use the initial conditions to determine the constants of integration.

$$y(0) = 1, \quad y'(0) = 1$$
$$c_1 + 4 = 1, \quad c_1 + c_2 = 1$$
$$c_1 = -3, \quad c_2 = 4$$

The solution of the initial value problem is

$$\boxed{y = \left(\frac{t^3}{6} + 4t - 3\right)\,e^t + 4.}$$

2. We consider the problem

$$y'' + 2y' + 5y = 4\,e^{-t}\cos(2t), \quad y(0) = 1, \quad y'(0) = 0.$$

First we solve the homogeneous equation with the substitution $y = \mathrm{e}^{\lambda t}$.

$$\lambda^2 + 2\lambda + 5 = 0$$
$$\lambda = -1 \pm \sqrt{1-5}$$
$$\lambda = -1 \pm i2$$

The homogeneous solution is

$$y_h = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t).$$

We guess a particular solution of the form

$$y_p = t\,\mathrm{e}^{-t}(a\cos(2t) + b\sin(2t))$$

We substitute this into the inhomogeneous differential equation to determine the coefficients.

$$y_p'' + 2y_p' + 5y_p = 4\,\mathrm{e}^{-t}\cos(2t)$$

$$\begin{aligned}\mathrm{e}^{-t}((-(2+3t)a + 4(1-t)b)\cos(2t) + (4(t-1)a - (2+3t)b)\sin(2t)) \\ + 2\,\mathrm{e}^{-t}(((1-t)a + 2tb)\cos(2t) + (-2ta + (1-t)b)\sin(2t)) \\ + 5(\mathrm{e}^{-t}(ta\cos(2t) + tb\sin(2t))) = 4\,\mathrm{e}^{-t}\cos(2t)\end{aligned}$$

$$4(b-1)\cos(2t) - 4a\sin(2t) = 0$$
$$a = 0, \quad b = 1$$

A particular solution is

$$y_p = t\,\mathrm{e}^{-t}\sin(2t).$$

The general solution of the differential equation is

$$y = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t) + t \mathrm{e}^{-t} \sin(2t).$$

We use the initial conditions to determine the constants of integration.

$$y(0) = 1, \quad y'(0) = 0$$
$$c_1 = 1, \quad -c_1 + 2c_2 = 0$$
$$c_1 = 1, \quad c_2 = \frac{1}{2}$$

The solution of the initial value problem is

$$\boxed{y = \frac{1}{2} \mathrm{e}^{-t} \left(2\cos(2t) + (2t+1)\sin(2t)\right).}$$

Variation of Parameters

Solution 23.3

1. We consider the equation

$$y'' - 5y' + 6y = 2 \mathrm{e}^{t}.$$

We find homogeneous solutions with the substitution $y = \mathrm{e}^{\lambda t}$.

$$\lambda^2 - 5\lambda + 6 = 0$$
$$\lambda = \{2, 3\}$$

The homogeneous solutions are

$$y_1 = \mathrm{e}^{2t}, \quad y_2 = \mathrm{e}^{3t}.$$

We compute the Wronskian of these solutions.

$$W(t) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2\,e^{2t} & 3\,e^{3t} \end{vmatrix} = e^{5t}$$

We find a particular solution with variation of parameters.

$$\begin{aligned} y_p &= -\,e^{2t} \int \frac{2\,e^t\,e^{3t}}{e^{5t}}\,dt + e^{3t} \int \frac{2\,e^t\,e^{2t}}{e^{5t}}\,dt \\ &= -2\,e^{2t} \int e^{-t}\,dt + 2\,e^{3t} \int e^{-2t}\,dt \\ &= 2\,e^t - e^t \end{aligned}$$

$$\boxed{y_p = e^t}$$

2. We consider the equation

$$y'' + y = \tan(t), \quad 0 < t < \frac{\pi}{2}.$$

We find homogeneous solutions with the substitution $y = e^{\lambda t}$.

$$\begin{gathered} \lambda^2 + 1 = 0 \\ \lambda = \pm i \end{gathered}$$

The homogeneous solutions are

$$y_1 = \cos(t), \quad y_2 = \sin(t).$$

We compute the Wronskian of these solutions.

$$W(t) = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = \cos^2(t) + \sin^2(t) = 1$$

We find a particular solution with variation of parameters.

$$\begin{aligned} y_p &= -\cos(t) \int \tan(t)\sin(t)\,\mathrm{d}t + \sin(t) \int \tan(t)\cos(t)\,\mathrm{d}t \\ &= -\cos(t) \int \frac{\sin^2(t)}{\cos(t)}\,\mathrm{d}t + \sin(t) \int \sin(t)\,\mathrm{d}t \\ &= \cos(t) \left(\ln \left(\frac{\cos(t/2) - \sin(t/2)}{\cos(t/2) + \sin(t/2)} + \sin(t) \right) \right) - \sin(t)\cos(t) \end{aligned}$$

$$\boxed{y_p = \cos(t) \ln \left(\frac{\cos(t/2) - \sin(t/2)}{\cos(t/2) + \sin(t/2)} \right)}$$

3. We consider the equation

$$y'' - 5y' + 6y = g(t).$$

The homogeneous solutions are

$$y_1 = \mathrm{e}^{2t}, \quad y_2 = \mathrm{e}^{3t}.$$

The Wronskian of these solutions is $W(t) = \mathrm{e}^{5t}$. We find a particular solution with variation of parameters.

$$y_p = -\mathrm{e}^{2t} \int \frac{g(t)\,\mathrm{e}^{3t}}{\mathrm{e}^{5t}}\,\mathrm{d}t + \mathrm{e}^{3t} \int \frac{g(t)\,\mathrm{e}^{2t}}{\mathrm{e}^{5t}}\,\mathrm{d}t$$

$$\boxed{y_p = -\mathrm{e}^{2t} \int g(t)\,\mathrm{e}^{-2t}\,\mathrm{d}t + \mathrm{e}^{3t} \int g(t)\,\mathrm{e}^{-3t}\,\mathrm{d}t}$$

Solution 23.4
Solve

$$y''(x) + y(x) = x, \quad y(0) = 1, \ y'(0) = 0.$$

The solutions of the homogeneous equation are

$$y_1(x) = \cos x, \quad y_2(x) = \sin x.$$

The Wronskian of these solutions is

$$\begin{aligned} W[\cos x, \sin x] &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= \cos^2 x + \sin^2 x \\ &= 1. \end{aligned}$$

The variation of parameters solution for the particular solution is

$$\begin{aligned} y_p &= -\cos x \int x \sin x \, \mathrm{d}x + \sin x \int x \cos x \, \mathrm{d}x \\ &= -\cos x \left(-x \cos x + \int \cos x \, \mathrm{d}x \right) + \sin x \left(x \sin x - \int \sin x \, \mathrm{d}x \right) \\ &= -\cos x \left(-x \cos x + \sin x \right) + \sin x \left(x \sin x + \cos x \right) \\ &= x \cos^2 x - \cos x \sin x + x \sin^2 x + \cos x \sin x \\ &= x \end{aligned}$$

The general solution of the differential equation is thus

$$y = c_1 \cos x + c_2 \sin x + x.$$

Applying the two initial conditions gives us the equations

$$c_1 = 1, \quad c_2 + 1 = 0.$$

The solution subject to the initial conditions is

$$\boxed{y = \cos x - \sin x + x.}$$

Solution 23.5
Solve

$$x^2 y''(x) - x y'(x) + y(x) = x.$$

The homogeneous equation is

$$x^2 y''(x) - x y'(x) + y(x) = 0.$$

Substituting $y = x^\lambda$ into the homogeneous differential equation yields

$$x^2 \lambda(\lambda - 1) x^{\lambda-2} - x\lambda x^\lambda + x^\lambda = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$$\lambda = 1.$$

The homogeneous solutions are

$$y_1 = x, \quad y_2 = x \log x.$$

The Wronskian of the homogeneous solutions is

$$\begin{aligned} W[x, x\log x] &= \begin{vmatrix} x & x \log x \\ 1 & 1 + \log x \end{vmatrix} \\ &= x + x \log x - x \log x \\ &= x. \end{aligned}$$

Writing the inhomogeneous equation in the standard form:

$$y''(x) - \frac{1}{x}y'(x) + \frac{1}{x^2}y(x) = \frac{1}{x}.$$

Using variation of parameters to find the particular solution,

$$\begin{aligned} y_p &= -x \int \frac{\log x}{x} \, \mathrm{d}x + x \log x \int \frac{1}{x} \, \mathrm{d}x \\ &= -x \frac{1}{2} \log^2 x + x \log x \log x \\ &= \frac{1}{2} x \log^2 x. \end{aligned}$$

Thus the general solution of the inhomogeneous differential equation is

$$\boxed{y = c_1 x + c_2 x \log x + \frac{1}{2} x \log^2 x.}$$

Solution 23.6

1. First we find the homogeneous solutions. We substitute $y = \mathrm{e}^{\lambda x}$ into the homogeneous differential equation.

$$\begin{gathered} y'' + y = 0 \\ \lambda^2 + 1 = 0 \\ \lambda = \pm i \\ y = \left\{ \mathrm{e}^{ix}, \, \mathrm{e}^{-ix} \right\} \end{gathered}$$

We can also write the solutions in terms of real-valued functions.

$$y = \{\cos x, \sin x\}$$

The Wronskian of the homogeneous solutions is

$$W[\cos x, \sin x] = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

We obtain a particular solution with the variation of parameters formula.

$$y_p = -\cos x \int \mathrm{e}^x \sin x \, \mathrm{d}x + \sin x \int \mathrm{e}^x \cos x \, \mathrm{d}x$$
$$y_p = -\cos x \frac{1}{2} \mathrm{e}^x (\sin x - \cos x) + \sin x \frac{1}{2} \mathrm{e}^x (\sin x + \cos x)$$
$$y_p = \frac{1}{2} \mathrm{e}^x$$

The general solution is the particular solution plus a linear combination of the homogeneous solutions.

$$\boxed{y = \frac{1}{2} \mathrm{e}^x + \cos x + \sin x}$$

2.

$$y'' + \lambda^2 y = \sin x, \qquad y(0) = y'(0) = 0$$

Assume that λ is positive. First we find the homogeneous solutions by substituting $y = \mathrm{e}^{\alpha x}$ into the homogeneous differential equation.

$$y'' + \lambda^2 y = 0$$
$$\alpha^2 + \lambda^2 = 0$$
$$\alpha = \pm i\lambda$$
$$y = \left\{ \mathrm{e}^{i\lambda x}, \mathrm{e}^{-i\lambda x} \right\}$$
$$y = \{\cos(\lambda x), \sin(\lambda x)\}$$

The Wronskian of these homogeneous solution is

$$W[\cos(\lambda x), \sin(\lambda x)] = \begin{vmatrix} \cos(\lambda x) & \sin(\lambda x) \\ -\lambda \sin(\lambda x) & \lambda \cos(\lambda x) \end{vmatrix} = \lambda \cos^2(\lambda x) + \lambda \sin^2(\lambda x) = \lambda.$$

We obtain a particular solution with the variation of parameters formula.

$$y_p = -\cos(\lambda x) \int \frac{\sin(\lambda x) \sin x}{\lambda} \, dx + \sin(\lambda x) \int \frac{\cos(\lambda x) \sin x}{\lambda} \, dx$$

We evaluate the integrals for $\lambda \neq 1$.

$$y_p = -\cos(\lambda x) \frac{\cos(x) \sin(\lambda x) - \lambda \sin x \cos(\lambda x)}{\lambda(\lambda^2 - 1)} + \sin(\lambda x) \frac{\cos(x) \cos(\lambda x) + \lambda \sin x \sin(\lambda x)}{\lambda(\lambda^2 - 1)}$$

$$y_p = \frac{\sin x}{\lambda^2 - 1}$$

The general solution for $\lambda \neq 1$ is

$$y = \frac{\sin x}{\lambda^2 - 1} + c_1 \cos(\lambda x) + c_2 \sin(\lambda x).$$

The initial conditions give us the constraints:

$$c_1 = 0,$$
$$\frac{1}{\lambda^2 - 1} + \lambda c_2 = 0,$$

For $\lambda \neq 1$, (non-resonant forcing), the solution subject to the initial conditions is

$$\boxed{y = \frac{\lambda \sin(x) - \sin(\lambda x)}{\lambda(\lambda^2 - 1)}}.$$

Now consider the case $\lambda = 1$. We obtain a particular solution with the variation of parameters formula.

$$y_p = -\cos(x) \int \sin^2(x)\,\mathrm{d}x + \sin(x) \int \cos(x) \sin x\,\mathrm{d}x$$
$$y_p = -\cos(x) \frac{1}{2}(x - \cos(x)\sin(x)) + \sin(x)\left(-\frac{1}{2}\cos^2(x)\right)$$
$$y_p = -\frac{1}{2} x \cos(x)$$

The general solution for $\lambda = 1$ is

$$y = -\frac{1}{2} x \cos(x) + c_1 \cos(x) + c_2 \sin(x).$$

The initial conditions give us the constraints:

$$c_1 = 0$$
$$-\frac{1}{2} + c_2 = 0$$

For $\lambda = 1$, (resonant forcing), the solution subject to the initial conditions is

$$\boxed{y = \frac{1}{2}(\sin(x) - x \cos x).}$$

Solution 23.7

1. A set of linearly independent, homogeneous solutions is $\{\cos t, \sin t\}$. The Wronskian of these solutions is

$$W(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1.$$

We use variation of parameters to find a particular solution.

$$y_p = -\cos t \int g(t) \sin t\,\mathrm{d}t + \sin t \int g(t) \cos t\,\mathrm{d}t$$

The general solution can be written in the form,

$$\boxed{y(t) = \left(c_1 - \int_a^t g(\tau)\sin\tau\,\mathrm{d}\tau\right)\cos t + \left(c_2 + \int_b^t g(\tau)\cos\tau\,\mathrm{d}\tau\right)\sin t.}$$

2. Since the initial conditions are given at $t = 0$ we choose the lower bounds of integration in the general solution to be that point.

$$y = \left(c_1 - \int_0^t g(\tau)\sin\tau\,\mathrm{d}\tau\right)\cos t + \left(c_2 + \int_0^t g(\tau)\cos\tau\,\mathrm{d}\tau\right)\sin t$$

The initial condition $y(0) = 0$ gives the constraint, $c_1 = 0$. The derivative of $y(t)$ is then,

$$y'(t) = -g(t)\sin t\cos t + \int_0^t g(\tau)\sin\tau\,\mathrm{d}\tau\sin t + g(t)\cos t\sin t + \left(c_2 + \int_0^t g(\tau)\cos\tau\,\mathrm{d}\tau\right)\cos t,$$

$$y'(t) = \int_0^t g(\tau)\sin\tau\,\mathrm{d}\tau\sin t + \left(c_2 + \int_0^t g(\tau)\cos\tau\,\mathrm{d}\tau\right)\cos t.$$

The initial condition $y'(0) = 0$ gives the constraint $c_2 = 0$. The solution subject to the initial conditions is

$$y = \int_0^t g(\tau)(\sin t\cos\tau - \cos t\sin\tau)\,\mathrm{d}\tau$$

$$\boxed{y = \int_0^t g(\tau)\sin(t-\tau)\,\mathrm{d}\tau}$$

3. The solution of the initial value problem

$$y'' + y = \sin(\lambda t), \quad y(0) = 0, \quad y'(0) = 0,$$

is

$$y = \int_0^t \sin(\lambda\tau)\sin(t-\tau)\,\mathrm{d}\tau.$$

For $\lambda \neq 1$, this is

$$
\begin{aligned}
y &= \frac{1}{2}\int_0^t \big(\cos(t-\tau-\lambda\tau) - \cos(t-\tau+\lambda\tau)\big)\,\mathrm{d}\tau \\
&= \frac{1}{2}\left[-\frac{\sin(t-\tau-\lambda\tau)}{1+\lambda} + \frac{\sin(t-\tau+\lambda\tau)}{1-\lambda}\right]_0^t \\
&= \frac{1}{2}\left(\frac{\sin(t)-\sin(-\lambda t)}{1+\lambda} + \frac{-\sin(t)+\sin(\lambda t)}{1-\lambda}\right)
\end{aligned}
$$

$$
\boxed{y = -\frac{\lambda \sin t}{1-\lambda^2} + \frac{\sin(\lambda t)}{1-\lambda^2}.} \tag{23.6}
$$

The solution is the sum of two periodic functions of period 2π and $2\pi/\lambda$. This solution is plotted in Figure 23.5 on the interval $t \in [0, 16\pi]$ for the values $\lambda = 1/4, 7/8, 5/2$.

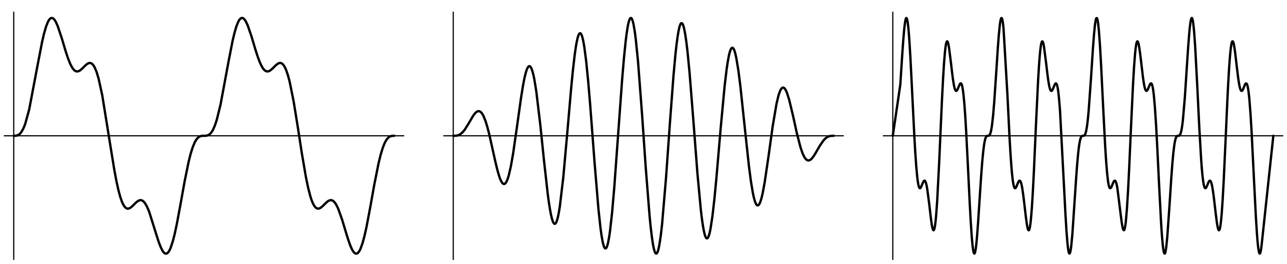

Figure 23.5: Non-resonant Forcing

For $\lambda = 1$, we have

$$\begin{aligned} y &= \frac{1}{2} \int_0^t \big(\cos(t - 2\tau) - \cos(tau) \big) \, \mathrm{d}\tau \\ &= \frac{1}{2} \left[-\frac{1}{2} \sin(t - 2\tau) - \tau \cos t \right]_0^t \end{aligned}$$

$$\boxed{y = \frac{1}{2} \left(\sin t - t \cos t \right).} \tag{23.7}$$

The solution has both a periodic and a transient term. This solution is plotted in Figure 23.5 on the interval $t \in [0, 16\pi]$.

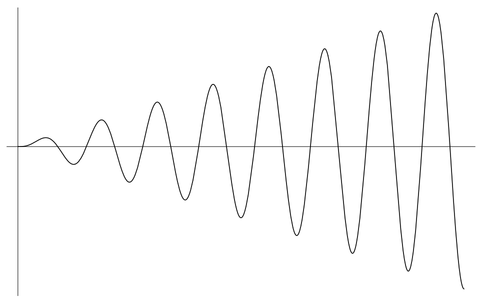

Figure 23.6: Resonant Forcing

Note that we can derive (23.7) from (23.6) by taking the limit as $\lambda \to 0$.

$$\begin{aligned} \lim_{\lambda \to 1} \frac{\sin(\lambda t) - \lambda \sin t}{1 - \lambda^2} &= \lim_{\lambda \to 1} \frac{t \cos(\lambda t) - \sin t}{-2\lambda} \\ &= \frac{1}{2} \left(\sin t - t \cos t \right) \end{aligned}$$

Solution 23.8

Let y_1, y_2 and y_3 be linearly independent homogeneous solutions to the differential equation

$$L[y] = y''' + p_2 y'' + p_1 y' + p_0 y = f(x).$$

We will look for a particular solution of the form

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3.$$

Since the u_j's are undetermined functions, we are free to impose two constraints. We choose the constraints to simplify the algebra.

$$\begin{aligned} u_1' y_1 + u_2' y_2 + u_3' y_3 &= 0 \\ u_1' y_1' + u_2' y_2' + u_3' y_3' &= 0 \end{aligned}$$

Differentiating the expression for y_p,

$$\begin{aligned} y_p' &= u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2' + u_3' y_3 + u_3 y_3' \\ &= u_1 y_1' + u_2 y_2' + u_3 y_3' \\ y_p'' &= u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + u_3' y_3' + u_3 y_3'' \\ &= u_1 y_1'' + u_2 y_2'' + u_3 y_3'' \\ y_p''' &= u_1' y_1'' + u_1 y_1''' + u_2' y_2'' + u_2 y_2''' + u_3' y_3'' + u_3 y_3''' \end{aligned}$$

Substituting the expressions for y_p and its derivatives into the differential equation,

$$\begin{aligned} u_1' y_1'' + u_1 y_1''' + u_2' y_2'' + u_2 y_2''' + u_3' y_3'' + u_3 y_3''' + p_2(u_1 y_1'' + u_2 y_2'' + u_3 y_3'') + p_1(u_1 y_1' + u_2 y_2' + u_3 y_3') \\ + p_0(u_1 y_1 + u_2 y_2 + u_3 y_3) = f(x) \end{aligned}$$

$$\begin{aligned} u_1' y_1'' + u_2' y_2'' + u_3' y_3'' + u_1 L[y_1] + u_2 L[y_2] + u_3 L[y_3] &= f(x) \\ u_1' y_1'' + u_2' y_2'' + u_3' y_3'' &= f(x). \end{aligned}$$

With the two constraints, we have the system of equations,

$$
\begin{aligned}
u_1' y_1 + u_2' y_2 + u_3' y_3 &= 0 \\
u_1' y_1' + u_2' y_2' + u_3' y_3' &= 0 \\
u_1' y_1'' + u_2' y_2'' + u_3' y_3'' &= f(x)
\end{aligned}
$$

We solve for the u_j' using Kramer's rule.

$$
u_1' = \frac{(y_2 y_3' - y_2' y_3) f(x)}{W(x)}, \quad u_2' = -\frac{(y_1 y_3' - y_1' y_3) f(x)}{W(x)}, \quad u_3' = \frac{(y_1 y_2' - y_1' y_2) f(x)}{W(x)}
$$

Here $W(x)$ is the Wronskian of $\{y_1, y_2, y_3\}$. Integrating the expressions for u_j', the particular solution is

$$
\boxed{y_p = y_1 \int \frac{(y_2 y_3' - y_2' y_3) f(x)}{W(x)} \, \mathrm{d}x + y_2 \int \frac{(y_3 y_1' - y_3' y_1) f(x)}{W(x)} \, \mathrm{d}x + y_3 \int \frac{(y_1 y_2' - y_1' y_2) f(x)}{W(x)} \, \mathrm{d}x.}
$$

Green Functions

Solution 23.9
We consider the Green function problem

$$
G'' = f(x), \quad G(-\infty|\xi) = G'(-\infty|\xi) = 0.
$$

The homogeneous solution is $y = c_1 + c_2 x$. The homogeneous solution that satisfies the boundary conditions is $y = 0$. Thus the Green function has the form

$$
G(x|\xi) = \begin{cases} 0 & x < \xi, \\ c_1 + c_2 x & x > \xi. \end{cases}
$$

The continuity and jump conditions are then

$$
G(\xi^+|\xi) = 0, \quad G'(\xi^+|\xi) = 1.
$$

Thus the Green function is

$$G(x|\xi) = \begin{cases} 0 & x < \xi, \\ x - \xi & x > \xi \end{cases} = (x - \xi)H(x - \xi).$$

The solution of the problem

$$y'' = f(x), \quad y(-\infty) = y'(-\infty) = 0.$$

is

$$y = \int_{-\infty}^{\infty} f(\xi)G(x|\xi)\,\mathrm{d}\xi$$
$$y = \int_{-\infty}^{\infty} f(\xi)(x - \xi)H(x - \xi)\,\mathrm{d}\xi$$
$$\boxed{y = \int_{-\infty}^{x} f(\xi)(x - \xi)\,\mathrm{d}\xi}$$

We differentiate this solution to verify that it satisfies the differential equation.

$$y' = [f(\xi)(x - \xi)]_{\xi = x} + \int_{-\infty}^{x} \frac{\partial}{\partial x}\left(f(\xi)(x - \xi)\right)\mathrm{d}\xi = \int_{-\infty}^{x} f(\xi)\,\mathrm{d}\xi$$
$$y'' = [f(\xi)]_{\xi = x} = f(x)$$

Solution 23.10

Since we are dealing with an Euler equation, we substitute $y = x^\lambda$ to find the homogeneous solutions.

$$\lambda(\lambda - 1) + \lambda - 1 = 0$$
$$(\lambda - 1)(\lambda + 1) = 0$$
$$y_1 = x, \qquad y_2 = \frac{1}{x}$$

Variation of Parameters. The Wronskian of the homogeneous solutions is

$$W(x) = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\frac{1}{x} - \frac{1}{x} = -\frac{2}{x}.$$

A particular solution is

$$\begin{aligned} y_p &= -x \int \frac{x^2(1/x)}{-2/x} \, \mathrm{d}x + \frac{1}{x} \int \frac{x^2 x}{-2/x} \, \mathrm{d}x \\ &= -x \int -\frac{x^2}{2} \, \mathrm{d}x + \frac{1}{x} \int -\frac{x^4}{2} \, \mathrm{d}x \\ &= \frac{x^4}{6} - \frac{x^4}{10} \\ &= \frac{x^4}{15}. \end{aligned}$$

The general solution is

$$y = \frac{x^4}{15} + c_1 x + c_2 \frac{1}{x}.$$

Applying the initial conditions,

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad c_2 = 0 \\ y'(0) = 0 \quad &\Rightarrow \quad c_1 = 1. \end{aligned}$$

Thus we have the solution

$$\boxed{y = \frac{x^4}{15} + x.}$$

Green Function. Since this problem has both an inhomogeneous term in the differential equation and inhomogeneous boundary conditions, we separate it into the two problems

$$u'' + \frac{1}{x}u' - \frac{1}{x^2}u = x^2, \qquad u(0) = u'(0) = 0,$$
$$v'' + \frac{1}{x}v' - \frac{1}{x^2}v = 0, \qquad v(0) = 0,\ v'(0) = 1.$$

First we solve the inhomogeneous differential equation with the homogeneous boundary conditions. The Green function for this problem satisfies

$$L[G(x|\xi)] = \delta(x - \xi), \quad G(0|\xi) = G'(0|\xi) = 0.$$

Since the Green function must satisfy the homogeneous boundary conditions, it has the form

$$G(x|\xi) = \begin{cases} 0 & \text{for } x < \xi \\ cx + d/x & \text{for } x > \xi. \end{cases}$$

From the continuity condition,

$$0 = c\xi + d/\xi.$$

The jump condition yields

$$c - d/\xi^2 = 1.$$

Solving these two equations, we obtain

$$G(x|\xi) = \begin{cases} 0 & \text{for } x < \xi \\ \frac{1}{2}x - \frac{\xi^2}{2x} & \text{for } x > \xi \end{cases}$$

Thus the solution is

$$
\begin{aligned}
u(x) &= \int_0^\infty G(x|\xi)\xi^2 \,\mathrm{d}\xi \\
&= \int_0^x \left(\frac{1}{2}x - \frac{\xi^2}{2x}\right) \xi^2 \,\mathrm{d}\xi \\
&= \frac{1}{6}x^4 - \frac{1}{10}x^4 \\
&= \frac{x^4}{15}.
\end{aligned}
$$

Now to solve the homogeneous differential equation with inhomogeneous boundary conditions. The general solution for v is

$$v = cx + d/x.$$

Applying the two boundary conditions gives

$$v = x.$$

Thus the solution for y is

$$\boxed{y = x + \frac{x^4}{15}.}$$

Solution 23.11

The Green function satisfies

$$G'''(x|\xi) + p_2(x)G''(x|\xi) + p_1(x)G'(x|\xi) + p_0(x)G(x|\xi) = \delta(x - \xi).$$

First note that only the $G'''(x|\xi)$ term can have a delta function singularity. If a lower derivative had a delta function type singularity, then $G'''(x|\xi)$ would be more singular than a delta function and there would be no other term in the equation to balance that behavior. Thus we see that $G'''(x|\xi)$ will have a delta function singularity; $G''(x|\xi)$ will have a jump discontinuity; $G'(x|\xi)$ will be continuous at $x = \xi$. Integrating the differential equation from ξ^- to ξ^+ yields

$$\int_{\xi^-}^{\xi^+} G'''(x|\xi)\,\mathrm{d}x = \int_{\xi^-}^{\xi^+} \delta(x-\xi)\,\mathrm{d}x$$

$$G''(\xi^+|\xi) - G''(\xi^-|\xi) = 1.$$

Thus we have the three continuity conditions:

$$\begin{aligned} G''(\xi^+|\xi) &= G''(\xi^-|\xi) + 1 \\ G'(\xi^+|\xi) &= G'(\xi^-|\xi) \\ G(\xi^+|\xi) &= G(\xi^-|\xi) \end{aligned}$$

Solution 23.12

Variation of Parameters. Consider the problem

$$x^2 y'' - 2xy' + 2y = \mathrm{e}^{-x}, \qquad y(1) = 0, \quad y'(1) = 1.$$

Previously we showed that two homogeneous solutions are

$$y_1 = x, \qquad y_2 = x^2.$$

The Wronskian of these solutions is

$$W(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2.$$

In the variation of parameters formula, we will choose 1 as the lower bound of integration. (This will simplify the algebra in applying the initial conditions.)

$$\begin{aligned}
y_p &= -x \int_1^x \frac{\mathrm{e}^{-\xi} \xi^2}{\xi^4} \,\mathrm{d}\xi + x^2 \int_1^x \frac{\mathrm{e}^{-\xi} \xi}{\xi^4} \,\mathrm{d}\xi \\
&= -x \int_1^x \frac{\mathrm{e}^{-\xi}}{\xi^2} \,\mathrm{d}\xi + x^2 \int_1^x \frac{\mathrm{e}^{-\xi}}{\xi^3} \,\mathrm{d}\xi \\
&= -x \left(\mathrm{e}^{-1} - \frac{\mathrm{e}^{-x}}{x} - \int_1^x \frac{\mathrm{e}^{-\xi}}{\xi} \,\mathrm{d}\xi \right) + x^2 \left(\frac{\mathrm{e}^{-x}}{2x} - \frac{\mathrm{e}^{-x}}{2x^2} + \frac{1}{2} \int_1^x \frac{\mathrm{e}^{-\xi}}{\xi} \,\mathrm{d}\xi \right) \\
&= -x \,\mathrm{e}^{-1} + \frac{1}{2}(1 + x) \,\mathrm{e}^{-x} + \left(\frac{x + x^2}{2} \right) \int_1^x \frac{\mathrm{e}^{-\xi}}{\xi} \,\mathrm{d}\xi
\end{aligned}$$

If you wanted to, you could write the last integral in terms of exponential integral functions.

The general solution is

$$y = c_1 x + c_2 x^2 - x \,\mathrm{e}^{-1} + \frac{1}{2}(1 + x) \,\mathrm{e}^{-x} + \left(x + \frac{x^2}{2} \right) \int_1^x \frac{\mathrm{e}^{-\xi}}{\xi} \,\mathrm{d}\xi$$

Applying the boundary conditions,

$$\begin{aligned}
y(1) &= 0 \qquad \Rightarrow \qquad c_1 + c_2 = 0 \\
y'(1) &= 1 \qquad \Rightarrow \qquad c_1 + 2c_2 = 1,
\end{aligned}$$

we find that $c_1 = -1$, $c_2 = 1$.

Thus the solution subject to the initial conditions is

$$\boxed{y = -(1 + \mathrm{e}^{-1}) x + x^2 + \frac{1}{2}(1 + x) \,\mathrm{e}^{-x} + \left(x + \frac{x^2}{2} \right) \int_1^x \frac{\mathrm{e}^{-\xi}}{\xi} \,\mathrm{d}\xi}$$

Green Functions. The solution to the problem is $y = u + v$ where

$$u'' - \frac{2}{x} u' + \frac{2}{x^2} u = \frac{\mathrm{e}^{-x}}{x^2}, \qquad u(1) = 0, \quad u'(1) = 0,$$

and

$$v'' - \frac{2}{x}v' + \frac{2}{x^2}v = 0, \qquad v(1) = 0, \quad v'(1) = 1.$$

The problem for v has the solution

$$v = -x + x^2.$$

The Green function for u is

$$G(x|\xi) = H(x - \xi)u_\xi(x)$$

where

$$u_\xi(\xi) = 0, \quad \text{and} \quad u'_\xi(\xi) = 1.$$

Thus the Green function is

$$G(x|\xi) = H(x - \xi)\left(-x + \frac{x^2}{\xi}\right).$$

The solution for u is then

$$\begin{aligned}
u &= \int_1^\infty G(x|\xi)\frac{\mathrm{e}^{-\xi}}{\xi^2}\,\mathrm{d}\xi \\
&= \int_1^x \left(-x + \frac{x^2}{\xi}\right)\frac{\mathrm{e}^{-\xi}}{\xi^2}\,\mathrm{d}\xi \\
&= -x\,\mathrm{e}^{-1} + \frac{1}{2}(1 + x)\,\mathrm{e}^{-x} + \left(x + \frac{x^2}{2}\right)\int_1^x \frac{\mathrm{e}^{-\xi}}{\xi}\,\mathrm{d}\xi.
\end{aligned}$$

Thus we find the solution for y is

$$\boxed{y = -(1 + \mathrm{e}^{-1})x + x^2 + \frac{1}{2}(1 + x)\,\mathrm{e}^{-x} + \left(x + \frac{x^2}{2}\right)\int_1^x \frac{\mathrm{e}^{-\xi}}{\xi}\,\mathrm{d}\xi}$$

Solution 23.13
The differential equation for the Green function is

$$G'' - G = \delta(x - \xi), \qquad G_x(0|\xi) = G(1|\xi) = 0.$$

Note that $\cosh(x)$ and $\sinh(x-1)$ are homogeneous solutions that satisfy the left and right boundary conditions, respectively. The Wronskian of these two solutions is

$$\begin{aligned}
W(x) &= \begin{vmatrix} \cosh(x) & \sinh(x-1) \\ \sinh(x) & \cosh(x-1) \end{vmatrix} \\
&= \cosh(x)\cosh(x-1) - \sinh(x)\sinh(x-1) \\
&= \frac{1}{4}\left(\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)\left(\mathrm{e}^{x-1} + \mathrm{e}^{-x+1}\right) - \left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right)\left(\mathrm{e}^{x-1} - \mathrm{e}^{-x+1}\right)\right) \\
&= \frac{1}{2}\left(\mathrm{e}^{1} + \mathrm{e}^{-1}\right) \\
&= \cosh(1).
\end{aligned}$$

The Green function for the problem is then

$$G(x|\xi) = \frac{\cosh(x_<)\sinh(x_> - 1)}{\cosh(1)},$$

$$\boxed{G(x|\xi) = \begin{cases} \frac{\cosh(x)\sinh(\xi-1)}{\cosh(1)} & \text{for } 0 \leq x \leq \xi, \\ \frac{\cosh(\xi)\sinh(x-1)}{\cosh(1)} & \text{for } \xi \leq x \leq 1. \end{cases}}$$

Solution 23.14
The differential equation for the Green function is

$$G'' - G = \delta(x - \xi), \qquad G(0|\xi) = G(\infty|\xi) = 0.$$

Note that $\sinh(x)$ and e^{-x} are homogeneous solutions that satisfy the left and right boundary conditions, respectively. The Wronskian of these two solutions is

$$\begin{aligned} W(x) &= \begin{vmatrix} \sinh(x) & \mathrm{e}^{-x} \\ \cosh(x) & -\mathrm{e}^{-x} \end{vmatrix} \\ &= -\sinh(x)\,\mathrm{e}^{-x} - \cosh(x)\,\mathrm{e}^{-x} \\ &= -\frac{1}{2}\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right)\mathrm{e}^{-x} - \frac{1}{2}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)\mathrm{e}^{-x} \\ &= -1 \end{aligned}$$

The Green function for the problem is then

$$G(x|\xi) = -\sinh(x_<)\,\mathrm{e}^{-x_>}$$

$$\boxed{G(x|\xi) = \begin{cases} -\sinh(x)\,\mathrm{e}^{-\xi} & \text{for } 0 \leq x \leq \xi, \\ -\sinh(\xi)\,\mathrm{e}^{-x} & \text{for } \xi \leq x \leq \infty. \end{cases}}$$

Solution 23.15

a) The Green function problem is

$$xG''(x|\xi) + G'(x|\xi) = \delta(x-\xi), \quad G(0|\xi) \text{ bounded}, \; G(1|\xi) = 0.$$

First we find the homogeneous solutions of the differential equation.

$$xy'' + y' = 0$$

This is an exact equation.

$$\frac{\mathrm{d}}{\mathrm{d}x}[xy'] = 0$$

$$y' = \frac{c_1}{x}$$

$$y = c_1 \log x + c_2$$

The homogeneous solutions $y_1 = 1$ and $y_2 = \log x$ satisfy the left and right boundary conditions, respectively. The Wronskian of these solutions is

$$W(x) = \begin{vmatrix} 1 & \log x \\ 0 & 1/x \end{vmatrix} = \frac{1}{x}.$$

The Green function is

$$G(x|\xi) = \frac{1 \cdot \log x_>}{\xi(1/\xi)},$$

$$\boxed{G(x|\xi) = \log x_>.}$$

b) The Green function problem is

$$G''(x|\xi) - G(x|\xi) = \delta(x - \xi), \quad G(-a|\xi) = G(a|\xi) = 0.$$

$\{\mathrm{e}^x, \mathrm{e}^{-x}\}$ and $\{\cosh x, \sinh x\}$ are both linearly independent sets of homogeneous solutions. $\sinh(x+a)$ and $\sinh(x-a)$ are homogeneous solutions that satisfy the left and right boundary conditions, respectively. The Wronskian of these two solutions is,

$$\begin{aligned} W(x) &= \begin{vmatrix} \sinh(x+a) & \sinh(x-a) \\ \cosh(x+a) & \cosh(x-a) \end{vmatrix} \\ &= \sinh(x+a)\cosh(x-a) - \sinh(x-a)\cosh(x+a) \\ &= \sinh(2a) \end{aligned}$$

The Green function is

$$\boxed{G(x|\xi) = \frac{\sinh(x_< + a)\sinh(x_> - a)}{\sinh(2a)}.}$$

c) The Green function problem is

$$G''(x|\xi) - G(x|\xi) = \delta(x-\xi), \quad G(x|\xi) \text{ bounded as } |x| \to \infty.$$

e^x and e^{-x} are homogeneous solutions that satisfy the left and right boundary conditions, respectively. The Wronskian of these solutions is

$$W(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

The Green function is

$$G(x|\xi) = \frac{e^{x_<} e^{-x_>}}{-2},$$

$$\boxed{G(x|\xi) = -\frac{1}{2} e^{x_< - x_>}.}$$

d) The Green function from part (b) is,

$$G(x|\xi) = \frac{\sinh(x_< + a)\sinh(x_> - a)}{\sinh(2a)}.$$

We take the limit as $a \to \infty$.

$$\begin{aligned}
\lim_{a\to\infty} \frac{\sinh(x_< + a)\sinh(x_> - a)}{\sinh(2a)} &= \lim_{a\to\infty} \frac{\left(e^{x_<+a} - e^{-x_<-a}\right)\left(e^{x_>-a} - e^{-x_>+a}\right)}{2\left(e^{2a} - e^{-2a}\right)} \\
&= \lim_{a\to\infty} \frac{-e^{x_<-x_>} + e^{x_<+x_>-2a} + e^{-x_<-x_>-2a} - e^{-x_<+x_>-4a}}{2 - 2e^{-4a}} \\
&= -\frac{e^{x_<-x_>}}{2}
\end{aligned}$$

Thus we see that the solution from part (b) approaches the solution from part (c) as $a \to \infty$.

Solution 23.16

1. The problem,

$$y'' + \lambda y = f(x), \quad y(0) = y(\pi) = 0,$$

has a Green function if and only if it has a unique solution. This inhomogeneous problem has a unique solution if and only if the homogeneous problem has only the trivial solution.

First consider the case $\lambda = 0$. We find the general solution of the homogeneous differential equation.

$$y = c_1 + c_2 x$$

Only the trivial solution satisfies the boundary conditions. The problem has a unique solution for $\lambda = 0$.

Now consider non-zero λ. We find the general solution of the homogeneous differential equation.

$$y = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right).$$

The solution that satisfies the left boundary condition is

$$y = c \sin\left(\sqrt{\lambda} x\right).$$

We apply the right boundary condition and find nontrivial solutions.

$$\sin\left(\sqrt{\lambda}\pi\right) = 0$$
$$\lambda = n^2, \quad n \in \mathbb{Z}^+$$

Thus the problem has a unique solution for all complex λ except $\lambda = n^2$, $n \in \mathbb{Z}^+$.

Consider the case $\lambda = 0$. We find solutions of the homogeneous equation that satisfy the left and right boundary conditions, respectively.

$$y_1 = x, \qquad y_2 = x - \pi.$$

We compute the Wronskian of these functions.

$$W(x) = \begin{vmatrix} x & x-\pi \\ 1 & 1 \end{vmatrix} = \pi.$$

The Green function for this case is

$$G(x|\xi) = \frac{x_<(x_> - \pi)}{\pi}.$$

We consider the case $\lambda \neq n^2$, $\lambda \neq 0$. We find the solutions of the homogeneous equation that satisfy the left and right boundary conditions, respectively.

$$y_1 = \sin\left(\sqrt{\lambda}x\right), \qquad y_2 = \sin\left(\sqrt{\lambda}(x-\pi)\right).$$

We compute the Wronskian of these functions.

$$W(x) = \begin{vmatrix} \sin\left(\sqrt{\lambda}x\right) & \sin\left(\sqrt{\lambda}(x-\pi)\right) \\ \sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) & \sqrt{\lambda}\cos\left(\sqrt{\lambda}(x-\pi)\right) \end{vmatrix} = \sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)$$

The Green function for this case is

$$G(x|\xi) = \frac{\sin\left(\sqrt{\lambda}x_<\right)\sin\left(\sqrt{\lambda}(x_> - \pi)\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)}.$$

2. Now we consider the problem

$$y'' + 9y = 1 + \alpha x, \quad y(0) = y(\pi) = 0.$$

The homogeneous solutions of the problem are constant multiples of $\sin(3x)$. Thus for each value of α, the problem either has no solution or an infinite number of solutions. There will be an infinite number of solutions if the inhomogeneity $1 + \alpha x$ is orthogonal to the homogeneous solution $\sin(3x)$ and no solution otherwise.

$$\int_0^\pi (1 + \alpha x) \sin(3x) \, \mathrm{d}x = \frac{\pi\alpha + 2}{3}$$

The problem has a solution only for $\alpha = -2/\pi$. For this case the general solution of the inhomogeneous differential equation is

$$y = \frac{1}{9}\left(1 - \frac{2x}{\pi}\right) + c_1 \cos(3x) + c_2 \sin(3x).$$

The one-parameter family of solutions that satisfies the boundary conditions is

$$y = \frac{1}{9}\left(1 - \frac{2x}{\pi} - \cos(3x)\right) + c \sin(3x).$$

3. For $\lambda = n^2$, $n \in \mathbb{Z}^+$, $y = \sin(nx)$ is a solution of the homogeneous equation that satisfies the boundary conditions. Equation 23.5 has a (non-unique) solution only if f is orthogonal to $\sin(nx)$.

$$\int_0^\pi f(x) \sin(nx) \, \mathrm{d}x = 0$$

The modified Green function satisfies

$$G'' + n^2 G = \delta(x - \xi) - \frac{\sin(nx)\sin(n\xi)}{\pi/2}.$$

We expand G in a series of the eigenfunctions.

$$G(x|\xi) = \sum_{k=1}^{\infty} g_k \sin(kx)$$

We substitute the expansion into the differential equation to determine the coefficients. This will not determine g_n. We choose $g_n = 0$, which is one of the choices that will make the modified Green function symmetric in x and ξ.

$$\sum_{k=1}^{\infty} g_k \left(n^2 - k^2\right) \sin(kx) = \frac{2}{\pi} \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \sin(kx) \sin(k\xi)$$

$$\boxed{G(x|\xi) = \frac{2}{\pi} \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{\sin(kx)\sin(k\xi)}{n^2 - k^2}}$$

The solution of the inhomogeneous problem is

$$y(x) = \int_0^{\pi} f(\xi) G(x|\xi) \, \mathrm{d}\xi.$$

Solution 23.17

We separate the problem for u into the two problems:

$$\begin{aligned} Lv &\equiv (pv')' + qv = f(x), \quad a < x < b, \quad v(a) = 0, \quad v(b) = 0 \\ Lw &\equiv (pw')' + qw = 0, \quad a < x < b, \quad w(a) = \alpha, \quad w(b) = \beta \end{aligned}$$

and note that the solution for u is $u = v + w$.

The problem for v has the solution,

$$v = \int_a^b g(x;\xi) f(\xi) \, \mathrm{d}\xi,$$

with the Green function,

$$g(x;\xi) = \frac{v_1(x_<) v_2(x_>)}{p(\xi) W(\xi)} \equiv \begin{cases} \frac{v_1(x) v_2(\xi)}{p(\xi) W(\xi)} & \text{for } a \leq x \leq \xi, \\ \frac{v_1(\xi) v_2(x)}{p(\xi) W(\xi)} & \text{for } \xi \leq x \leq b. \end{cases}$$

Here v_1 and v_2 are homogeneous solutions that respectively satisfy the left and right homogeneous boundary conditions.

Since $g(x;\xi)$ is a solution of the homogeneous equation for $x \neq \xi$, $g_\xi(x;\xi)$ is a solution of the homogeneous equation for $x \neq \xi$. This is because for $x \neq \xi$,

$$L\left[\frac{\partial}{\partial \xi} g\right] = \frac{\partial}{\partial \xi} L[g] = \frac{\partial}{\partial \xi} \delta(x - \xi) = 0.$$

If ξ is outside of the domain, (a, b), then $g(x;\xi)$ and $g_\xi(x;\xi)$ are homogeneous solutions on that domain. In particular $g_\xi(x;a)$ and $g_\xi(x;b)$ are homogeneous solutions,

$$L\left[g_\xi(x;a)\right] = L\left[g_\xi(x;b)\right] = 0.$$

Now we use the definition of the Green function and $v_1(a) = v_2(b) = 0$ to determine simple expressions for these homogeneous solutions.

$$\begin{aligned}
g_\xi(x;a) &= \frac{v_1'(a)v_2(x)}{p(a)W(a)} - \frac{(p'(a)W(a) + p(a)W'(a))v_1(a)v_2(x)}{(p(a)W(a))^2} \\
&= \frac{v_1'(a)v_2(x)}{p(a)W(a)} \\
&= \frac{v_1'(a)v_2(x)}{p(a)(v_1(a)v_2'(a) - v_1'(a)v_2(a))} \\
&= -\frac{v_1'(a)v_2(x)}{p(a)v_1'(a)v_2(a)} \\
&= -\frac{v_2(x)}{p(a)v_2(a)}
\end{aligned}$$

We note that this solution has the boundary values,

$$g_\xi(a;a) = -\frac{v_2(a)}{p(a)v_2(a)} = -\frac{1}{p(a)}, \qquad g_\xi(b;a) = -\frac{v_2(b)}{p(a)v_2(a)} = 0.$$

We examine the second solution.

$$
\begin{aligned}
g_\xi(x;b) &= \frac{v_1(x)v_2'(b)}{p(b)W(b)} - \frac{(p'(b)W(b)+p(b)W'(b))v_1(x)v_2(b)}{(p(b)W(b))^2} \\
&= \frac{v_1(x)v_2'(b)}{p(b)W(b)} \\
&= \frac{v_1(x)v_2'(b)}{p(b)(v_1(b)v_2'(b)-v_1'(b)v_2(b))} \\
&= \frac{v_1(x)v_2'(b)}{p(b)v_1(b)v_2'(b)} \\
&= \frac{v_1(x)}{p(b)v_1(b)}
\end{aligned}
$$

This solution has the boundary values,

$$
g_\xi(a;b) = \frac{v_1(a)}{p(b)v_1(b)} = 0, \qquad g_\xi(b;b) = \frac{v_1(b)}{p(b)v_1(b)} = \frac{1}{p(b)}.
$$

Thus we see that the solution of

$$
Lw = (pw')' + qw = 0, \quad a < x < b, \quad w(a) = \alpha, \quad w(b) = \beta,
$$

is

$$
w = -\alpha p(a) g_\xi(x;a) + \beta p(b) g_\xi(x;b).
$$

Therefore the solution of the problem for u is

$$
\boxed{u = \int_a^b g(x;\xi) f(\xi)\, \mathrm{d}\xi - \alpha p(a) g_\xi(x;a) + \beta p(b) g_\xi(x;b).}
$$

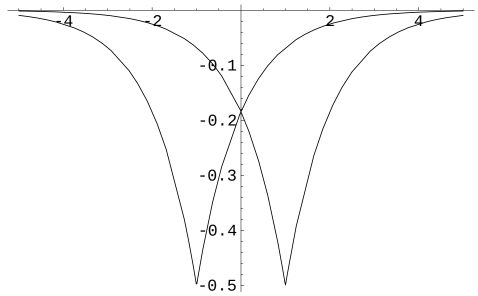

Figure 23.7: $G(x;1)$ and $G(x;-1)$

Solution 23.18
Figure 23.7 shows a plot of $G(x;1)$ and $G(x;-1)$ for $k=1$.

First we consider the boundary condition $u(0)=0$. Note that the solution of

$$G'' - k^2 G = \delta(x-\xi) - \delta(x+\xi), |G(\pm\infty;\xi)| < \infty,$$

satisfies the condition $G(0;\xi)=0$. Thus the Green function which satisfies $G(0;\xi)=0$ is

$$G(x;\xi) = -\frac{1}{2k}\,\mathrm{e}^{-k|x-\xi|} + \frac{1}{2k}\,\mathrm{e}^{-k|x+\xi|}.$$

Since $x,\xi>0$ we can write this as

$$\begin{aligned}
G(x;\xi) &= -\frac{1}{2k}\,\mathrm{e}^{-k|x-\xi|} + \frac{1}{2k}\,\mathrm{e}^{-k(x+\xi)} \\
&= \begin{cases} -\frac{1}{2k}\,\mathrm{e}^{-k(\xi-x)} + \frac{1}{2k}\,\mathrm{e}^{-k(x+\xi)}, & \text{for } x<\xi \\ -\frac{1}{2k}\,\mathrm{e}^{-k(x-\xi)} + \frac{1}{2k}\,\mathrm{e}^{-k(x+\xi)}, & \text{for } \xi<x \end{cases} \\
&= \begin{cases} -\frac{1}{k}\,\mathrm{e}^{-k\xi}\sinh(kx), & \text{for } x<\xi \\ -\frac{1}{k}\,\mathrm{e}^{-kx}\sinh(k\xi), & \text{for } \xi<x \end{cases}
\end{aligned}$$

$$\boxed{G(x;\xi) = -\frac{1}{k}\,\mathrm{e}^{-kx_>}\sinh(kx_<)}$$

Now consider the boundary condition $u'(0) = 0$. Note that the solution of

$$G'' - k^2 G = \delta(x-\xi) + \delta(x+\xi), \quad |G(\pm\infty;\xi)| < \infty,$$

satisfies the boundary condition $G'(x;\xi) = 0$. Thus the Green function is

$$G(x;\xi) = -\frac{1}{2k}\,\mathrm{e}^{-k|x-\xi|} - \frac{1}{2k}\,\mathrm{e}^{-k|x+\xi|}.$$

Since $x, \xi > 0$ we can write this as

$$\begin{aligned}
G(x;\xi) &= -\frac{1}{2k}\,\mathrm{e}^{-k|x-\xi|} - \frac{1}{2k}\,\mathrm{e}^{-k(x+\xi)} \\
&= \begin{cases} -\frac{1}{2k}\,\mathrm{e}^{-k(\xi-x)} - \frac{1}{2k}\,\mathrm{e}^{-k(x+\xi)}, & \text{for } x < \xi \\ -\frac{1}{2k}\,\mathrm{e}^{-k(x-\xi)} - \frac{1}{2k}\,\mathrm{e}^{-k(x+\xi)}, & \text{for } \xi < x \end{cases} \\
&= \begin{cases} -\frac{1}{k}\,\mathrm{e}^{-k\xi}\cosh(kx), & \text{for } x < \xi \\ -\frac{1}{k}\,\mathrm{e}^{-kx}\cosh(k\xi), & \text{for } \xi < x \end{cases}
\end{aligned}$$

$$\boxed{G(x;\xi) = -\frac{1}{k}\,\mathrm{e}^{-kx_>}\cosh(kx_<)}$$

The Green functions which satisfies $G(0;\xi) = 0$ and $G'(0;\xi) = 0$ are shown in Figure 23.8.

Solution 23.19

1. The Green function satisfies

$$g'' - a^2 g = \delta(x-\xi), \quad g(0;\xi) = g'(L;\xi) = 0.$$

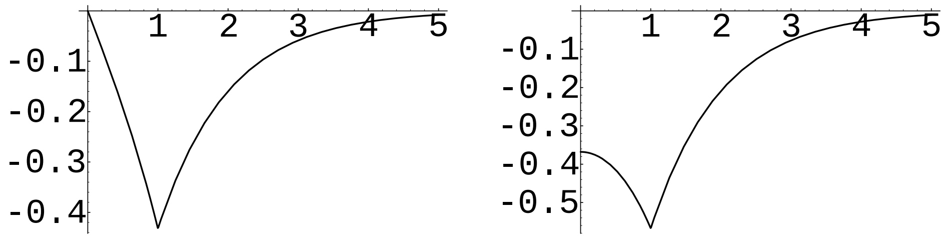

Figure 23.8: $G(x; 1)$ and $G(x; -1)$

We can write the set of homogeneous solutions as

$$\left\{ \mathrm{e}^{ax}, \mathrm{e}^{-ax} \right\} \text{ or } \left\{ \cosh(ax), \sinh(ax) \right\}.$$

The solutions that respectively satisfy the left and right boundary conditions are

$$u_1 = \sinh(ax), \quad u_2 = \cosh(a(x - L)).$$

The Wronskian of these solutions is

$$W(x) = \begin{pmatrix} \sinh(ax) & \cosh(a(x-L)) \\ a\cosh(ax) & a\sinh(a(x-L)) \end{pmatrix} = -a\cosh(aL).$$

Thus the Green function is

$$\boxed{g(x; \xi) = \begin{cases} -\frac{\sinh(ax)\cosh(a(\xi-L))}{a\cosh(aL)} & \text{for } x \leq \xi, \\ -\frac{\sinh(a\xi)\cosh(a(x-L))}{a\cosh(aL)} & \text{for } \xi \leq x. \end{cases} = -\frac{\sinh(ax_<)\cosh(a(x_> - L))}{a\cosh(aL)}.}$$

2. We take the limit as $L \to \infty$.

$$g(x;\xi) = \lim_{L\to\infty} -\frac{\sinh(ax_<)\cosh(a(x_> - L))}{a\cosh(aL)}$$
$$= \lim_{L\to\infty} -\frac{\sinh(ax_<)}{a}\frac{\cosh(ax_>)\cosh(aL) - \sinh(ax_>)\sinh(aL)}{\cosh(aL)}$$
$$= -\frac{\sinh(ax_<)}{a}(\cosh(ax_>) - \sinh(ax_>))$$

$$\boxed{g(x;\xi) = -\frac{1}{a}\sinh(ax_<)\,\mathrm{e}^{-ax_>}}$$

The solution of

$$y'' - a^2 y = \mathrm{e}^{-x}, \quad y(0) = y'(\infty) = 0$$

is

$$y = \int_0^\infty g(x;\xi)\,\mathrm{e}^{-\xi}\,\mathrm{d}\xi$$
$$= -\frac{1}{a}\int_0^\infty \sinh(ax_<)\,\mathrm{e}^{-ax_>}\,\mathrm{e}^{-\xi}\,\mathrm{d}\xi$$
$$= -\frac{1}{a}\left(\int_0^x \sinh(a\xi)\,\mathrm{e}^{-ax}\,\mathrm{e}^{-\xi}\,\mathrm{d}\xi + \int_x^\infty \sinh(ax)\,\mathrm{e}^{-a\xi}\,\mathrm{e}^{-\xi}\,\mathrm{d}\xi\right)$$

We first consider the case that $a \neq 1$.

$$= -\frac{1}{a}\left(\frac{\mathrm{e}^{-ax}}{a^2-1}\left(-a + \mathrm{e}^{-x}(a\cosh(ax) + \sinh(ax))\right) + \frac{1}{a+1}\,\mathrm{e}^{-(a+1)x}\sinh(ax)\right)$$
$$= \frac{\mathrm{e}^{-ax} - \mathrm{e}^{-x}}{a^2-1}$$

For $a = 1$, we have

$$
\begin{aligned}
y &= -\left(\frac{1}{4}\,\mathrm{e}{-x}\left(-1 + 2x + \mathrm{e}^{-2x}\right) + \frac{1}{2}\,\mathrm{e}^{-2x}\sinh(x)\right) \\
&= -\frac{1}{2} x\,\mathrm{e}^{-x}.
\end{aligned}
$$

Thus the solution of the problem is

$$
\boxed{y = \begin{cases} \frac{\mathrm{e}^{-ax} - \mathrm{e}^{-x}}{a^2 - 1} & \text{for } a \neq 1, \\ -\frac{1}{2} x\,\mathrm{e}^{-x} & \text{for } a = 1. \end{cases}}
$$

We note that this solution satisfies the differential equation and the boundary conditions.

Chapter 24

Difference Equations

Televisions should have a dial to turn up the intelligence. There is a brightness knob, but it doesn't work.

-?

24.1 Introduction

Example 24.1.1 Gambler's ruin problem. Consider a gambler that initially has n dollars. He plays a game in which he has a probability p of winning a dollar and q of losing a dollar. (Note that $p+q=1$.) The gambler has decided that if he attains N dollars he will stop playing the game. In this case we will say that he has succeeded. Of course if he runs out of money before that happens, we will say that he is ruined. What is the probability of the gambler's ruin? Let us denote this probability by a_n. We know that if he has no money left, then his ruin is certain, so $a_0 = 1$. If he reaches N dollars he will quit the game, so that $a_N = 0$. If he is somewhere in between ruin and success then the probability of his ruin is equal to p times the probability of his ruin if he had $n+1$ dollars plus q times the probability of his ruin if he had $n-1$ dollars. Writing this in an equation,

$$a_n = pa_{n+1} + qa_{n-1} \quad \text{subject to} \quad a_0 = 1, \quad a_N = 0.$$

This is an example of a difference equation. You will learn how to solve this particular problem in the section on constant coefficient equations.

Consider the sequence $a_1, a_2, a_3, \ldots$ Analogous to a derivative of a continuous function, we can define a discrete derivative on the sequence

$$Da_n = a_{n+1} - a_n.$$

The second discrete derivative is then defined as

$$D^2 a_n = D[a_{n+1} - a_n] = a_{n+2} - 2a_{n+1} + a_n.$$

The discrete integral of a_n is

$$\sum_{i=n_0}^{n} a_i.$$

Corresponding to

$$\int_\alpha^\beta \frac{\mathrm{d}f}{\mathrm{d}x}\, dx = f(\beta) - f(\alpha),$$

in the discrete realm we have

$$\sum_{n=\alpha}^{\beta-1} D[a_n] = \sum_{n=\alpha}^{\beta-1} (a_{n+1} - a_n) = a_\beta - a_\alpha.$$

Linear difference equations have the form

$$D^r a_n + p_{r-1}(n) D^{r-1} a_n + \cdots + p_1(n) D a_n + p_0(n) a_n = f(n).$$

From the definition of the discrete derivative an equivalent form is

$$a_{n+r} + q_{r-1}(n) a_{n_r-1} + \cdots + q_1(n) a_{n+1} + q_0(n) a_n = f(n).$$

Besides being important in their own right, we will need to solve difference equations in order to develop series solutions of differential equations. Also, some methods of solving differential equations numerically are based on approximating them with difference equations.

There are many similarities between differential and difference equations. Like differential equations, an r^{th} order homogeneous difference equation has r linearly independent solutions. The general solution to the r^{th} order inhomogeneous equation is the sum of the particular solution and an arbitrary linear combination of the homogeneous solutions.

For an r^{th} order difference equation, the initial condition is given by specifying the values of the first r a_n's.

Example 24.1.2 Consider the difference equation $a_{n-2}-a_{n-1}-a_n = 0$ subject to the initial condition $a_1 = a_2 = 1$. Note that although we may not know a closed-form formula for the a_n we can calculate the a_n in order by substituting into the difference equation. The first few a_n are $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ We recognize this as the Fibonacci sequence.

24.2 Exact Equations

Consider the sequence $a_1, a_2, \ldots$. Exact difference equations on this sequence have the form

$$D[F(a_n, a_{n+1}, \ldots, n)] = g(n).$$

We can reduce the order of, (or solve for first order), this equation by summing from 1 to $n-1$.

$$\sum_{j=1}^{n-1} D[F(a_j, a_{j+1}, \ldots, j)] = \sum_{j=1}^{n-1} g(j)$$

$$F(a_n, a_{n+1}, \ldots, n) - F(a_1, a_2, \ldots, 1) = \sum_{j=1}^{n-1} g(j)$$

$$F(a_n, a_{n+1}, \ldots, n) = \sum_{j=1}^{n-1} g(j) + F(a_1, a_2, \ldots, 1)$$

Result 24.2.1 We can reduce the order of the exact difference equation

$$D[F(a_n, a_{n+1}, \dots, n)] = g(n), \qquad \text{for } n \geq 1$$

by summing both sides of the equation to obtain

$$F(a_n, a_{n+1}, \dots, n) = \sum_{j=1}^{n-1} g(j) + F(a_1, a_2, \dots, 1).$$

Example 24.2.1 Consider the difference equation, $D[na_n] = 1$. Summing both sides of this equation

$$\sum_{j=1}^{n-1} D[ja_j] = \sum_{j=1}^{n-1} 1$$

$$na_n - a_1 = n - 1$$

$$\boxed{a_n = \frac{n + a_1 - 1}{n}}.$$

24.3 Homogeneous First Order

Consider the homogeneous first order difference equation

$$a_{n+1} = p(n)a_n, \qquad \text{for } n \geq 1.$$

We can directly solve for a_n.

$$\begin{aligned} a_n &= a_n \frac{a_{n-1}}{a_{n-1}} \frac{a_{n-2}}{a_{n-2}} \cdots \frac{a_1}{a_1} \\ &= a_1 \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_2}{a_1} \\ &= a_1 p(n-1) p(n-2) \cdots p(1) \\ &= a_1 \prod_{j=1}^{n-1} p(j) \end{aligned}$$

Alternatively, we could solve this equation by making it exact. Analogous to an integrating factor for differential equations, we multiply the equation by the summing factor

$$S(n) = \left[\prod_{j=1}^{n} p(j) \right]^{-1}.$$

$$\begin{aligned} a_{n+1} - p(n) a_n &= 0 \\ \frac{a_{n+1}}{\prod_{j=1}^{n} p(j)} - \frac{a_n}{\prod_{j=1}^{n-1} p(j)} &= 0 \\ D \left[\frac{a_n}{\prod_{j=1}^{n-1} p(j)} \right] &= 0 \end{aligned}$$

Now we sum from 1 to $n-1$.

$$\begin{aligned} \frac{a_n}{\prod_{j=1}^{n-1} p(j)} - a_1 &= 0 \\ a_n &= a_1 \prod_{j=1}^{n-1} p(j) \end{aligned}$$

Result 24.3.1 The solution of the homogeneous first order difference equation

$$a_{n+1} = p(n) a_n, \qquad \text{for } n \geq 1,$$

is

$$a_n = a_1 \prod_{j=1}^{n-1} p(j).$$

Example 24.3.1 Consider the equation $a_{n+1} = n a_n$ with the initial condition $a_1 = 1$.

$$a_n = a_1 \prod_{j=1}^{n-1} j = (1)(n-1)! = \Gamma(n)$$

Recall that $\Gamma(z)$ is the generalization of the factorial function. For positive integral values of the argument, $\Gamma(n) = (n-1)!$.

24.4 Inhomogeneous First Order

Consider the equation

$$a_{n+1} = p(n) a_n + q(n) \quad \text{for} \quad n \geq 1.$$

Multiplying by $S(n) = \left[\prod_{j=1}^{n} p(j)\right]^{-1}$ yields

$$\frac{a_{n+1}}{\prod_{j=1}^{n} p(j)} - \frac{a_n}{\prod_{j=1}^{n-1} p(j)} = \frac{q(n)}{\prod_{j=1}^{n} p(j)}.$$

The left hand side is a discrete derivative.

$$D\left[\frac{a_n}{\prod_{j=1}^{n-1} p(j)}\right] = \frac{q(n)}{\prod_{j=1}^{n} p(j)}$$

Summing both sides from 1 to $n-1$,

$$\frac{a_n}{\prod_{j=1}^{n-1} p(j)} - a_1 = \sum_{k=1}^{n-1}\left[\frac{q(k)}{\prod_{j=1}^{k} p(j)}\right]$$

$$a_n = \left[\prod_{m=1}^{n-1} p(m)\right]\left[\sum_{k=1}^{n-1}\left[\frac{q(k)}{\prod_{j=1}^{k} p(j)}\right] + a_1\right].$$

Result 24.4.1 The solution of the inhomogeneous first order difference equation

$$a_{n+1} = p(n)a_n + q(n) \quad \text{for} \quad n \geq 1$$

is

$$a_n = \left[\prod_{m=1}^{n-1} p(m)\right]\left[\sum_{k=1}^{n-1}\left[\frac{q(k)}{\prod_{j=1}^{k} p(j)}\right] + a_1\right].$$

Example 24.4.1 Consider the equation $a_{n+1} = na_n + 1$ for $n \geq 1$. The summing factor is

$$S(n) = \left[\prod_{j=1}^{n} j\right]^{-1} = \frac{1}{n!}.$$

Multiplying the difference equation by the summing factor,

$$\frac{a_{n+1}}{n!} - \frac{a_n}{(n-1)!} = \frac{1}{n!}$$
$$D\left[\frac{a_n}{(n-1)!}\right] = \frac{1}{n!}$$
$$\frac{a_n}{(n-1)!} - a_1 = \sum_{k=1}^{n-1} \frac{1}{k!}$$
$$\boxed{a_n = (n-1)!\left[\sum_{k=1}^{n-1} \frac{1}{k!} + a_1\right].}$$

Example 24.4.2 Consider the equation

$$a_{n+1} = \lambda a_n + \mu, \quad \text{for } n \geq 0.$$

From the above result, (with the products and sums starting at zero instead of one), the solution is

$$\begin{aligned}
a_0 &= \left[\prod_{m=0}^{n-1} \lambda\right]\left[\sum_{k=0}^{n-1}\left[\frac{\mu}{\prod_{j=0}^{k} \lambda}\right] + a_0\right] \\
&= \lambda^n \left[\sum_{k=0}^{n-1}\left[\frac{\mu}{\lambda^{k+1}}\right] + a_0\right] \\
&= \lambda^n \left[\mu \frac{\lambda^{-n-1} - \lambda^{-1}}{\lambda^{-1} - 1} + a_0\right] \\
&= \lambda^n \left[\mu \frac{\lambda^{-n} - 1}{1 - \lambda} + a_0\right] \\
&= \mu \frac{1 - \lambda^n}{1 - \lambda} + a_0 \lambda^n.
\end{aligned}$$

24.5 Homogeneous Constant Coefficient Equations

Homogeneous constant coefficient equations have the form

$$a_{n+N} + p_{N-1} a_{n+N-1} + \cdots + p_1 a_{n+1} + p_0 a_n = 0.$$

The substitution $a_n = r^n$ yields

$$\begin{aligned} r^N + p_{N-1} r^{N-1} + \cdots + p_1 r + p_0 &= 0 \\ (r - r_1)^{m_1} \cdots (r - r_k)^{m_k} &= 0. \end{aligned}$$

If r_1 is a distinct root then the associated linearly independent solution is r_1^n. If r_1 is a root of multiplicity $m > 1$ then the associated solutions are $r_1^n, n r_1^n, n^2 r_1^n, \ldots, n^{m-1} r_1^n$.

Result 24.5.1 Consider the homogeneous constant coefficient difference equation

$$a_{n+N} + p_{N-1} a_{n+N-1} + \cdots + p_1 a_{n+1} + p_0 a_n = 0.$$

The substitution $a_n = r^n$ yields the equation

$$(r - r_1)^{m_1} \cdots (r - r_k)^{m_k} = 0.$$

A set of linearly independent solutions is

$$\{r_1^n, n r_1^n, \ldots, n^{m_1 - 1} r_1^n, \ldots, r_k^n, n r_k^n, \ldots, n^{m_k - 1} r_k^n\}.$$

Example 24.5.1 Consider the equation $a_{n+2} - 3a_{n+1} + 2a_n = 0$ with the initial conditions $a_1 = 1$ and $a_2 = 3$. The substitution $a_n = r^n$ yields

$$r^2 - 3r + 2 = (r - 1)(r - 2) = 0.$$

Thus the general solution is

$$a_n = c_1 1^n + c_2 2^n.$$

The initial conditions give the two equations,

$$\begin{aligned} a_1 &= 1 = c_1 + 2c_2 \\ a_2 &= 3 = c_1 + 4c_2 \end{aligned}$$

Since $c_1 = -1$ and $c_2 = 1$, the solution to the difference equation subject to the initial conditions is

$$a_n = 2^n - 1.$$

Example 24.5.2 Consider the gambler's ruin problem that was introduced in Example 24.1.1. The equation for the probability of the gambler's ruin at n dollars is

$$a_n = p a_{n+1} + q a_{n-1} \quad \text{subject to} \quad a_0 = 1, \quad a_N = 0.$$

We assume that $0 < p < 1$. With the substitution $a_n = r^n$ we obtain

$$r = pr^2 + q.$$

The roots of this equation are

$$\begin{aligned} r &= \frac{1 \pm \sqrt{1 - 4pq}}{2p} \\ &= \frac{1 \pm \sqrt{1 - 4p(1-p)}}{2p} \\ &= \frac{1 \pm \sqrt{(1-2p)^2}}{2p} \\ &= \frac{1 \pm |1 - 2p|}{2p}. \end{aligned}$$

We will consider the two cases $p \neq 1/2$ and $p = 1/2$.

$\mathbf{p \neq 1/2}$. If $p < 1/2$, the roots are

$$r = \frac{1 \pm (1 - 2p)}{2p}$$
$$r_1 = \frac{1 - p}{p} = \frac{q}{p}, \qquad r_2 = 1.$$

If $p > 1/2$ the roots are

$$r = \frac{1 \pm (2p - 1)}{2p}$$
$$r_1 = 1, \qquad r_2 = \frac{-p + 1}{p} = \frac{q}{p}.$$

Thus the general solution for $p \neq 1/2$ is

$$a_n = c_1 + c_2 \left(\frac{q}{p} \right)^n.$$

The boundary condition $a_0 = 1$ requires that $c_1 + c_2 = 1$. From the boundary condition $a_N = 0$ we have

$$(1 - c_2) + c_2 \left(\frac{q}{p} \right)^N = 0$$
$$c_2 = \frac{-1}{-1 + (q/p)^N}$$
$$c_2 = \frac{p^N}{p^N - q^N}.$$

Solving for c_1,

$$c_1 = 1 - \frac{p^N}{p^N - q^N}$$
$$c_1 = \frac{-q^N}{p^N - q^N}.$$

Thus we have

$$\boxed{a_n = \frac{-q^N}{p^N - q^N} + \frac{p^N}{p^N - q^N}\left(\frac{q}{p}\right)^n.}$$

p = 1/2. In this case, the two roots of the polynomial are both 1. The general solution is

$$a_n = c_1 + c_2 n.$$

The left boundary condition demands that $c_1 = 1$. From the right boundary condition we obtain

$$1 + c_2 N = 0$$
$$c_2 = -\frac{1}{N}.$$

Thus the solution for this case is

$$\boxed{a_n = 1 - \frac{n}{N}.}$$

As a check that this formula makes sense, we see that for $n = N/2$ the probability of ruin is $1 - \frac{N/2}{N} = \frac{1}{2}$.

24.6 Reduction of Order

Consider the difference equation

$$(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n = 0 \quad \text{for} \quad n \geq 0 \tag{24.1}$$

We see that one solution to this equation is $a_n = 1/n!$. Analogous to the reduction of order for differential equations, the substitution $a_n = b_n/n!$ will reduce the order of the difference equation.

$$\frac{(n+1)(n+2)b_{n+2}}{(n+2)!} - \frac{3(n+1)b_{n+1}}{(n+1)!} + \frac{2b_n}{n!} = 0$$
$$b_{n+2} - 3b_{n+1} + 2b_n = 0 \tag{24.2}$$

At first glance it appears that we have not reduced the order of the equation, but writing it in terms of discrete derivatives

$$D^2 b_n - D b_n = 0$$

shows that we now have a first order difference equation for $D b_n$. The substitution $b_n = r^n$ in equation 24.2 yields the algebraic equation

$$r^2 - 3r + 2 = (r - 1)(r - 2) = 0.$$

Thus the solutions are $b_n = 1$ and $b_n = 2^n$. Only the $b_n = 2^n$ solution will give us another linearly independent solution for a_n. Thus the second solution for a_n is $a_n = b_n/n! = 2^n/n!$. The general solution to equation 24.1 is then

$$\boxed{a_n = c_1 \frac{1}{n!} + c_2 \frac{2^n}{n!}.}$$

Result 24.6.1 Let $a_n = s_n$ be a homogeneous solution of a linear difference equation. The substitution $a_n = s_n b_n$ will yield a difference equation for b_n that is of order one less than the equation for a_n.

24.7 Exercises

Exercise 24.1
Find a formula for the n^{th} term in the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$.
Hint, Solution

Exercise 24.2
Solve the difference equation

$$a_{n+2} = \frac{2}{n} a_n, \quad a_1 = a_2 = 1.$$

Hint, Solution

24.8 Hints

Hint 24.1
The difference equation corresponding to the Fibonacci sequence is

$$a_{n+2} - a_{n+1} - a_n = 0, \qquad a_1 = a_2 = 1.$$

Hint 24.2
Consider this exercise as two first order difference equations; one for the even terms, one for the odd terms.

24.9 Solutions

Solution 24.1
We can describe the Fibonacci sequence with the difference equation

$$a_{n+2} - a_{n+1} - a_n = 0, \qquad a_1 = a_2 = 1.$$

With the substitution $a_n = r^n$ we obtain the equation

$$r^2 - r - 1 = 0.$$

This equation has the two distinct roots

$$r_1 = \frac{1+\sqrt{5}}{2}, \qquad r_2 = \frac{1-\sqrt{5}}{2}.$$

Thus the general solution is

$$a_n = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

From the initial conditions we have

$$\begin{aligned} c_1 r_1 + c_2 r_2 &= 1 \\ c_1 r_1^2 + c_2 r_2^2 &= 1. \end{aligned}$$

Solving for c_2 in the first equation,

$$c_2 = \frac{1}{r_2}(1 - c_1 r_1).$$

We substitute this into the second equation.

$$
\begin{aligned}
c_1 r_1^2 + \frac{1}{r_2}(1 - c_1 r_1) r_2^2 = 1 \\
c_1 (r_1^2 - r_1 r_2) = 1 - r_2
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= \frac{1 - r_2}{r_1^2 - r_1 r_2} \\
&= \frac{1 - \frac{1-\sqrt{5}}{2}}{\frac{1+\sqrt{5}}{2}\sqrt{5}} \\
&= \frac{\frac{1+\sqrt{5}}{2}}{\frac{1+\sqrt{5}}{2}\sqrt{5}} \\
&= \frac{1}{\sqrt{5}}
\end{aligned}
$$

Substitute this result into the equation for c_2.

$$
\begin{aligned}
c_2 &= \frac{1}{r_2}\left(1 - \frac{1}{\sqrt{5}} r_1\right) \\
&= \frac{2}{1 - \sqrt{5}}\left(1 - \frac{1}{\sqrt{5}} \frac{1 + \sqrt{5}}{2}\right) \\
&= -\frac{2}{1 - \sqrt{5}}\left(\frac{1 - \sqrt{5}}{2\sqrt{5}}\right) \\
&= -\frac{1}{\sqrt{5}}
\end{aligned}
$$

Thus the n^{th} term in the Fibonacci sequence has the formula

$$\boxed{a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.}$$

It is interesting to note that although the Fibonacci sequence is defined in terms of integers, one cannot express the formula form the n^{th} element in terms of rational numbers.

Solution 24.2
We can consider

$$a_{n+2} = \frac{2}{n}a_n, \quad a_1 = a_2 = 1$$

to be a first order difference equation. First consider the odd terms.

$$\begin{aligned}
a_1 &= 1 \\
a_3 &= \frac{2}{1} \\
a_5 &= \frac{2}{3}\frac{2}{1} \\
a_n &= \frac{2^{(n-1)/2}}{(n-2)(n-4)\cdots(1)}
\end{aligned}$$

For the even terms,

$$\begin{aligned}
a_2 &= 1 \\
a_4 &= \frac{2}{2} \\
a_6 &= \frac{2}{4}\frac{2}{2} \\
a_n &= \frac{2^{(n-2)/2}}{(n-2)(n-4)\cdots(2)}.
\end{aligned}$$

Thus

$$\boxed{a_n = \begin{cases} \frac{2^{(n-1)/2}}{(n-2)(n-4)\cdots(1)} & \text{for odd } n \\ \frac{2^{(n-2)/2}}{(n-2)(n-4)\cdots(2)} & \text{for even } n. \end{cases}}$$

Chapter 25

Series Solutions of Differential Equations

Skill beats honesty any day.

-?

25.1 Ordinary Points

Big $\mathcal{O}$ and Little o Notation. The notation $\mathcal{O}(z^n)$ means "terms no bigger than z^n." This gives us a convenient shorthand for manipulating series. For example,

$$\sin z = z - \frac{z^3}{6} + \mathcal{O}(z^5)$$

$$\frac{1}{1-z} = 1 + \mathcal{O}(z)$$

The notation $o(z^n)$ means "terms smaller that z^n." For example,

$$\cos z = 1 + o(1)$$

$$\mathrm{e}^z = 1 + z + o(z)$$

Example 25.1.1 Consider the equation

$$w''(z) - 3w'(z) + 2w(z) = 0.$$

The general solution to this constant coefficient equation is

$$w = c_1\,\mathrm{e}^z + c_2\,\mathrm{e}^{2z}.$$

The functions e^z and e^{2z} are analytic in the finite complex plane. Recall that a function is analytic at a point z_0 if and only if the function has a Taylor series about z_0 with a nonzero radius of convergence. If we substitute the Taylor series expansions about $z = 0$ of e^z and e^{2z} into the general solution, we obtain

$$w = c_1 \sum_{n=0}^{\infty} \frac{z^n}{n!} + c_2 \sum_{n=0}^{\infty} \frac{2^n z^n}{n!}.$$

Thus we have a series solution of the differential equation.

Alternatively, we could try substituting a Taylor series into the differential equation and solving for the coefficients. Substituting $w = \sum_{n=0}^{\infty} a_n z^n$ into the differential equation yields

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} \sum_{n=0}^{\infty} a_n z^n - 3\frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=0}^{\infty} a_n z^n + 2\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} - 3\sum_{n=1}^{\infty} n a_n z^{n-1} + 2\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2} z^n - 3\sum_{n=0}^{\infty}(n+1)a_{n+1} z^n + 2\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + 2a_n\right] z^n = 0.$$

Equating powers of z, we obtain the difference equation

$$(n+2)(n+1)a_{n+2} - 3(n+1)a_{n+1} + 2a_n = 0, \qquad n \geq 0.$$

We see that $a_n = 1/n!$ is one solution since

$$\frac{(n+2)(n+1)}{(n+2)!} - 3\frac{n+1}{(n+1)!} + 2\frac{1}{n!} = \frac{1-3+2}{n!} = 0.$$

We use reduction of order for difference equations to find the other solution. Substituting $a_n = b_n/n!$ into the difference equation yields

$$(n+2)(n+1)\frac{b_{n+2}}{(n+2)!} - 3(n+1)\frac{b_{n+1}}{(n+1)!} + 2\frac{b_n}{n!} = 0$$
$$b_{n+2} - 3b_{n+1} + 2b_n = 0.$$

At first glance it appears that we have not reduced the order of the difference equation. However writing this equation in terms of discrete derivatives,

$$D^2 b_n - Db_n = 0$$

we see that this is a first order difference equation for Db_n. Since this is a constant coefficient difference equation we substitute $b_n = r^n$ into the equation to obtain an algebraic equation for r.

$$r^2 - 3r + 2 = (r-1)(r-2) = 0$$

Thus the two solutions are $b_n = 1^n b_0$ and $b_n = 2^n b_0$. Only $b_n = 2^n b_0$ will give us a second independent solution for a_n. Thus the two solutions for a_n are

$$a_n = \frac{a_0}{n!} \qquad \text{and} \qquad a_n = \frac{2^n a_0}{n!}.$$

Thus we can write the general solution to the differential equation as

$$\boxed{w = c_1 \sum_{n=0}^{\infty} \frac{z^n}{n!} + c_2 \sum_{n=0}^{\infty} \frac{2^n z^n}{n!}.}$$

We recognize these two sums as the Taylor expansions of e^z and e^{2z}. Thus we obtain the same result as we did solving the differential equation directly.

Of course it would be pretty silly to go through all the grunge involved in developing a series expansion of the solution in a problem like Example 25.1.1 since we can solve the problem exactly. However if we could not solve a differential equation, then having a Taylor series expansion of the solution about a point z_0 would be useful in determining the behavior of the solutions near that point.

For this method of substituting a Taylor series into the differential equation to be useful we have to know at what points the solutions are analytic. Let's say we were considering a second order differential equation whose solutions were

$$w_1 = \frac{1}{z}, \quad \text{and} \quad w_2 = \log z.$$

Trying to find a Taylor series expansion of the solutions about the point $z = 0$ would fail because the solutions are not analytic at $z = 0$. This brings us to two important questions.

1. Can we tell if the solutions to a linear differential equation are analytic at a point without knowing the solutions?

2. If there are Taylor series expansions of the solutions to a differential equation, what are the radii of convergence of the series?

In order to answer these questions, we will introduce the concept of an ordinary point. Consider the n^{th} order linear homogeneous equation

$$\frac{\mathrm{d}^n w}{\mathrm{d}z^n} + p_{n-1}(z)\frac{\mathrm{d}^{n-1} w}{\mathrm{d}z^{n-1}} + \cdots + p_1(z)\frac{\mathrm{d}w}{\mathrm{d}z} + p_0(z)w = 0.$$

If each of the coefficient functions $p_i(z)$ are analytic at $z = z_0$ then z_0 is an **ordinary point** of the differential equation.

For reasons of typography we will restrict our attention to second order equations and the point $z_0 = 0$ for a while. The generalization to an n^{th} order equation will be apparent. Considering the point $z_0 \neq 0$ is only trivially more general as we could introduce the transformation $z - z_0 \to z$ to move the point to the origin.

In the chapter on first order differential equations we showed that the solution is analytic at ordinary points. One would guess that this remains true for higher order equations. Consider the second order equation

$$y'' + p(z)y' + q(z)y = 0,$$

where p and q are analytic at the origin.

$$p(z) = \sum_{n=0}^{\infty} p_n z^n, \quad \text{and} \quad q(z) = \sum_{n=0}^{\infty} q_n z^n$$

Assume that one of the solutions is not analytic at the origin and behaves like z^α at $z = 0$ where $\alpha \neq 0, 1, 2, \ldots$. That is, we can approximate the solution with $w(z) = z^\alpha + o(z^\alpha)$. Let's substitute $w = z^\alpha + o(z^\alpha)$ into the differential equation and look at the lowest power of z in each of the terms.

$$\left[\alpha(\alpha-1)z^{\alpha-2} + o(z^{\alpha-2})\right] + \left[\alpha z^{\alpha-1} + o(z^{\alpha-1})\right] \sum_{n=0}^{\infty} p_n z^n + \left[z^\alpha + o(z^\alpha)\right] \sum_{n=0}^{\infty} q_n z^n = 0.$$

We see that the solution could not possibly behave like z^α, $\alpha \neq 0, 1, 2, \cdots$ because there is no term on the left to cancel out the $z^{\alpha-2}$ term. The terms on the left side could not add to zero.

You could also check that a solution could not possibly behave like $\log z$ at the origin. Though we will not prove it, if z_0 is an ordinary point of a homogeneous differential equation, then all the solutions are analytic at the point z_0. Since the solution is analytic at z_0 we can expand it in a Taylor series.

Now we are prepared to answer our second question. From complex variables, we know that the radius of convergence of the Taylor series expansion of a function is the distance to the nearest singularity of that function. Since the solutions to a differential equation are analytic at ordinary points of the equation, the series expansion about an ordinary point will have a radius of convergence at least as large as the distance to the nearest singularity of the coefficient functions.

Example 25.1.2 Consider the equation

$$w'' + \frac{1}{\cos z} w' + z^2 w = 0.$$

If we expand the solution to the differential equation in Taylor series about $z = 0$, the radius of convergence will be at least $\pi/2$. This is because the coefficient functions are analytic at the origin, and the nearest singularities of $1/\cos z$ are at $z = \pm\pi/2$.

25.1.1 Taylor Series Expansion for a Second Order Differential Equation

Consider the differential equation

$$w'' + p(z)w' + q(z)w = 0$$

where $p(z)$ and $q(z)$ are analytic in some neighborhood of the origin.

$$p(z) = \sum_{n=0}^{\infty} p_n z^n \quad \text{and} \quad q(z) = \sum_{n=0}^{\infty} q_n z^n$$

We substitute a Taylor series and it's derivatives

$$\begin{aligned} w &= \sum_{n=0}^{\infty} a_n z^n \\ w' &= \sum_{n=1}^{\infty} n z_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n \\ w'' &= \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n \end{aligned}$$

into the differential equation to obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n + \left(\sum_{n=0}^{\infty} p_n z^n \right) \left(\sum_{n=0}^{\infty} (n+1) a_{n+1} z^n \right) + \left(\sum_{n=0}^{\infty} q_n z^n \right) \left(\sum_{n=0}^{\infty} a_n z^n \right) = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n + \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(m+1)a_{m+1}p_{n-m}\right)z^n + \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}a_m q_{n-m}\right)z^n = 0$$
$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + \sum_{m=0}^{n}\big((m+1)a_{m+1}p_{n-m} + a_m q_{n-m}\big)\right]z^n = 0.$$

Equating coefficients of powers of z,

$$(n+2)(n+1)a_{n+2} + \sum_{m=0}^{n}\big((m+1)a_{m+1}p_{n-m} + a_m q_{n-m}\big) = 0 \quad \text{for } n \geq 0.$$

We see that a_0 and a_1 are arbitrary and the rest of the coefficients are determined by the recurrence relation

$$a_{n+2} = -\frac{1}{(n+1)(n+2)}\sum_{m=0}^{n}\left((m+1)a_{m+1}p_{n-m} + a_m q_{n-m}\right) \quad \text{for } n \geq 0.$$

Example 25.1.3 Consider the problem

$$y'' + \frac{1}{\cos x}y' + \mathrm{e}^x y = 0, \quad y(0) = y'(0) = 1.$$

Let's expand the solution in a Taylor series about the origin.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Since $y(0) = a_0$ and $y'(0) = a_1$, we see that $a_0 = a_1 = 1$. The Taylor expansions of the coefficient functions are

$$\frac{1}{\cos x} = 1 + \mathcal{O}(x), \quad \text{and} \quad \mathrm{e}^x = 1 + \mathcal{O}(x).$$

Now we can calculate a_2 from the recurrence relation.

$$\begin{aligned} a_2 &= -\frac{1}{1 \cdot 2} \sum_{m=0}^{0} \left((m+1) a_{m+1} p_{0-m} + a_m q_{0-m} \right) \\ &= -\frac{1}{2}(1 \cdot 1 \cdot 1 + 1 \cdot 1) \\ &= -1 \end{aligned}$$

Thus the solution to the problem is

$$y(x) = 1 + x - x^2 + \mathcal{O}(x^3).$$

In Figure 25.1 the numerical solution is plotted in a solid line and the sum of the first three terms of the Taylor series is plotted in a dashed line.

The general recurrence relation for the a_n's is useful if you only want to calculate the first few terms in the Taylor expansion. However, for many problems substituting the Taylor series for the coefficient functions into the differential equation will enable you to find a simpler form of the solution. We consider the following example to illustrate this point.

Example 25.1.4 Develop a series expansion of the solution to the initial value problem

$$w'' + \frac{1}{(z^2+1)} w = 0, \qquad w(0) = 1, \quad w'(0) = 0.$$

Solution using the General Recurrence Relation. The coefficient function has the Taylor expansion

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}.$$

From the initial condition we obtain $a_0 = 1$ and $a_1 = 0$. Thus we see that the solution is

$$w = \sum_{n=0}^{\infty} a_n z^n,$$

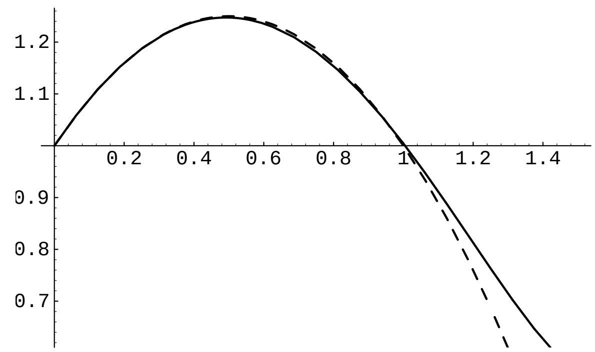

Figure 25.1: Plot of the Numerical Solution and the First Three Terms in the Taylor Series.

where

$$a_{n+2} = -\frac{1}{(n+1)(n+2)} \sum_{m=0}^{n} a_m q_{n-m}$$

and

$$q_n = \begin{cases} 0 & \text{for odd } n \\ (-1)^{(n/2)} & \text{for even } n. \end{cases}$$

Although this formula is fine if you only want to calculate the first few a_n's, it is just a tad unwieldy to work with. Let's see if we can get a better expression for the solution.

Substitute the Taylor Series into the Differential Equation. Substituting a Taylor series for w yields

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\sum_{n=0}^{\infty} a_n z^n + \frac{1}{(z^2+1)}\sum_{n=0}^{\infty} a_n z^n = 0.$$

Note that the algebra will be easier if we multiply by z^2+1. The polynomial z^2+1 has only two terms, but the Taylor series for $1/(z^2+1)$ has an infinite number of terms.

$$(z^2+1)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n z^n + \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} + \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n z^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2} z^n + \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty}\Big[(n+2)(n+1)a_{n+2} + n(n-1)a_n + a_n\Big] z^n = 0$$

Equating powers of z gives us the difference equation

$$a_{n+2} = -\frac{n^2-n+1}{(n+2)(n+1)}a_n, \qquad \text{for } n \geq 0.$$

From the initial conditions we see that $a_0 = 1$ and $a_1 = 0$. All of the odd terms in the series will be zero. For the even terms, it is easier to reformulate the problem with the change of variables $b_n = a_{2n}$. In terms of b_n the difference equation is

$$b_{n+1} = -\frac{(2n)^2-2n+1}{(2n+2)(2n+1)}b_n, \qquad b_0 = 1.$$

This is a first order difference equation with the solution

$$b_n = \prod_{j=0}^{n} \left(-\frac{4j^2 - 2j + 1}{(2j+2)(2j+1)} \right).$$

Thus we have that

$$a_n = \begin{cases} \prod_{j=0}^{n/2} \left(-\frac{4j^2-2j+1}{(2j+2)(2j+1)} \right) & \text{for even } n, \\ 0 & \text{for odd } n. \end{cases}$$

Note that the nearest singularities of $1/(z^2+1)$ in the complex plane are at $z = \pm i$. Thus the radius of convergence must be at least 1. Applying the ratio test, the series converges for values of $|z|$ such that

$$\lim_{n\to\infty} \left| \frac{a_{n+2} z^{n+2}}{a_n z^n} \right| < 1$$
$$\lim_{n\to\infty} \left| -\frac{n^2 - n + 1}{(n+2)(n+1)} \right| |z|^2 < 1$$
$$|z|^2 < 1.$$

The radius of convergence is 1.

The first few terms in the Taylor expansion are

$$\boxed{w = 1 - \frac{1}{2} z^2 + \frac{1}{8} z^4 - \frac{13}{240} z^6 + \cdots.}$$

In Figure 25.2 the plot of the first two nonzero terms is shown in a short dashed line, the plot of the first four nonzero terms is shown in a long dashed line, and the numerical solution is shown in a solid line.

In general, if the coefficient functions are rational functions, that is they are fractions of polynomials, multiplying the equations by the quotient will reduce the algebra involved in finding the series solution.

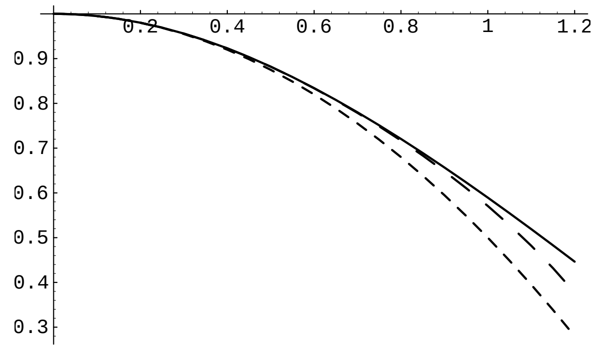

Figure 25.2: Plot of the solution and approximations.

Example 25.1.5 If we were going to find the Taylor series expansion about $z = 0$ of the solution to

$$w'' + \frac{z}{1+z}w' + \frac{1}{1-z^2}w = 0,$$

we would first want to multiply the equation by $1 - z^2$ to obtain

$$(1-z^2)w'' + z(1-z)w'' + w = 0.$$

Example 25.1.6 Find the series expansions about $z = 0$ of the fundamental set of solutions for

$$w'' + z^2 w = 0.$$

Recall that the fundamental set of solutions $\{w_1, w_2\}$ satisfy

$$\begin{aligned} w_1(0) &= 1 & w_2(0) &= 0 \\ w_1'(0) &= 0 & w_2'(0) &= 1. \end{aligned}$$

Thus if

$$w_1 = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad w_2 = \sum_{n=0}^{\infty} b_n z^n,$$

then the coefficients must satisfy

$$a_0 = 1, \quad a_1 = 0, \qquad \text{and} \qquad b_0 = 0, \quad b_1 = 1.$$

Substituting the Taylor expansion $w = \sum_{n=0}^{\infty} c_n z^n$ into the differential equation,

$$\sum_{n=2}^{\infty} n(n-1) c_n z^{n-2} + \sum_{n=0}^{\infty} c_n z^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} z^n + \sum_{n=2}^{\infty} c_{n-2} z^n = 0$$

$$2c_2 + 6c_3 z + \sum_{n=2}^{\infty} \left[(n+2)(n+1) c_{n+2} + c_{n-2} \right] z^n = 0$$

Equating coefficients of powers of z,

$$\begin{aligned} z^0 : &\quad c_2 = 0 \\ z^1 : &\quad c_3 = 0 \\ z^n : &\quad (n+2)(n+1) c_{n+2} + c_{n-2} = 0, \quad \text{for } n \geq 2 \\ &\quad c_{n+4} = -\frac{c_n}{(n+4)(n+3)} \end{aligned}$$

For our first solution we have the difference equation

$$a_0 = 1,\ a_1 = 0,\ a_2 = 0,\ a_3 = 0, \qquad a_{n+4} = -\frac{a_n}{(n+4)(n+3)}.$$

For our second solution,

$$b_0 = 0,\ b_1 = 1,\ b_2 = 0,\ b_3 = 0, \qquad b_{n+4} = -\frac{b_n}{(n+4)(n+3)}.$$

The first few terms in the fundamental set of solutions are

$$\boxed{w_1 = 1 - \frac{1}{12}z^4 + \frac{1}{672}z^8 - \cdots, \qquad w_2 = z - \frac{1}{20}z^5 + \frac{1}{1440}z^9 - \cdots.}$$

In Figure 25.3 the five term approximation is graphed in a coarse dashed line, the ten term approximation is graphed in a fine dashed line, and the numerical solution of w_1 is graphed in a solid line. The same is done for w_2.

Result 25.1.1 Consider the n^{th} order linear homogeneous equation

$$\frac{d^n w}{dz^n} + p_{n-1}(z)\frac{d^{n-1}w}{dz^{n-1}} + \cdots + p_1(z)\frac{\mathrm{d}w}{\mathrm{d}z} + p_0(z)w = 0.$$

If each of the coefficient functions $p_i(z)$ are analytic at $z = z_0$ then z_0 is an ordinary point of the differential equation. The solution is analytic in some region containing z_0 and can be expanded in a Taylor series. The radius of convergence of the series will be at least the distance to the nearest singularity of the coefficient functions in the complex plane.

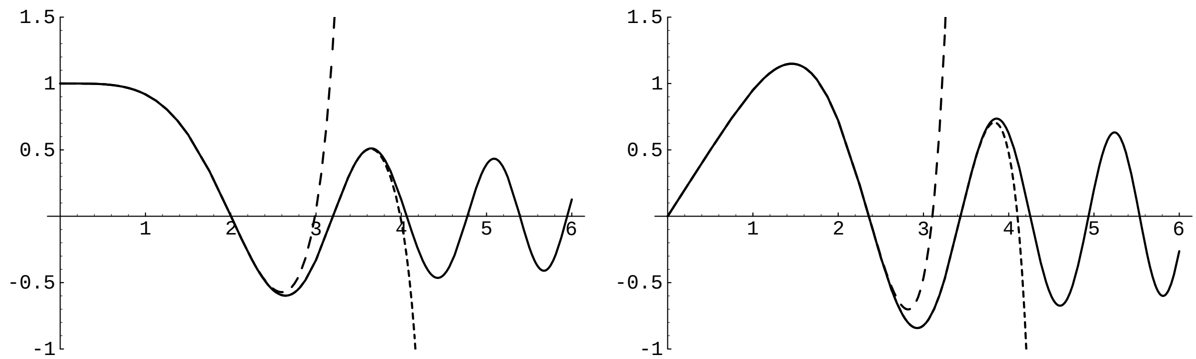

Figure 25.3: The graph of approximations and numerical solution of w_1 and w_2.

25.2 Regular Singular Points of Second Order Equations

Consider the differential equation

$$w'' + \frac{p(z)}{z - z_0} w' + \frac{q(z)}{(z - z_0)^2} w = 0.$$

If $z = z_0$ is not an ordinary point but both $p(z)$ and $q(z)$ are analytic at $z = z_0$ then z_0 is a **regular singular point** of the differential equation. The following equations have a regular singular point at $z = 0$.

- $w'' + \frac{1}{z} w' + z^2 w = 0$
- $w'' + \frac{1}{\sin z} w' - w = 0$

- $w'' - zw' + \frac{1}{z \sin z} w = 0$

Concerning regular singular points of second order linear equations there is good news and bad news.

The Good News. We will find that with the use of the Frobenius method we can always find series expansions of two linearly independent solutions at a regular singular point. We will illustrate this theory with several examples.

The Bad News. Instead of a tidy little theory like we have for ordinary points, the solutions can be of several different forms. Also, for some of the problems the algebra can get pretty ugly.

Example 25.2.1 Consider the equation

$$w'' + \frac{3(1+z)}{16z^2} w = 0.$$

We wish to find series solutions about the point $z = 0$. First we try a Taylor series $w = \sum_{n=0}^{\infty} a_n z^n$. Substituting this into the differential equation,

$$z^2 \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \frac{3}{16}(1+z) \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n z^n + \frac{3}{16} \sum_{n=0}^{\infty} a_n z^n + \frac{3}{16} \sum_{n=1}^{\infty} a_{n+1} z^n = 0.$$

Equating powers of z,

$$\begin{aligned}
z^0 : \qquad & a_0 = 0 \\
z^n : \qquad & \left[n(n-1) + \frac{3}{16} \right] a_n + \frac{3}{16} a_{n+1} = 0 \\
& a_{n+1} = \left[\frac{16}{3} n(n-1) + 1 \right] a_n.
\end{aligned}$$

This difference equation has the solution $a_n = 0$ for all n. Thus we have obtained only the trivial solution to the differential equation. We must try an expansion of a more general form. We recall that for regular singular points of first order equations we can always find a solution in the form of a Frobenius series $w = z^\alpha \sum_{n=0}^\infty a_n z^n$, $a_0 \neq 0$. We substitute this series into the differential equation.

$$z^2 \sum_{n=0}^\infty \left[\alpha(\alpha-1) + 2\alpha n + n(n-1)\right] a_n z^{n+\alpha-2} + \frac{3}{16}(1+z) z^\alpha \sum_{n=0}^\infty a_n z^n = 0$$

$$\sum_{n=0}^\infty \left[\alpha(\alpha-1) + 2n + n(n-1)\right] a_n z^n + \frac{3}{16} \sum_{n=0}^\infty a_n z^n + \frac{3}{16} \sum_{n=1}^\infty a_{n-1} z^n = 0$$

Equating the z^0 term to zero yields the equation

$$\left(\alpha(\alpha-1) + \frac{3}{16}\right) a_0 = 0.$$

Since we have assumed that $a_0 \neq 0$, the polynomial in α must be zero. The two roots of the polynomial are

$$\alpha_1 = \frac{1 + \sqrt{1 - 3/4}}{2} = \frac{3}{4}, \qquad \alpha_2 = \frac{1 - \sqrt{1 - 3/4}}{2} = \frac{1}{4}.$$

Thus our two series solutions will be of the form

$$w_1 = z^{3/4} \sum_{n=0}^\infty a_n z^n, \qquad w_2 = z^{1/4} \sum_{n=0}^\infty b_n z^n.$$

Substituting the first series into the differential equation,

$$\sum_{n=0}^\infty \left[-\frac{3}{16} + 2n + n(n-1) + \frac{3}{16}\right] a_n z^n + \frac{3}{16} \sum_{n=1}^\infty a_{n-1} z^n = 0.$$

Equating powers of z, we see that a_0 is arbitrary and

$$a_n = -\frac{3}{16n(n+1)} a_{n-1} \quad \text{for } n \geq 1.$$

This difference equation has the solution

$$a_n = a_0 \prod_{j=1}^{n} \left(-\frac{3}{16j(j+1)} \right)$$
$$= a_0 \left(-\frac{3}{16} \right)^n \prod_{j=1}^{n} \frac{1}{j(j+1)}$$
$$= a_0 \left(-\frac{3}{16} \right)^n \frac{1}{n!(n+1)!} \quad \text{for } n \geq 1.$$

Substituting the second series into the differential equation,

$$\sum_{n=0}^{\infty} \left[-\frac{3}{16} + 2n + n(n-1) + \frac{3}{16} \right] b_n z^n + \frac{3}{16} \sum_{n=1}^{\infty} b_{n-1} z^n = 0.$$

We see that the difference equation for b_n is the same as the equation for a_n. Thus we can write the general solution to the differential equation as

$$w = c_1 z^{3/4} \left(1 + \sum_{n=1}^{\infty} \left(-\frac{3}{16} \right)^n \frac{1}{n!(n+1)!} z^n \right) + c_2 z^{1/4} \left(1 + \sum_{n=1}^{\infty} \left(-\frac{3}{16} \right)^n \frac{1}{n!(n+1)!} z^n \right)$$

$$\boxed{\left(c_1 z^{3/4} + c_2 z^{1/4} \right) \left(1 + \sum_{n=1}^{\infty} \left(-\frac{3}{16} \right)^n \frac{1}{n!(n+1)!} z^n \right)}.$$

25.2.1 Indicial Equation

Now let's consider the general equation for a regular singular point at $z = 0$

$$w'' + \frac{p(z)}{z} w' + \frac{q(z)}{z^2} w = 0.$$

Since $p(z)$ and $q(z)$ are analytic at $z = 0$ we can expand them in Taylor series.

$$p(z) = \sum_{n=0}^{\infty} p_n z^n, \qquad q(z) = \sum_{n=0}^{\infty} q_n z^n$$

Substituting a Frobenius series $w = z^\alpha \sum_{n=0}^{\infty} a_n z^n$, $a_0 \neq 0$ and the Taylor series for $p(z)$ and $q(z)$ into the differential equation yields

$$\sum_{n=0}^{\infty} \Big[(\alpha+n)(\alpha+n-1)\Big] a_n z^n + \left(\sum_{n=0}^{\infty} p_n z^n\right)\left(\sum_{n=0}^{\infty}(\alpha+n) a_n z^n\right) + \left(\sum_{n=0}^{\infty} q_n z^n\right)\left(\sum_{n=0}^{\infty} a_n z^n\right) = 0$$

$$\sum_{n=0}^{\infty} \Big[(\alpha+n)^2 - (\alpha+n) + p_0(\alpha+n) + q_0\Big] a_n z^n \\ + \left(\sum_{n=1}^{\infty} p_n z^n\right)\left(\sum_{n=0}^{\infty}(\alpha+n) a_n z^n\right) + \left(\sum_{n=1}^{\infty} q_n z^n\right)\left(\sum_{n=0}^{\infty} a_n z^n\right) = 0$$

$$\sum_{n=0}^{\infty} \Big[(\alpha+n)^2 + (p_0 - 1)(\alpha_n) + q_0\Big] a_n z^n + \sum_{n=1}^{\infty}\left(\sum_{j=0}^{n-1}(\alpha+j) a_j p_{n-j}\right) z^n + \sum_{n=1}^{\infty}\left(\sum_{j=0}^{n-1} a_j q_{n-j}\right) z^n = 0$$

Equating powers of z,

$$z^0: \quad \Big[\alpha^2 + (p_0 - 1)\alpha + q_0\Big] a_0 = 0$$

$$z^n: \quad \Big[(\alpha+n)^2 + (p_0 - 1)(\alpha+n) + q_0\Big] a_n = -\sum_{j=0}^{n-1} \Big[(\alpha+j) p_{n-j} + q_{n-j}\Big] a_j.$$

Let

$$I(\alpha) = \alpha^2 + (p_0 - 1)\alpha + q_0 = 0.$$

This is known as the **indicial equation**. The indicial equation gives us the form of the solutions. The equation for a_0 is $I(\alpha)a_0 = 0$. Since we assumed that a_0 is nonzero, $I(\alpha) = 0$. Let the two roots of $I(\alpha)$ be α_1 and α_2 where $\Re(\alpha_1) \geq \Re(\alpha_2)$.

Rewriting the difference equation for $a_n(\alpha)$,

$$I(\alpha + n)a_n(\alpha) = -\sum_{j=0}^{n-1} \left[(\alpha + j)p_{n-j} + q_{n-j}\right] a_j(\alpha) \quad \text{for } n \geq 1. \tag{25.1}$$

If the roots are distinct and do not differ by an integer then we can use Equation 25.1 to solve for $a_n(\alpha_1)$ and $a_n(\alpha_2)$, which will give us the two solutions

$$w_1 = z^{\alpha_1} \sum_{n=0}^{\infty} a_n(\alpha_1) z^n, \quad \text{and} \quad w_2 = z^{\alpha_2} \sum_{n=0}^{\infty} a_n(\alpha_2) z^n.$$

If the roots are not distinct, $\alpha_1 = \alpha_2$, we will only have one solution and will have to generate another. If the roots differ by an integer, $\alpha_1 - \alpha_2 = N$, there is one solution corresponding to α_1, but when we try to solve Equation 25.1 for $a_n(\alpha_2)$, we will encounter the equation

$$I(\alpha_2 + N)a_N(\alpha_2) = I(\alpha_1)a_N(\alpha_2) = 0 \cdot a_N(\alpha_2) = -\sum_{j=0}^{N-1} \left[(\alpha + n)p_{n-j} + q_{n-j}\right] a_j(\alpha_2).$$

If the right side of the equation is nonzero, then $a_N(\alpha_2)$ is undefined. On the other hand, if the right side is zero then $a_N(\alpha_2)$ is arbitrary. The rest of this section is devoted to considering the cases $\alpha_1 = \alpha_2$ and $\alpha_1 - \alpha_2 = N$.

25.2.2 The Case: Double Root

Consider a second order equation $L[w] = 0$ with a regular singular point at $z = 0$. Suppose the indicial equation has a double root.

$$I(\alpha) = (\alpha - \alpha_1)^2 = 0$$

One solution has the form

$$w_1 = z^{\alpha_1} \sum_{n=0}^{\infty} a_n z^n.$$

In order to find the second solution, we will differentiate with respect to the parameter, α. Let $a_n(\alpha)$ satisfy Equation 25.1 Substituting the Frobenius expansion into the differential equation,

$$L\left[z^{\alpha} \sum_{n=0}^{\infty} a_n(\alpha) z^n\right] = 0.$$

Setting $\alpha = \alpha_1$ will make the left hand side of the equation zero. Differentiating this equation with respect to α,

$$\frac{\partial}{\partial \alpha} L\left[z^{\alpha} \sum_{n=0}^{\infty} a_n(\alpha) z^n\right] = 0.$$

Interchanging the order of differentiation,

$$L\left[\log z\, z^{\alpha} \sum_{n=0}^{\infty} a_n(\alpha) z^n + z^{\alpha} \sum_{n=0}^{\infty} \frac{\mathrm{d}a_n(\alpha)}{\mathrm{d}\alpha} z^n\right] = 0.$$

Since setting $\alpha = \alpha_1$ will make the left hand side of this equation zero, the second linearly independent solution is

$$w_2 = \log z\, z^{\alpha_1} \sum_{n=0}^{\infty} a_n(\alpha_1) z^n + z^{\alpha_1} \sum_{n=0}^{\infty} \left.\frac{\mathrm{d}a_n(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_1} z^n$$

$$\boxed{w_2 = w_1 \log z + z^{\alpha_1} \sum_{n=0}^{\infty} a_n'(\alpha_1) z^n.}$$

Example 25.2.2 Consider the differential equation

$$w'' + \frac{1+z}{4z^2} w = 0.$$

There is a regular singular point at $z = 0$. The indicial equation is

$$\alpha(\alpha - 1) + \frac{1}{4} = \left(\alpha - \frac{1}{2}\right)^2 = 0.$$

One solution will have the form

$$w_1 = z^{1/2} \sum_{n=0}^{\infty} a_n z^n, \quad a_0 \neq 0.$$

Substituting the Frobenius expansion

$$z^\alpha \sum_{n=0}^{\infty} a_n(\alpha) z^n$$

into the differential equation yields

$$z^2 w'' + \frac{1}{4}(1+z) w = 0$$

$$\sum_{n=0}^{\infty} \left[\alpha(\alpha-1) + 2\alpha n + n(n-1)\right] a_n(\alpha) z^{n+\alpha} + \frac{1}{4} \sum_{n=0}^{\infty} a_n(\alpha) z^{n+\alpha} + \frac{1}{4} \sum_{n=0}^{\infty} a_n(\alpha) z^{n+\alpha+1} = 0.$$

Divide by z^α and adjust the summation indices.

$$\sum_{n=0}^{\infty} \left[\alpha(\alpha-1) + 2\alpha n + n(n-1)\right] a_n(\alpha) z^n + \frac{1}{4} \sum_{n=0}^{\infty} a_n(\alpha) z^n + \frac{1}{4} \sum_{n=1}^{\infty} a_{n-1}(\alpha) z^n = 0$$

$$\left[\alpha(\alpha-1)a_0 + \frac{1}{4}\right] a_0 + \sum_{n=1}^{\infty} \left(\left[\alpha(\alpha-1) + 2n + n(n-1) + \frac{1}{4}\right] a_n(\alpha) + \frac{1}{4} a_{n-1}(\alpha) \right) z^n = 0$$

Equating the coefficient of z^0 to zero yields $I(\alpha)a_0 = 0$. Equating the coefficients of z^n to zero yields the difference equation

$$\left[\alpha(\alpha-1) + 2n + n(n-1) + \frac{1}{4}\right] a_n(\alpha) + \frac{1}{4} a_{n-1}(\alpha) = 0$$
$$a_n(\alpha) = -\left(\frac{n(n+1)}{4} + \frac{\alpha(\alpha-1)}{4} + \frac{1}{16}\right) a_{n-1}(\alpha).$$

The first few a_n's are

$$a_0, \quad -\left(\alpha(\alpha-1) + \frac{9}{16}\right) a_0, \quad \left(\alpha(\alpha-1) + \frac{25}{16}\right)\left(\alpha(\alpha-1) + \frac{9}{16}\right) a_0, \dots$$

Setting $\alpha = 1/2$, the coefficients for the first solution are

$$a_0, \quad -\frac{5}{16} a_0, \quad \frac{105}{16} a_0, \quad \dots$$

The second solution has the form

$$w_2 = w_1 \log z + z^{1/2} \sum_{n=0}^{\infty} a_n'(1/2) z^n.$$

Differentiating the $a_n(\alpha)$,

$$\frac{\mathrm{d}a_0}{\mathrm{d}\alpha} = 0, \quad \frac{\mathrm{d}a_1(\alpha)}{\mathrm{d}\alpha} = -(2\alpha-1)a_0, \quad \frac{\mathrm{d}a_2(\alpha)}{\mathrm{d}\alpha} = (2\alpha-1)\left[\left(\alpha(\alpha-1) + \frac{9}{16}\right) + \left(\alpha(\alpha-1) + \frac{25}{16}\right)\right] a_0, \quad \dots$$

Setting $\alpha = 1/2$ in this equation yields

$$a_0' = 0, \quad a_1'(1/2) = 0, \quad a_2'(1/2) = 0, \quad \dots$$

Thus the second solution is

$$w_2 = w_1 \log z.$$

The first few terms in the general solution are

$$\boxed{(c_1 + c_2 \log z)\left(1 - \frac{5}{16}z + \frac{105}{16}z^2 - \cdots\right).}$$

25.2.3 The Case: Roots Differ by an Integer

Consider the case in which the roots of the indicial equation α_1 and α_2 differ by an integer. ($\alpha_1 - \alpha_2 = N$) Recall the equation that determines $a_n(\alpha)$

$$I(\alpha + n)a_n = \left[(\alpha + n)^2 + (p_0 - 1)(\alpha + n) + q_0\right]a_n = -\sum_{j=0}^{n-1}\left[(\alpha + j)p_{n-j} + q_{n-j}\right]a_j.$$

When $\alpha = \alpha_2$ the equation for a_N is

$$I(\alpha_2 + N)a_N(\alpha_2) = 0 \cdot a_N(\alpha_2) = -\sum_{j=0}^{N-1}\left[(\alpha + j)p_{N-j} + q_{N-j}\right]a_j.$$

If the right hand side of this equation is zero, then a_N is arbitrary. There will be two solutions of the Frobenius form.

$$w_1 = z^{\alpha_1}\sum_{n=0}^{\infty} a_n(\alpha_1)z^n \quad \text{and} \quad w_2 = z^{\alpha_2}\sum_{n=0}^{\infty} a_n(\alpha_2)z^n.$$

If the right hand side of the equation is nonzero then $a_N(\alpha_2)$ will be undefined. We will have to generate the second solution. Let

$$w(z, \alpha) = z^{\alpha}\sum_{n=0}^{\infty} a_n(\alpha)z^n,$$

where $a_n(\alpha)$ satisfies the recurrence formula. Substituting this series into the differential equation yields

$$L[w(z,\alpha)] = 0.$$

We will multiply by $(\alpha-\alpha_2)$, differentiate this equation with respect to α and then set $\alpha = \alpha_2$. This will generate a linearly independent solution.

$$\begin{aligned}
\frac{\partial}{\partial \alpha} L[(\alpha-\alpha_2) w(z,\alpha)] &= L\left[\frac{\partial}{\partial \alpha}(\alpha-\alpha_2) w(z,\alpha)\right] \\
&= L\left[\frac{\partial}{\partial \alpha}(\alpha-\alpha_2) z^\alpha \sum_{n=0}^{\infty} a_n(\alpha) z^n\right] \\
&= L\left[\log z \; z^\alpha \sum_{n=0}^{\infty} (\alpha-\alpha_2) a_n(\alpha) z^n + z^\alpha \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}\alpha}[(\alpha-\alpha_2) a_n(\alpha)] z^n\right]
\end{aligned}$$

Setting $\alpha = \alpha_2$ with make this expression zero, thus

$$\log z \; z^\alpha \sum_{n=0}^{\infty} \lim_{\alpha\to\alpha_2} \{(\alpha-\alpha_2) a_n(\alpha)\} z^n + z^{\alpha_2} \sum_{n=0}^{\infty} \lim_{\alpha\to\alpha_2} \left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}[(\alpha-\alpha_2) a_n(\alpha)]\right\} z^n$$

is a solution. Now let's look at the first term in this solution

$$\log z \; z^\alpha \sum_{n=0}^{\infty} \lim_{\alpha\to\alpha_2} \{(\alpha-\alpha_2) a_n(\alpha)\} z^n.$$

The first N terms in the sum will be zero. That is because $a_0, \ldots, a_{N-1}$ are finite, so multiplying by $(\alpha-\alpha_2)$ and taking the limit as $\alpha \to \alpha_2$ will make the coefficients vanish. The equation for $a_N(\alpha)$ is

$$I(\alpha+N) a_N(\alpha) = -\sum_{j=0}^{N-1} \left[(\alpha+j) p_{N-j} + q_{N-j}\right] a_j(\alpha).$$

Thus the coefficient of the N^{th} term is

$$\lim_{\alpha\to\alpha_2}(\alpha-\alpha_2)a_N(\alpha) = -\lim_{\alpha\to\alpha_2}\left[\frac{(\alpha-\alpha_2)}{I(\alpha+N)}\sum_{j=0}^{N-1}\left[(\alpha+j)p_{N-j}+q_{N-j}\right]a_j(\alpha)\right]$$

$$= -\lim_{\alpha\to\alpha_2}\left[\frac{(\alpha-\alpha_2)}{(\alpha+N-\alpha_1)(\alpha+N-\alpha_2)}\sum_{j=0}^{N-1}\left[(\alpha+j)p_{N-j}+q_{N-j}\right]a_j(\alpha)\right]$$

Since $\alpha_1 = \alpha_2 + N$, $\lim_{\alpha\to\alpha_2}\frac{\alpha-\alpha_2}{\alpha+N-\alpha_1} = 1$.

$$= -\frac{1}{(\alpha_1-\alpha_2)}\sum_{j=0}^{N-1}\left[(\alpha_2+j)p_{N-j}+q_{N-j}\right]a_j(\alpha_2).$$

Using this you can show that the first term in the solution can be written

$$d_{-1}\log z\, w_1,$$

where d_{-1} is a constant. Thus the second linearly independent solution is

$$\boxed{w_2 = d_{-1}\log z\, w_1 + z^{\alpha_2}\sum_{n=0}^{\infty} d_n z^n,}$$

where

$$d_{-1} = -\frac{1}{a_0}\frac{1}{(\alpha_1-\alpha_2)}\sum_{j=0}^{N-1}\left[(\alpha_2+j)p_{N-j}+q_{N-j}\right]a_j(\alpha_2)$$

and

$$d_n = \lim_{\alpha\to\alpha_2}\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha-\alpha_2)a_n(\alpha)\right]\right\} \quad \text{for } n \geq 0.$$

Example 25.2.3 Consider the differential equation

$$w'' + \left(1 - \frac{2}{z}\right) w' + \frac{2}{z^2} w = 0.$$

The point $z = 0$ is a regular singular point. In order to find series expansions of the solutions, we first calculate the indicial equation. We can write the coefficient functions in the form

$$\frac{p(z)}{z} = \frac{1}{z}(-2 + z), \quad \text{and} \quad \frac{q(z)}{z^2} = \frac{1}{z^2}(2).$$

Thus the indicial equation is

$$\alpha^2 + (-2 - 1)\alpha + 2 = 0$$
$$(\alpha - 1)(\alpha - 2) = 0.$$

The First Solution. The first solution will have the Frobenius form

$$w_1 = z^2 \sum_{n=0}^{\infty} a_n(\alpha_1) z^n.$$

Substituting a Frobenius series into the differential equation,

$$z^2 w'' + (z^2 - 2z) w' + 2w = 0$$
$$\sum_{n=0}^{\infty} (n + \alpha)(n + \alpha - 1) z^{n+\alpha} + (z^2 - 2z) \sum_{n=0}^{\infty} (n + \alpha) z^{n+\alpha-1} + 2 \sum_{n=0}^{\infty} a_n z^n = 0$$
$$[\alpha^2 - 3\alpha + 2] a_0 + \sum_{n=1}^{\infty} \left[(n + \alpha)(n + \alpha - 1) a_n + (n + \alpha - 1) a_{n-1} - 2(n + \alpha) a_n + 2a_n\right] z^n = 0.$$

Equating powers of z,

$$\left[(n + \alpha)(n + \alpha - 1) - 2(n + \alpha) + 2\right] a_n = -(n + \alpha - 1) a_{n-1}$$
$$a_n = -\frac{a_{n-1}}{n + \alpha - 2}.$$

Setting $\alpha = \alpha_1 = 2$, the recurrence relation becomes

$$\begin{aligned} a_n(\alpha_1) &= -\frac{a_{n-1}(\alpha_1)}{n} \\ &= a_0 \frac{(-1)^n}{n!}. \end{aligned}$$

The first solution is

$$\boxed{w_1 = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n = a_0 \, \mathrm{e}^{-z}.}$$

The Second Solution. The equation for $a_1(\alpha_2)$ is

$$0 \cdot a_1(\alpha_2) = 2a_0.$$

Since the right hand side of this equation is not zero, the second solution will have the form

$$w_2 = d_{-1} \log z \; w_1 + z^{\alpha_2} \sum_{n=0}^{\infty} \lim_{\alpha \to \alpha_2} \left\{ \frac{\mathrm{d}}{\mathrm{d}\alpha} [(\alpha - \alpha_2) a_n(\alpha)] \right\} z^n$$

First we will calculate d_{-1} as we defined it previously.

$$d_{-1} = -\frac{1}{a_0} \frac{1}{2-1} a_0 = -1.$$

The expression for $a_n(\alpha)$ is

$$a_n(\alpha) = \frac{(-1)^n a_0}{(\alpha + n - 2)(\alpha + n - 1) \cdots (\alpha - 1)}.$$

The first few $a_n(\alpha)$ are

$$a_1(\alpha) = -\frac{a_0}{\alpha - 1}$$
$$a_2(\alpha) = \frac{a_0}{\alpha(\alpha - 1)}$$
$$a_3(\alpha) = -\frac{a_0}{(\alpha + 1)\alpha(\alpha - 1)}.$$

We would like to calculate

$$d_n = \lim_{\alpha \to 1} \left\{ \frac{\mathrm{d}}{\mathrm{d}\alpha} \Big[(\alpha - 1) a_n(\alpha) \Big] \right\}.$$

The first few d_n are

$$
\begin{aligned}
d_0 &= \lim_{\alpha\to 1}\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}\Big[(\alpha-1)a_0\Big]\right\}\\
&= a_0\\
d_1 &= \lim_{\alpha\to 1}\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha-1)\left(-\frac{a_0}{\alpha-1}\right)\right]\right\}\\
&= \lim_{\alpha\to 1}\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}\Big[-a_0\Big]\right\}\\
&= 0\\
d_2 &= \lim_{\alpha\to 1}\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha-1)\left(\frac{a_0}{\alpha(\alpha-1)}\right)\right]\right\}\\
&= \lim_{\alpha\to 1}\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{a_0}{\alpha}\right]\right\}\\
&= -a_0\\
d_3 &= \lim_{\alpha\to 1}\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha-1)\left(-\frac{a_0}{(\alpha+1)\alpha(\alpha-1)}\right)\right]\right\}\\
&= \lim_{\alpha\to 1}\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[-\frac{a_0}{(\alpha+1)\alpha}\right]\right\}\\
&= \frac{3}{4}a_0.
\end{aligned}
$$

It will take a little work to find the general expression for d_n. We will need the following relations.

$$
\Gamma(n)=(n-1)!,\quad \Gamma'(z)=\Gamma(z)\psi(z),\quad \psi(n)=-\gamma+\sum_{k=1}^{n-1}\frac{1}{k}.
$$

See the chapter on the Gamma function for explanations of these equations.

$$\begin{aligned}
d_n &= \lim_{\alpha\to 1}\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha-1)\frac{(-1)^n a_0}{(\alpha+n-2)(\alpha+n-1)\cdots(\alpha-1)}\right]\right\} \\
&= \lim_{\alpha\to 1}\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{(-1)^n a_0}{(\alpha+n-2)(\alpha+n-1)\cdots(\alpha)}\right]\right\} \\
&= \lim_{\alpha\to 1}\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{(-1)^n a_0\Gamma(\alpha)}{\Gamma(\alpha+n-1)}\right]\right\} \\
&= (-1)^n a_0 \lim_{\alpha\to 1}\left\{\frac{\Gamma(\alpha)\psi(\alpha)}{\Gamma(\alpha+n-1)} - \frac{\Gamma(\alpha)\psi(\alpha+n-1)}{\Gamma(\alpha+n-1)}\right\} \\
&= (-1)^n a_0 \lim_{\alpha\to 1}\left\{\frac{\Gamma(\alpha)[\psi(\alpha)-\psi(\alpha+n-1)]}{\Gamma(\alpha+n-1)}\right\} \\
&= (-1)^n a_0 \frac{\psi(1)-\psi(n)}{(n-1)!} \\
&= \frac{(-1)^{n+1}a_0}{(n-1)!}\sum_{k=0}^{n-1}\frac{1}{k}
\end{aligned}$$

Thus the second solution is

$$\boxed{w_2 = -\log z\ w_1 + z\sum_{n=0}^{\infty}\left(\frac{(-1)^{n+1}a_0}{(n-1)!}\sum_{k=0}^{n-1}\frac{1}{k}\right)z^n.}$$

The general solution is

$$\boxed{w = c_1\,\mathrm{e}^{-z} - c_2\log z\ \ \mathrm{e}^{-z} + c_2 z\sum_{n=0}^{\infty}\left(\frac{(-1)^{n+1}}{(n-1)!}\sum_{k=0}^{n-1}\frac{1}{k}\right)z^n.}$$

We see that even in problems that are chosen for their simplicity, the algebra involved in the Frobenius method can be pretty involved.

Example 25.2.4 Consider a series expansion about the origin of the equation

$$w'' + \frac{1-z}{z} w' - \frac{1}{z^2} w = 0.$$

The indicial equation is

$$\alpha^2 - 1 = 0$$
$$\alpha = \pm 1.$$

Substituting a Frobenius series into the differential equation,

$$z^2 \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n z^{n-2} + (z - z^2) \sum_{n=0}^{\infty} (n+\alpha) a_n z^{n-1} - \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n z^n + \sum_{n=0}^{\infty} (n+\alpha) a_n z^n - \sum_{n=1}^{\infty} (n+\alpha-1) a_{n-1} z^n - \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\left[\alpha(\alpha-1) + \alpha - 1\right] a_0 + \sum_{n=1}^{\infty} \left[n+\alpha)(n+\alpha-1) a_n + (n+\alpha-1) a_n - (n+\alpha-1) a_{n-1}\right] z^n = 0.$$

Equating powers of z to zero,

$$a_n(\alpha) = \frac{a_{n-1}(\alpha)}{n+\alpha+1}.$$

We know that the first solution has the form

$$w_1 = z \sum_{n=0}^{\infty} a_n z^n.$$

Setting $\alpha = 1$ in the reccurence formula,

$$a_n = \frac{a_{n-1}}{n+2} = \frac{2 a_0}{(n+2)!}.$$

Thus the first solution is

$$\begin{aligned}
w_1 &= z \sum_{n=0}^{\infty} \frac{2a_0}{(n+2)!} z^n \\
&= 2a_0 \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^{n+2}}{(n+2)!} \\
&= \frac{2a_0}{z} \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 - z \right) \\
&= \frac{2a_0}{z} (\mathrm{e}^z - 1 - z).
\end{aligned}$$

Now to find the second solution. Setting $\alpha = -1$ in the reccurence formula,

$$a_n = \frac{a_{n-1}}{n} = \frac{a_0}{n!}.$$

We see that in this case there is no trouble in defining $a_2(\alpha_2)$. The second solution is

$$w_2 = \frac{a_0}{z} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \frac{a_0}{z} \mathrm{e}^z.$$

Thus we see that the general solution is

$$w = \frac{c_1}{z}(\mathrm{e}^z - 1 - z) + \frac{c_2}{z} \mathrm{e}^z$$

$$\boxed{w = \frac{d_1}{z} \mathrm{e}^z + d_2 \left(1 + \frac{1}{z} \right).}$$

25.3 Irregular Singular Points

If a point z_0 of a differential equation is not ordinary or regular singular, then it is an **irregular singular point**. At least one of the solutions at an irregular singular point will not be of the Frobenius form. We will examine how to obtain series expansions about an irregular singular point in the chapter on asymptotic expansions.

25.4 The Point at Infinity

If we want to determine the behavior of a function $f(z)$ at infinity, we can make the transformation $t = 1/z$ and examine the point $t = 0$.

Example 25.4.1 Consider the behavior of $f(z) = \sin z$ at infinity. This is the same as considering the point $t = 0$ of $\sin(1/t)$, which has the series expansion

$$\sin\left(\frac{1}{t}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!t^{2n+1}}.$$

Thus we see that the point $t = 0$ is an essential singularity of $\sin(1/t)$. Hence $\sin z$ has an essential singularity at $z = \infty$.

Example 25.4.2 Consider the behavior at infinity of $z\,\mathrm{e}^{1/z}$. With the transformation $t = 1/z$ the function is

$$\frac{1}{t}\,\mathrm{e}^t = \frac{1}{t}\sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

Thus $z\,\mathrm{e}^{1/z}$ has a pole of order 1 at infinity.

In order to classify the point at infinity of a differential equation in $w(z)$, we apply the transformation $t = 1/z$, $u(t) = w(z)$. Writing the derivatives with respect to z in terms if t yields

$$\begin{aligned} z &= \frac{1}{t} \\ dz &= -\frac{1}{t^2} dt \\ \frac{\mathrm{d}}{\mathrm{d}z} &= -t^2 \frac{\mathrm{d}}{\mathrm{d}t} \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}z^2} &= -t^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(-t^2 \frac{\mathrm{d}}{\mathrm{d}t}\right) \\ &= t^4 \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2t^3 \frac{\mathrm{d}}{\mathrm{d}t}. \end{aligned}$$

Applying the transformation to the differential equation

$$w'' + p(z)w' + q(z)w = 0$$

yields

$$\begin{gathered} t^4 u'' + 2t^3 u' + p(1/t)(-t^2)u' + q(1/t)u = 0 \\ u'' + \left(\frac{2}{t} - \frac{p(1/t)}{t^2}\right) u' + \frac{q(1/t)}{t^4} u = 0. \end{gathered}$$

Example 25.4.3 Classify the singular points of the differential equation

$$w'' + \frac{1}{z} w' + 2w = 0.$$

There is a regular singular point at $z = 0$. To examine the point at infinity we make the transformation $t = 1/z$, $u(t) = w(z)$. The equation in u is

$$u'' + \left(\frac{2}{t} - \frac{1}{t}\right) u' + \frac{2}{t^4} u = 0$$

$$u'' + \frac{1}{t}u' + \frac{2}{t^4}u = 0.$$

Thus we see that the differential equation for $w(z)$ has an irregular singular point at infinity.

25.5 Exercises

Exercise 25.1 (mathematica/ode/series/series.nb)
$f(x)$ satisfies the Hermite equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}f}{\mathrm{d}x} + 2\lambda f = 0.$$

Construct two linearly independent solutions of the equation as Taylor series about $x = 0$. For what values of x do the series converge?

Show that for certain values of λ, called eigenvalues, one of the solutions is a polynomial, called an eigenfunction. Calculate the first four eigenfunctions $H_0(x)$, $H_1(x)$, $H_2(x)$, $H_3(x)$, ordered by degree.
Hint, Solution

Exercise 25.2
Consider the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

1. Find two linearly independent solutions in the form of power series about $x = 0$.

2. Compute the radius of convergence of the series. Explain why it is possible to predict the radius of convergence without actually deriving the series.

3. Show that if $\alpha = 2n$, with n an integer and $n \geq 0$, the series for one of the solutions reduces to an even polynomial of degree $2n$.

4. Show that if $\alpha = 2n + 1$, with n an integer and $n \geq 0$, the series for one of the solutions reduces to an odd polynomial of degree $2n + 1$.

5. Show that the first 4 polynomial solutions $P_n(x)$ (known as *Legendre* polynomials) ordered by their degree and normalized so that $P_n(1) = 1$ are

$$\begin{aligned} P_0 &= 1 & P_1 &= x \\ P_2 &= \frac{1}{2}(3x^2 - 1) & P_4 &= \frac{1}{2}(5x^3 - 3x) \end{aligned}$$

6. Show that the Legendre equation can also be written as

$$((1 - x^2)y')' = -\alpha(\alpha + 1)y.$$

Note that two Legendre polynomials $P_n(x)$ and $P_m(x)$ must satisfy this relation for $\alpha = n$ and $\alpha = m$ respectively. By multiplying the first relation by $P_m(x)$ and the second by $P_n(x)$ and integrating by parts show that Legendre polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x = 0 \text{ if } n \neq m.$$

If $n = m$, it can be shown that the value of the integral is $2/(2n + 1)$. Verify this for the first three polynomials (but you needn't prove it in general).

Hint, Solution

Exercise 25.3
Find the forms of two linearly independent series expansions about the point $z = 0$ for the differential equation

$$w'' + \frac{1}{\sin z}w' + \frac{1 - z}{z^2}w = 0,$$

such that the series are real-valued on the positive real axis. Do not calculate the coefficients in the expansions.
Hint, Solution

Exercise 25.4
Classify the singular points of the equation

$$w'' + \frac{w'}{z-1} + 2w = 0.$$

Hint, Solution

Exercise 25.5
Find the series expansions about $z = 0$ for

$$w'' + \frac{5}{4z}w' + \frac{z-1}{8z^2}w = 0.$$

Hint, Solution

Exercise 25.6
Find the series expansions about $z = 0$ of the fundamental solutions of

$$w'' + zw' + w = 0.$$

Hint, Solution

Exercise 25.7
Find the series expansions about $z = 0$ of the two linearly independent solutions of

$$w'' + \frac{1}{2z}w' + \frac{1}{z}w = 0.$$

Hint, Solution

Exercise 25.8
Classify the singularity at infinity of the differential equation

$$w'' + \left(\frac{2}{z} + \frac{3}{z^2} \right) w' + \frac{1}{z^2} w = 0.$$

Find the forms of the series solutions of the differential equation about infinity that are real-valued when z is real-valued and positive. Do not calculate the coefficients in the expansions.
Hint, Solution

Exercise 25.9
Consider the second order differential equation

$$x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (b - x) \frac{\mathrm{d}y}{\mathrm{d}x} - a y = 0,$$

where a, b are real constants.

1. Show that $x = 0$ is a regular singular point. Determine the location of any additional singular points and classify them. Include the point at infinity.

2. Compute the indicial equation for the point $x = 0$.

3. By solving an appropriate recursion relation, show that one solution has the form

$$y_1(x) = 1 + \frac{ax}{b} + \frac{(a)_2 x^2}{(b)_2 2!} + \cdots + \frac{(a)_n x^n}{(b)_n n!} + \cdots$$

   where the notation $(a)_n$ is defined by

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad (a)_0 = 1.$$

   Assume throughout this problem that $b \neq n$ where n is a non-negative integer.

4. Show that when $a = -m$, where m is a non-negative integer, that there are polynomial solutions to this equation. Compute the radius of convergence of the series above when $a \neq -m$. Verify that the result you get is in accord with the Frobenius theory.

5. Show that if $b = n+1$ where $n = 0, 1, 2, \ldots$, then the second solution of this equation has logarithmic terms. Indicate the *form* of the second solution in this case. You need not compute any coefficients.

Hint, Solution

Exercise 25.10

Consider the equation

$$xy'' + 2xy' + 6\,\mathrm{e}^x y = 0.$$

Find the first three non-zero terms in each of two linearly independent series solutions about $x = 0$.

Hint, Solution

25.6 Hints

Hint 25.1

Hint 25.2

Hint 25.3

Hint 25.4

Hint 25.5

Hint 25.6

Hint 25.7

Hint 25.8

Hint 25.9

Hint 25.10

25.7 Solutions

Solution 25.1

$f(x)$ is a Taylor series about $x = 0$.

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n x^n \\
f'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
&= \sum_{n=0}^{\infty} n a_n x^{n-1} \\
f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n
\end{aligned}$$

We substitute the Taylor series into the differential equation.

$$f''(x) - 2x f'(x) + 2\lambda f = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2 \sum_{n=0}^{\infty} n a_n x^n + 2\lambda \sum_{n=0}^{\infty} a_n x^n$$

Equating coefficients gives us a difference equation for a_n:

$$(n+2)(n+1) a_{n+2} - 2n a_n + 2\lambda a_n = 0$$

$$a_{n+2} = 2 \frac{n - \lambda}{(n+1)(n+2)} a_n.$$

The first two coefficients, a_0 and a_1 are arbitrary. The remaining coefficients are determined by the recurrence relation. We will find the fundamental set of solutions at $x = 0$. That is, for the first solution we choose $a_0 = 1$ and $a_1 = 0$; for the second solution we choose $a_0 = 0$, $a_1 = 1$. The difference equation for y_1 is

$$a_{n+2} = 2\frac{n - \lambda}{(n+1)(n+2)} a_n, \quad a_0 = 1, \quad a_1 = 0,$$

which has the solution

$$a_{2n} = \frac{2^n \prod_{k=0}^{n} (2(n-k) - \lambda)}{(2n)!}, \quad a_{2n+1} = 0.$$

The difference equation for y_2 is

$$a_{n+2} = 2\frac{n - \lambda}{(n+1)(n+2)} a_n, \quad a_0 = 0, \quad a_1 = 1,$$

which has the solution

$$a_{2n} = 0, \quad a_{2n+1} = \frac{2^n \prod_{k=0}^{n-1} (2(n-k) - 1 - \lambda)}{(2n+1)!}.$$

A set of linearly independent solutions, (in fact the fundamental set of solutions at $x = 0$), is

$$\boxed{y_1(x) = \sum_{n=0}^{\infty} \frac{2^n \prod_{k=0}^{n} (2(n-k) - \lambda)}{(2n)!} x^{2n}, \quad y_2(x) = \sum_{n=0}^{\infty} \frac{2^n \prod_{k=0}^{n-1} (2(n-k) - 1 - \lambda)}{(2n+1)!} x^{2n+1}.}$$

Since the coefficient functions in the differential equation do not have any singularities in the finite complex plane, the radius of convergence of the series is infinite.

If $\lambda = n$ is a positive even integer, then the first solution, y_1, is a polynomial of order n. If $\lambda = n$ is a positive odd integer, then the second solution, y_2, is a polynomial of order n. For $\lambda = 0, 1, 2, 3$, we have

$$H_0(x) = 1$$
$$H_1(x) = x$$
$$H_2(x) = 1 - 2x^2$$
$$H_3(x) = x - \frac{2}{3}x^3$$

Solution 25.2

1. First we write the differential equation in the standard form.

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0 \quad (25.2)$$

$$y'' - \frac{2x}{1 - x^2} + \frac{\alpha(\alpha + 1)}{1 - x^2}y = 0. \quad (25.3)$$

Since the coefficients of y' and y are analytic in a neighborhood of $x = 0$, We can find two Taylor series solutions about that point. We find the Taylor series for y and its derivatives.

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} (n - 1) n a_n x^{n-2}$$
$$= \sum_{n=0}^{\infty} (n + 1)(n + 2) a_{n+2} x^n$$

Here we used index shifting to explicitly write the two forms that we will need for y''. Note that we can take the lower bound of summation to be $n = 0$ for all above sums. The terms added by this operation are zero. We substitute the Taylor series into Equation 25.2.

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty}(n-1)na_nx^n - 2\sum_{n=0}^{\infty}na_nx^n + \alpha(\alpha+1)\sum_{n=0}^{\infty}a_nx^n = 0$$

$$\sum_{n=0}^{\infty}\Big((n+1)(n+2)a_{n+2} - \big((n-1)n + 2n - \alpha(\alpha+1)\big)a_n\Big)x^n = 0$$

We equate coefficients of x^n to obtain a recurrence relation.

$$(n+1)(n+2)a_{n+2} = (n(n+1) - \alpha(\alpha+1))a_n$$

$$a_{n+2} = \frac{n(n+1) - \alpha(\alpha+1)}{(n+1)(n+2)}a_n, \quad n \geq 0.$$

We can solve this difference equation to determine the a_n's. (a_0 and a_1 are arbitrary.)

$$a_n = \begin{cases} \dfrac{a_0}{n!}\displaystyle\prod_{\substack{k=0\\ \text{even } k}}^{n-2}\big(k(k+1) - \alpha(\alpha+1)\big), & \text{even } n,\\ \dfrac{a_1}{n!}\displaystyle\prod_{\substack{k=1\\ \text{odd } k}}^{n-2}\big(k(k+1) - \alpha(\alpha+1)\big), & \text{odd } n \end{cases}$$

We will find the fundamental set of solutions at $x = 0$, that is the set $\{y_1, y_2\}$ that satisfies

$$y_1(0) = 1 \quad y_1'(0) = 0$$
$$y_2(0) = 0 \quad y_2'(0) = 1.$$

For y_1 we take $a_0 = 1$ and $a_1 = 0$; for y_2 we take $a_0 = 0$ and $a_1 = 1$. The rest of the coefficients are determined from the recurrence relation.

$$\boxed{y_1 = \sum_{\substack{n=0 \\ \text{even } n}}^{\infty} \left(\frac{1}{n!} \prod_{\substack{k=0 \\ \text{even } k}}^{n-2} \big(k(k+1) - \alpha(\alpha+1)\big) \right) x^n}$$

$$\boxed{y_2 = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \left(\frac{1}{n!} \prod_{\substack{k=1 \\ \text{odd } k}}^{n-2} \big(k(k+1) - \alpha(\alpha+1)\big) \right) x^n}$$

2. We determine the radius of convergence of the series solutions with the ratio test.

$$\lim_{n \to \infty} \left| \frac{a_{n+2} x^{n+2}}{a_n x^n} \right| < 1$$

$$\lim_{n \to \infty} \left| \frac{\frac{n(n+1) - \alpha(\alpha+1)}{(n+1)(n+2)} a_n x^{n+2}}{a_n x^n} \right| < 1$$

$$\lim_{n \to \infty} \left| \frac{n(n+1) - \alpha(\alpha+1)}{(n+1)(n+2)} \right| \left| x^2 \right| < 1$$

$$\left| x^2 \right| < 1$$

Thus we see that the radius of convergence of the series is 1. We knew that the radius of convergence would be at least one, because the nearest singularities of the coefficients of (25.3) occur at $x = \pm 1$, a distance of 1 from the origin. This implies that the solutions of the equation are analytic in the unit circle about $x = 0$. The radius of convergence of the Taylor series expansion of an analytic function is the distance to the nearest singularity.

3. If $\alpha = 2n$ then $a_{2n+2} = 0$ in our first solution. From the recurrence relation, we see that all subsequent

coefficients are also zero. The solution becomes an even polynomial.

$$\boxed{y_1 = \sum_{\substack{m=0 \\ \text{even } m}}^{2n} \left(\frac{1}{m!} \prod_{\substack{k=0 \\ \text{even } k}}^{m-2} \left(k(k+1) - \alpha(\alpha+1)\right) \right) x^m}$$

4. If $\alpha = 2n+1$ then $a_{2n+3} = 0$ in our second solution. From the recurrence relation, we see that all subsequent coefficients are also zero. The solution becomes an odd polynomial.

$$\boxed{y_2 = \sum_{\substack{m=1 \\ \text{odd } m}}^{2n+1} \left(\frac{1}{m!} \prod_{\substack{k=1 \\ \text{odd } k}}^{m-2} \left(k(k+1) - \alpha(\alpha+1)\right) \right) x^m}$$

5. From our solutions above, the first four polynomials are

$$1$$
$$x$$
$$1 - 3x^2$$
$$x - \frac{5}{3}x^3$$

To obtain the Legendre polynomials we normalize these to have value unity at $x = 1$

$$\begin{aligned} P_0 &= 1 \\ P_1 &= x \\ P_2 &= \frac{1}{2}\left(3x^2 - 1\right) \\ P_3 &= \frac{1}{2}\left(5x^3 - 3x\right) \end{aligned}$$

These four Legendre polynomials are plotted in Figure 25.4.

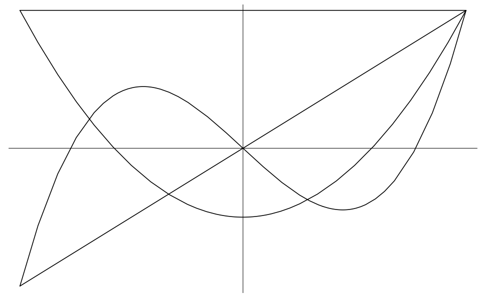

Figure 25.4: The First Four Legendre Polynomials

6. We note that the first two terms in the Legendre equation form an exact derivative. Thus the Legendre equation can also be written as

$$\left((1-x^2)y'\right)' = -\alpha(\alpha+1)y.$$

P_n and P_m are solutions of the Legendre equation.

$$\left((1-x^2)P_n'\right)' = -n(n+1)P_n, \quad \left((1-x^2)P_m'\right)' = -m(m+1)P_m \tag{25.4}$$

We multiply the first relation of Equation 25.4 by P_m and integrate by parts.

$$\left((1-x^2)P_n'\right)' P_m = -n(n+1)P_nP_m$$

$$\int_{-1}^{1} \left((1-x^2)P_n'\right)' P_m \,\mathrm{d}x = -n(n+1)\int_{-1}^{1} P_nP_m \,\mathrm{d}x$$

$$\left[\left((1-x^2)P_n'\right) P_m\right]_{-1}^{1} - \int_{-1}^{1} (1-x^2)P_n'P_m' \,\mathrm{d}x = -n(n+1)\int_{-1}^{1} P_nP_m \,\mathrm{d}x$$

$$\int_{-1}^{1} (1-x^2)P_n'P_m' \,\mathrm{d}x = n(n+1)\int_{-1}^{1} P_nP_m \,\mathrm{d}x$$

We multiply the secord relation of Equation 25.4 by P_n and integrate by parts. To obtain a different expression for $\int_{-1}^{1}(1-x^2)P_m' P_n' \, dx$.

$$\int_{-1}^{1}(1-x^2)P_m' P_n' \, dx = m(m+1)\int_{-1}^{1} P_m P_n \, dx$$

We equate the two expressions for $\int_{-1}^{1}(1-x^2)P_m' P_n' \, dx$. to obtain an orthogonality relation.

$$(n(n+1) - m(m+1))\int_{-1}^{1} P_n P_m \, dx = 0$$

$$\boxed{\int_{-1}^{1} P_n(x) P_m(x) \, dx = 0 \text{ if } n \neq m.}$$

We verify that for the first four polynomials the value of the integral is $2/(2n+1)$ for $n = m$.

$$\int_{-1}^{1} P_0(x)P_0(x) \, dx = \int_{-1}^{1} 1 \, dx = 2$$

$$\int_{-1}^{1} P_1(x)P_1(x) \, dx = \int_{-1}^{1} x^2 \, dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

$$\int_{-1}^{1} P_2(x)P_2(x) \, dx = \int_{-1}^{1} \frac{1}{4}\left(9x^4 - 6x^2 + 1\right) \, dx = \left[\frac{1}{4}\left(\frac{9x^5}{5} - 2x^3 + x\right)\right]_{-1}^{1} = \frac{2}{5}$$

$$\int_{-1}^{1} P_3(x)P_3(x) \, dx = \int_{-1}^{1} \frac{1}{4}\left(25x^6 - 30x^4 + 9x^2\right) \, dx = \left[\frac{1}{4}\left(\frac{25x^7}{7} - 6x^5 + 3x^3\right)\right]_{-1}^{1} = \frac{2}{7}$$

Solution 25.3

The indicial equation for this problem is

$$\alpha^2 + 1 = 0.$$

Since the two roots $\alpha_1 = i$ and $\alpha_2 = -i$ are distinct and do not differ by an integer, there are two solutions in the Frobenius form.

$$w_1 = z^i \sum_{n=0}^{\infty} a_n z^n, \qquad w_1 = z^{-i} \sum_{n=0}^{\infty} b_n z^n$$

However, these series are not real-valued on the positive real axis. Recalling that

$$z^i = \mathrm{e}^{i \log z} = \cos(\log z) + i \sin(\log z), \quad \text{and} \quad z^{-i} = \mathrm{e}^{-i \log z} = \cos(\log z) - i \sin(\log z),$$

we can write a new set of solutions that are real-valued on the positive real axis as linear combinations of w_1 and w_2.

$$u_1 = \frac{1}{2}(w_1 + w_2), \qquad u_2 = \frac{1}{2i}(w_1 - w_2)$$

$$\boxed{u_1 = \cos(\log z) \sum_{n=0}^{\infty} c_n z^n, \qquad u_1 = \sin(\log z) \sum_{n=0}^{\infty} d_n z^n}$$

Solution 25.4

Consider the equation $w'' + w'/(z-1) + 2w = 0$.

We see that there is a regular singular point at $z = 1$. All other finite values of z are ordinary points of the equation. To examine the point at infinity we introduce the transformation $z = 1/t$, $w(z) = u(t)$. Writing the derivatives with respect to z in terms of t yields

$$\frac{d}{dz} = -t^2 \frac{d}{dt}, \qquad \frac{d^2}{dz^2} = t^4 \frac{d^2}{dt^2} + 2t^3 \frac{d}{dt}.$$

Substituting into the differential equation gives us

$$t^4 u'' + 2t^3 u' - \frac{t^2 u'}{1/t - 1} + 2u = 0$$

$$u'' + \left(\frac{2}{t} - \frac{1}{t(1-t)} \right) u' + \frac{2}{t^4} u = 0.$$

Since $t = 0$ is an irregular singular point in the equation for $u(t)$, $z = \infty$ is an irregular singular point in the equation for $w(z)$.

Solution 25.5

Find the series expansions about $z = 0$ for

$$w'' + \frac{5}{4z}w' + \frac{z-1}{8z^2}w = 0.$$

We see that $z = 0$ is a regular singular point of the equation. The indicial equation is

$$\alpha^2 + \frac{1}{4}\alpha - \frac{1}{8} = 0$$
$$\left(\alpha + \frac{1}{2}\right)\left(\alpha - \frac{1}{4}\right) = 0.$$

Since the roots are distinct and do not differ by an integer, there will be two solutions in the Frobenius form.

$$w_1 = z^{1/4}\sum_{n=0}^{\infty} a_n(\alpha_1)z^n, \qquad w_2 = z^{-1/2}\sum_{n=0}^{\infty} a_n(\alpha_2)z^n$$

We multiply the differential equation by $8z^2$ to put it in a better form. Substituting a Frobenius series into the differential equation,

$$8z^2\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n z^{n+\alpha-2} + 10z\sum_{n=0}^{\infty}(n+\alpha)a_n z^{n+\alpha-1} + (z-1)\sum_{n=0}^{\infty} a_n z^{n+\alpha}$$
$$8\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n z^n + 10\sum_{n=0}^{\infty}(n+\alpha)a_n z^n + \sum_{n=1}^{\infty} a_{n-1}z^n - \sum_{n=0}^{\infty} a_n z^n.$$

Equating coefficients of powers of z,

$$\left[8(n+\alpha)(n+\alpha-1) + 10(n+\alpha) - 1\right]a_n = -a_{n-1}$$
$$a_n = -\frac{a_{n-1}}{8(n+\alpha)^2 + 2(n+\alpha) - 1}.$$

The First Solution. Setting $\alpha = 1/4$ in the recurrence formula,

$$a_n(\alpha_1) = -\frac{a_{n-1}}{8(n+1/4)^2 + 2(n+1/4) - 1}$$
$$a_n(\alpha_1) = -\frac{a_{n-1}}{2n(4n+3)}.$$

Thus the first solution is

$$\boxed{w_1 = z^{1/4} \sum_{n=0}^{\infty} a_n(\alpha_1) z^n = a_0 z^{1/4} \left(1 - \frac{1}{14} z + \frac{1}{616} z^2 + \cdots \right).}$$

The Second Solution. Setting $\alpha = -1/2$ in the recurrence formula,

$$a_n = -\frac{a_{n-1}}{8(n-1/2)^2 + 2(n-1/2) - 1}$$
$$a_n = -\frac{a_{n-1}}{2n(4n-3)}$$

Thus the second linearly independent solution is

$$\boxed{w_2 = z^{-1/2} \sum_{n=0}^{\infty} a_n(\alpha_2) z^n = a_0 z^{-1/2} \left(1 - \frac{1}{2} z + \frac{1}{40} z^2 + \cdots \right).}$$

Solution 25.6
We consider the series solutions of,

$$w'' + z w' + w = 0.$$

We would like to find the expansions of the fundamental set of solutions about $z = 0$. Since $z = 0$ is a regular point, (the coefficient functions are analytic there), we expand the solutions in Taylor series. Differentiating the series expansions for $w(z)$,

$$\begin{aligned} w &= \sum_{n=0}^{\infty} a_n z^n \\ w' &= \sum_{n=1}^{\infty} n a_n z^{n-1} \\ w'' &= \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n \end{aligned}$$

We may take the lower limit of summation to be zero without changing the sums. Substituting these expressions into the differential equation,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n + \sum_{n=0}^{\infty} n a_n z^n + \sum_{n=0}^{\infty} a_n z^n = 0$$
$$\sum_{n=0}^{\infty} \big((n+2)(n+1) a_{n+2} + (n+1) a_n\big) z^n = 0.$$

Equating the coefficient of the z^n term gives us

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0, \qquad n \geq 0$$
$$a_{n+2} = -\frac{a_n}{n+2}, \qquad n \geq 0.$$

a_0 and a_1 are arbitrary. We determine the rest of the coefficients from the recurrence relation. We consider the cases for even and odd n separately.

$$\begin{aligned} a_{2n} &= -\frac{a_{2n-2}}{2n} \\ &= \frac{a_{2n-4}}{(2n)(2n-2)} \\ &= (-1)^n \frac{a_0}{(2n)(2n-2)\cdots 4\cdot 2} \\ &= (-1)^n \frac{a_0}{\prod_{m=1}^n 2m}, \quad n \geq 0 \end{aligned}$$

$$\begin{aligned} a_{2n+1} &= -\frac{a_{2n-1}}{2n+1} \\ &= \frac{a_{2n-3}}{(2n+1)(2n-1)} \\ &= (-1)^n \frac{a_1}{(2n+1)(2n-1)\cdots 5\cdot 3} \\ &= (-1)^n \frac{a_1}{\prod_{m=1}^n (2m+1)}, \quad n \geq 0 \end{aligned}$$

If $\{w_1, w_2\}$ is the fundamental set of solutions, then the initial conditions demand that $w_1 = 1 + 0 \cdot z + \cdots$ and $w_2 = 0 + z + \cdots$. We see that w_1 will have only even powers of z and w_2 will have only odd powers of z.

$$\boxed{w_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{\prod_{m=1}^n 2m} z^{2n}, \qquad w_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{\prod_{m=1}^n (2m+1)} z^{2n+1}}$$

Since the coefficient functions in the differential equation are entire, (analytic in the finite complex plane), the radius of convergence of these series solutions is infinite.

Solution 25.7

$$w'' + \frac{1}{2z} w' + \frac{1}{z} w = 0.$$

We can find the indicial equation by substituting $w = z^\alpha + \mathcal{O}(z^{\alpha+1})$ into the differential equation.

$$\alpha(\alpha - 1) z^{\alpha-2} + \frac{1}{2} \alpha z^{\alpha-2} + z^{\alpha-1} = \mathcal{O}(z^{\alpha-1})$$

Equating the coefficient of the $z^{\alpha-2}$ term,

$$\alpha(\alpha - 1) + \frac{1}{2}\alpha = 0$$
$$\alpha = 0, \frac{1}{2}.$$

Since the roots are distinct and do not differ by an integer, the solutions are of the form

$$w_1 = \sum_{n=0}^{\infty} a_n z^n, \qquad w_2 = z^{1/2} \sum_{n=0}^{\infty} b_n z^n.$$

Differentiating the series for the first solution,

$$w_1 = \sum_{n=0}^{\infty} a_n z^n$$
$$w_1' = \sum_{n=1}^{\infty} n a_n z^{n-1}$$
$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$
$$w_1'' = \sum_{n=1}^{\infty} n(n+1) a_{n+1} z^{n-1}.$$

Substituting this series into the differential equation,

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}z^{n-1} + \frac{1}{2z}\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n + \frac{1}{z}\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1} + \frac{1}{2}(n+1)a_{n+1} + a_n\right] z^{n-1} + \frac{1}{2z}a_1 + \frac{1}{z}a_0 = 0.$$

Equating powers of z,

$$z^{-1}: \frac{a_1}{2} + a_0 = 0 \quad \Rightarrow \quad a_1 = -2a_0$$

$$z^{n-1}: \left(n + \frac{1}{2}\right)(n+1)a_{n+1} + a_n = 0 \quad \Rightarrow \quad a_{n+1} = -\frac{a_n}{(n+1/2)(n+1)}.$$

We can combine the above two equations for a_n.

$$a_{n+1} = -\frac{a_n}{(n+1/2)(n+1)}, \quad \text{for } n \geq 0$$

Solving this difference equation for a_n,

$$a_n = a_0 \prod_{j=0}^{n-1} \frac{-1}{(j+1/2)(j+1)}$$

$$\boxed{a_n = a_0 \frac{(-1)^n}{n!} \prod_{j=0}^{n-1} \frac{1}{j+1/2}}$$

Now let's find the second solution. Differentiating w_2,

$$w_2' = \sum_{n=0}^{\infty} (n+1/2) b_n z^{n-1/2}$$
$$w_2'' = \sum_{n=0}^{\infty} (n+1/2)(n-1/2) b_n z^{n-3/2}.$$

Substituting these expansions into the differential equation,

$$\sum_{n=0}^{\infty} (n+1/2)(n-1/2) b_n z^{n-3/2} + \frac{1}{2}\sum_{n=0}^{\infty} (n+1/2) b_n z^{n-3/2} + \sum_{n=1}^{\infty} b_{n-1} z^{n-3/2} = 0.$$

Equating the coefficient of the $z^{-3/2}$ term,

$$\frac{1}{2}\left(-\frac{1}{2}\right) b_0 + \frac{1}{2}\frac{1}{2} b_0 = 0,$$

we see that b_0 is arbitrary. Equating the other coefficients of powers of z,

$$(n+1/2)(n-1/2) b_n + \frac{1}{2}(n+1/2) b_n + b_{n-1} = 0$$
$$b_n = -\frac{b_{n-1}}{n(n+1/2)}$$

Calculating the b_n's,

$$b_1 = -\frac{b_0}{1 \cdot \frac{3}{2}}$$
$$b_2 = \frac{b_0}{1 \cdot 2 \cdot \frac{3}{2} \cdot \frac{5}{2}}$$
$$b_n = \frac{(-1)^n 2^n b_0}{n! \cdot 3 \cdot 5 \cdots (2n+1)}$$

Thus the second solution is

$$\boxed{w_2 = b_0 z^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n z^n}{n!\, 3 \cdot 5 \cdots (2n+1)}.}$$

Solution 25.8

$$w'' + \left(\frac{2}{z} + \frac{3}{z^2} \right) w' + \frac{1}{z^2} w = 0.$$

In order to analyze the behavior at infinity we make the change of variables $t = 1/z$, $u(t) = w(z)$ and examine the point $t = 0$. Writing the derivatives with respect to z in terms if t yields

$$\begin{aligned} z &= \frac{1}{t} \\ dz &= -\frac{1}{t^2} dt \\ \frac{\mathrm{d}}{\mathrm{d}z} &= -t^2 \frac{\mathrm{d}}{\mathrm{d}t} \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}z^2} &= -t^2 \frac{\mathrm{d}}{\mathrm{d}t} \left(-t^2 \frac{\mathrm{d}}{\mathrm{d}t} \right) \\ &= t^4 \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2t^3 \frac{\mathrm{d}}{\mathrm{d}t}. \end{aligned}$$

The equation for u is then

$$\begin{aligned} t^4 u'' + 2t^3 u' + (2t + 3t^2)(-t^2) u' + t^2 u = 0 \\ u'' + -3u' + \frac{1}{t^2} u = 0 \end{aligned}$$

We see that $t = 0$ is a regular singular point. To find the indicial equation, we substitute $u = t^\alpha + \mathcal{O}(t^{\alpha+1})$ into the differential equation.

$$\alpha(\alpha - 1)t^{\alpha-2} - 3\alpha t^{\alpha-1} + t^{\alpha-2} = \mathcal{O}(t^{\alpha-1})$$

Equating the coefficients of the $t^{\alpha-2}$ terms,

$$\alpha(\alpha - 1) + 1 = 0$$
$$\alpha = \frac{1 \pm i\sqrt{3}}{2}$$

Since the roots of the indicial equation are distinct and do not differ by an integer, a set of solutions has the form

$$\left\{ t^{(1+i\sqrt{3})/2} \sum_{n=0}^{\infty} a_n t^n, \quad t^{(1-i\sqrt{3})/2} \sum_{n=0}^{\infty} b_n t^n \right\}.$$

Noting that

$$t^{(1+i\sqrt{3})/2} = t^{1/2} \exp\left(\frac{i\sqrt{3}}{2} \log t\right), \quad \text{and} \quad t^{(1-i\sqrt{3})/2} = t^{1/2} \exp\left(-\frac{i\sqrt{3}}{2} \log t\right).$$

We can take the sum and difference of the above solutions to obtain the form

$$u_1 = t^{1/2} \cos\left(\frac{\sqrt{3}}{2} \log t\right) \sum_{n=0}^{\infty} a_n t^n, \qquad u_1 = t^{1/2} \sin\left(\frac{\sqrt{3}}{2} \log t\right) \sum_{n=0}^{\infty} b_n t^n.$$

Putting the answer in terms of z, we have the form of the two Frobenius expansions about infinity.

$$\boxed{w_1 = z^{-1/2} \cos\left(\frac{\sqrt{3}}{2} \log z\right) \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \qquad w_1 = z^{-1/2} \sin\left(\frac{\sqrt{3}}{2} \log z\right) \sum_{n=0}^{\infty} \frac{b_n}{z^n}.}$$

Solution 25.9

1. We write the equation in the standard form.

$$y'' + \frac{b-x}{x} y' - \frac{a}{x} y = 0$$

Since $\frac{b-x}{x}$ has no worse than a first order pole and $\frac{a}{x}$ has no worse than a second order pole at $x = 0$, that is a regular singular point. Since the coefficient functions have no other singularities in the finite complex plane, all the other points in the finite complex plane are regular points.

Now to examine the point at infinity. We make the change of variables $u(\xi) = y(x)$, $\xi = 1/x$.

$$y' = \frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} u = -\frac{1}{x^2} u' = -\xi^2 u'$$
$$y'' = -\xi^2 \frac{\mathrm{d}}{\mathrm{d}\xi}\left(-\xi^2 \frac{\mathrm{d}}{\mathrm{d}\xi}\right) u = \xi^4 u'' + 2\xi^3 u'$$

The differential equation becomes

$$xy'' + (b-x)y' - ay$$
$$\frac{1}{\xi}\left(\xi^4 u'' + 2\xi^3 u'\right) + \left(b - \frac{1}{\xi}\right)\left(-\xi^2 u'\right) - au = 0$$
$$\xi^3 u'' + \left((2-b)\xi^2 + \xi\right) u' - au = 0$$
$$u'' + \left(\frac{2-b}{\xi} + \frac{1}{\xi^2}\right) - \frac{a}{\xi^3} u = 0$$

Since this equation has an irregular singular point at $\xi = 0$, the equation for $y(x)$ has an irregular singular point at infinity.

2. The coefficient functions are

$$p(x) \equiv \frac{1}{x} \sum_{n=1}^{\infty} p_n x^n = \frac{1}{x}(b - x),$$
$$q(x) \equiv \frac{1}{x^2} \sum_{n=1}^{\infty} q_n x^n = \frac{1}{x^2}(0 - ax).$$

The indicial equation is

$$\alpha^2 + (p_0 - 1)\alpha + q_0 = 0$$
$$\alpha^2 + (b - 1)\alpha + 0 = 0$$
$$\boxed{\alpha(\alpha + b - 1) = 0.}$$

3. Since one of the roots of the indicial equation is zero, and the other root is not a negative integer, one of

the solutions of the differential equation is a Taylor series.

$$
\begin{aligned}
y_1 &= \sum_{k=0}^{\infty} c_k x^k \\
y_1' &= \sum_{k=1}^{\infty} k c_k x^{k-1} \\
&= \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k \\
&= \sum_{k=0}^{\infty} k c_k x^{k-1} \\
y_1'' &= \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} \\
&= \sum_{k=1}^{\infty} (k+1) k c_{k+1} x^{k-1} \\
&= \sum_{k=0}^{\infty} (k+1) k c_{k+1} x^{k-1}
\end{aligned}
$$

We substitute the Taylor series into the differential equation.

$$
xy'' + (b-x)y' - ay = 0
$$

$$
\sum_{k=0}^{\infty} (k+1) k c_{k+1} x^k + b \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k - \sum_{k=0}^{\infty} k c_k x^k - a \sum_{k=0}^{\infty} c_k x^k = 0
$$

We equate coefficients to determine a recurrence relation for the coefficients.

$$
(k+1) k c_{k+1} + b(k+1) c_{k+1} - k c_k - a c_k = 0
$$

$$
c_{k+1} = \frac{k+a}{(k+1)(k+b)} c_k
$$

For $c_0 = 1$, the recurrence relation has the solution

$$c_k = \frac{(a)_k x^k}{(b)_k k!}.$$

Thus one solution is

$$\boxed{y_1(x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k k!} x^k.}$$

4. If $a = -m$, where m is a non-negative integer, then $(a)_k = 0$ for $k > m$. This makes y_1 a polynomial:

$$\boxed{y_1(x) = \sum_{k=0}^{m} \frac{(a)_k}{(b)_k k!} x^k.}$$

5. If $b = n + 1$, where n is a non-negative integer, the indicial equation is

$$\alpha(\alpha + n) = 0.$$

For the case $n = 0$, the indicial equation has a double root at zero. Thus the solutions have the form:

$$\boxed{y_1(x) = \sum_{k=0}^{m} \frac{(a)_k}{(b)_k k!} x^k, \qquad y_2(x) = y_1(x) \log x + \sum_{k=0}^{\infty} d_k x^k}$$

For the case $n > 0$ the roots of the indicial equation differ by an integer. The solutions have the form:

$$\boxed{y_1(x) = \sum_{k=0}^{m} \frac{(a)_k}{(b)_k k!} x^k, \qquad y_2(x) = d_{-1} y_1(x) \log x + x^{-n} \sum_{k=0}^{\infty} d_k x^k}$$

The form of the solution for y_2 can be substituted into the equation to determine the coefficients d_k.

Solution 25.10
We write the equation in the standard form.

$$xy'' + 2xy' + 6\,\mathrm{e}^x y = 0$$
$$y'' + 2y' + 6\frac{\mathrm{e}^x}{x} y = 0$$

We see that $x = 0$ is a regular singular point. The indicial equation is

$$\alpha^2 - \alpha = 0$$
$$\alpha = 0, 1.$$

The first solution has the Frobenius form.

$$y_1 = x + a_2 x^2 + a_3 x^3 + \mathcal{O}(x^4)$$

We substitute y_1 into the differential equation and equate coefficients of powers of x.

$$xy'' + 2xy' + 6\,\mathrm{e}^x y = 0$$

$$x(2a_2 + 6a_3 x + \mathcal{O}(x^2)) + 2x(1 + 2a_2 x + 3a_3 x^2 + \mathcal{O}(x^3)) \\ + 6(1 + x + x^2/2 + \mathcal{O}(x^3))(x + a_2 x^2 + a_3 x^3 + \mathcal{O}(x^4)) = 0$$

$$(2a_2 x + 6a_3 x^2) + (2x + 4a_2 x^2) + (6x + 6(1 + a_2)x^2) = \mathcal{O}(x^3) = 0$$
$$a_2 = -4, \quad a_3 = \frac{17}{3}$$
$$\boxed{y_1 = x - 4x^2 + \frac{17}{3}x^3 + \mathcal{O}(x^4)}$$

Now we see if the second solution has the Frobenius form. There is no $a_1 x$ term because y_2 is only determined up to an additive constant times y_1.

$$y_2 = 1 + \mathcal{O}(x^2)$$

We substitute y_2 into the differential equation and equate coefficients of powers of x.

$$\begin{gathered} xy'' + 2xy' + 6\,\mathrm{e}^x y = 0 \\ \mathcal{O}(x) + \mathcal{O}(x) + 6(1 + \mathcal{O}(x))(1 + \mathcal{O}(x^2)) = 0 \\ 6 = \mathcal{O}(x) \end{gathered}$$

The substitution $y_2 = 1 + \mathcal{O}(x)$ has yielded a contradiction. Since the second solution is not of the Frobenius form, it has the following form:

$$y_2 = y_1 \ln(x) + a_0 + a_2 x^2 + \mathcal{O}(x^3)$$

The first three terms in the solution are

$$y_2 = a_0 + x \ln x - 4x^2 \ln x + \mathcal{O}(x^2).$$

We calculate the derivatives of y_2.

$$\begin{aligned} y_2' &= \ln(x) + \mathcal{O}(1) \\ y_2'' &= \frac{1}{x} + \mathcal{O}(\ln(x)) \end{aligned}$$

We substitute y_2 into the differential equation and equate coefficients.

$$\begin{gathered} xy'' + 2xy' + 6\,\mathrm{e}^x y = 0 \\ (1 + \mathcal{O}(x \ln x)) + 2\,(\mathcal{O}(x \ln x)) + 6\,(a_0 + \mathcal{O}(x \ln x)) = 0 \\ 1 + 6a_0 = 0 \end{gathered}$$

$$\boxed{y_2 = -\frac{1}{6} + x \ln x - 4x^2 \ln x + \mathcal{O}(x^2)}$$

Chapter 26

Asymptotic Expansions

The more you sweat in practice, the less you bleed in battle.

-Navy Seal Saying

26.1 Asymptotic Relations

The $\ll$ and $\sim$ symbols. First we will introduce two new symbols used in asymptotic relations.

$$f(x) \ll g(x) \quad \text{as } x \to x_0,$$

is read, "$f(x)$ is much smaller than $g(x)$ as x tends to x_0". This means

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

The notation

$$f(x) \sim g(x) \quad \text{as } x \to x_0,$$

is read "$f(x)$ is asymptotic to $g(x)$ as x tends to x_0"; which means

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1.$$

A few simple examples are

- $-\mathrm{e}^x \gg x \quad \text{as } x \to +\infty$
- $\sin x \sim x \quad \text{as } x \to 0$
- $1/x \ll 1 \quad \text{as } x \to +\infty$
- $\mathrm{e}^{-1/x} \ll x^{-n} \quad \text{as } x \to 0^+$ for all n

An equivalent definition of $f(x) \sim g(x)$ as $x \to x_0$ is

$$f(x) - g(x) \ll g(x) \quad \text{as } x \to x_0.$$

Note that it does not make sense to say that a function $f(x)$ is asymptotic to zero. Using the above definition this would imply

$$f(x) \ll 0 \quad \text{as } x \to x_0.$$

If you encounter an expression like $f(x) + g(x) \sim 0$, take this to mean $f(x) \sim -g(x)$.

The Big $\mathcal{O}$ and Little o Notation. If $|f(x)| \leq m|g(x)|$ for some constant m in some neighborhood of the point $x = x_0$, then we say that

$$f(x) = \mathcal{O}(g(x)) \quad \text{as } x \to x_0.$$

We read this as "f is big $\mathcal{O}$ of g as x goes to x_0". If $g(x)$ does not vanish, an equivalent definition is that $f(x)/g(x)$ is bounded as $x \to x_0$.

If for any given positive δ there exists a neighborhood of $x = x_0$ in which $|f(x)| \leq \delta |g(x)|$ then

$$f(x) = \mathsf{o}(g(x)) \quad \text{as } x \to x_0.$$

This is read, "f is little o of g as x goes to x_0."

For a few examples of the use of this notation,

- $\mathrm{e}^{-x} = \mathsf{o}(x^{-n})$ as $x \to \infty$ for any n.
- $\sin x = \mathcal{O}(x)$ as $x \to 0$.
- $\cos x - 1 = \mathsf{o}(1)$ as $x \to 0$.
- $\log x = \mathsf{o}(x^\alpha)$ as $x \to +\infty$ for any positive α.

Operations on Asymptotic Relations. You can perform the ordinary arithmetic operations on asymptotic relations. Addition, multiplication, and division are valid.

You can always integrate an asymptotic relation. Integration is a smoothing operation. However, it is necessary to exercise some care.

Example 26.1.1 Consider

$$f'(x) \sim \frac{1}{x^2} \quad \text{as } x \to \infty.$$

This does not imply that

$$f(x) \sim \frac{-1}{x} \quad \text{as } x \to \infty.$$

We have forgotten the constant of integration. Integrating the asymptotic relation for $f'(x)$ yields

$$f(x) \sim \frac{-1}{x} + c \quad \text{as } x \to \infty.$$

If c is nonzero then

$$f(x) \sim c \quad \text{as } x \to \infty.$$

It is not always valid to differentiate an asymptotic relation.

Example 26.1.2 Consider $f(x) = \frac{1}{x} + \frac{1}{x^2}\sin(x^3)$.

$$f(x) \sim \frac{1}{x} \quad \text{as } x \to \infty.$$

Differentiating this relation yields

$$f'(x) \sim -\frac{1}{x^2} \quad \text{as } x \to \infty.$$

However, this is not true since

$$\begin{aligned} f'(x) &= -\frac{1}{x^2} - \frac{2}{x^3}\sin(x^3) + 2\cos(x^3) \\ &\not\sim -\frac{1}{x^2} \quad \text{as } x \to \infty. \end{aligned}$$

The Controlling Factor. The controlling factor is the most rapidly varying factor in an asymptotic relation. Consider a function $f(x)$ that is asymptotic to $x^2\,\mathrm{e}^x$ as x goes to infinity. The controlling factor is e^x. For a few examples of this,

- $x \log x$ has the controlling factor x as $x \to \infty$.
- $x^{-2}\,\mathrm{e}^{1/x}$ has the controlling factor $\mathrm{e}^{1/x}$ as $x \to 0$.
- $x^{-1}\sin x$ has the controlling factor $\sin x$ as $x \to \infty$.

The Leading Behavior. Consider a function that is asymptotic to a sum of terms.

$$f(x) \sim a_0(x) + a_1(x) + a_2(x) + \cdots, \quad \text{as } x \to x_0.$$

where

$$a_0(x) \gg a_1(x) \gg a_2(x) \gg \cdots, \quad \text{as } x \to x_0.$$

The first term in the sum is the leading order behavior. For a few examples,

- For $\sin x \sim x - x^3/6 + x^5/120 - \cdots$ as $x \to 0$, the leading order behavior is x.
- For $f(x) \sim \mathrm{e}^x(1 - 1/x + 1/x^2 - \cdots)$ as $x \to \infty$, the leading order behavior is e^x.

26.2 Leading Order Behavior of Differential Equations

It is often useful to know the leading order behavior of the solutions to a differential equation. If we are considering a regular point or a regular singular point, the approach is straight forward. We simply use a Taylor expansion or the Frobenius method. However, if we are considering an irregular singular point, we will have to be a little more creative. Instead of an all encompassing theory like the Frobenius method which always gives us the solution, we will use a heuristic approach that usually gives us the solution.

Example 26.2.1 Consider the Airy equation

$$y'' = xy.$$

We [1] would like to know how the solutions of this equation behave as $x \to +\infty$. First we need to classify the point at infinity. The change of variables

$$x = \frac{1}{t}, \quad y(x) = u(t), \quad \frac{\mathrm{d}}{\mathrm{d}x} = -t^2 \frac{\mathrm{d}}{\mathrm{d}t}, \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2} = t^4 \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2t^3 \frac{\mathrm{d}}{\mathrm{d}t}$$

[1] Using "We" may be a bit presumptuous on my part. Even if you don't particularly want to know how the solutions behave, I urge you to just play along. This is an interesting section, I promise.

yields

$$t^4 u'' + 2t^3 u' = \frac{1}{t} u$$
$$u'' + \frac{2}{t} u' - \frac{1}{t^5} u = 0.$$

Since the equation for u has an irregular singular point at zero, the equation for y has an irregular singular point at infinity.

The Controlling Factor. Since the solutions at irregular singular points often have exponential behavior, we make the substitution $y = \mathrm{e}^{s(x)}$ into the differential equation for y.

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\mathrm{e}^s\right] = x\,\mathrm{e}^s$$
$$\left[s'' + (s')^2\right]\mathrm{e}^s = x\,\mathrm{e}^s$$
$$s'' + (s')^2 = x$$

The Dominant Balance. Now we have a differential equation for s that appears harder to solve than our equation for y. However, we did not introduce the substitution in order to obtain an equation that we could solve exactly. We are looking for an equation that we can solve approximately in the limit as $x \to \infty$. If one of the terms in the equation for s is much smaller that the other two as $x \to \infty$, then dropping that term and solving the simpler equation may give us an approximate solution. If one of the terms in the equation for s is much smaller than the others then we say that the remaining terms form a **dominant balance** in the limit as $x \to \infty$.

Assume that the s'' term is much smaller that the others, $s'' \ll (s')^2, x$ as $x \to \infty$. This gives us

$$(s')^2 \sim x$$
$$s' \sim \pm\sqrt{x}$$
$$s \sim \pm\frac{2}{3}x^{3/2} \quad \text{as } x \to \infty.$$

Now let's check our assumption that the s'' term is small. Assuming that we can differentiate the asymptotic relation $s' \sim \pm\sqrt{x}$, we obtain $s'' \sim \pm\frac{1}{2}x^{-1/2}$ as $x \to \infty$.

$$s'' \ll (s')^2, x \qquad \Rightarrow \qquad x^{-1/2} \ll x \quad \text{as } x \to \infty$$

Thus we see that the behavior we found for s is consistent with our assumption. The controlling factors for solutions to the Airy equation are $\exp(\pm\frac{2}{3}x^{3/2})$ as $x \to \infty$.

The Leading Order Behavior of the Decaying Solution. Let's find the leading order behavior as x goes to infinity of the solution with the controlling factor $\exp(-\frac{2}{3}x^{3/2})$. We substitute

$$s(x) = -\frac{2}{3}x^{3/2} + t(x), \qquad \text{where } t(x) \ll x^{3/2} \text{ as } x \to \infty$$

into the differential equation for s.

$$\begin{gathered} s'' + (s')^2 = x \\ -\frac{1}{2}x^{-1/2} + t'' + (-x^{1/2} + t')^2 = x \\ t'' + (t')^2 - 2x^{1/2}t' - \frac{1}{2}x^{-1/2} = 0 \end{gathered}$$

Assume that we can differentiate $t \ll x^{3/2}$ to obtain

$$t' \ll x^{1/2}, \qquad t'' \ll x^{-1/2} \quad \text{as } x \to \infty.$$

Since $t'' \ll -\frac{1}{2}x^{-1/2}$ we drop the t'' term. Also, $t' \ll x^{1/2}$ implies that $(t')^2 \ll -2x^{1/2}t'$, so we drop the $(t')^2$ term. This gives us

$$
\begin{gathered}
-2x^{1/2}t' - \frac{1}{2}x^{-1/2} \sim 0 \\
t' \sim -\frac{1}{4}x^{-1} \\
t \sim -\frac{1}{4}\log x + c \\
t \sim -\frac{1}{4}\log x \quad \text{as } x \to \infty.
\end{gathered}
$$

Checking our assumptions about t,

$$
\begin{aligned}
t' \ll x^{1/2} \qquad &\Rightarrow \qquad x^{-1} \ll x^{1/2} \\
t'' \ll x^{-1/2} \qquad &\Rightarrow \qquad x^{-2} \ll x^{-1/2}
\end{aligned}
$$

we see that the behavior of t is consistent with our assumptions.

So far we have

$$
y(x) \sim \exp\left(-\frac{2}{3}x^{3/2} - \frac{1}{4}\log x + u(x)\right) \quad \text{as } x \to \infty,
$$

where $u(x) \ll \log x$ as $x \to \infty$. To continue, we substitute $t(x) = -\frac{1}{4}\log x + u(x)$ into the differential equation for $t(x)$.

$$
\begin{gathered}
t'' + (t')^2 - 2x^{1/2}t' - \frac{1}{2}x^{-1/2} = 0 \\
\frac{1}{4}x^{-2} + u'' + \left(-\frac{1}{4}x^{-1} + u'\right)^2 - 2x^{1/2}\left(-\frac{1}{4}x^{-1} + u'\right) - \frac{1}{2}x^{-1/2} = 0 \\
u'' + (u')^2 + \left(-\frac{1}{2}x^{-1} - 2x^{1/2}\right)u' + \frac{5}{16}x^{-2} = 0
\end{gathered}
$$

Assume that we can differentiate the asymptotic relation for u to obtain

$$u' \ll x^{-1}, \qquad u'' \ll x^{-2} \quad \text{as } x \to \infty.$$

We know that $-\frac{1}{2}x^{-1}u' \ll -2x^{1/2}u'$. Using our assumptions,

$$u'' \ll x^{-2} \qquad \Rightarrow \qquad u'' \ll \frac{5}{16}x^{-2}$$
$$u' \ll x^{-1} \qquad \Rightarrow \qquad (u')^2 \ll \frac{5}{16}x^{-2}.$$

Thus we obtain

$$-2x^{1/2}u' + \frac{5}{16}x^{-2} \sim 0$$
$$u' \sim \frac{5}{32}x^{-5/2}$$
$$u \sim -\frac{5}{48}x^{-3/2} + c$$
$$u \sim c \quad \text{as } x \to \infty.$$

Since $u = c + \mathsf{o}(1)$, $\mathrm{e}^u = \mathrm{e}^c + \mathsf{o}(1)$. The behavior of y is

$$y \sim x^{-1/4} \exp\left(-\frac{2}{3}x^{3/2}\right)\left(\mathrm{e}^c + \mathsf{o}(1)\right) \quad \text{as } x \to \infty.$$

Thus the full leading order behavior of the decaying solution is

$$\boxed{y \sim (\text{const})x^{-1/4} \exp\left(-\frac{2}{3}x^{3/2}\right) \quad \text{as } x \to \infty.}$$

You can show that the leading behavior of the exponentially growing solution is

$$y \sim (\text{const})x^{-1/4} \exp\left(\frac{2}{3}x^{3/2}\right) \quad \text{as } x \to \infty.$$

Example 26.2.2 The Modified Bessel Equation. Consider the modified Bessel equation

$$x^2 y'' + x y' - (x^2 + \nu^2) y = 0.$$

We would like to know how the solutions of this equation behave as $x \to +\infty$. First we need to classify the point at infinity. The change of variables $x = \frac{1}{t}$, $y(x) = u(t)$ yields

$$\frac{1}{t^2}(t^4 u'' + 2t^3 u') + \frac{1}{t}(-t^2 u') - \left(\frac{1}{t^2} + \nu^2\right) u = 0$$
$$u'' + \frac{1}{t} u' - \left(\frac{1}{t^4} + \frac{\nu^2}{t^2}\right) u = 0$$

Since $u(t)$ has an irregular singular point at $t = 0$, $y(x)$ has an irregular singular point at infinity.

The Controlling Factor. Since the solutions at irregular singular points often have exponential behavior, we make the substitution $y = \mathrm{e}^{s(x)}$ into the differential equation for y.

$$x^2 (s'' + (s')^2) \mathrm{e}^s + x s' \mathrm{e}^s - (x^2 + \nu^2) \mathrm{e}^s = 0$$
$$s'' + (s')^2 + \frac{1}{x} s' - (1 + \frac{\nu^2}{x^2}) = 0$$

We make the assumption that $s'' \ll (s')^2$ as $x \to \infty$ and we know that $\nu^2/x^2 \ll 1$ as $x \to \infty$. Thus we drop these two terms from the equation to obtain an approximate equation for s.

$$(s')^2 + \frac{1}{x} s' - 1 \sim 0$$

This is a quadratic equation for s', so we can solve it exactly. However, let us try to simplify the equation even further. Assume that as x goes to infinity one of the three terms is much smaller that the other two. If this is the case, there will be a balance between the two dominant terms and we can neglect the third. Let's check the three possibilities.

1.

$$1 \text{ is small.} \qquad \Rightarrow \qquad (s')^2 + \frac{1}{x}s' \sim 0 \qquad \Rightarrow \qquad s' \sim -\frac{1}{x}, 0$$

$1 \not\ll \frac{1}{x^2}, 0$ as $x \to \infty$ so this balance is inconsistent.

2.

$$\frac{1}{x}s' \text{ is small.} \qquad \Rightarrow \qquad (s')^2 - 1 \sim 0 \qquad \Rightarrow \qquad s' \sim \pm 1$$

This balance is consistent as $\frac{1}{x} \ll 1$ as $x \to \infty$.

3.

$$(s')^2 \text{ is small.} \qquad \Rightarrow \qquad \frac{1}{x}s' - 1 \sim 0 \qquad \Rightarrow \qquad s' \sim x$$

This balance is not consistent as $x^2 \not\ll 1$ as $x \to \infty$.

The only dominant balance that makes sense leads to $s' \sim \pm 1$ as $x \to \infty$. Integrating this relationship,

$$\begin{aligned} s &\sim \pm x + c \\ &\sim \pm x \quad \text{as } x \to \infty. \end{aligned}$$

Now let's see if our assumption that we made to get the simplified equation for s is valid. Assuming that we can differentiate $s' \sim \pm 1$, $s'' \ll (s')^2$ becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[\pm 1 + \mathsf{o}(1)\right] &\ll \left[\pm 1 + \mathsf{o}(1)\right]^2 \\ 0 + \mathsf{o}(1/x) &\ll 1 \end{aligned}$$

Thus we see that the behavior we obtained for s is consistent with our initial assumption.

We have found two controlling factors, e^x and e^{-x}. This is a good sign as we know that there must be two linearly independent solutions to the equation.

Leading Order Behavior. Now let's find the full leading behavior of the solution with the controlling factor e^{-x}. In order to find a better approximation for s, we substitute $s(x) = -x + t(x)$, where $t(x) \ll x$ as $x \to \infty$, into the differential equation for s.

$$s'' + (s')^2 + \frac{1}{x}s' - \left(1 + \frac{\nu^2}{x^2}\right) = 0$$

$$t'' + (-1 + t')^2 + \frac{1}{x}(-1 + t') - \left(1 + \frac{\nu^2}{x^2}\right) = 0$$

$$t'' + (t')^2 + \left(\frac{1}{x} - 2\right)t' - \left(\frac{1}{x} + \frac{\nu^2}{x^2}\right) = 0$$

We know that $\frac{1}{x} \ll 2$ and $\frac{\nu^2}{x^2} \ll \frac{1}{x}$ as $x \to \infty$. Dropping these terms from the equation yields

$$t'' + (t')^2 - 2t' - \frac{1}{x} \sim 0.$$

Assuming that we can differentiate the asymptotic relation for t, we obtain $t' \ll 1$ and $t'' \ll \frac{1}{x}$ as $x \to \infty$. We can drop t''. Since t' vanishes as x goes to infinity, $(t')^2 \ll t'$. Thus we are left with

$$-2t' - \frac{1}{x} \sim 0, \quad \text{as } x \to \infty.$$

Integrating this relationship,

$$t \sim -\frac{1}{2}\log x + c$$
$$\sim -\frac{1}{2}\log x \quad \text{as } x \to \infty.$$

Checking our assumptions about the behavior of t,

$$t' \ll 1 \quad \Rightarrow \quad -\frac{1}{2x} \ll 1$$
$$t'' \ll \frac{1}{x} \quad \Rightarrow \quad \frac{1}{2x^2} \ll \frac{1}{x}$$

we see that the solution is consistent with our assumptions.

The leading order behavior to the solution with controlling factor e^{-x} is

$$y(x) \sim \exp\left(-x - \frac{1}{2}\log x + u(x)\right) = x^{-1/2}\,\mathrm{e}^{-x+u(x)} \quad \text{as } x \to \infty,$$

where $u(x) \ll \log x$. We substitute $t = -\frac{1}{2}\log x + u(x)$ into the differential equation for t in order to find the asymptotic behavior of u.

$$t'' + (t')^2 + \left(\frac{1}{x} - 2\right) t' - \left(\frac{1}{x} + \frac{\nu^2}{x^2}\right) = 0$$

$$\frac{1}{2x^2} + u'' + \left(-\frac{1}{2x} + u'\right)^2 + \left(\frac{1}{x} - 2\right)\left(-\frac{1}{2x} + u'\right) - \left(\frac{1}{x} + \frac{\nu^2}{x^2}\right) = 0$$

$$u'' + (u')^2 - 2u' + \frac{1}{4x^2} - \frac{\nu^2}{x^2} = 0$$

Assuming that we can differentiate the asymptotic relation for u, $u' \ll \frac{1}{x}$ and $u'' \ll \frac{1}{x^2}$ as $x \to \infty$. Thus we see that we can neglect the u'' and $(u')^2$ terms.

$$-2u' + \left(\frac{1}{4} - \nu^2\right)\frac{1}{x^2} \sim 0$$

$$u' \sim \frac{1}{2}\left(\frac{1}{4} - \nu^2\right)\frac{1}{x^2}$$

$$u \sim \frac{1}{2}\left(\nu^2 - \frac{1}{4}\right)\frac{1}{x} + c$$

$$u \sim c \quad \text{as } x \to \infty$$

Since $u = c + \mathsf{o}(1)$, we can expand e^{u} as $\mathrm{e}^{c} + \mathsf{o}(1)$. Thus we can write the leading order behavior as

$$y \sim x^{-1/2}\,\mathrm{e}^{-x}(\,\mathrm{e}^{c} + \mathsf{o}(1)).$$

Thus the full leading order behavior is

$$\boxed{y \sim (\text{const}) x^{-1/2}\, \mathrm{e}^{-x} \quad \text{as } x \to \infty.}$$

You can verify that the solution with the controlling factor e^{x} has the leading order behavior

$$y \sim (\text{const}) x^{-1/2}\, \mathrm{e}^{x} \quad \text{as } x \to \infty.$$

Two linearly independent solutions to the modified Bessel equation are the modified Bessel functions, $I_\nu(x)$ and $K_\nu(x)$. These functions have the asymptotic behavior

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}}\, \mathrm{e}^{x}, \qquad K_\nu(x) \sim \frac{\sqrt{\pi}}{\sqrt{2x}}\, \mathrm{e}^{-x} \quad \text{as } x \to \infty.$$

In Figure 26.1 $K_0(x)$ is plotted in a solid line and $\frac{\sqrt{\pi}}{\sqrt{2x}}\, \mathrm{e}^{-x}$ is plotted in a dashed line. We see that the leading order behavior of the solution as x goes to infinity gives a good approximation to the behavior even for fairly small values of x.

26.3 Integration by Parts

Example 26.3.1 The complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2}\, dt$$

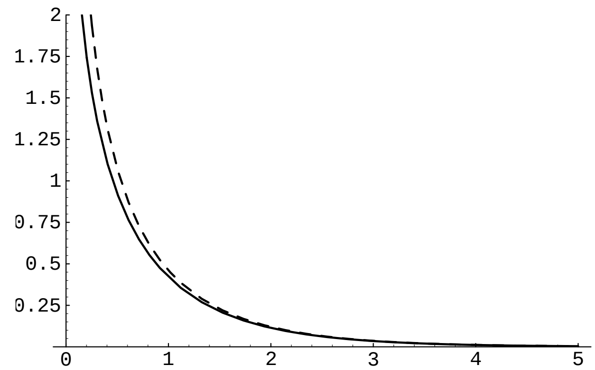

Figure 26.1: Plot of $K_0(x)$ and it's leading order behavior.

is used in statistics for its relation to the normal probability distribution. We would like to find an approximation to $\operatorname{erfc}(x)$ for large x. Using integration by parts,

$$
\begin{aligned}
\operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty \left(\frac{-1}{2t} \right) \left(-2t \, \mathrm{e}^{-t^2} \right) dt \\
&= \frac{2}{\sqrt{\pi}} \left[\frac{-1}{2t} \mathrm{e}^{-t^2} \right]_x^\infty - \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{1}{2} t^{-2} \, \mathrm{e}^{-t^2} \, dt \\
&= \frac{1}{\sqrt{\pi}} x^{-1} \, \mathrm{e}^{-x^2} - \frac{1}{\sqrt{\pi}} \int_x^\infty t^{-2} \, \mathrm{e}^{-t^2} \, dt.
\end{aligned}
$$

We examine the residual integral in this expression.

$$\begin{aligned}
\frac{1}{\sqrt{\pi}} \int_x^\infty t^{-2} \mathrm{e}^{-t^2} \, dt &\leq \frac{-1}{2\sqrt{\pi}} x^{-3} \int_x^\infty -2t \, \mathrm{e}^{-t^2} \, dt \\
&= \frac{1}{2\sqrt{\pi}} x^{-3} \, \mathrm{e}^{-x^2}.
\end{aligned}$$

Thus we see that

$$\frac{1}{\sqrt{\pi}} x^{-1} \mathrm{e}^{-x^2} \gg \frac{1}{\sqrt{\pi}} \int_x^\infty t^{-2} \mathrm{e}^{-t^2} \, dt \quad \text{as } x \to \infty.$$

Therefore,

$$\operatorname{erfc}(x) \sim \frac{1}{\sqrt{\pi}} x^{-1} \mathrm{e}^{-x^2} \quad \text{as } x \to \infty,$$

and we expect that $\frac{1}{\sqrt{\pi}} x^{-1} \mathrm{e}^{-x^2}$ would be a good approximation to $\operatorname{erfc}(x)$ for large x. In Figure 26.2 $\log(\operatorname{erfc}(x))$ is graphed in a solid line and $\log\left(\frac{1}{\sqrt{\pi}} x^{-1} \mathrm{e}^{-x^2}\right)$ is graphed in a dashed line. We see that this first approximation to the error function gives very good results even for moderate values of x. Table 26.1 gives the error in this first approximation for various values of x.

If we continue integrating by parts, we might get a better approximation to the complementary error function.

$$\begin{aligned}
\operatorname{erfc}(x) &= \frac{1}{\sqrt{\pi}} x^{-1} \mathrm{e}^{-x^2} - \frac{1}{\sqrt{\pi}} \int_x^\infty t^{-2} \mathrm{e}^{-t^2} \, dt \\
&= \frac{1}{\sqrt{\pi}} x^{-1} \mathrm{e}^{-x^2} - \frac{1}{\sqrt{\pi}} \left[-\frac{1}{2} t^{-3} \mathrm{e}^{-t^2} \right]_x^\infty + \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{3}{2} t^{-4} \mathrm{e}^{-t^2} \, dt \\
&= \frac{1}{\sqrt{\pi}} \mathrm{e}^{-x^2} \left(x^{-1} - \frac{1}{2} x^{-3} \right) + \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{3}{2} t^{-4} \mathrm{e}^{-t^2} \, dt \\
&= \frac{1}{\sqrt{\pi}} \mathrm{e}^{-x^2} \left(x^{-1} - \frac{1}{2} x^{-3} \right) + \frac{1}{\sqrt{\pi}} \left[-\frac{3}{4} t^{-5} \mathrm{e}^{-t^2} \right]_x^\infty - \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{15}{4} t^{-6} \mathrm{e}^{-t^2} \, dt \\
&= \frac{1}{\sqrt{\pi}} \mathrm{e}^{-x^2} \left(x^{-1} - \frac{1}{2} x^{-3} + \frac{3}{4} x^{-5} \right) - \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{15}{4} t^{-6} \mathrm{e}^{-t^2} \, dt
\end{aligned}$$

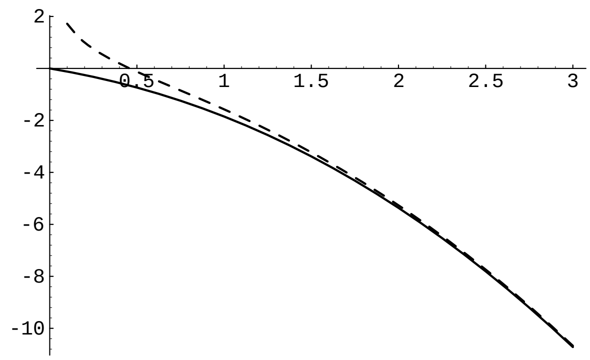

Figure 26.2: Logarithm of the Approximation to the Complementary Error Function.

The error in approximating $\operatorname{erfc}(x)$ with the first three terms is given in Table 26.1. We see that for $x \geq 2$ the three terms give a much better approximation to $\operatorname{erfc}(x)$ than just the first term.

At this point you might guess that you could continue this process indefinitely. By repeated application of integration by parts, you can obtain the series expansion

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \mathrm{e}^{-x^2} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{n!(2x)^{2n+1}}.$$

x	erfc (x)	One Term Relative Error	Three Term Relative Error
1	0.157	0.3203	0.6497
2	0.00468	0.1044	0.0182
3	2.21×10^{-5}	0.0507	0.0020
4	1.54×10^{-8}	0.0296	$3.9 \cdot 10^{-4}$
5	1.54×10^{-12}	0.0192	$1.1 \cdot 10^{-4}$
6	2.15×10^{-17}	0.0135	$3.7 \cdot 10^{-5}$
7	4.18×10^{-23}	0.0100	$1.5 \cdot 10^{-5}$
8	1.12×10^{-29}	0.0077	$6.9 \cdot 10^{-6}$
9	4.14×10^{-37}	0.0061	$3.4 \cdot 10^{-6}$
10	2.09×10^{-45}	0.0049	$1.8 \cdot 10^{-6}$

Table 26.1:

This is a Taylor expansion about infinity. Let's find the radius of convergence.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}(x)}{a_n(x)}\right| < 1 &\Rightarrow \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(2(n+1))!}{(n+1)!(2x)^{2(n+1)+1}}\frac{n!(2x)^{2n+1}}{(-1)^n(2n)!}\right| < 1 \\
&\Rightarrow \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)}{(n+1)(2x)^2}\right| < 1 \\
&\Rightarrow \lim_{n\to\infty}\left|\frac{2(2n+1)}{(2x)^2}\right| < 1 \\
&\Rightarrow \left|\frac{1}{x}\right| = 0
\end{aligned}$$

Thus we see that our series diverges for all x. Our conventional mathematical sense would tell us that this series is useless, however we will see that this series is very useful as an asymptotic expansion of $\operatorname{erfc}(x)$.

Say we are working with a convergent series expansion of some function $f(x)$.

$$f(x) = \sum_{n=0}^{\infty} a_n(x)$$

For fixed $x = x_0$,

$$f(x_0) - \sum_{n=0}^{N} a_n(x_0) \to 0 \quad \text{as } N \to \infty.$$

For an asymptotic series we have a quite different behavior. If $g(x)$ is asymptotic to $\sum_{n=0}^{\infty} b_n(x)$ as $x \to x_0$ then for fixed N,

$$g(x) - \sum_{0}^{N} b_n(x) \ll b_N(x) \quad \text{as } x \to x_0.$$

For the complementary error function,

$$\text{For fixed } N, \ \operatorname{erfc}(x) - \frac{2}{\sqrt{\pi}} \mathrm{e}^{-x^2} \sum_{n=0}^{N} \frac{(-1)^n (2n)!}{n!(2x)^{2n+1}} \ll x^{-2N-1} \quad \text{as } x \to \infty.$$

We say that the error function is asymptotic to the series as x goes to infinity.

$$\operatorname{erfc}(x) \sim \frac{2}{\sqrt{\pi}} \mathrm{e}^{-x^2} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{n!(2x)^{2n+1}} \quad \text{as } x \to \infty$$

In Figure 26.3 the logarithm of the difference between the one term, ten term and twenty term approximations and the complementary error function are graphed in coarse, medium, and fine dashed lines, respectively.

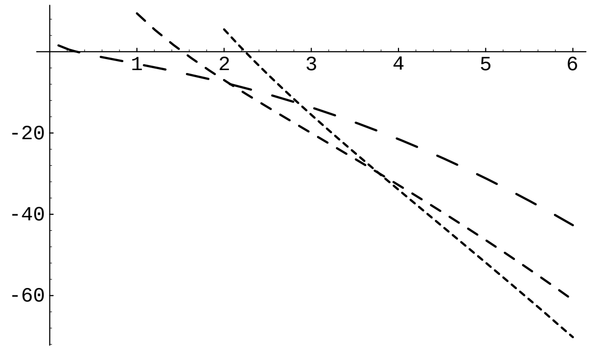

Figure 26.3: log(error in approximation)

***Optimal Asymptotic Series.** Of the three approximations, the one term is best for $x \lesssim 2$, the ten term is best for $2 \lesssim x \lesssim 4$, and the twenty term is best for $4 \lesssim x$. This leads us to the concept of an optimal asymptotic approximation. An optimal asymptotic approximation contains the number of terms in the series that best approximates the true behavior.

In Figure 26.4 we see a plot of the number of terms in the approximation versus the logarithm of the error for $x = 3$. Thus we see that the optimal asymptotic approximation is the first nine terms. After nine terms the error gets larger. It was inevitable that the error would start to grow after some point as the series diverges for all x.

A good rule of thumb for finding the optimal series is to find the smallest term in the series and take all of the terms up to but not including the smallest term as the optimal approximation. This makes sense, because the n^{th} term is an approximation of the error incurred by using the first $n-1$ terms. In Figure 26.5 there is a plot

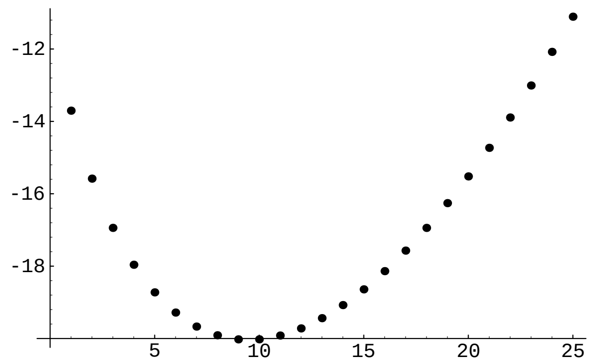

Figure 26.4: The logarithm of the error in using n terms.

of n versus the logarithm of the n^{th} term in the asymptotic expansion of $\operatorname{erfc}(3)$. We see that the tenth term is the smallest. Thus, in this case, our rule of thumb predicts the actual optimal series.

26.4 Asymptotic Series

A function $f(x)$ has an asymptotic series expansion about $x = x_0$, $\sum_{n=0}^{\infty} a_n(x)$, if

$$f(x) - \sum_{n=0}^{N} a_n(x) \ll a_N(x) \quad \text{as } x \to x_0 \quad \text{for all } N.$$

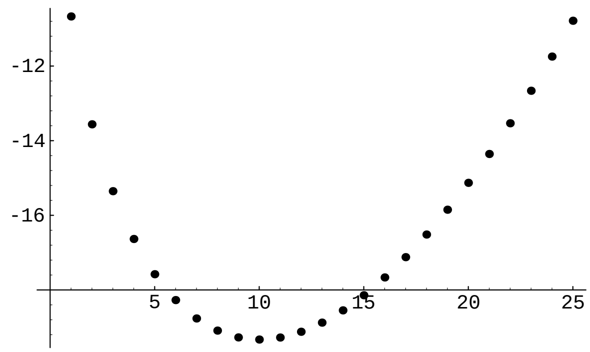

Figure 26.5: The logarithm of the n^{th} term in the expansion for $x = 3$.

An asymptotic series may be convergent or divergent. Most of the asymptotic series you encounter will be divergent. If the series is convergent, then we have that

$$f(x) - \sum_{n=0}^{N} a_n(x) \to 0 \quad \text{as } N \to \infty \quad \text{for fixed } x.$$

Let $\epsilon_n(x)$ be some set of gauge functions. The example that we are most familiar with is $\epsilon_n(x) = x^n$. If we say that

$$\sum_{n=0}^{\infty} a_n \epsilon_n(x) \sim \sum_{n=0}^{\infty} b_n \epsilon_n(x),$$

then this means that $a_n = b_n$.

26.5 Asymptotic Expansions of Differential Equations

26.5.1 The Parabolic Cylinder Equation.

Controlling Factor. Let us examine the behavior of the bounded solution of the parabolic cylinder equation as $x \to +\infty$.

$$y'' + \left(\nu + \frac{1}{2} - \frac{1}{4}x^2\right) y = 0$$

This equation has an irregular singular point at infinity. With the substitution $y = \mathrm{e}^s$, the equation becomes

$$s'' + (s')^2 + \nu + \frac{1}{2} - \frac{1}{4}x^2 = 0.$$

We know that

$$\nu + \frac{1}{2} \ll \frac{1}{4}x^2 \quad \text{as } x \to +\infty$$

so we drop this term from the equation. Let us make the assumption that

$$s'' \ll (s')^2 \quad \text{as } x \to +\infty.$$

Thus we are left with the equation

$$\begin{aligned}
(s')^2 &\sim \frac{1}{4}x^2 \\
s' &\sim \pm\frac{1}{2}x \\
s &\sim \pm\frac{1}{4}x^2 + c \\
s &\sim \pm\frac{1}{4}x^2 \quad \text{as } x \to +\infty
\end{aligned}$$

Now let's check if our assumption is consistent. Substituting into $s'' \ll (s')^2$ yields $1/2 \ll x^2/4 \quad \text{as } x \to +\infty$ which is true. Since the equation for y is second order, we would expect that there are two different behaviors as $x \to +\infty$. This is confirmed by the fact that we found two behaviors for s. $s \sim -x^2/4$ corresponds to the solution that is bounded at $+\infty$. Thus the controlling factor of the leading behavior is $\mathrm{e}^{-x^2/4}$.

Leading Order Behavior. Now we attempt to get a better approximation to s. We make the substitution $s = -\frac{1}{4}x^2 + t(x)$ into the equation for s where $t \ll x^2$ as $x \to +\infty$.

$$-\frac{1}{2} + t'' + \frac{1}{4}x^2 - xt' + (t')^2 + \nu + \frac{1}{2} - \frac{1}{4}x^2 = 0$$

$$t'' - xt' + (t')^2 + \nu = 0$$

Since $t \ll x^2$, we assume that $t' \ll x$ and $t'' \ll 1$ as $x \to +\infty$. Note that this in only an assumption since it is not always valid to differentiate an asymptotic relation. Thus $(t')^2 \ll xt'$ and $t'' \ll xt'$ as $x \to +\infty$; we drop these terms from the equation.

$$\begin{aligned} t' &\sim \frac{\nu}{x} \\ t &\sim \nu \log x + c \\ t &\sim \nu \log x \quad \text{as } x \to +\infty \end{aligned}$$

Checking our assumptions for the derivatives of t,

$$t' \ll x \quad \Rightarrow \quad \frac{1}{x} \ll x \qquad\qquad t'' \ll 1 \quad \Rightarrow \quad \frac{1}{x^2} \ll 1,$$

we see that they were consistent. Now we wish to refine our approximation for t with the substitution $t(x) = \nu \log x + u(x)$. So far we have that

$$y \sim \exp\left[-\frac{x^2}{4} + \nu \log x + u(x)\right] = x^\nu \exp\left[-\frac{x^2}{4} + u(x)\right] \quad \text{as } x \to +\infty.$$

We can try and determine $u(x)$ by substituting the expression $t(x) = \nu \log x + u(x)$ into the equation for t.

$$-\frac{\nu}{x^2} + u'' - (\nu + xu') + \frac{\nu^2}{x^2} + \frac{2\nu}{x}u' + (u')^2 + \nu = 0$$

After suitable simplification, this equation becomes

$$u' \sim \frac{\nu^2 - \nu}{x^3} \quad \text{as } x \to +\infty$$

Integrating this asymptotic relation,

$$u \sim \frac{\nu - \nu^2}{2x^2} + c \quad \text{as } x \to +\infty.$$

Notice that $\frac{\nu-\nu^2}{2x^2} \ll c$ as $x \to +\infty$; thus this procedure fails to give us the behavior of $u(x)$. Further refinements to our approximation for s go to a constant value as $x \to +\infty$. Thus we have that the leading behavior is

$$y \sim cx^\nu \exp\left[-\frac{x^2}{4}\right] \quad \text{as } x \to +\infty$$

Asymptotic Expansion Since we have factored off the singular behavior of y, we might expect that what is left over is well behaved enough to be expanded in a Taylor series about infinity. Let us assume that we can expand the solution for y in the form

$$y(x) \sim x^\nu \exp\left(-\frac{x^2}{4}\right)\sigma(x) = x^\nu \exp\left(-\frac{x^2}{4}\right)\sum_{n=0}^{\infty} a_n x^{-n} \text{as } x \to +\infty$$

where $a_0 = 1$. Differentiating $y = x^\nu \exp\left(-\frac{x^2}{4}\right)\sigma(x)$,

$$y' = \left[\nu x^{\nu-1} - \frac{1}{2}x^{\nu+1}\right] \mathrm{e}^{-x^2/4}\sigma(x) + x^\nu \,\mathrm{e}^{-x^2/4}\sigma'(x)$$

$$y'' = \left[\nu(\nu-1)x^{\nu-2} - \frac{1}{2}\nu x^{\nu} - \frac{1}{2}(\nu+1)x^{\nu} + \frac{1}{4}x^{\nu+2}\right] \mathrm{e}^{-x^2/4}\sigma(x) + 2\left[\nu x^{\nu-1} - \frac{1}{2}x^{\nu+1}\right] \mathrm{e}^{-x^2/4}\sigma'(x)$$
$$+ x^{\nu}\,\mathrm{e}^{-x^2/4}\sigma''(x).$$

Substituting this into the differential equation for y,

$$\left[\nu(\nu-1)x^{-2} - (\nu+\frac{1}{2}) + \frac{1}{4}x^2\right]\sigma(x) + 2\left[\nu x^{-1} - \frac{1}{2}x\right]\sigma'(x) + \sigma''(x) + \left[\nu + \frac{1}{2} - \frac{1}{4}x^2\right]\sigma(x) = 0$$
$$\sigma''(x) + (2\nu x^{-1} - x)\sigma'(x) + \nu(\nu-1)x^{-2}\sigma = 0$$
$$x^2\sigma''(x) + (2\nu x - x^3)\sigma'(x) + \nu(\nu-1)\sigma(x) = 0.$$

Differentiating the expression for $\sigma(x)$,

$$\sigma(x) = \sum_{n=0}^{\infty} a_n x^{-n}$$
$$\sigma'(x) = \sum_{n=1}^{\infty} -n a_n x^{-n-1} = \sum_{n=-1}^{\infty} -(n+2)a_{n+2}x^{-n-3}$$
$$\sigma''(x) = \sum_{n=1}^{\infty} n(n+1)a_n x^{-n-2}.$$

Substituting this into the differential equation for $\sigma(x)$,

$$\sum_{n=1}^{\infty} n(n+1)a_n x^{-n} + 2\nu\sum_{n=1}^{\infty} -n a_n x^{-n} - \sum_{n=-1}^{\infty} -(n+2)a_{n+2}x^{-n} + \nu(\nu-1)\sum_{n=0}^{\infty} a_n x^{-n} = 0.$$

Equating the coefficient of x^1 to zero yields

$$a_1 x = 0 \quad \Rightarrow \quad a_1 = 0.$$

Equating the coefficient of x^0,

$$2a_2 + \nu(\nu-1)a_0 = 0 \quad \Rightarrow \quad a_2 = -\frac{1}{2}\nu(\nu-1).$$

From the coefficient of x^{-n} for $n > 0$,

$$\begin{aligned} n(n+1)a_n - 2\nu n a_n + (n+2)a_{n+2} + \nu(\nu-1)a_n &= 0 \\ (n+2)a_{n+2} &= -[n(n+1) - 2\nu n + \nu(\nu-1)]a_n \\ (n+2)a_{n+2} &= -[n^2 + n - 2\nu n + \nu(\nu-1)]a_n \\ (n+2)a_{n+2} &= -(n-\nu)(n-\nu+1)a_n. \end{aligned}$$

Thus the recursion formula for the a_n's is

$$a_{n+2} = -\frac{(n-\nu)(n-\nu+1)}{n+2}a_n, \quad a_0 = 1, \quad a_1 = 0.$$

The first few terms in $\sigma(x)$ are

$$\sigma(x) \sim 1 - \frac{\nu(\nu-1)}{2^1 1!}x^{-2} + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2^2 2!}x^{-4} - \cdots \quad \text{as } x \to +\infty$$

If we check the radius of convergence of this series

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+2}x^{-n-2}}{a_n x^{-n}}\right| < 1 \quad &\Rightarrow \quad \lim_{n\to\infty}\left|-\frac{(n-\nu)(n-\nu+1)}{n+2}x^{-2}\right| < 1 \\ &\Rightarrow \quad \frac{1}{x} = 0 \end{aligned}$$

we see that the radius of convergence is zero. Thus if $\nu \neq 0, 1, 2, \ldots$ our asymptotic expansion for y

$$y \sim x^\nu \, \mathrm{e}^{-x^2/4}\left[1 - \frac{\nu(\nu-1)}{2^1 1!}x^{-2} + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2^2 2!}x^{-4} - \cdots\right]$$

diverges for all x. However this solution is still very useful. If we only use a finite number of terms, we will get a very good numerical approximation for large x.

In Figure 26.6 the one term, two term, and three term asymptotic approximations are shown in rough, medium, and fine dashing, respectively. The numerical solution is plotted in a solid line.

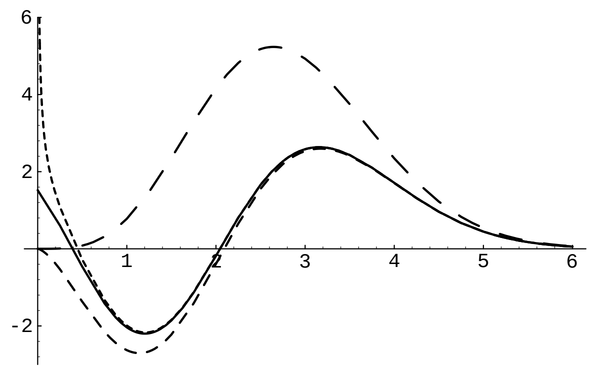

Figure 26.6: Asymptotic Approximations to the Parabolic Cylinder Function.

Chapter 27

Hilbert Spaces

An expert is a man who has made all the mistakes which can be made, in a narrow field.

- Niels Bohr

WARNING: UNDER HEAVY CONSTRUCTION.

In this chapter we will introduce Hilbert spaces. We develop the two important examples: l_2, the space of square summable infinite vectors and L_2, the space of square integrable functions.

27.1 Linear Spaces

A *linear space* is a set of elements $\{x, y, z, \dots\}$ that is closed under addition and scalar multiplication. By closed under addition we mean: if x and y are elements, then $z = x + y$ is an element. The addition is commutative and associative.

$$x + y = y + x$$
$$(x + y) + z = x + (y + z)$$

Scalar multiplication is associative and distributive. Let a and b be scalars, $a, b \in \mathbb{C}$.

$$(ab)x = a(bx)$$
$$(a + b)x = ax + bx$$
$$a(x + y) = ax + ay$$

All the linear spaces that we will work with have additional properties: The zero element 0 is the additive identity.

$$x + 0 = x$$

Multiplication by the scalar 1 is the multiplicative identity.

$$1x = x$$

Each element x and the additive inverse, $-x$.

$$x + (-x) = 0$$

Consider a set of elements $\{x_1, x_2, \ldots\}$. Let the c_i be scalars. If

$$y = c_1 x_1 + c_2 x_2 + \cdots$$

then y is a *linear combination* of the x_i. A set of elements $\{x_1, x_2, \ldots\}$ is *linearly independent* if the equation

$$c_1 x_1 + c_2 x_2 + \cdots = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = 0$. Otherwise the set is *linearly dependent.*

Let $\{e_1, e_2, \cdots\}$ be a linearly independent set of elements. If every element x can be written as a linear combination of the e_i then the set $\{e_i\}$ is a *basis* for the space. The e_i are called *base elements.*

$$x = \sum_i c_i e_i$$

The set $\{e_i\}$ is also called a *coordinate system.* The scalars c_i are the *coordinates* or *components* of x. If the set $\{e_i\}$ is a basis, then we say that the set is *complete.*

27.2 Inner Products

$\langle x|y\rangle$ is an *inner product* of two elements x and y if it satisfies the properties:

1. Conjugate-commutative.

$$\langle x|y\rangle = \overline{\langle x|y\rangle}$$

2. Linearity in the second argument.

$$\langle x|ay + bz\rangle = a\langle x|y\rangle + b\langle x|y\rangle$$

3. Positive definite.

$$\langle x|x\rangle \geq 0$$
$$\langle x|x\rangle = 0 \text{ if and only if } x = 0$$

From these properties one can derive the properties:

1. Conjugate linearity in the first argument.

$$\langle ax + by|z\rangle = \overline{a}\langle x|z\rangle + \overline{b}\langle x|z\rangle$$

2. Schwarz Inequality.

$$|\langle x|y\rangle|^2 \leq \langle x|x\rangle\langle y|y\rangle$$

One inner product of vectors is the *Euclidean inner product.*

$$\langle \mathbf{x}|\mathbf{y}\rangle \equiv \mathbf{x}\cdot\mathbf{y} = \sum_{i=0}^{n} \overline{x_i} y_i.$$

One inner product of functions defined on $(a \dots b)$ is

$$\langle u|v\rangle = \int_a^b \overline{u(x)} v(x) \, \mathrm{d}x.$$

If $\sigma(x)$ is a positive-valued function, then we can define the inner product:

$$\langle u|\sigma|v\rangle = \int_a^b \overline{u(x)} \sigma(x) v(x) \, \mathrm{d}x.$$

This is called the inner product with respect to the weighting function $\sigma(x)$. It is also denoted $\langle u|v\rangle_\sigma$.

27.3 Norms

A *norm* is a real-valued function on a space which satisfies the following properties.

1. Positive.

$$\|x\| \geq 0$$

2. Definite.

$$\|x\| = 0 \text{ if and only if } x = 0$$

3. Multiplication my a scalar, $c \in \mathbb{C}$.

$$\|cx\| = |c|\|x\|$$

4. Triangle inequality.

$$\|x + y\| \leq \|x\| + \|y\|$$

Example 27.3.1 Consider a vector space, (finite or infinite dimension), with elements $x = (x_1, x_2, x_3, \dots)$. Here are some common norms.

- Norm generated by the inner product.

$$\|x\| = \sqrt{\langle x|x\rangle}$$

- The l_p norm.

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$$

  There are three common cases of the l_p norm.

  - Euclidian norm, or l_2 norm.

$$\|x\|_2 = \sqrt{\sum_{k=1}^{\infty} |x_k|^2}$$

  - l_1 norm.

$$\|x\|_1 = \sum_{k=1}^{\infty} |x_k|$$

  - l_∞ norm.

$$\|x\|_\infty = \max_k |x_k|$$

Example 27.3.2 Consider a space of functions defined on the interval $(a \dots b)$. Here are some common norms.

- Norm generated by the inner product.

$$\|u\| = \sqrt{\langle u|u\rangle}$$

- The L_p norm.

$$\|u\|_p = \left(\int_a^b |u(x)|^p \, \mathrm{d}x \right)^{1/p}$$

There are three common cases of the L_p norm.

  – Euclidian norm, or L_2 norm.

$$\|u\|_2 = \sqrt{\int_a^b |u(x)|^2 \, \mathrm{d}x}$$

  – L_1 norm.

$$\|u\|_1 = \int_a^b |u(x)| \, \mathrm{d}x$$

  – L_∞ norm.

$$\|u\|_\infty = \limsup_{x \in (a \ldots b)} |u(x)|$$

Distance. Using the norm, we can define the distance between elements u and v.

$$d(u, v) \equiv \|u - v\|$$

Note that $d(u, v) = 0$ does not necessarily imply that $u = v$. CONTINUE.

27.4 Linear Independence.

27.5 Orthogonality

Orthogonality.

$$\langle \phi_j | \phi_k \rangle = 0 \text{ if } j \neq k$$

Orthonormality.

$$\langle \phi_j | \phi_k \rangle = \delta_{jk}$$

Example 27.5.1 Infinite vectors. e_j has all zeros except for a 1 in the j^{th} position.

$$e_j = (0, 0, \ldots 0, 1, 0, \ldots)$$

Example 27.5.2 L_2 functions on $(0 \ldots 2\pi)$.

$$\phi_j = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{ijx}, \quad j \in \mathbb{Z}$$

$$\phi_0 = \frac{1}{\sqrt{2\pi}}, \quad \phi_j^{(1)} = \frac{1}{\sqrt{\pi}} \cos(jx), \quad \phi_j^{(1)} = \frac{1}{\sqrt{\pi}} \sin(jx), \quad j \in \mathbb{Z}^+$$

27.6 Gramm-Schmidt Orthogonalization

Let $\{\psi_1(x), \ldots, \psi_n(x)\}$ be a set of linearly independent functions. Using the Gramm-Schmidt orthogonalization process we can construct a set of orthogonal functions $\{\phi_1(x), \ldots, \phi_n(x)\}$ that has the same span as the set of

ψ_n's with the formulas

$$\phi_1 = \psi_1$$
$$\phi_2 = \psi_2 - \frac{\langle \phi_1 | \psi_2 \rangle}{\| \phi_1 \|^2} \phi_1$$
$$\phi_3 = \psi_3 - \frac{\langle \phi_1 | \psi_3 \rangle}{\| \phi_1 \|^2} \phi_1 - \frac{\langle \phi_2 | \psi_3 \rangle}{\| \phi_2 \|^2} \phi_2$$
$$\cdots$$
$$\phi_n = \psi_n - \sum_{j=1}^{n-1} \frac{\langle \phi_j | \psi_n \rangle}{\| \phi_j \|^2} \phi_j.$$

You could verify that the ϕ_m are orthogonal with a proof by induction.

Example 27.6.1 Suppose we would like a polynomial approximation to $\cos(\pi x)$ in the domain $[-1, 1]$. One way to do this is to find the Taylor expansion of the function about $x = 0$. Up to terms of order x^4, this is

$$\cos(\pi x) = 1 - \frac{(\pi x)^2}{2} + \frac{(\pi x)^4}{24} + O(x^6).$$

In the first graph of Figure 27.1 $\cos(\pi x)$ and this fourth degree polynomial are plotted. We see that the approximation is very good near $x = 0$, but deteriorates as we move away from that point. This makes sense because the Taylor expansion only makes use of information about the function's behavior at the point $x = 0$.

As a second approach, we could find the least squares fit of a fourth degree polynomial to $\cos(\pi x)$. The set of functions $\{1, x, x^2, x^3, x^4\}$ is independent, but not orthogonal in the interval $[-1, 1]$. Using Gramm-Schmidt orthogonalization,

$$\begin{aligned}
\phi_0 &= 1 \\
\phi_1 &= x - \frac{\langle 1|x\rangle}{\langle 1|1\rangle} = x \\
\phi_2 &= x^2 - \frac{\langle 1|x^2\rangle}{\langle 1|1\rangle} - \frac{\langle x|x^2\rangle}{\langle x|x\rangle}x = x^2 - \frac{1}{3} \\
\phi_3 &= x^3 - \frac{3}{5}x \\
\phi_4 &= x^4 - \frac{6}{7}x^2 - \frac{3}{35}
\end{aligned}$$

A widely used set of functions in mathematics is the set of **Legendre polynomials** $\{P_0(x), P_1(x), \dots\}$. They differ from the ϕ_n's that we generated only by constant factors. The first few are

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{3x^2 - 1}{2} \\
P_3(x) &= \frac{5x^3 - 3x}{2} \\
P_4(x) &= \frac{35x^4 - 30x^2 + 3}{8}.
\end{aligned}$$

Expanding $\cos(\pi x)$ in Legendre polynomials

$$\cos(\pi x) \approx \sum_{n=0}^{4} c_n P_n(x),$$

and calculating the generalized Fourier coefficients with the formula

$$c_n = \frac{\langle P_n|\cos(\pi x)\rangle}{\langle P_n|P_n\rangle},$$

yields

$$\cos(\pi x) \approx -\frac{15}{\pi^2} P_2(x) + \frac{45(2\pi^2 - 21)}{\pi^4} P_4(x)$$
$$= \frac{105}{8\pi^4}[(315 - 30\pi^2)x^4 + (24\pi^2 - 270)x^2 + (27 - 2\pi^2)]$$

The cosine and this polynomial are plotted in the second graph in Figure 27.1. The least squares fit method uses information about the function on the entire interval. We see that the least squares fit does not give as good an approximation close to the point $x = 0$ as the Taylor expansion. However, the least squares fit gives a good approximation on the entire interval.

In order to expand a function in a Taylor series, the function must be analytic in some domain. One advantage of using the method of least squares is that the function being approximated does not even have to be continuous.

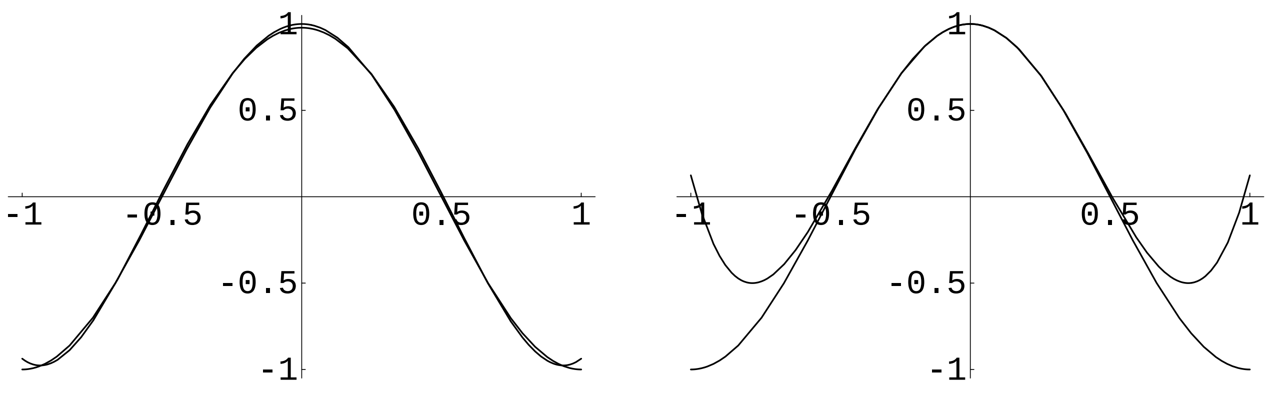

Figure 27.1: Polynomial Approximations to $\cos(\pi x)$.

27.7 Orthonormal Function Expansion

Let $\{\phi_j\}$ be an orthonormal set of functions on the interval (a, b). We expand a function $f(x)$ in the ϕ_j.

$$f(x) = \sum_j c_j \phi_j$$

We choose the coefficients to minimize the norm of the error.

$$\begin{aligned}
\left\| f - \sum_j c_j \phi_j \right\|^2 &= \left\langle f - \sum_j c_j \phi_j \middle| f - \sum_j c_j \phi_j \right\rangle \\
&= \|f\|^2 - \left\langle f \middle| \sum_j c_j \phi_j \right\rangle - \left\langle \sum_j c_j \phi_j \middle| f \right\rangle + \left\langle \sum_j c_j \phi_j \middle| \sum_j c_j \phi_j \right\rangle \\
&= \|f\|^2 + \sum_j |c_j|^2 - \sum_j c_j \langle f|\phi_j\rangle - \sum_j \overline{c_j} \langle \phi_j|f\rangle
\end{aligned}$$

$$\left\| f - \sum_j c_j \phi_j \right\|^2 = \|f\|^2 + \sum_j |c_j|^2 - \sum_j c_j \overline{\langle \phi_j|f\rangle} - \sum_j \overline{c_j} \langle \phi_j|f\rangle \tag{27.1}$$

To complete the square, we add the constant $\sum_j \langle \phi_j|f\rangle \overline{\langle \phi_j|f\rangle}$. We see the values of c_j which minimize

$$\|f\|^2 + \sum_j \left| c_j - \langle \phi_j|f\rangle \right|^2.$$

Clearly the unique minimum occurs for

$$c_j = \langle \phi_j|f\rangle.$$

We substitute this value for c_j into the right side of Equation 27.1 and note that this quantity, the squared norm of the error, is non-negative.

$$\|f\|^2 + \sum_j |c_j|^2 - \sum_j |c_j|^2 - \sum_j |c_j|^2 \geq 0$$
$$\|f\|^2 \geq \sum_j |c_j|^2$$

This is known as *Bessel's Inequality*. If the set of $\{\phi_j\}$ is complete then the norm of the error is zero and we obtain *Bessel's Equality*.

$$\|f\|^2 = \sum_j |c_j|^2$$

27.8 Sets Of Functions

Orthogonality. Consider two complex valued functions of a real variable $\phi_1(x)$ and $\phi_2(x)$ defined on the interval $a \leq x \leq b$. The inner product of the two functions is defined

$$\langle\phi_1|\phi_2\rangle = \int_a^b \overline{\phi_1}(x)\phi_2(x)\,\mathrm{d}x.$$

The two functions are orthogonal if $\langle\phi_1|\phi_2\rangle = 0$. The L_2 norm of a function is defined $\|\phi\| = \sqrt{\langle\phi|\phi\rangle}$.

Let $\{\phi_1, \phi_2, \phi_3, \dots\}$ be a set of complex valued functions. The set of functions is orthogonal if each pair of functions is orthogonal. That is,

$$\langle\phi_n|\phi_m\rangle = 0 \quad \text{if } n \neq m.$$

If in addition the norm of each function is 1, then the set is orthonormal. That is,

$$\langle\phi_n|\phi_m\rangle = \delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Example 27.8.1 The set of functions

$$\left\{ \sqrt{\frac{2}{\pi}} \sin(x), \sqrt{\frac{2}{\pi}} \sin(2x), \sqrt{\frac{2}{\pi}} \sin(3x), \dots \right\}$$

is orthonormal on the interval $[0, \pi]$. To verify this,

$$\begin{aligned} \left\langle \sqrt{\frac{2}{\pi}} \sin(nx) \middle| \sqrt{\frac{2}{\pi}} \sin(nx) \right\rangle &= \frac{2}{\pi} \int_0^\pi \sin^2(nx)\, \mathrm{d}x \\ &= 1 \end{aligned}$$

If $n \neq m$ then

$$\begin{aligned} \left\langle \sqrt{\frac{2}{\pi}} \sin(nx) \middle| \sqrt{\frac{2}{\pi}} \sin(mx) \right\rangle &= \frac{2}{\pi} \int_0^\pi \sin(nx) \sin(mx)\, \mathrm{d}x \\ &= \frac{1}{\pi} \int_0^\pi (\cos[(n-m)x] - \cos[(n+m)x])\, \mathrm{d}x \\ &= 0. \end{aligned}$$

Example 27.8.2 The set of functions

$$\{\dots, \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-ix}, \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} \mathrm{e}^{ix}, \frac{1}{\sqrt{2\pi}} \mathrm{e}^{2ix}, \dots\},$$

is orthonormal on the interval $[-\pi, \pi]$. To verify this,

$$\begin{aligned} \left\langle \frac{1}{\sqrt{2\pi}} \mathrm{e}^{inx} \middle| \frac{1}{\sqrt{2\pi}} \mathrm{e}^{inx} \right\rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-inx} \mathrm{e}^{inx}\, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}x \\ &= 1. \end{aligned}$$

If $n \neq m$ then

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{2\pi}} \mathrm{e}^{inx} \middle| \frac{1}{\sqrt{2\pi}} \mathrm{e}^{imx} \right\rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-inx} \, \mathrm{e}^{imx} \, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{i(m-n)x} \, \mathrm{d}x \\
&= 0.
\end{aligned}$$

Orthogonal with Respect to a Weighting Function. Let $\sigma(x)$ be a real-valued, positive function on the interval $[a, b]$. We introduce the notation

$$\langle \phi_n | \sigma | \phi_m \rangle \equiv \int_a^b \overline{\phi_n} \sigma \phi_m \, dx.$$

If the set of functions $\{\phi_1, \phi_2, \phi_3, \ldots\}$ satisfy

$$\langle \phi_n | \sigma | \phi_m \rangle = 0 \quad \text{if } n \neq m$$

then the functions are orthogonal with respect to the weighting function $\sigma(x)$.

If the functions satisfy

$$\langle \phi_n | \sigma | \phi_m \rangle = \delta_{nm}$$

then the set is orthonormal with respect to $\sigma(x)$.

Example 27.8.3 We know that the set of functions

$$\left\{ \sqrt{\frac{2}{\pi}} \sin(x), \sqrt{\frac{2}{\pi}} \sin(2x), \sqrt{\frac{2}{\pi}} \sin(3x), \ldots \right\}$$

is orthonormal on the interval $[0, \pi]$. That is,

$$\int_0^\pi \sqrt{\frac{2}{\pi}} \sin(nx) \sqrt{\frac{2}{\pi}} \sin(mx) \, \mathrm{d}x = \delta_{nm}.$$

If we make the change of variables $x = \sqrt{t}$ in this integral, we obtain

$$\int_0^{\pi^2} \frac{1}{2\sqrt{t}} \sqrt{\frac{2}{\pi}} \sin(n\sqrt{t}) \sqrt{\frac{2}{\pi}} \sin(m\sqrt{t}) \, dt = \delta_{nm}.$$

Thus the set of functions

$$\left\{ \sqrt{\frac{1}{\pi}} \sin(\sqrt{t}), \sqrt{\frac{1}{\pi}} \sin(2\sqrt{t}), \sqrt{\frac{1}{\pi}} \sin(3\sqrt{t}), \ldots \right\}$$

is orthonormal with respect to $\sigma(t) = \frac{1}{2\sqrt{t}}$ on the interval $[0, \pi^2]$.

Orthogonal Series. Suppose that a function $f(x)$ defined on $[a, b]$ can be written as a uniformly convergent sum of functions that are orthogonal with respect to $\sigma(x)$.

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

We can solve for the c_n by taking the inner product of $\phi_m(x)$ and each side of the equation with respect to $\sigma(x)$.

$$\langle \phi_m | \sigma | f \rangle = \left\langle \phi_m \middle| \sigma \middle| \sum_{n=1}^{\infty} c_n \phi_n \right\rangle$$

$$\langle \phi_m | \sigma | f \rangle = \sum_{n=1}^{\infty} c_n \langle \phi_m | \sigma | \phi_n \rangle$$

$$\langle \phi_m | \sigma | f \rangle = c_m \langle \phi_m | \sigma | \phi_m \rangle$$

$$c_m = \frac{\langle \phi_m | \sigma | f \rangle}{\langle \phi_m | \sigma | \phi_m \rangle}$$

The c_m are known as **Generalized Fourier coefficients**. If the functions in the expansion are orthonormal, the formula simplifies to

$$c_m = \langle \phi_m | \sigma | f \rangle.$$

Example 27.8.4 The function $f(x) = x(\pi - x)$ has a uniformly convergent series expansion in the domain $[0, \pi]$ of the form

$$x(\pi - x) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{\pi}} \sin(nx).$$

The Fourier coefficients are

$$\begin{aligned}
c_n &= \left\langle \sqrt{\frac{2}{\pi}} \sin(nx) \middle| x(\pi - x) \right\rangle \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\pi} x(\pi - x) \sin(nx) \, \mathrm{d}x \\
&= \sqrt{\frac{2}{\pi}} \frac{2}{n^3} (1 - (-1)^n) \\
&= \begin{cases} \sqrt{\frac{2}{\pi}} \frac{4}{n^3} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}
\end{aligned}$$

Thus the expansion is

$$x(\pi - x) = \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \frac{8}{\pi n^3} \sin(nx) \quad \text{for } x \in [0, \pi].$$

In the first graph of Figure 27.2 the first term in the expansion is plotted in a dashed line and $x(\pi - x)$ is plotted in a solid line. The second graph shows the two term approximation.

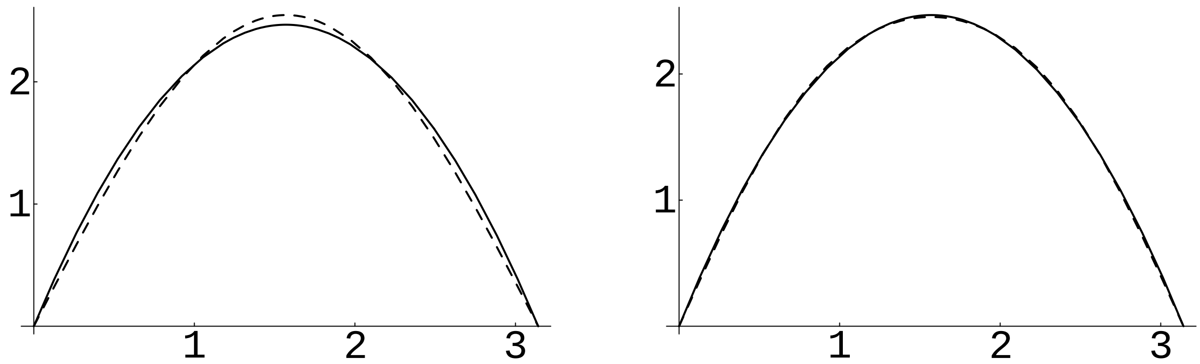

Figure 27.2: Series Expansions of $x(\pi - x)$.

Example 27.8.5 The set $\{\dots, 1/\sqrt{2\pi}\,\mathrm{e}^{-ix}, 1/\sqrt{2\pi}, 1/\sqrt{2\pi}\,\mathrm{e}^{ix}, 1/\sqrt{2\pi}\,\mathrm{e}^{2ix}, \dots\}$ is orthonormal on the interval $[-\pi, \pi]$. $f(x) = \operatorname{sign}(x)$ has the expansion

$$\begin{aligned}
\operatorname{sign}(x) &\sim \sum_{n=-\infty}^{\infty} \left\langle \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{in\xi} \middle| \operatorname{sign}(\xi) \right\rangle \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{inx} \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \mathrm{e}^{-in\xi} \operatorname{sign}(\xi)\,\mathrm{d}\xi\,\mathrm{e}^{inx} \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-\pi}^{0} -\,\mathrm{e}^{-in\xi}\,\mathrm{d}\xi + \int_{0}^{\pi} \mathrm{e}^{-in\xi}\,\mathrm{d}\xi \right) \mathrm{e}^{inx} \\
&= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1-(-1)^n}{in}\,\mathrm{e}^{inx}.
\end{aligned}$$

In terms of real functions, this is

$$= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1-(-1)^n}{in} (\cos(nx) + i \sin(nx))$$
$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{in} \sin(nx)$$

$$\boxed{\operatorname{sign}(x) \sim \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \frac{1}{n} \sin(nx).}$$

27.9 Least Squares Fit to a Function and Completeness

Let $\{\phi_1, \phi_2, \phi_3, \dots\}$ be a set of real, square integrable functions that are orthonormal with respect to the weighting function $\sigma(x)$ on the interval $[a, b]$. That is,

$$\langle \phi_n | \sigma | \phi_m \rangle = \delta_{nm}.$$

Let $f(x)$ be some square integrable function defined on the same interval. We would like to approximate the function $f(x)$ with a finite orthonormal series.

$$f(x) \approx \sum_{n=1}^{N} \alpha_n \phi_n(x)$$

$f(x)$ may or may not have a uniformly convergent expansion in the orthonormal functions.

We would like to choose the α_n so that we get the best possible approximation to $f(x)$. The most common measure of how well a series approximates a function is the least squares measure. The error is defined as the integral of the weighting function times the square of the deviation.

$$E = \int_a^b \sigma(x) \left(f(x) - \sum_{n=1}^{N} \alpha_n \phi_n(x) \right)^2 \, \mathrm{d}x$$

The "best" fit is found by choosing the α_n that minimize E. Let c_n be the Fourier coefficients of $f(x)$.

$$c_n = \langle \phi_n | \sigma | f \rangle$$

we expand the integral for E.

$$\begin{aligned}
E(\alpha) &= \int_a^b \sigma(x) \left(f(x) - \sum_{n=1}^{N} \alpha_n \phi_n(x) \right)^2 \, \mathrm{d}x \\
&= \left\langle f - \sum_{n=1}^{N} \alpha_n \phi_n \,\middle|\, \sigma \,\middle|\, f - \sum_{n=1}^{N} \alpha_n \phi_n \right\rangle \\
&= \langle f | \sigma | f \rangle - 2 \left\langle \sum_{n=1}^{N} \alpha_n \phi_n \middle| \sigma \middle| f \right\rangle + \left\langle \sum_{n=1}^{N} \alpha_n \phi_n \middle| \sigma \middle| \sum_{n=1}^{N} \alpha_n \phi_n \right\rangle \\
&= \langle f | \sigma | f \rangle - 2 \sum_{n=1}^{N} \alpha_n \langle \phi_n | \sigma | f \rangle + \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \alpha_m \langle \phi_n | \sigma | \phi_m \rangle \\
&= \langle f | \sigma | f \rangle - 2 \sum_{n=1}^{N} \alpha_n c_n + \sum_{n=1}^{N} \alpha_n^2 \\
&= \langle f | \sigma | f \rangle + \sum_{n=1}^{N} (\alpha_n - c_n)^2 - \sum_{n=1}^{N} c_n^2
\end{aligned}$$

Each term involving α_n in non-negative and is minimized for $\alpha_n = c_n$. The Fourier coefficients give the least squares approximation to a function. The least squares fit to $f(x)$ is thus

$$f(x) \approx \sum_{n=1}^{N} \langle \phi_n | \sigma | f \rangle \, \phi_n(x).$$

Result 27.9.1 If $\{\phi_1, \phi_2, \phi_3, \dots\}$ is a set of real, square integrable functions that are orthogonal with respect to $\sigma(x)$ then the least squares fit of the first N orthogonal functions to the square integrable function $f(x)$ is

$$f(x) \approx \sum_{n=1}^{N} \frac{\langle \phi_n | \sigma | f \rangle}{\langle \phi_n | \sigma | \phi_n \rangle} \phi_n(x).$$

If the set is orthonormal, this formula reduces to

$$f(x) \approx \sum_{n=1}^{N} \langle \phi_n | \sigma | f \rangle \, \phi_n(x).$$

Since the error in the approximation E is a nonnegative number we can obtain on inequality on the sum of the squared coefficients.

$$E = \langle f | \sigma | f \rangle - \sum_{n=1}^{N} c_n^2$$

$$\boxed{\sum_{n=1}^{N} c_n^2 \leq \langle f | \sigma | f \rangle}$$

This equation is known as **Bessel's Inequality**. Since $\langle f | \sigma | f \rangle$ is just a nonnegative number, independent of N, the sum $\sum_{n=1}^{\infty} c_n^2$ is convergent and $c_n \to 0$ as $n \to \infty$

Convergence in the Mean. If the error E goes to zero as N tends to infinity

$$\lim_{N \to \infty} \int_a^b \sigma(x) \left(f(x) - \sum_{n=1}^{N} c_n \phi_n(x) \right)^2 \, dx = 0,$$

then the sum converges in the mean to $f(x)$ relative to the weighting function $\sigma(x)$. This implies that

$$\lim_{N\to\infty}\left(\langle f|\sigma|f\rangle - \sum_{n=1}^{N} c_n^2\right) = 0$$

$$\boxed{\sum_{n=1}^{\infty} c_n^2 = \langle f|\sigma|f\rangle.}$$

This is known as **Parseval's identity**.

Completeness. Consider a set of functions $\{\phi_1, \phi_2, \phi_3, \ldots\}$ that is orthogonal with respect to the weighting function $\sigma(x)$. If every function $f(x)$ that is square integrable with respect to $\sigma(x)$ has an orthogonal series expansion

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$$

that converges in the mean to $f(x)$, then the set is **complete**.

27.10 Closure Relation

Let $\{\phi_1, \phi_2, \ldots\}$ be an orthonormal, complete set on the domain $[a, b]$. For any square integrable function $f(x)$ we can write

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Here the c_n are the generalized Fourier coefficients and the sum converges in the mean to $f(x)$. Substituting the expression for the Fourier coefficients into the sum yields

$$\begin{aligned} f(x) &\sim \sum_{n=1}^{\infty} \langle \phi_n | f \rangle \phi_n(x) \\ &= \sum_{n=1}^{\infty} \left(\int_a^b \overline{\phi_n(\xi)} f(\xi) \, \mathrm{d}\xi \right) \phi_n(x). \end{aligned}$$

Since the sum is not necessarily uniformly convergent, we are not justified in exchanging the order of summation and integration ... but what the heck, let's do it anyway.

$$\begin{aligned} &= \int_a^b \left(\sum_{n=1}^{\infty} \overline{\phi_n(\xi)} f(\xi) \phi_n(x) \right) \mathrm{d}\xi \\ &= \int_a^b \left(\sum_{n=1}^{\infty} \overline{\phi_n(\xi)} \phi_n(x) \right) f(\xi) \, \mathrm{d}\xi \end{aligned}$$

The sum behaves like a Dirac delta function. Recall that $\delta(x - \xi)$ satisfies the equation

$$f(x) = \int_a^b \delta(x - \xi) f(\xi) \, \mathrm{d}\xi \quad \text{for } x \in (a, b).$$

Thus we could say that the sum is a representation of $\delta(x - \xi)$. Note that a series representation of the delta function could not be convergent, hence the necessity of throwing caution to the wind when we interchanged the summation and integration in deriving the series. The **closure relation** for an orthonormal, complete set states

$$\sum_{n=1}^{\infty} \phi_n(x) \overline{\phi_n(\xi)} \sim \delta(x - \xi).$$

Alternatively, you can derive the closure relation by computing the generalized Fourier coefficients of the delta function.

$$\delta(x-\xi) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$\begin{aligned} c_n &= \langle \phi_n | \delta(x-\xi) \rangle \\ &= \int_a^b \overline{\phi_n(x)} \delta(x-\xi) \, \mathrm{d}x \\ &= \overline{\phi_n(\xi)} \end{aligned}$$

$$\delta(x-\xi) \sim \sum_{n=1}^{\infty} \phi_n(x) \overline{\phi_n(\xi)}$$

Result 27.10.1 If $\{\phi_1, \phi_2, \ldots\}$ is an orthogonal, complete set on the domain $[a, b]$, then

$$\sum_{n=1}^{\infty} \frac{\phi_n(x)\overline{\phi_n(\xi)}}{\|\phi_n\|^2} \sim \delta(x-\xi).$$

If the set is orthonormal, then

$$\sum_{n=1}^{\infty} \phi_n(x)\overline{\phi_n(\xi)} \sim \delta(x-\xi).$$

Example 27.10.1 The integral of the Dirac delta function is the Heaviside function. On the interval $x \in (-\pi, \pi)$

$$\int_{-\pi}^{x} \delta(t) \, \mathrm{d}t = H(x) = \begin{cases} 1 & \text{for } 0 < x < \pi \\ 0 & \text{for } -\pi < x < 0. \end{cases}$$

Consider the orthonormal, complete set $\{\dots, \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-ix}, \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} \mathrm{e}^{ix}, \dots\}$ on the domain $[-\pi, \pi]$. The delta function has the series

$$\delta(t) \sim \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{int} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-in0} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \mathrm{e}^{int}.$$

We will find the series expansion of the Heaviside function first by expanding directly and then by integrating the expansion for the delta function.

Finding the series expansion of H(x) directly. The generalized Fourier coefficients of $H(x)$ are

$$\begin{aligned}
c_0 &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} H(x) \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\pi} \mathrm{d}x \\
&= \sqrt{\frac{\pi}{2}} \\
c_n &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-inx} H(x) \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\pi} \mathrm{e}^{-inx} \,\mathrm{d}x \\
&= \frac{1 - (-1)^n}{in\sqrt{2\pi}}.
\end{aligned}$$

Thus the Heaviside function has the expansion

$$H(x) \sim \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi}} + \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} \frac{1-(-1)^n}{in\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{inx}$$
$$= \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n}\sin(nx)$$

$$\boxed{H(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{1}{n}\sin(nx).}$$

Integrating the series for $\boldsymbol{\delta}(\mathbf{t})$.

$$\int_{-\pi}^{x} \delta(t)\,\mathrm{d}t \sim \frac{1}{2\pi}\int_{-\pi}^{x}\sum_{n=-\infty}^{\infty} \mathrm{e}^{int}\,\mathrm{d}t$$
$$= \frac{1}{2\pi}\left((x+\pi) + \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left[\frac{1}{in}\,\mathrm{e}^{int}\right]_{-\pi}^{x}\right)$$
$$= \frac{1}{2\pi}\left((x+\pi) + \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{1}{in}\left(\mathrm{e}^{inx} - (-1)^n\right)\right)$$
$$= \frac{x}{2\pi} + \frac{1}{2} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{in}\left(\mathrm{e}^{inx} - \mathrm{e}^{-inx} - (-1)^n + (-1)^n\right)$$
$$= \frac{x}{2\pi} + \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(nx)$$

Expanding $\frac{x}{2\pi}$ in the orthonormal set,

$$\frac{x}{2\pi} \sim \sum_{n=-\infty}^{\infty} c_n \frac{1}{\sqrt{2\pi}} \mathrm{e}^{inx}.$$

$$c_0 = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \frac{x}{2\pi} \, \mathrm{d}x = 0$$
$$c_n = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-inx} \frac{x}{2\pi} \, \mathrm{d}x = \frac{i(-1)^n}{n\sqrt{2\pi}}$$

$$\frac{x}{2\pi} \sim \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i(-1)^n}{n\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{inx} = -\frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \sin(nx)$$

Substituting the series for $\frac{x}{2\pi}$ into the expression for the integral of the delta function,

$$\int_{-\pi}^{x} \delta(t) \, \mathrm{d}t \sim \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin(nx)$$

$$\boxed{\int_{-\pi}^{x} \delta(t) \, \mathrm{d}t \sim \frac{1}{2} + \frac{2}{\pi} \sum_{\substack{n=1 \\ \mathrm{odd}\, n}}^{\infty} \frac{1}{n} \sin(nx).}$$

Thus we see that the series expansions of the Heaviside function and the integral of the delta function are the same.

27.11 Linear Operators

27.12 Exercises

Exercise 27.1

1. Suppose $\{\phi_k(x)\}_{k=0}^{\infty}$ is an orthogonal system on $[a, b]$. Show that any finite set of the $\phi_j(x)$ is a linearly independent set on $[a, b]$. That is, if $\{\phi_{j_1}(x), \phi_{j_2}(x), \dots, \phi_{j_n}(x)\}$ is the set and all the j_ν are distinct, then

$$a_1\phi_{j_1}(x) + a_2\phi_{j_2}(x) + \cdots + a_n\phi_{j_n}(x) = 0 \quad \text{on} \quad a \leq x \leq b$$

   is true iff: $a_1 = a_2 = \cdots = a_n = 0$.

2. Show that the complex functions $\phi_k(x) \equiv \mathrm{e}^{ik\pi x/L}$, $k = 0, 1, 2, \dots$ are orthogonal in the sense that $\int_{-L}^{L} \phi_k(x)\phi_n^*(x)$ 0, for $n \neq k$. Here $\phi_n^*(x)$ is the complex conjugate of $\phi_n(x)$.

Hint, Solution

27.13 Hints

Hint 27.1

27.14 Solutions

Solution 27.1

1.

$$a_1\phi_{j_1}(x) + a_2\phi_{j_2}(x) + \cdots + a_n\phi_{j_n}(x) = 0$$
$$\sum_{k=1}^{n} a_k\phi_{j_k}(x) = 0$$

We take the inner product with ϕ_{j_ν} for any $\nu = 1, \ldots, n$. ($\langle \phi, \psi \rangle \equiv \int_a^b \phi(x)\psi^*(x)\,\mathrm{d}x$.)

$$\left\langle \sum_{k=1}^{n} a_k\phi_{j_k}, \phi_{j_\nu} \right\rangle = 0$$

We interchange the order of summation and integration.

$$\sum_{k=1}^{n} a_k \langle \phi_{j_k}, \phi_{j_\nu} \rangle = 0$$

$\langle \phi_{j_k}\phi_{j_\nu} \rangle = 0$ for $j \neq \nu$.

$$a_\nu \langle \phi_{j_\nu}\phi_{j_\nu} \rangle = 0$$

$\langle \phi_{j_\nu}\phi_{j_\nu} \rangle \neq 0$.

$$a_\nu = 0$$

Thus we see that $a_1 = a_2 = \cdots = a_n = 0$.

2. For $k \neq n$, $\langle \phi_k, \phi_n \rangle = 0$.

$$\begin{aligned}
\langle \phi_k, \phi_n \rangle &\equiv \int_{-L}^{L} \phi_k(x)\phi_n^*(x)\,\mathrm{d}x \\
&= \int_{-L}^{L} \mathrm{e}^{ik\pi x/L}\,\mathrm{e}^{-in\pi x/L}\,\mathrm{d}x \\
&= \int_{-L}^{L} \mathrm{e}^{i(k-n)\pi x/L}\,\mathrm{d}x \\
&= \left[\frac{\mathrm{e}^{i(k-n)\pi x/L}}{i(k-n)\pi/L}\right]_{-L}^{L} \\
&= \frac{\mathrm{e}^{i(k-n)\pi} - \mathrm{e}^{-i(k-n)\pi}}{i(k-n)\pi/L} \\
&= \frac{2L\sin((k-n)\pi)}{(k-n)\pi} \\
&= 0
\end{aligned}$$

Chapter 28

Self Adjoint Linear Operators

28.1 Adjoint Operators

The *adjoint* of an operator, L^*, satisfies

$$\langle v|Lu\rangle - \langle L^* v|u\rangle = 0$$

for all elements u an v. This is known as *Green's Identity*.

The adjoint of a matrix. For vectors, one can represent linear operators L with matrix multiplication.

$$L\mathbf{x} \equiv \mathbf{A}\mathbf{x}$$

Let $\mathbf{B} = \mathbf{A}^*$ be the adjoint of the matrix $\mathbf{A}$. We determine the adjoint of $\mathbf{A}$ from Green's Identity.

$$\langle \mathbf{x} | \mathbf{A}\mathbf{y} \rangle - \langle \mathbf{B}\mathbf{x} | \mathbf{y} \rangle = 0$$
$$\overline{\mathbf{x}} \cdot \mathbf{A}\mathbf{y} = \overline{\mathbf{B}\mathbf{x}} \cdot \mathbf{y}$$
$$\overline{\mathbf{x}}^T \mathbf{A}\mathbf{y} = \overline{\mathbf{B}\mathbf{x}}^T \mathbf{y}$$
$$\overline{\mathbf{x}}^T \mathbf{A}\mathbf{y} = \overline{\mathbf{x}}^T \overline{\mathbf{B}}^T \mathbf{y}$$
$$\overline{\mathbf{y}}^T \overline{\mathbf{A}}^T \mathbf{x} = \overline{\mathbf{y}}^T \mathbf{B}\mathbf{x}\mathbf{B} = \overline{\mathbf{A}}^T$$

Thus we see that the adjoint of a matrix is the *conjugate transpose* of the matrix, $\mathbf{A}^* = \overline{\mathbf{A}}^T$. The conjugate transpose is also called the *Hermitian transpose* and is denoted $\mathbf{A}^H$.

The adjoint of a differential operator. Consider a second order linear differential operator acting on C^2 functions defined on $(a \dots b)$ which satisfy certain boundary conditions.

$$Lu \equiv p_2(x)u'' + p_1(x)u' + p_0(x)u$$

28.2 Self-Adjoint Operators

Matrices. A matrix is self-adjoint if it is equal to its conjugate transpose $\mathbf{A} = \mathbf{A}^H \equiv \overline{\mathbf{A}}^T$. Such matrices are called *Hermitian*. For a Hermitian matrix $\mathbf{H}$, Green's identity is

$$\langle \mathbf{y} | \mathbf{H}\mathbf{x} \rangle = \langle \mathbf{H}\mathbf{y} | \mathbf{x} \rangle$$
$$\overline{\mathbf{y}} \cdot \mathbf{H}\mathbf{x} = \overline{\mathbf{H}\mathbf{y}} \cdot \mathbf{x}$$

The eigenvalues of a Hermitian matrix are real. Let $\mathbf{x}$ be an eigenvector with eigenvalue λ.

$$\begin{gathered}
\langle \mathbf{x} | \mathbf{H}\mathbf{x} \rangle = \langle \mathbf{H}\mathbf{x} | \mathbf{x} \rangle \\
\langle \mathbf{x} | \lambda \mathbf{x} \rangle - \langle \lambda \mathbf{x} | \mathbf{x} \rangle = 0 \\
(\lambda - \overline{\lambda}) \langle \mathbf{x} | \mathbf{x} \rangle = 0 \\
\lambda = \overline{\lambda}
\end{gathered}$$

The eigenvectors corresponding to distinct eigenvalues are distinct. Let $\mathbf{x}$ and $\mathbf{y}$ be eigenvectors with distinct eigenvalues λ and μ.

$$\begin{gathered}
\langle \mathbf{y} | \mathbf{H}\mathbf{x} \rangle = \langle \mathbf{H}\mathbf{y} | \mathbf{x} \rangle \\
\langle \mathbf{y} | \lambda \mathbf{x} \rangle - \langle \mu \mathbf{y} | \mathbf{x} \rangle = 0 \\
(\lambda - \overline{\mu}) \langle \mathbf{y} | \lambda \mathbf{x} \rangle = 0 \\
(\lambda - \mu) \langle \mathbf{y} | \mathbf{x} \rangle = 0 \\
\langle \mathbf{y} | \mathbf{x} \rangle = 0
\end{gathered}$$

Furthermore, all Hermitian matrices are similar to a diagonal matrix and have a complete set of orthogonal eigenvectors.

Trigonometric Series. Consider the problem

$$-y'' = \lambda y, \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi).$$

We verify that the differential operator $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with periodic boundary conditions is self-adjoint.

$$\begin{aligned}
\langle v|Lu\rangle &= \langle v| - u''\rangle \\
&= [-\overline{v}u']_0^{2\pi} - \langle v'| - u'\rangle \\
&= \langle v'|u'\rangle \\
&= \left[\overline{v'}u\right]_0^{2\pi} - \langle v''|u\rangle \\
&= \langle -v''|u\rangle \\
&= \langle Lv|u\rangle
\end{aligned}$$

The eigenvalues and eigenfunctions of this problem are

$$\lambda_0 = 0, \quad \phi_0 = 1$$
$$\lambda_n = n^2, \quad \phi_n^{(1)} = \cos(nx), \quad \phi_n^{(2)} = \sin(nx), \quad n \in \mathbb{Z}^+$$

28.3 Exercises

28.4 Hints

28.5 Solutions

Chapter 29

Self-Adjoint Boundary Value Problems

Seize the day and throttle it.

-Calvin

29.1 Summary of Adjoint Operators

The adjoint of the operator

$$L[y] = p_n \frac{d^n y}{dx^n} + p_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_0 y,$$

is defined

$$L^*[y] = (-1)^n \frac{d^n}{dx^n}(\overline{p_n} y) + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(\overline{p_{n-1}} y) + \cdots + \overline{p_0} y$$

If each of the p_k is k times continuously differentiable and u and v are n times continuously differentiable on some interval, then on that interval Lagrange's identity states

$$\overline{v} L[u] - u \overline{L^*[v]} = \frac{\mathrm{d}}{\mathrm{d}x} B[u, v]$$

where $B[u,v]$ is the bilinear form

$$B[u,v] = \sum_{m=1}^{n} \sum_{\substack{j+k=m-1 \\ j \geq 0, k \geq 0}} (-1)^j u^{(k)} (p_m \overline{v})^{(j)}.$$

If L is a second order operator then

$$\overline{v} L[u] - u \overline{L^*[v]} = u'' p_2 \overline{v} + u' p_1 \overline{v} + u \big[-p_2 \overline{v}'' + (-2p_2' + p_1) \overline{v}' + (-p_2'' + p_1') \overline{v} \big].$$

Integrating Lagrange's identity on its interval of validity gives us Green's formula.

$$\int_a^b \left(\overline{v} L[u] - u \overline{L^*[v]} \right) dx = \langle v | L[u] \rangle - \langle L^*[v] | u \rangle = B[u,v]\big|_{x=b} - B[u,v]\big|_{x=a}$$

29.2 Formally Self-Adjoint Operators

Example 29.2.1 The linear operator

$$L[y] = x^2 y'' + 2xy' + 3y$$

has the adjoint operator

$$\begin{aligned} L^*[y] &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2 y) - \frac{\mathrm{d}}{\mathrm{d}x}(2xy) + 3y \\ &= x^2 y'' + 4xy' + 2y - 2xy' - 2y + 3y \\ &= x^2 y'' + 2xy' + 3y. \end{aligned}$$

In Example 29.2.1, the adjoint operator is the same as the operator. If $L = L^*$, the operator is said to be **formally self-adjoint**.

Most of the differential equations that we study in this book are second order, formally self-adjoint, with real-valued coefficient functions. Thus we wish to find the general form of this operator. Consider the operator

$$L[y] = p_2 y'' + p_1 y' + p_0 y,$$

where the p_j's are real-valued functions. The adjoint operator then is

$$\begin{aligned} L^*[y] &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}(p_2 y) - \frac{\mathrm{d}}{\mathrm{d}x}(p_1 y) + p_0 y \\ &= p_2 y'' + 2p_2' y' + p_2'' y - p_1 y' - p_1' y + p_0 y \\ &= p_2 y'' + (2p_2' - p_1) y' + (p_2'' - p_1' + p_0) y. \end{aligned}$$

Equating L and L^* yields the two equations,

$$\begin{aligned} 2p_2' - p_1 &= p_1, \qquad & p_2'' - p_1' + p_0 &= p_0 \\ p_2' &= p_1, & p_2'' &= p_1'. \end{aligned}$$

Thus second order, formally self-adjoint operators with real-valued coefficient functions have the form

$$L[y] = p_2 y'' + p_2' y' + p_0 y,$$

which is equivalent to the form

$$L[y] = \frac{\mathrm{d}}{\mathrm{d}x}(p y') + q y.$$

Any linear differential equation of the form

$$L[y] = y'' + p_1 y' + p_0 y = f(x),$$

where each p_j is j times continuously differentiable and real-valued, can be written as a formally self adjoint equation. We just multiply by the factor,

$$\mathrm{e}^{P(x)} = \exp\left(\int^x p_1(\xi)\, d\xi\right)$$

to obtain

$$\exp\left[P(x)\right](y'' + p_1 y' + p_0 y) = \exp\left[P(x)\right] f(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left[P(x)\right] y'\right) + \exp\left[P(x)\right] p_0 y = \exp\left[P(x)\right] f(x).$$

Example 29.2.2 Consider the equation

$$y'' + \frac{1}{x} y' + y = 0.$$

Multiplying by the factor

$$\exp\left(\int^x \frac{1}{\xi}\, d\xi\right) = \mathrm{e}^{\log x} = x$$

will make the equation formally self-adjoint.

$$xy'' + y' + xy = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(xy') + xy = 0$$

Result 29.2.1 If $L = L^*$ then the linear operator L is formally self-adjoint. Second order formally self-adjoint operators have the form

$$L[y] = \frac{\mathrm{d}}{\mathrm{d}x}(py') + qy.$$

Any differential equation of the form

$$L[y] = y'' + p_1 y' + p_0 y = f(x),$$

where each p_j is j times continuously differentiable and real-valued, can be written as a formally self adjoint equation by multiplying the equation by the factor $\exp(\int^x p_1(\xi)\, d\xi)$.

29.3 Self-Adjoint Problems

Consider the n^{th} order formally self-adjoint equation $L[y] = 0$, on the domain $a \leq x \leq b$ subject to the boundary conditions, $B_j[y] = 0$ for $j = 1, \dots, n$. where the boundary conditions can be written

$$B_j[y] = \sum_{k=1}^{n} \alpha_{jk} y^{(k-1)}(a) + \beta_{jk} y^{(k-1)}(b) = 0.$$

If the boundary conditions are such that Green's formula reduces to

$$\langle v | L[u] \rangle - \langle L[v] | u \rangle = 0$$

then the problem is **self-adjoint**

Example 29.3.1 Consider the formally self-adjoint equation $-y'' = 0$, subject to the boundary conditions $y(0) =$

$y(\pi) = 0$. Green's formula is

$$\begin{aligned} \langle v | -u'' \rangle - \langle -v'' | u \rangle &= [u'(-\overline{v}) - u(-\overline{v})']_0^\pi \\ &= [u\overline{v}' - u'\overline{v}]_0^\pi \\ &= 0. \end{aligned}$$

Thus this problem is self-adjoint.

29.4 Self-Adjoint Eigenvalue Problems

Associated with the self-adjoint problem

$$L[y] = 0, \quad \text{subject to} \quad B_j[y] = 0,$$

is the eigenvalue problem

$$L[y] = \lambda y, \quad \text{subject to} \quad B_j[y] = 0.$$

This is called a self-adjoint eigenvalue problem. The values of λ for which there exist nontrivial solutions to this problem are called eigenvalues. The functions that satisfy the equation when λ is an eigenvalue are called eigenfunctions.

Example 29.4.1 Consider the self-adjoint eigenvalue problem

$$-y'' = \lambda y, \quad \text{subject to} \quad y(0) = y(\pi) = 0.$$

First consider the case $\lambda = 0$. The general solution is

$$y = c_1 + c_2 x.$$

Only the trivial solution satisfies the boundary conditions. $\lambda = 0$ is not an eigenvalue. Now consider $\lambda \neq 0$. The general solution is

$$y = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right).$$

The solution that satisfies the left boundary condition is

$$y = c \sin\left(\sqrt{\lambda}x\right).$$

For non-trivial solutions, we must have

$$\sin\left(\sqrt{\lambda}\pi\right) = 0,$$

$$\lambda = n^2, \quad n \in \mathbb{N}.$$

Thus the eigenvalues λ_n and eigenfunctions ϕ_n are

$$\boxed{\lambda_n = n^2, \qquad \phi_n = \sin(nx), \qquad \text{for } n = 1, 2, 3, \ldots}$$

Self-adjoint eigenvalue problems have a number a interesting properties. We will devote the rest of this section to developing some of these properties.

Real Eigenvalues. The eigenvalues of a self-adjoint problem are real. Let λ be an eigenvalue with the eigenfunction ϕ. Green's formula states

$$\langle \phi | L[\phi] \rangle - \langle L[\phi] | \phi \rangle = 0$$
$$\langle \phi | \lambda\phi \rangle - \langle \lambda\phi | \phi \rangle = 0$$
$$(\lambda - \overline{\lambda}) \langle \phi | \phi \rangle = 0$$

Since $\phi \not\equiv 0$, $\langle \phi | \phi \rangle > 0$. Thus $\lambda = \overline{\lambda}$ and λ is real.

Orthogonal Eigenfunctions. The eigenfunctions corresponding to distinct eigenvalues are orthogonal. Let λ_n and λ_m be distinct eigenvalues with the eigenfunctions ϕ_n and ϕ_m. Using Green's formula,

$$\langle \phi_n | L[\phi_m] \rangle - \langle L[\phi_n] | \phi_m \rangle = 0$$
$$\langle \phi_n | \lambda_m \phi_m \rangle - \langle \lambda_n \phi_n | \phi_m \rangle = 0$$
$$(\lambda_m - \overline{\lambda_n}) \langle \phi_n | \phi_m \rangle = 0.$$

Since the eigenvalues are real,

$$(\lambda_m - \lambda_n) \langle \phi_n | \phi_m \rangle = 0.$$

Since the two eigenvalues are distinct, $\langle \phi_n | \phi_m \rangle = 0$ and thus ϕ_n and ϕ_m are orthogonal.

***Enumerable Set of Eigenvalues.** The eigenvalues of a self-adjoint eigenvalue problem form an enumerable set with no finite cluster point. Consider the problem

$$L[y] = \lambda y \text{ on } a \leq x \leq b, \qquad \text{subject to } B_j[y] = 0.$$

Let $\{\psi_1, \psi_2, \ldots, \psi_n\}$ be a fundamental set of solutions at $x = x_0$ for some $a \leq x_0 \leq b$. That is,

$$\psi_j^{(k-1)}(x_0) = \delta_{jk}.$$

The key to showing that the eigenvalues are enumerable, is that the ψ_j are entire functions of λ. That is, they are analytic functions of λ for all finite λ. We will not prove this.

The boundary conditions are

$$B_j[y] = \sum_{k=1}^{n} \left[\alpha_{jk} y^{(k-1)}(a) + \beta_{jk} y^{(k-1)}(b) \right] = 0.$$

The eigenvalue problem has a solution for a given value of λ if $y = \sum_{k=1}^{n} c_k \psi_k$ satisfies the boundary conditions. That is,

$$B_j \left[\sum_{k=1}^{n} c_k \psi_k \right] = \sum_{k=1}^{n} c_k B_j[\psi_k] = 0 \quad \text{for } j = 1, \ldots, n.$$

Define an $n \times n$ matrix M such that $M_{jk} = B_k[\psi_j]$. Then if $\vec{c} = (c_1, c_2, \ldots, c_n)$, the boundary conditions can be written in terms of the matrix equation $M\vec{c} = 0$. This equation has a solution if and only if the determinant of the matrix is zero. Since the ψ_j are entire functions of λ, $\Delta[M]$ is an entire function of λ. The eigenvalues are real, so $\Delta[M]$ has only real roots. Since $\Delta[M]$ is an entire function, (that is not identically zero), with only real roots, the roots of $\Delta[M]$ can only cluster at infinity. Thus the eigenvalues of a self-adjoint problem are enumerable and can only cluster at infinity.

An example of a function whose roots have a finite cluster point is $\sin(1/x)$. This function, (graphed in Figure 29.1), is clearly not analytic at the cluster point $x = 0$.

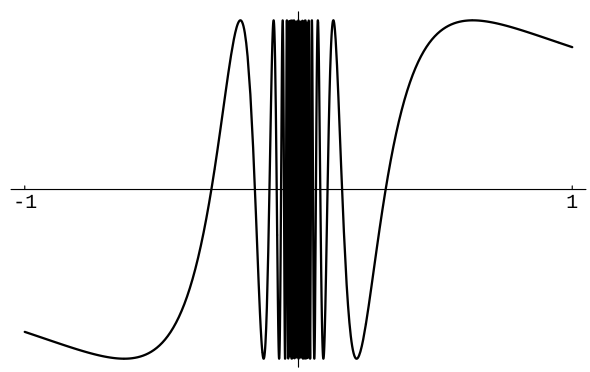

Figure 29.1: Graph of $\sin(1/x)$.

Infinite Number of Eigenvalues. Though we will not show it, self-adjoint problems have an infinite number of eigenvalues. Thus the eigenfunctions form an infinite orthogonal set.

Eigenvalues of Second Order Problems. Consider the second order, self-adjoint eigenvalue problem

$$L[y] = (py')' + qy = \lambda y, \quad \text{on } a \leq x \leq b, \quad \text{subject to } B_j[y] = 0.$$

Let λ_n be an eigenvalue with the eigenfunction ϕ_n.

$$\langle \phi_n | L[\phi_n] \rangle = \langle \phi_n | \lambda_n \phi_n \rangle$$
$$\langle \phi_n | (p\phi_n')' + q\phi_n \rangle = \lambda_n \langle \phi_n | \phi_n \rangle$$
$$\int_a^b \overline{\phi_n} (p\phi_n')' \, dx + \langle \phi_n | q | \phi_n \rangle = \lambda_n \langle \phi_n | \phi_n \rangle$$
$$\left[\overline{\phi_n} p \phi_n' \right]_a^b - \int_a^b \overline{\phi_n}' p \phi_n' \, dx + \langle \phi_n | q | \phi_n \rangle = \lambda_n \langle \phi_n | \phi_n \rangle$$
$$\boxed{\lambda_n = \frac{[p\overline{\phi_n}\phi_n']_a^b - \langle \phi_n' | p | \phi_n' \rangle + \langle \phi_n | q | \phi_n \rangle}{\langle \phi_n | \phi_n \rangle}}$$

Thus we can express each eigenvalue in terms of its eigenfunction. You might think that this formula is just a shade less than worthless. When solving an eigenvalue problem you have to find the eigenvalues before you determine the eigenfunctions. Thus this formula could not be used to compute the eigenvalues. However, we can often use the formula to obtain information about the eigenvalues before we solve a problem.

Example 29.4.2 Consider the self-adjoint eigenvalue problem

$$-y'' = \lambda y, \qquad y(0) = y(\pi) = 0.$$

The eigenvalues are given by the formula

$$\begin{aligned}\lambda_n &= \frac{\left[(-1)\overline{\phi}\phi'\right]_a^b - \langle \phi_n' | (-1) | \phi_n' \rangle + \langle \phi_n | 0 | \phi_n \rangle}{\langle \phi_n | \phi_n \rangle} \\ &= \frac{0 + \langle \phi_n' | \phi_n' \rangle + 0}{\langle \phi_n | \phi_n \rangle}.\end{aligned}$$

We see that $\lambda_n \geq 0$. If $\lambda_n = 0$ then $\langle \phi_n' | \phi_n' \rangle = 0$,which implies that $\phi_n = \text{const}$. The only constant that satisfies the boundary conditions is $\phi_n = 0$ which is not an eigenfunction since it is the trivial solution. Thus the eigenvalues are positive.

29.5 Inhomogeneous Equations

Let the problem,

$$L[y] = 0, \quad B_k[y] = 0,$$

be self-adjoint. If the inhomogeneous problem,

$$L[y] = f, \quad B_k[y] = 0,$$

has a solution, then we we can write this solution in terms of the eigenfunction of the associated eigenvalue problem,

$$L[y] = \lambda y, \quad B_k[y] = 0.$$

We denote the eigenvalues as λ_n and the eigenfunctions as ϕ_n for $n \in \mathbb{Z}^+$. For the moment we assume that $\lambda = 0$ is not an eigenvalue and that the eigenfunctions are real-valued. We expand the function $f(x)$ in a series of the eigenfunctions.

$$f(x) = \sum f_n \phi_n(x), \quad f_n = \frac{\langle \phi_n | f \rangle}{\|\phi_n\|}$$

We expand the inhomogeneous solution in a series of eigenfunctions and substitute it into the differential equation.

$$L[y] = f$$
$$L\left[\sum y_n \phi_n(x)\right] = \sum f_n \phi_n(x)$$
$$\sum \lambda_n y_n \phi_n(x) = \sum f_n \phi_n(x)$$
$$y_n = \frac{f_n}{\lambda_n}$$

The inhomogeneous solution is

$$y(x) = \sum \frac{\langle \phi_n | f \rangle}{\lambda_n \|\phi_n\|} \phi_n(x). \tag{29.1}$$

As a special case we consider the Green function problem,

$$L[G] = \delta(x - \xi), \quad B_k[G] = 0,$$

We expand the Dirac delta function in an eigenfunction series.

$$\delta(x - \xi) = \sum \frac{\langle \phi_n | \delta \rangle}{\|\phi_n\|} \phi_n(x) = \sum \frac{\phi_n(\xi) \phi_n(x)}{\|\phi_n\|}$$

The Green function is

$$G(x|\xi) = \sum \frac{\phi_n(\xi) \phi_n(x)}{\lambda_n \|\phi_n\|}.$$

We corroborate Equation 29.1 by solving the inhomogeneous equation in terms of the Green function.

$$y = \int_a^b G(x|\xi) f(\xi)\,\mathrm{d}\xi$$
$$y = \int_a^b \sum \frac{\phi_n(\xi)\phi_n(x)}{\lambda_n \|\phi_n\|} f(\xi)\,\mathrm{d}\xi$$
$$y = \sum \frac{\int_a^b \phi_n(\xi) f(\xi)\,\mathrm{d}\xi}{\lambda_n \|\phi_n\|} \phi_n(x)$$
$$y = \sum \frac{\langle \phi_n | f \rangle}{\lambda_n \|\phi_n\|} \phi_n(x)$$

Example 29.5.1 Consider the Green function problem

$$G'' + G = \delta(x - \xi), \quad G(0|\xi) = G(1|\xi) = 0.$$

First we examine the associated eigenvalue problem.

$$\phi'' + \phi = \lambda\phi, \quad \phi(0) = \phi(1) = 0$$
$$\phi'' + (1 - \lambda)\phi = 0, \quad \phi(0) = \phi(1) = 0$$
$$\lambda_n = 1 - (n\pi)^2, \quad \phi_n = \sin(n\pi x), \quad n \in \mathbb{Z}^+$$

We write the Green function as a series of the eigenfunctions.

$$G(x|\xi) = 2 \sum_{n=1}^{\infty} \frac{\sin(n\pi\xi)\sin(n\pi x)}{1 - (n\pi)^2}$$

29.6 Exercises

Exercise 29.1
Show that the operator adjoint to

$$Ly = y^{(n)} + p_1(z)y^{(n-1)} + p_2(z)y^{(n-2)} + \cdots + p_n(z)y$$

is given by

$$My = (-1)^n u^{(n)} + (-1)^{n-1}(\overline{p_1(z)}u)^{(n-1)} + (-1)^{n-2}(\overline{p_2(z)}u)^{(n-2)} + \cdots + \overline{p_n(z)}u.$$

Hint, Solution

29.7 Hints

Hint 29.1

29.8 Solutions

Solution 29.1

Consider $u(x), v(x) \in C^n$. (C^n is the set of n times continuously differentiable functions). First we prove the preliminary result

$$uv^{(n)} - (-1)^n u^{(n)} v = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{n-1} (-1)^k u^{(k)} v^{(n-k-1)} \tag{29.2}$$

by simplifying the right side.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{n-1} (-1)^k u^{(k)} v^{(n-k-1)} &= \sum_{k=0}^{n-1} (-1)^k \left(u^{(k)} v^{(n-k)} + u^{(k+1)} v^{(n-k-1)} \right) \\
&= \sum_{k=0}^{n-1} (-1)^k u^{(k)} v^{(n-k)} - \sum_{k=0}^{n-1} (-1)^{k+1} u^{(k+1)} v^{(n-k-1)} \\
&= \sum_{k=0}^{n-1} (-1)^k u^{(k)} v^{(n-k)} - \sum_{k=1}^{n} (-1)^k u^{(k)} v^{(n-k)} \\
&= (-1)^0 u^{(0)} v^{n-0} - (-1)^n u^{(n)} v^{(n-n)} \\
&= uv^{(n)} - (-1)^n u^{(n)} v
\end{aligned}$$

We define $p_0(x) = 1$ so that we can write the operators in a nice form.

$$Ly = \sum_{m=0}^{n} p_m(z) y^{(n-m)}, \quad Mu = \sum_{m=0}^{n} (-1)^m (\overline{p_m(z)} u)^{(n-m)}$$

Now we show that M is the adjoint to L.

$$\begin{aligned}\overline{u}Ly - y\overline{Mu} &= \overline{u}\sum_{m=0}^{n} p_m(z)y^{(n-m)} - y\sum_{m=0}^{n}(-1)^m(p_m(z)\overline{u})^{(n-m)} \\ &= \sum_{m=0}^{n}\left(\overline{u}p_m(z)y^{(n-m)} - (p_m(z)\overline{u})^{(n-m)}y\right)\end{aligned}$$

We use Equation 29.2.

$$= \sum_{m=0}^{n}\frac{\mathrm{d}}{\mathrm{d}z}\sum_{k=0}^{n-m-1}(-1)^k(\overline{u}p_m(z))^{(k)}y^{(n-m-k-1)}$$

$$\boxed{\overline{u}Ly - y\overline{Mu} = \frac{\mathrm{d}}{\mathrm{d}z}\sum_{m=0}^{n}\sum_{k=0}^{n-m-1}(-1)^k(\overline{u}p_m(z))^{(k)}y^{(n-m-k-1)}}$$

Chapter 30

Fourier Series

Every time I close my eyes
The noise inside me amplifies
I can't escape
I relive every moment of the day
Every misstep I have made
Finds a way it can invade
My every thought
And this is why I find myself awake

-*Failure*
-Tom Shear (Assemblage 23)

30.1 An Eigenvalue Problem.

A self adjoint eigenvalue problem. Consider the eigenvalue problem

$$y'' + \lambda y = 0, \qquad y(-\pi) = y(\pi), \qquad y'(-\pi) = y'(\pi).$$

We rewrite the equation so the eigenvalue is on the right side.

$$L[y] \equiv -y'' = \lambda y$$

We demonstrate that this eigenvalue problem is self adjoint.

$$\begin{aligned}
\langle v|L[u]\rangle - \langle L[v]|u\rangle &= \langle v| - u''\rangle - \langle -v''|u\rangle \\
&= [-\bar{v}u']_{-\pi}^{\pi} + \langle v'|u'\rangle - [-\bar{v}'u]{-\pi}^{\pi} - \langle v'|u'\rangle \\
&= -\overline{v(\pi)}u'(\pi) + \overline{v(-\pi)}u'(-\pi) + \overline{v'(\pi)}u(\pi) - \overline{v'(-\pi)}u(-\pi) \\
&= -\overline{v(\pi)}u'(\pi) + \overline{v(\pi)}u'(\pi) + \overline{v'(\pi)}u(\pi) - \overline{v'(\pi)}u(\pi) \\
&= 0
\end{aligned}$$

Since Green's Identity reduces to $\langle v|L[u]\rangle - \langle L[v]|u\rangle = 0$, the problem is self adjoint. This means that the eigenvalues are real and that eigenfunctions corresponding to distinct eigenvalues are orthogonal. We compute the Rayleigh quotient for an eigenvalue λ with eigenfunction ϕ.

$$\begin{aligned}
\lambda &= \frac{-[\bar{\phi}\phi']_{-\pi}^{\pi} + \langle \phi'|\phi'\rangle}{\langle \phi|\phi\rangle} \\
&= \frac{-\overline{\phi(\pi)}\phi'(\pi) + \overline{\phi(-\pi)}\phi'(-\pi) + \langle \phi'|\phi'\rangle}{\langle \phi|\phi\rangle} \\
&= \frac{-\overline{\phi(\pi)}\phi'(\pi) + \overline{\phi(\pi)}\phi'(\pi) + \langle \phi'|\phi'\rangle}{\langle \phi|\phi\rangle} \\
&= \frac{\langle \phi'|\phi'\rangle}{\langle \phi|\phi\rangle}
\end{aligned}$$

We see that the eigenvalues are non-negative.

Computing the eigenvalues and eigenfunctions. Now we find the eigenvalues and eigenfunctions. First we consider the case $\lambda = 0$. The general solution of the differential equation is

$$y = c_1 + c_2 x.$$

The solution that satisfies the boundary conditions is $y = \text{const}$.

Now consider $\lambda > 0$. The general solution of the differential equation is

$$y = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right).$$

We apply the first boundary condition.

$$\begin{gathered}
y(-\pi) = y(\pi) \\
c_1 \cos\left(-\sqrt{\lambda}\pi\right) + c_2 \sin\left(-\sqrt{\lambda}\pi\right) = c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(\sqrt{\lambda}\pi\right) \\
c_1 \cos\left(\sqrt{\lambda}\pi\right) - c_2 \sin\left(\sqrt{\lambda}\pi\right) = c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(\sqrt{\lambda}\pi\right) \\
c_2 \sin\left(\sqrt{\lambda}\pi\right) = 0
\end{gathered}$$

Then we apply the second boundary condition.

$$\begin{gathered}
y'(-\pi) = y'(\pi) \\
-c_1\sqrt{\lambda} \sin\left(-\sqrt{\lambda}\pi\right) + c_2\sqrt{\lambda} \cos\left(-\sqrt{\lambda}\pi\right) = -c_1\sqrt{\lambda} \sin\left(\sqrt{\lambda}\pi\right) + c_2\sqrt{\lambda} \cos\left(\sqrt{\lambda}\pi\right) \\
c_1 \sin\left(\sqrt{\lambda}\pi\right) + c_2 \cos\left(\sqrt{\lambda}\pi\right) = -c_1 \sin\left(\sqrt{\lambda}\pi\right) + c_2 \cos\left(\sqrt{\lambda}\pi\right) \\
c_1 \sin\left(\sqrt{\lambda}\pi\right) = 0
\end{gathered}$$

To satisify the two boundary conditions either $c_1 = c_2 = 0$ or $\sin\left(\sqrt{\lambda}\pi\right) = 0$. The former yields the trivial solution. The latter gives us the eigenvalues $\lambda_n = n^2$, $n \in \mathbb{Z}^+$. The corresponding solution is

$$y_n = c_1 \cos(nx) + c_2 \sin(nx).$$

There are two eigenfunctions for each of the positive eigenvalues.

We choose the eigenvalues and eigenfunctions.

$$\begin{aligned}
&\lambda_0 = 0, &&\phi_0 = \frac{1}{2} \\
&\lambda_n = n^2, &&\phi_{2n-1} = \cos(nx), \quad \phi_{2n} = \sin(nx), \quad \text{for } n = 1, 2, 3, \ldots
\end{aligned}$$

Orthogonality of Eigenfunctions. We know that the eigenfunctions of distinct eigenvalues are orthogonal. In addition, the two eigenfunctions of each positive eigenvalue are orthogonal.

$$\int_{-\pi}^{\pi} \cos(nx)\sin(nx)\,\mathrm{d}x = \left[\frac{1}{2n}\sin^2(nx)\right]_{-\pi}^{\pi} = 0$$

Thus the eigenfunctions $\{\frac{1}{2}, \cos(x), \sin(x), \cos(2x), \sin(2x)\}$ are an orthogonal set.

30.2 Fourier Series.

A series of the eigenfunctions

$$\phi_0 = \frac{1}{2}, \quad \phi_n^{(1)} = \cos(nx), \quad \phi_n^{(2)} = \sin(nx), \quad \text{for } n \geq 1$$

is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big).$$

This is known as a *Fourier series.* (We choose $\phi_0 = \frac{1}{2}$ so all of the eigenfunctions have the same norm.) A fairly general class of functions can be expanded in Fourier series. Let $f(x)$ be a function defined on $-\pi < x < \pi$. Assume that $f(x)$ can be expanded in a Fourier series

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big). \tag{30.1}$$

Here the "$\sim$" means "has the Fourier series". We have not said if the series converges yet. For now let's assume that the series converges uniformly so we can replace the $\sim$ with an $=$.

We integrate Equation 30.1 from $-\pi$ to π to determine a_0.

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \mathrm{d}x + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)\,\mathrm{d}x$$

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \pi a_0 + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x + b_n \int_{-\pi}^{\pi} \sin(nx)\,\mathrm{d}x \right)$$

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \pi a_0$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

Multiplying by $\cos(mx)$ and integrating will enable us to solve for a_m.

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(mx)\,\mathrm{d}x + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,\mathrm{d}x + b_n \int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,\mathrm{d}x \right)$$

All but one of the terms on the right side vanishes due to the orthogonality of the eigenfunctions.

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = a_m \int_{-\pi}^{\pi} \cos(mx)\cos(mx)\,\mathrm{d}x$$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = a_m \int_{-\pi}^{\pi} \left(\frac{1}{2} + \cos(2mx) \right)\,\mathrm{d}x$$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = \pi a_m$$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x.$$

Note that this formula is valid for $m = 0, 1, 2, \dots$.

Similarly, we can multiply by $\sin(mx)$ and integrate to solve for b_m. The result is

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, \mathrm{d}x.$$

a_n and b_n are called *Fourier coefficients.*

Although we will not show it, Fourier series converge for a fairly general class of functions. Let $f(x^-)$ denote the left limit of $f(x)$ and $f(x^+)$ denote the right limit.

Example 30.2.1 For the function defined

$$f(x) = \begin{cases} 0 & \text{for } x < 0, \\ x + 1 & \text{for } x \geq 0, \end{cases}$$

the left and right limits at $x = 0$ are

$$f(0^-) = 0, \qquad f(0^+) = 1.$$

Result 30.2.1 Let $f(x)$ be a 2π-periodic function for which $\int_{-\pi}^{\pi} |f(x)| \, \mathrm{d}x$ exists. Define the Fourier coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, \mathrm{d}x.$$

If x is an interior point of an interval on which $f(x)$ has limited total fluctuation, then the Fourier series of $f(x)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big),$$

converges to $\frac{1}{2}(f(x^-) + f(x^+))$. If f is continuous at x, then the series converges to $f(x)$.

Periodic Extension of a Function. Let $g(x)$ be a function that is arbitrarily defined on $-\pi \leq x < \pi$. The Fourier series of $g(x)$ will represent the periodic extension of $g(x)$. The periodic extension, $f(x)$, is defined by the two conditions:

$$f(x) = g(x) \quad \text{for } -\pi \leq x < \pi,$$
$$f(x + 2\pi) = f(x).$$

The periodic extension of $g(x) = x^2$ is shown in Figure 30.1.

Limited Fluctuation. A function that has limited total fluctuation can be written $f(x) = \psi_+(x) - \psi_-(x)$, where ψ_+ and ψ_- are bounded, nondecreasing functions. An example of a function that does not have limited total fluctuation is $\sin(1/x)$, whose fluctuation is unlimited at the point $x = 0$.

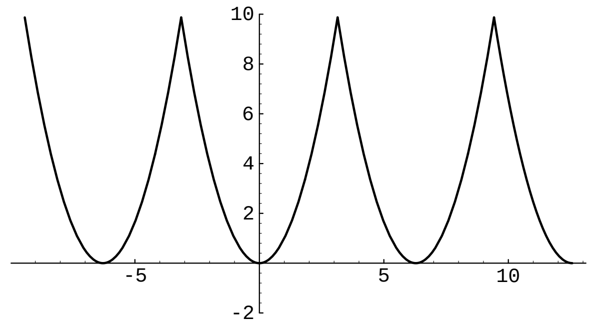

Figure 30.1: The Periodic Extension of $g(x) = x^2$.

Functions with Jump Discontinuities. Let $f(x)$ be a discontinuous function that has a convergent Fourier series. Note that the series does not necessarily converge to $f(x)$. Instead it converges to $\hat{f}(x) = \frac{1}{2}(f(x^-)+f(x^+))$.

Example 30.2.2 Consider the function defined by

$$f(x) = \begin{cases} -x & \text{for } -\pi \leq x < 0 \\ \pi - 2x & \text{for } 0 \leq x < \pi. \end{cases}$$

The Fourier series converges to the function defined by

$$\hat{f}(x) = \begin{cases} 0 & \text{for } x = -\pi \\ -x & \text{for } -\pi < x < 0 \\ \pi/2 & \text{for } x = 0 \\ \pi - 2x & \text{for } 0 < x < \pi. \end{cases}$$

The function $\hat{f}(x)$ is plotted in Figure 30.2.

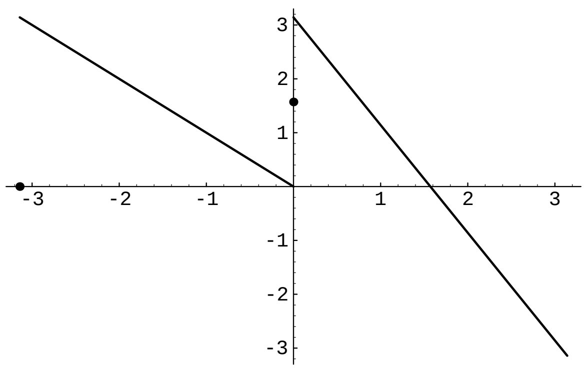

Figure 30.2: Graph of $\hat{f}(x)$.

30.3 Least Squares Fit

Approximating a function with a Fourier series. Suppose we want to approximate a 2π-periodic function $f(x)$ with a finite Fourier series.

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx))$$

Here the coefficients are computed with the familiar formulas. Is this the best approximation to the function? That is, is it possible to choose coefficients α_n and β_n such that

$$f(x) \approx \frac{\alpha_0}{2} + \sum_{n=1}^{N} (\alpha_n \cos(nx) + \beta_n \sin(nx))$$

would give a better approximation?

Least squared error fit. The most common criterion for finding the best fit to a function is the least squares fit. The best approximation to a function is defined as the one that minimizes the integral of the square of the deviation. Thus if $f(x)$ is to be approximated on the interval $a \leq x \leq b$ by a series

$$f(x) \approx \sum_{n=1}^{N} c_n \phi_n(x), \tag{30.2}$$

the best approximation is found by choosing values of c_n that minimize the error E.

$$E \equiv \int_a^b \left| f(x) - \sum_{n=1}^{N} c_n \phi_n(x) \right|^2 \, \mathrm{d}x$$

Generalized Fourier coefficients. We consider the case that the ϕ_n are orthogonal. For simplicity, we also assume that the ϕ_n are real-valued. Then most of the terms will vanish when we interchange the order of integration and summation.

$$E = \int_a^b \left(f^2 - 2f \sum_{n=1}^N c_n \phi_n + \sum_{n=1}^N c_n \phi_n \sum_{m=1}^N c_m \phi_m \right) \mathrm{d}x$$

$$E = \int_a^b f^2 \,\mathrm{d}x - 2 \sum_{n=1}^N c_n \int_a^b f \phi_n \,\mathrm{d}x + \sum_{n=1}^N \sum_{m=1}^N c_n c_m \int_a^b \phi_n \phi_m \,\mathrm{d}x$$

$$E = \int_a^b f^2 \,\mathrm{d}x - 2 \sum_{n=1}^N c_n \int_a^b f \phi_n \,\mathrm{d}x + \sum_{n=1}^N c_n^2 \int_a^b \phi_n^2 \,\mathrm{d}x$$

$$E = \int_a^b f^2 \,\mathrm{d}x + \sum_{n=1}^N \left(c_n^2 \int_a^b \phi_n^2 \,\mathrm{d}x - 2 c_n \int_a^b f \phi_n \,\mathrm{d}x \right)$$

We complete the square for each term.

$$E = \int_a^b f^2 \,\mathrm{d}x + \sum_{n=1}^N \left(\int_a^b \phi_n^2 \,\mathrm{d}x \left(c_n - \frac{\int_a^b f \phi_n \,\mathrm{d}x}{\int_a^b \phi_n^2 \,\mathrm{d}x} \right)^2 - \left(\frac{\int_a^b f \phi_n \,\mathrm{d}x}{\int_a^b \phi_n^2 \,\mathrm{d}x} \right)^2 \right)$$

Each term involving c_n is non-negative, and is minimized for

$$c_n = \frac{\int_a^b f \phi_n \,\mathrm{d}x}{\int_a^b \phi_n^2 \,\mathrm{d}x}. \tag{30.3}$$

We call these the *generalized Fourier coefficients*.

For such a choice of the c_n, the error is

$$E = \int_a^b f^2 \,\mathrm{d}x - \sum_{n=1}^N c_n^2 \int_a^b \phi_n^2 \,\mathrm{d}x.$$

Since the error is non-negative, we have

$$\int_a^b f^2 \,\mathrm{d}x \geq \sum_{n=1}^{N} c_n^2 \int_a^b \phi_n^2 \,\mathrm{d}x.$$

This is known as *Bessel's Inequality*. If the series in Equation 30.2 converges in the mean to $f(x)$, $\lim N \to \infty E = 0$, then we have equality as $N \to \infty$.

$$\int_a^b f^2 \,\mathrm{d}x = \sum_{n=1}^{\infty} c_n^2 \int_a^b \phi_n^2 \,\mathrm{d}x.$$

This is *Parseval's equality.*

Fourier coefficients. Previously we showed that if the series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx),$$

converges uniformly then the coefficients in the series are the Fourier coefficients,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \,\mathrm{d}x, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \,\mathrm{d}x.$$

Now we show that by choosing the coefficients to minimize the squared error, we obtain the same result. We apply Equation 30.3 to the Fourier eigenfunctions.

$$a_0 = \frac{\int_{-\pi}^{\pi} f \frac{1}{2} \,\mathrm{d}x}{\int_{-\pi}^{\pi} \frac{1}{4} \,\mathrm{d}x} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \,\mathrm{d}x$$

$$a_n = \frac{\int_{-\pi}^{\pi} f \cos(nx) \,\mathrm{d}x}{\int_{-\pi}^{\pi} \cos^2(nx) \,\mathrm{d}x} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \,\mathrm{d}x$$

$$b_n = \frac{\int_{-\pi}^{\pi} f \sin(nx) \,\mathrm{d}x}{\int_{-\pi}^{\pi} \sin^2(nx) \,\mathrm{d}x} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \,\mathrm{d}x$$

30.4 Fourier Series for Functions Defined on Arbitrary Ranges

If $f(x)$ is defined on $c - d \leq x < c + d$ and $f(x + 2d) = f(x)$, then $f(x)$ has a Fourier series of the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi(x+c)}{d}\right) + b_n \sin\left(\frac{n\pi(x+c)}{d}\right).$$

Since

$$\int_{c-d}^{c+d} \cos^2\left(\frac{n\pi(x+c)}{d}\right) \mathrm{d}x = \int_{c-d}^{c+d} \sin^2\left(\frac{n\pi(x+c)}{d}\right) \mathrm{d}x = d,$$

the Fourier coefficients are given by the formulas

$$a_n = \frac{1}{d}\int_{c-d}^{c+d} f(x)\cos\left(\frac{n\pi(x+c)}{d}\right) \mathrm{d}x$$
$$b_n = \frac{1}{d}\int_{c-d}^{c+d} f(x)\sin\left(\frac{n\pi(x+c)}{d}\right) \mathrm{d}x.$$

Example 30.4.1 Consider the function defined by

$$f(x) = \begin{cases} x+1 & \text{for } -1 \leq x < 0 \\ x & \text{for } 0 \leq x < 1 \\ 3-2x & \text{for } 1 \leq x < 2. \end{cases}$$

This function is graphed in Figure 30.3.

The Fourier series converges to $\hat{f}(x) = (f(x^-) + f(x^+))/2$,

$$\hat{f}(x) = \begin{cases} -\frac{1}{2} & \text{for } x = -1 \\ x+1 & \text{for } -1 < x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ x & \text{for } 0 < x < 1 \\ 3-2x & \text{for } 1 \leq x < 2. \end{cases}$$

$\hat{f}(x)$ is also graphed in Figure 30.3.

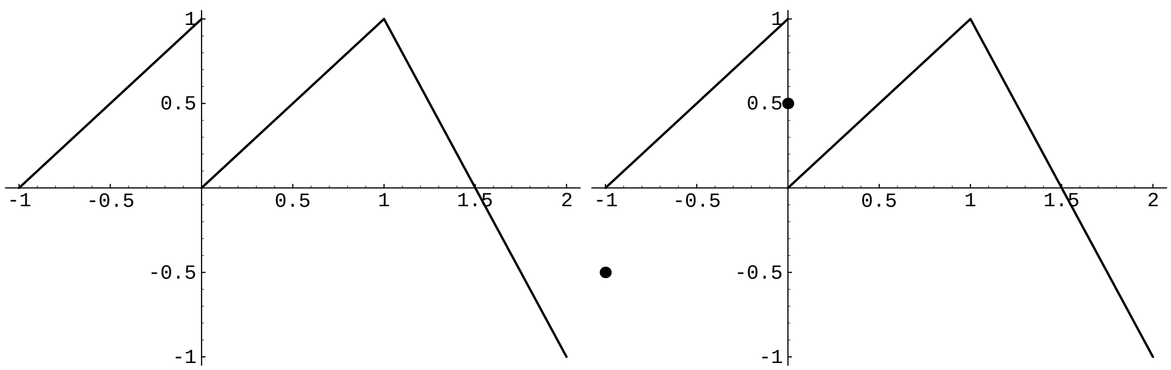

Figure 30.3: A Function Defined on the range $-1 \leq x < 2$ and the Function to which the Fourier Series Converges.

The Fourier coefficients are

$$
\begin{aligned}
a_n &= \frac{1}{3/2}\int_{-1}^{2} f(x)\cos\left(\frac{2n\pi(x+1/2)}{3}\right)\,\mathrm{d}x \\
&= \frac{2}{3}\int_{-1/2}^{5/2} f(x-1/2)\cos\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x \\
&= \frac{2}{3}\int_{-1/2}^{1/2}(x+1/2)\cos\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x + \frac{2}{3}\int_{1/2}^{3/2}(x-1/2)\cos\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x \\
&\qquad + \frac{2}{3}\int_{3/2}^{5/2}(4-2x)\cos\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x \\
&= -\frac{1}{(n\pi)^2}\sin\left(\frac{2n\pi}{3}\right)\left[2(-1)^n n\pi + 9\sin\left(\frac{n\pi}{3}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{3/2}\int_{-1}^{2} f(x)\sin\left(\frac{2n\pi(x+1/2)}{3}\right)\,\mathrm{d}x \\
&= \frac{2}{3}\int_{-1/2}^{5/2} f(x-1/2)\sin\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x \\
&= \frac{2}{3}\int_{-1/2}^{1/2}(x+1/2)\sin\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x + \frac{2}{3}\int_{1/2}^{3/2}(x-1/2)\sin\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x \\
&\qquad + \frac{2}{3}\int_{3/2}^{5/2}(4-2x)\sin\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x \\
&= -\frac{2}{(n\pi)^2}\sin^2\left(\frac{n\pi}{3}\right)\left[2(-1)^n n\pi + 4n\pi\cos\left(\frac{n\pi}{3}\right) - 3\sin\left(\frac{n\pi}{3}\right)\right]
\end{aligned}
$$

30.5 Fourier Cosine Series

If $f(x)$ is an even function, $(f(-x) = f(x))$, then there will not be any sine terms in the Fourier series for $f(x)$. The Fourier sine coefficient is

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, \mathrm{d}x.$$

Since $f(x)$ is an even function and $\sin(nx)$ is odd, $f(x)\sin(nx)$ is odd. b_n is the integral of an odd function from $-\pi$ to π and is thus zero. We can rewrite the cosine coefficients,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x \\ &= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos(nx) \, \mathrm{d}x. \end{aligned}$$

Example 30.5.1 Consider the function defined on $[0, \pi)$ by

$$f(x) = \begin{cases} x & \text{for } 0 \leq x < \pi/2 \\ \pi - x & \text{for } \pi/2 \leq x < \pi. \end{cases}$$

The Fourier cosine coefficients for this function are

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_{0}^{\pi/2} x \cos(nx) \, \mathrm{d}x + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \cos(nx) \, \mathrm{d}x \\ &= \begin{cases} \frac{\pi}{4} & \text{for } n = 0, \\ \frac{8}{\pi n^2} \cos\left(\frac{n\pi}{2}\right) \sin^2\left(\frac{n\pi}{4}\right) & \text{for } n \geq 1. \end{cases} \end{aligned}$$

In Figure 30.4 the even periodic extension of $f(x)$ is plotted in a dashed line and the sum of the first five nonzero terms in the Fourier cosine series are plotted in a solid line.

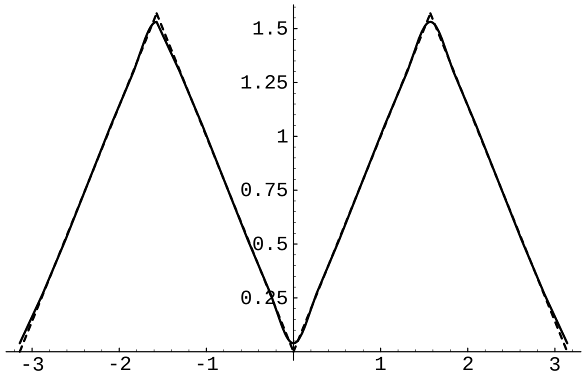

Figure 30.4: Fourier Cosine Series.

30.6 Fourier Sine Series

If $f(x)$ is an odd function, $(f(-x) = -f(x))$, then there will not be any cosine terms in the Fourier series. Since $f(x)\cos(nx)$ is an odd function, the cosine coefficients will be zero. Since $f(x)\sin(nx)$ is an even function,we can rewrite the sine coefficients

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) \, \mathrm{d}x.$$

Example 30.6.1 Consider the function defined on $[0, \pi)$ by

$$f(x) = \begin{cases} x & \text{for } 0 \leq x < \pi/2 \\ \pi - x & \text{for } \pi/2 \leq x < \pi. \end{cases}$$

The Fourier sine coefficients for this function are

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi/2} x \sin(nx)\, \mathrm{d}x + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \sin(nx)\, \mathrm{d}x \\ &= \frac{16}{\pi n^2} \cos\left(\frac{n\pi}{4}\right) \sin^3\left(\frac{n\pi}{4}\right) \end{aligned}$$

In Figure 30.5 the odd periodic extension of $f(x)$ is plotted in a dashed line and the sum of the first five nonzero terms in the Fourier sine series are plotted in a solid line.

30.7 Complex Fourier Series and Parseval's Theorem

By writing $\sin(nx)$ and $\cos(nx)$ in terms of e^{inx} and e^{-inx} we can obtain the complex form for a Fourier series.

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{1}{2}(\mathrm{e}^{inx} + \mathrm{e}^{-inx}) + b_n \frac{1}{2i}(\mathrm{e}^{inx} - \mathrm{e}^{-inx})\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{2}(a_n - ib_n)\, \mathrm{e}^{inx} + \frac{1}{2}(a_n + ib_n)\, \mathrm{e}^{-inx}\right) \\ &= \sum_{n=-\infty}^{\infty} c_n\, \mathrm{e}^{inx} \end{aligned}$$

where

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & \text{for } n \geq 1 \\ \frac{a_0}{2} & \text{for } n = 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & \text{for } n \leq -1. \end{cases}$$

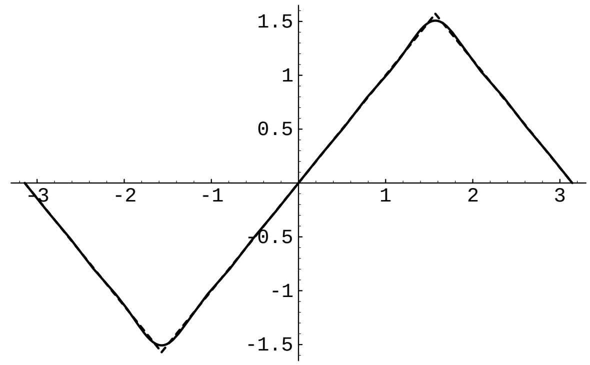

Figure 30.5: Fourier Sine Series.

The functions $\{\dots, e^{-ix}, 1, e^{ix}, e^{2ix}, \dots\}$, satisfy the relation

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx} \, dx = \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx$$
$$= \begin{cases} 2\pi & \text{for } n = m \\ 0 & \text{for } n \neq m. \end{cases}$$

Starting with the complex form of the Fourier series of a function $f(x)$,

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx},$$

we multiply by e^{-imx} and integrate from $-\pi$ to π to obtain

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{-imx}\,\mathrm{d}x = \int_{-\pi}^{\pi} \sum_{-\infty}^{\infty} c_n\,\mathrm{e}^{inx}\,\mathrm{e}^{-imx}\,\mathrm{d}x$$
$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{-imx}\,\mathrm{d}x$$

If $f(x)$ is real-valued then

$$c_{-m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{imx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{(\,\mathrm{e}^{-imx})}\,\mathrm{d}x = \overline{c_m}$$

where $\bar{z}$ denotes the complex conjugate of z.

Assume that $f(x)$ has a uniformly convergent Fourier series.

$$\begin{aligned}
\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x &= \int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} c_m\,\mathrm{e}^{imx}\right)\left(\sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{inx}\right)\mathrm{d}x \\
&= 2\pi \sum_{n=-\infty}^{\infty} c_n c_{-n} \\
&= 2\pi\left(\sum_{n=-\infty}^{-1}\left[\frac{1}{4}(a_{-n}+ib_{-n})(a_{-n}-ib_{-n})\right] + \frac{a_0}{2}\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{4}(a_n-ib_n)(a_n+ib_n)\right]\right) \\
&= 2\pi\left(\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)\right)
\end{aligned}$$

This yields a result known as Parseval's theorem which holds even when the Fourier series of $f(x)$ is not uniformly convergent.

Result 30.7.1 Parseval's Theorem. If $f(x)$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

then

$$\int_{-\pi}^{\pi} f^2(x) \,\mathrm{d}x = \frac{\pi}{2} a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

30.8 Behavior of Fourier Coefficients

Before we jump hip-deep into the grunge involved in determining the behavior of the Fourier coefficients, let's take a step back and get some perspective on what we should be looking for.

One of the important questions is whether the Fourier series converges uniformly. From Result 14.2.1 we know that a uniformly convergent series represents a continuous function. Thus we know that the Fourier series of a discontinuous function cannot be uniformly convergent. From Section 14.2 we know that a series is uniformly convergent if it can be bounded by a series of positive terms. If the Fourier coefficients, a_n and b_n, are $O(1/n^\alpha)$ where $\alpha > 1$ then the series can be bounded by (const) $\sum_{n=1}^{\infty} 1/n^\alpha$ and will thus be uniformly convergent.

Let $f(x)$ be a function that meets the conditions for having a Fourier series and in addition is bounded. Let $(-\pi, p_1), (p_1, p_2), (p_2, p_3), \ldots, (p_m, \pi)$ be a partition into a finite number of intervals of the domain, $(-\pi, \pi)$ such that on each interval $f(x)$ and all it's derivatives are continuous. Let $f(p^-)$ denote the left limit of $f(p)$ and $f(p^+)$ denote the right limit.

$$f(p^-) = \lim_{\epsilon \to 0^+} f(p - \epsilon), \qquad f(p^+) = \lim_{\epsilon \to 0^+} f(p + \epsilon)$$

Example 30.8.1 The function shown in Figure 30.6 would be partitioned into the intervals

$$(-2, -1), (-1, 0), (0, 1), (1, 2).$$

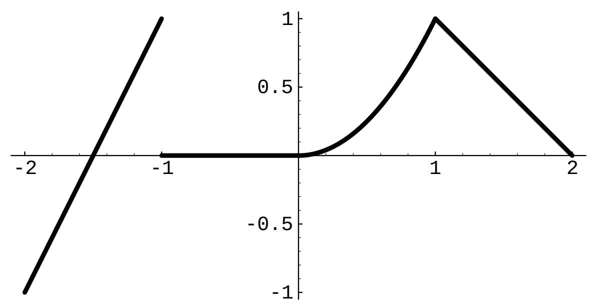

Figure 30.6: A Function that can be Partitioned.

Suppose $f(x)$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

We can use the integral formula to find the a_n's.

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{p_1} f(x)\cos(nx)\,\mathrm{d}x + \int_{p_1}^{p_2} f(x)\cos(nx)\,\mathrm{d}x + \cdots + \int_{p_m}^{\pi} f(x)\cos(nx)\,\mathrm{d}x\right)
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
&= \frac{1}{n\pi}\left(\Big[f(x)\sin(nx)\Big]_{-\pi}^{p_1} + \Big[f(x)\sin(nx)\Big]_{p_1}^{p_2} + \cdots + \Big[f(x)\sin(nx)\Big]_{p_m}^{\pi}\right) \\
&\qquad - \frac{1}{n\pi}\left(\int_{-\pi}^{p_1} f'(x)\sin(nx)\,\mathrm{d}x + \int_{p_1}^{p_2} f'(x)\sin(nx)\,\mathrm{d}x + \int_{p_m}^{\pi} f'(x)\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{n\pi}\left\{\left[f(p_1^-) - f(p_1^+)\right]\sin(np_1) + \cdots + \left[f(p_m^-) - f(p_m^+)\right]\sin(np_m)\right\} \\
&\qquad - \frac{1}{n}\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{n}A_n - \frac{1}{n}b'_n
\end{aligned}$$

where

$$A_n = \frac{1}{\pi}\sum_{j=1}^{m}\sin(np_j)\left[f(p_j^-) - f(p_j^+)\right]$$

and the b'_n are the sine coefficients of $f'(x)$.

Since $f(x)$ is bounded, $A_n = O(1)$. Since $f'(x)$ is bounded,

$$b'_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,\mathrm{d}x = O(1).$$

Thus $a_n = O(1/n)$ as $n \to \infty$. (Actually, from the Riemann-Lebesgue Lemma, $b'_n = \mathcal{O}(1/n)$.)

Now we repeat this analysis for the sine coefficients.

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, \mathrm{d}x \\
&= \frac{1}{\pi} \left(\int_{-\pi}^{p_1} f(x) \sin(nx) \, \mathrm{d}x + \int_{p_1}^{p_2} f(x) \sin(nx) \, \mathrm{d}x + \cdots + \int_{p_m}^{\pi} f(x) \sin(nx) \, \mathrm{d}x \right) \\
&= \frac{-1}{n\pi} \left\{ \left[f(x) \cos(nx) \right]_{-\pi}^{p_1} + \left[f(x) \cos(nx) \right]_{p_1}^{p_2} + \cdots + \left[f(x) \cos(nx) \right]_{p_m}^{\pi} \right\} \\
&\qquad + \frac{1}{n\pi} \left(\int_{-\pi}^{p_1} f'(x) \cos(nx) \, \mathrm{d}x + \int_{p_1}^{p_2} f'(x) \cos(nx) \, \mathrm{d}x + \int_{p_m}^{\pi} f'(x) \cos(nx) \, \mathrm{d}x \right) \\
&= -\frac{1}{n} B_n + \frac{1}{n} a'_n
\end{aligned}$$

where

$$B_n = \frac{(-1)^n}{\pi} \left[f(-\pi) - f(\pi) \right] - \frac{1}{\pi} \sum_{j=1}^{m} \cos(np_j) \left[f(p_j^-) - f(p_j^+) \right]$$

and the a'_n are the cosine coefficients of $f'(x)$.

Since $f(x)$ and $f'(x)$ are bounded, $B_n, a'_n = O(1)$ and thus $b_n = O(1/n)$ as $n \to \infty$.

With integration by parts on the Fourier coefficients of $f'(x)$ we could find that

$$a'_n = \frac{1}{n} A'_n - \frac{1}{n} b''_n$$

where $A'_n = \frac{1}{\pi} \sum_{j=1}^{m} \sin(np_j)[f'(p_j^-) - f'(p_j^+)]$ and the b''_n are the sine coefficients of $f''(x)$, and

$$b'_n = -\frac{1}{n} B'_n + \frac{1}{n} a''_n$$

where $B'_n = \frac{(-1)^n}{\pi}[f'(-\pi) - f'(\pi)] - \frac{1}{\pi} \sum_{j=1}^{m} \cos(np_j)[f'(p_j^-) - f'(p_j^+)]$ and the a''_n are the cosine coefficients of $f''(x)$.

Now we can rewrite a_n and b_n as

$$a_n = \frac{1}{n} A_n + \frac{1}{n^2} B'_n - \frac{1}{n^2} a''_n$$
$$b_n = -\frac{1}{n} B_n + \frac{1}{n^2} A'_n - \frac{1}{n^2} b''_n.$$

Continuing this process we could define $A_n^{(j)}$ and $B_n^{(j)}$ so that

$$a_n = \frac{1}{n} A_n + \frac{1}{n^2} B'_n - \frac{1}{n^3} A''_n - \frac{1}{n^4} B'''_n + \cdots$$
$$b_n = -\frac{1}{n} B_n + \frac{1}{n^2} A'_n + \frac{1}{n^3} B''_n - \frac{1}{n^4} A'''_n - \cdots .$$

For any bounded function, the Fourier coefficients satisfy $a_n, b_n = O(1/n)$ as $n \to \infty$. If A_n and B_n are zero then the Fourier coefficients will be $O(1/n^2)$. A sufficient condition for this is that the periodic extension of $f(x)$ is continuous. We see that if the periodic extension of $f'(x)$ is continuous then A'_n and B'_n will be zero and the Fourier coefficients will be $O(1/n^3)$.

Result 30.8.1 Let $f(x)$ be a bounded function for which there is a partition of the range $(-\pi, \pi)$ into a finite number of intervals such that $f(x)$ and all it's derivatives are continuous on each of the intervals. If $f(x)$ is not continuous then the Fourier coefficients are $O(1/n)$. If $f(x), f'(x), \ldots, f^{(k-2)}(x)$ are continuous then the Fourier coefficients are $O(1/n^k)$.

If the periodic extension of $f(x)$ is continuous, then the Fourier coefficients will be $O(1/n^2)$. The series $\sum_{n=1}^{\infty} |a_n \cos(nx) b_n \sin(nx)|$ can be bounded by $M \sum_{n=1}^{\infty} 1/n^2$ where $M = \max_n(|a_n| + |b_n|)$. Thus the Fourier series converges to $f(x)$ uniformly.

Result 30.8.2 If the periodic extension of $f(x)$ is continuous then the Fourier series of $f(x)$ will converge uniformly for all x.

If the periodic extension of $f(x)$ is not continuous, we have the following result.

Result 30.8.3 If $f(x)$ is continuous in the interval $c < x < d$, then the Fourier series is uniformly convergent in the interval $c + \delta \leq x \leq d - \delta$ for any $\delta > 0$.

Example 30.8.2 Different Rates of Convergence.

A Discontinuous Function. Consider the function defined by

$$f_1(x) = \begin{cases} -1 & \text{for } -1 < x < 0 \\ 1, & \text{for } 0 < x < 1. \end{cases}$$

This function has jump discontinuities, so we know that the Fourier coefficients are $O(1/n)$.

Since this function is odd, there will only be sine terms in it's Fourier expansion. Furthermore, since the function is symmetric about $x = 1/2$, there will be only odd sine terms. Computing these terms,

$$\begin{aligned} b_n &= 2 \int_0^1 \sin(n\pi x) \, \mathrm{d}x \\ &= 2 \left[\frac{-1}{n\pi} \cos(n\pi x) \right]_0^1 \\ &= 2 \left(-\frac{(-1)^n}{n\pi} - \frac{-1}{n\pi} \right) \\ &= \begin{cases} \frac{4}{n\pi} & \text{for odd } n \\ 0 & \text{for even } n. \end{cases} \end{aligned}$$

The function and the sum of the first three terms in the expansion are plotted, in dashed and solid lines respectively, in Figure 30.7. Although the three term sum follows the general shape of the function, it is clearly not a good approximation.

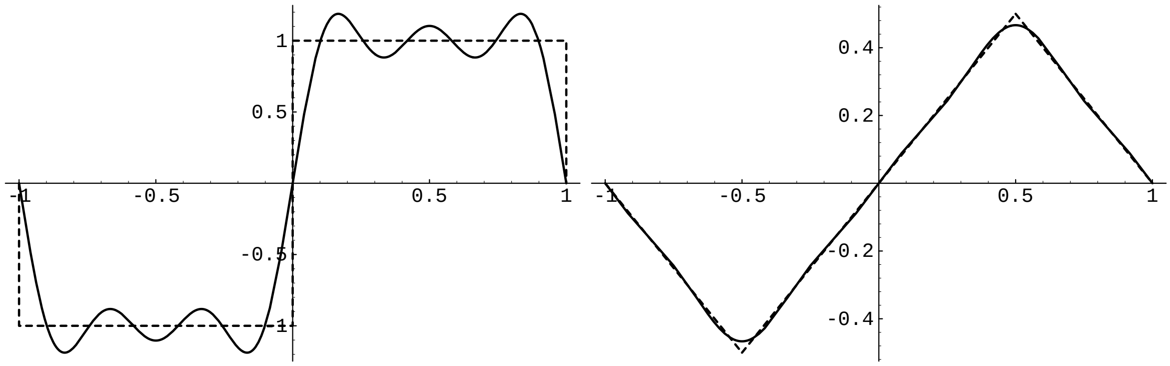

Figure 30.7: Three Term Approximation for a Function with Jump Discontinuities and a Continuous Function.

A Continuous Function. Consider the function defined by

$$f_2(x) = \begin{cases} -x - 1 & \text{for } -1 < x < -1/2 \\ x & \text{for } -1/2 < x < 1/2 \\ -x + 1 & \text{for } 1/2 < x < 1. \end{cases}$$

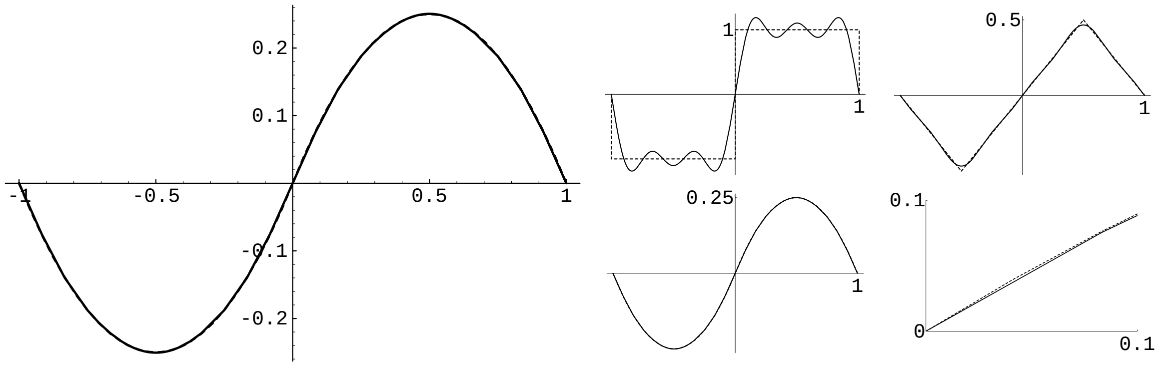

Figure 30.8: Three Term Approximation for a Function with Continuous First Derivative and Comparison of the Rates of Convergence.

Since this function is continuous, the Fourier coefficients will be $O(1/n^2)$. Also we see that there will only be odd sine terms in the expansion.

$$
\begin{aligned}
b_n &= \int_{-1}^{-1/2} (-x-1)\sin(n\pi x)\,\mathrm{d}x + \int_{-1/2}^{1/2} x\sin(n\pi x)\,\mathrm{d}x + \int_{1/2}^{1} (-x+1)\sin(n\pi x)\,\mathrm{d}x \\
&= 2\int_0^{1/2} x\sin(n\pi x)\,\mathrm{d}x + 2\int_{1/2}^{1} (1-x)\sin(n\pi x)\,\mathrm{d}x \\
&= \frac{4}{(n\pi)^2}\sin(n\pi/2) \\
&= \begin{cases} \frac{4}{(n\pi)^2}(-1)^{(n-1)/2} & \text{for odd } n \\ 0 & \text{for even } n. \end{cases}
\end{aligned}
$$

The function and the sum of the first three terms in the expansion are plotted, in dashed and solid lines respectively, in Figure 30.7. We see that the convergence is much better than for the function with jump discontinuities.

A Function with a Continuous First Derivative. Consider the function defined by

$$f_3(x) = \begin{cases} x(1+x) & \text{for } -1 < x < 0 \\ x(1-x) & \text{for } 0 < x < 1. \end{cases}$$

Since the periodic extension of this function is continuous and has a continuous first derivative, the Fourier coefficients will be $O(1/n^3)$. We see that the Fourier expansion will contain only odd sine terms.

$$\begin{aligned} b_n &= \int_{-1}^{0} x(1+x)\sin(n\pi x)\,\mathrm{d}x + \int_0^1 x(1-x)\sin(n\pi x)\,\mathrm{d}x \\ &= 2\int_0^1 x(1-x)\sin(n\pi x)\,\mathrm{d}x \\ &= \frac{4(1-(-1)^n)}{(n\pi)^3} \\ &= \begin{cases} \frac{4}{(n\pi)^3} & \text{for odd } n \\ 0 & \text{for even } n. \end{cases} \end{aligned}$$

The function and the sum of the first three terms in the expansion are plotted in Figure 30.8. We see that the first three terms give a very good approximation to the function. The plots of the function, (in a dashed line), and the three term approximation, (in a solid line), are almost indistinguishable.

In Figure 30.8 the convergence of the of the first three terms to $f_1(x)$, $f_2(x)$, and $f_3(x)$ are compared. In the last graph we see a closeup of $f_3(x)$ and it's Fourier expansion to show the error.

30.9 Gibb's Phenomenon

The Fourier expansion of

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ -1 & \text{for } -1 \leq x < 0 \end{cases}$$

is

$$f(x) \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi x).$$

For any fixed x, the series converges to $\frac{1}{2}(f(x^-) + f(x^+))$. For any $\delta > 0$, the convergence is uniform in the intervals $-1 + \delta \leq x \leq -\delta$ and $\delta \leq x \leq 1 - \delta$. How will the nonuniform convergence at integral values of x affect the Fourier series? Finite Fourier series are plotted in Figure 30.9 for 5, 10, 50 and 100 terms. (The plot for 100 terms is closeup of the behavior near $x = 0$.) Note that at each discontinuous point there is a series of overshoots and undershoots that are pushed closer to the discontinuity by increasing the number of terms, but do not seem to decrease in height. In fact, as the number of terms goes to infinity, the height of the overshoots and undershoots does not vanish. This is known as Gibb's phenomenon.

30.10 Integrating and Differentiating Fourier Series

Integrating Fourier Series. Since integration is a smoothing operation, any convergent Fourier series can be integrated term by term to yield another convergent Fourier series.

Example 30.10.1 Consider the step function

$$f(x) = \begin{cases} \pi & \text{for } 0 \leq x < \pi \\ -\pi & \text{for } -\pi \leq x < 0. \end{cases}$$

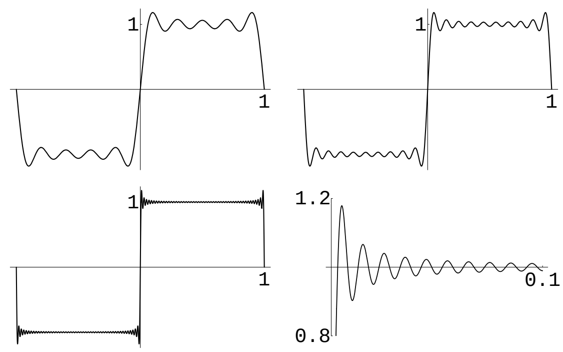

Figure 30.9:

Since this is an odd function, there are no cosine terms in the Fourier series.

$$
\begin{aligned}
b_n &= \frac{2}{\pi} \int_0^\pi \pi \sin(nx) \, \mathrm{d}x \\
&= 2 \left[-\frac{1}{n} \cos(nx) \right]_0^\pi \\
&= \frac{2}{n} (1 - (-1)^n) \\
&= \begin{cases} \frac{4}{n} & \text{for odd } n \\ 0 & \text{for even } n. \end{cases}
\end{aligned}
$$

$$f(x) \sim \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{4}{n} \sin nx$$

Integrating this relation,

$$\int_{-\pi}^{x} f(t)\,dt \sim \int_{-\pi}^{x} \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{4}{n} \sin(nt)\,dt$$

$$\begin{aligned}
F(x) &\sim \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{4}{n} \int_{-\pi}^{x} \sin(nt)\,dt \\
&= \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{4}{n} \left[-\frac{1}{n} \cos(nt) \right]_{-\pi}^{x} \\
&= \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{4}{n^2} (-\cos(nx) + (-1)^n) \\
&= 4 \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{-1}{n^2} - 4 \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{\cos(nx)}{n^2}
\end{aligned}$$

Since this series converges uniformly,

$$4 \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{-1}{n^2} - 4 \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{\cos(nx)}{n^2} = F(x) = \begin{cases} -x - \pi & \text{for } -\pi \leq x < 0 \\ x - \pi & \text{for } 0 \leq x < \pi. \end{cases}$$

The value of the constant term is

$$4 \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{-1}{n^2} = \frac{2}{\pi} \int_{0}^{\pi} F(x)\,\mathrm{d}x = -\frac{1}{\pi}.$$

Thus

$$-\frac{1}{\pi} - 4\sum_{\substack{n=1\\ \text{odd}n}}^{\infty} \frac{\cos(nx)}{n^2} = \begin{cases} -x-\pi & \text{for } -\pi \leq x < 0 \\ x-\pi & \text{for } 0 \leq x < \pi. \end{cases}$$

Differentiating Fourier Series. Recall that in general, a series can only be differentiated if it is uniformly convergent. The necessary and sufficient condition that a Fourier series be uniformly convergent is that the periodic extension of the function is continuous.

Result 30.10.1 The Fourier series of a function $f(x)$ can be differentiated only if the periodic extension of $f(x)$ is continuous.

Example 30.10.2 Consider the function defined by

$$f(x) = \begin{cases} \pi & \text{for } 0 \leq x < \pi \\ -\pi & \text{for } -\pi \leq x < 0. \end{cases}$$

$f(x)$ has the Fourier series

$$f(x) \sim \sum_{\substack{n=1\\ \text{odd}n}}^{\infty} \frac{4}{n} \sin nx.$$

The function has a derivative except at the points $x = n\pi$. Differentiating the Fourier series yields

$$f'(x) \sim 4\sum_{\substack{n=1\\ \text{odd}n}}^{\infty} \cos(nx).$$

For $x \neq n\pi$, this implies

$$0 = 4 \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \cos(nx),$$

which is false. The series does not converge. This is as we expected since the Fourier series for $f(x)$ is not uniformly convergent.

30.11 Exercises

Exercise 30.1

1. Consider a 2π periodic function $f(x)$ expressed as a Fourier series with partial sums

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nt).$$

   Assuming that the Fourier series converges in the mean, i.e.

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} \left(f(x) - S_N(x)\right)^2 \, \mathrm{d}x = 0,$$

   show

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, \mathrm{d}x.$$

   This is called Parseval's equation.

2. Find the Fourier series for $f(x) = x$ on $-\pi \leq x < \pi$ (and repeating periodically). Use this to show

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3. Similarly, by choosing appropriate functions $f(x)$, use Parseval's equation to determine

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^6}.$$

Exercise 30.2
Consider the Fourier series of $f(x) = x$ on $-\pi \leq x < \pi$ as found above. Investigate the convergence at the points of discontinuity.

1. Let S_N be the sum of the first N terms in the Fourier series. Show that
$$\frac{\mathrm{d}S_N}{\mathrm{d}x} = 1 - (-1)^N \frac{\cos\left(\left(N + \frac{1}{2}\right)x\right)}{\cos\left(\frac{x}{2}\right)}.$$
2. Now use this to show that
$$x - S_N = \int_0^x \frac{\sin\left(\left(N + \frac{1}{2}\right)(\xi - \pi)\right)}{\sin\left(\frac{\xi - \pi}{2}\right)}\,\mathrm{d}\xi.$$
3. Finally investigate the maxima of this difference around $x = \pi$ and provide an estimate (good to two decimal places) of the overshoot in the limit $N \to \infty$.

Exercise 30.3
Consider the boundary value problem on the interval $0 < x < 1$
$$y'' + 2y = 1 \quad y(0) = y(1) = 0.$$

1. Choose an appropriate periodic extension and find a Fourier series solution.
2. Solve directly and find the Fourier series of the solution (using the same extension). Compare the result to the previous step and verify the series agree.

Exercise 30.4
Consider the boundary value problem on $0 < x < \pi$
$$y'' + 2y = \sin x \quad y'(0) = y'(\pi) = 0.$$

1. Find a Fourier series solution.

2. Suppose the ODE is slightly modified: $y'' + 4y = \sin x$ with the same boundary conditions. Attempt to find a Fourier series solution and discuss in as much detail as possible what goes wrong.

Exercise 30.5
Find the Fourier cosine and sine series for $f(x) = x^2$ on $0 \leq x < \pi$. Are the series differentiable?

Exercise 30.6
Find the Fourier series of $\cos^n(x)$.

Exercise 30.7
For what values of x does the Fourier series

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx = x^2$$

converge? What is the value of the above Fourier series for all x? From this relation show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

Exercise 30.8

1. Compute the Fourier sine series for the function

$$f(x) = \cos x - 1 + \frac{2x}{\pi}, \qquad 0 \leq x \leq \pi.$$

2. How fast do the Fourier coefficients a_n where

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx$$

decrease with increasing n? Explain this rate of decrease.

Exercise 30.9
Determine the cosine and sine series of

$$f(x) = x \sin x, \qquad (0 < x < \pi).$$

Estimate before doing the calculation the rate of decrease of Fourier coefficients, a_n, b_n, for large n.

Exercise 30.10
Determine the Fourier cosine series of the function

$$f(x) = \cos \nu x, \qquad 0 \leq x \leq \pi,$$

where ν is an arbitrary real number. From this series deduce that for $\nu \neq n$

$$\frac{\pi}{\sin \pi\nu} = \frac{1}{\nu} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\nu - n} + \frac{1}{\nu + n} \right)$$

$$\pi \cot \pi\nu = \frac{1}{\nu} + \sum_{n=1}^{\infty} \left(\frac{1}{\nu - n} + \frac{1}{\nu + n} \right)$$

Integrate the last formula with respect to ν from $\nu = 0$ to $\nu = \theta$, $(0 < \theta < 1)$, to show that

$$\frac{\sin \pi\theta}{\pi\theta} = \prod_{n=1}^{\infty} \left(1 - \frac{\theta^2}{n^2} \right)$$

The symbol $\prod_1^\infty u_n$ denotes the infinite product $u_1 u_2 u_3 \cdots$.

Exercise 30.11

1. Show that

$$\log\cos\left(\frac{x}{2}\right) = -\log 2 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos nx, \qquad -\pi < x < \pi$$

   Hint: use the identity

$$\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{for } |z| \leq 1,\ z \neq 1.$$

2. From this series deduce that

$$\int_0^\pi \log\left(\cos\frac{x}{2}\right) dx = -\pi \log 2.$$

3. Show that

$$\frac{1}{2}\log\left|\frac{\sin((x+\xi)/2)}{\sin((x-\xi)/2)}\right| = \sum_{n=1}^{\infty} \frac{\sin nx \sin n\xi}{n}, \quad x \neq \pm\xi + 2k\pi.$$

Exercise 30.12

Solve the problem

$$y'' + \alpha y = f(x), \quad y(a) = y(b) = 0,$$

with an eigenfunction expansion. Assume that $\alpha \neq n\pi/(b-a)$, $n \in \mathbb{N}$.

Exercise 30.13

Solve the problem

$$y'' + \alpha y = f(x), \quad y(a) = A, \quad y(b) = B,$$

with an eigenfunction expansion. Assume that $\alpha \neq n\pi/(b-a)$, $n \in \mathbb{N}$.

Exercise 30.14
Find the trigonometric series and the simple closed form expressions for $A(r,x)$ and $B(r,x)$ where $z = r\,\mathrm{e}^{ix}$ and $|r| < 1$.

$$\text{a)} \qquad A + iB \equiv \frac{1}{1-z^2} = 1 + z^2 + z^4 + \cdots$$

$$\text{b)} \qquad A + iB \equiv \log(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \cdots$$

Find A_n and B_n, and the trigonometric sum for them where:

$$\text{c)} \qquad A_n + iB_n = 1 + z + z^2 + \cdots + z^n.$$

Exercise 30.15
1. Is the trigonometric system
$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \dots\}$$
orthogonal on the interval $[0, \pi]$? Is the system orthogonal on any interval of length π? Why, in each case?

2. Show that each of the systems
$$\{1, \cos x, \cos 2x, \dots\}, \quad \text{and} \quad \{\sin x, \sin 2x, \dots\}$$
are orthogonal on $[0, \pi]$. Make them orthonormal too.

Exercise 30.16
Let $S_N(x)$ be the N^{th} partial sum of the Fourier series for $f(x) \equiv |x|$ on $-\pi < x < \pi$. Find N such that $|f(x) - S_N(x)| < 10^{-1}$ on $|x| < \pi$.

Exercise 30.17
The set $\{\sin(nx)\}_{n=1}^{\infty}$ is orthogonal and complete on $[0, \pi]$.

1. Find the Fourier sine series for $f(x) \equiv 1$ on $0 \leq x \leq \pi$.
2. Find a convergent series for $g(x) = x$ on $0 \leq x \leq \pi$ by integrating the series for part (a).
3. Apply Parseval's relation to the series in (a) to find:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Check this result by evaluating the series in (b) at $x = \pi$.

Exercise 30.18

1. Show that the Fourier cosine series expansion on $[0, \pi]$ of:

$$f(x) \equiv \begin{cases} 1, & 0 \leq x < \frac{\pi}{2}, \\ \frac{1}{2}, & x = \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < x \leq \pi, \end{cases}$$

is

$$S(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos((2n+1)x).$$

2. Show that the N^{th} partial sum of the series in (a) is

$$S_N(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^{x-\pi/2} \frac{\sin((2(N+1)t)}{\sin t}\, dt.$$

(Hint: Consider the difference of $\sum_{n=1}^{2N+1} (\mathrm{e}^{iy})^n$ and $\sum_{n=1}^{N} (\mathrm{e}^{i2y})^n$, where $y = x - \pi/2$.)

3. Show that $dS_N(x)/dx = 0$ at $x = x_n = \frac{n\pi}{2(N+1)}$ for $n = 0, 1, \ldots, N, N+2, \ldots, 2N+2$.

4. Show that at $x = x_N$, the maximum of $S_N(x)$ nearest to $\pi/2$ in $(0, \pi/2)$ is

$$S_N(x_N) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\frac{\pi N}{2(N+1)}} \frac{\sin(2(N+1)t)}{\sin t} \, dt.$$

Clearly $x_N \uparrow \pi/2$ as $N \to \infty$.

5. Show that also in this limit,

$$S_N(x_N) \to \frac{1}{2} + \frac{1}{\pi} \int_0^{\pi} \frac{\sin t}{t} \, dt \approx 1.0895.$$

How does this compare with $f(\pi/2 - 0)$? This overshoot is the Gibbs phenomenon that occurs at each discontinuity. It is a manifestation of the non-uniform convergence of the Fourier series for $f(x)$ on $[0, \pi]$.

Exercise 30.19

Prove the Isoperimetric Inequality: $L^2 \geq 4\pi A$ where L is the length of the perimeter and A the area of any piecewise smooth plane figure. Show that equality is attained only for the circle. (Hints: The closed curve is represented parametrically as

$$x = x(s), \quad y = y(s), \quad 0 \leq s \leq L$$

where s is the arclength. In terms of $t = 2\pi s/L$ we have

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \left(\frac{L}{2\pi}\right)^2.$$

Integrate this relation over $[0, 2\pi]$. The area is given by

$$A = \int_0^{2\pi} x \frac{\mathrm{d}y}{\mathrm{d}t} \, dt.$$

Express $x(t)$ and $y(t)$ as Fourier series and use the completeness and orthogonality relations to show that $L^2 - 4\pi A \geq 0$.)

Exercise 30.20

1. Find the Fourier sine series expansion and the Fourier cosine series expansion of

$$g(x) = x(1-x), \text{ on } 0 \le x \le 1.$$

Which is better and why over the indicated interval?

2. Use these expansions to show that:

$$\text{i) } \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \text{ii) } \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}, \quad \text{iii) } \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)^2} = -\frac{\pi^3}{32}.$$

Note: Some useful integration by parts formulas are:

$$\int x\sin(nx) = \frac{1}{n^2}\sin(nx) - \frac{x}{n}\cos(nx); \quad \int x\cos(nx) = \frac{1}{n^2}\cos(nx) + \frac{x}{n}\sin(nx)$$

$$\int x^2\sin(nx) = \frac{2x}{n^2}\sin(nx) - \frac{n^2x^2-2}{n^3}\cos(nx)$$

$$\int x^2\cos(nx) = \frac{2x}{n^2}\cos(nx) + \frac{n^2x^2-2}{n^3}\sin(nx)$$

30.12 Hints

Hint 30.1

Hint 30.2

Hint 30.3

Hint 30.4

Hint 30.5

Hint 30.6
Expand

$$\cos^n(x) = \left[\frac{1}{2}(\mathrm{e}^{ix} + \mathrm{e}^{-ix})\right]^n$$

Using Newton's binomial formula.

Hint 30.7

Hint 30.8

Hint 30.9

Hint 30.10

Hint 30.11

Hint 30.12

Hint 30.13

Hint 30.14

Hint 30.15

Hint 30.16

Hint 30.17

Hint 30.18

Hint 30.19

Hint 30.20

30.13 Solutions

Solution 30.1

1. We start by assuming that the Fourier series converges in the mean.

$$\int_{-\pi}^{\pi} \left(f(x) - \frac{a_0}{2} - \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \right)^2 = 0$$

We interchange the order of integration and summation.

$$\int_{-\pi}^{\pi} (f(x))^2 \,\mathrm{d}x - a_0 \int_{-\pi}^{\pi} f(x) \,\mathrm{d}x - 2 \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} f(x) \cos(nx) \,\mathrm{d}x + b_n \int_{-\pi}^{\pi} f(x) \sin(nx) \right)$$
$$+ \frac{\pi a_0^2}{2} + a_0 \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos(nx) + b_n \sin(nx)) \,\mathrm{d}x$$
$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos(nx) + b_n \sin(nx))(a_m \cos(mx) + b_m \sin(mx)) \,\mathrm{d}x = 0$$

Most of the terms vanish because the eigenfunctions are orthogonal.

$$\int_{-\pi}^{\pi} (f(x))^2 \,\mathrm{d}x - a_0 \int_{-\pi}^{\pi} f(x) \,\mathrm{d}x - 2 \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} f(x) \cos(nx) \,\mathrm{d}x + b_n \int_{-\pi}^{\pi} f(x) \sin(nx) \right)$$
$$+ \frac{\pi a_0^2}{2} + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n^2 \cos^2(nx) + b_n^2 \sin^2(nx)) \,\mathrm{d}x = 0$$

We use the definition of the Fourier coefficients to evaluate the integrals in the last sum.

$$\int_{-\pi}^{\pi} (f(x))^2 \,\mathrm{d}x - \pi a_0^2 - 2\pi \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) + \frac{\pi a_0^2}{2} + \pi \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = 0$$

$$\boxed{\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \,\mathrm{d}x}$$

2. We determine the Fourier coefficients for $f(x) = x$. Since $f(x)$ is odd, all of the a_n are zero.

$$\begin{aligned} b_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) \,\mathrm{d}x \\ &= \frac{1}{\pi} \left[-\frac{1}{n} x \cos(nx) \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{1}{n} \cos(nx) \,\mathrm{d}x \\ &= \frac{2(-1)^{n+1}}{n} \end{aligned}$$

The Fourier series is

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) \quad \text{for } x \in (-\pi \ldots \pi).$$

We apply Parseval's theorem for this series to find the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{4}{n^2} &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \,\mathrm{d}x \\ \sum_{n=1}^{\infty} \frac{4}{n^2} &= \frac{2\pi^2}{3} \\ \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}} \end{aligned}$$

3. Consider $f(x) = x^2$. Since the function is even, there are no sine terms in the Fourier series. The coefficients in the cosine series are

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi x^2 \,\mathrm{d}x \\ &= \frac{2\pi^2}{3} \\ a_n &= \frac{2}{\pi} \int_0^\pi x^2 \cos(nx) \,\mathrm{d}x \\ &= \frac{4(-1)^n}{n^2}. \end{aligned}$$

Thus the Fourier series is

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) \quad \text{for } x \in (-\pi \ldots \pi).$$

We apply Parseval's theorem for this series to find the value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

$$\frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \,\mathrm{d}x$$

$$\frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{2\pi^4}{5}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$

Now we integrate the series for $f(x) = x^2$.

$$\int_0^x \left(\xi^2 - \frac{\pi^2}{3}\right) \mathrm{d}\xi = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \int_0^x \cos(n\xi)\,\mathrm{d}\xi$$
$$\frac{x^3}{3} - \frac{\pi^2}{3} x = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nx)$$

We apply Parseval's theorem for this series to find the value of $\sum_{n=1}^{\infty} \frac{1}{n^6}$.

$$16 \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{x^3}{3} - \frac{\pi^2}{3} x\right)^2 \mathrm{d}x$$
$$16 \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{16\pi^6}{945}$$
$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}}$$

Solution 30.2

1. We differentiate the partial sum of the Fourier series and evaluate the sum.

$$S_N = \sum_{n=1}^{N} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

$$S_N' = 2\sum_{n=1}^{N} (-1)^{n+1} \cos(nx)$$

$$S_N' = 2\Re\left(\sum_{n=1}^{N} (-1)^{n+1} \mathrm{e}^{\mathrm{i}nx}\right)$$

$$S_N' = 2\Re\left(\frac{1 - (-1)^{N+2} \mathrm{e}^{\mathrm{i}(N+1)x}}{1 + \mathrm{e}^{\mathrm{i}x}}\right)$$

$$S_N' = \Re\left(\frac{1 + \mathrm{e}^{-\mathrm{i}x} - (-1)^N \mathrm{e}^{\mathrm{i}(N+1)x} - (-1)^N \mathrm{e}^{\mathrm{i}Nx}}{1 + \cos(x)}\right)$$

$$S_N' = 1 - (-1)^N \frac{\cos((N+1)x) + \cos(Nx)}{1 + \cos(x)}$$

$$S_N' = 1 - (-1)^N \frac{\cos\left(\left(N + \frac{1}{2}\right)x\right) \cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}$$

$$\boxed{\frac{\mathrm{d}S_N}{\mathrm{d}x} = 1 - (-1)^N \frac{\cos\left(\left(N + \frac{1}{2}\right)x\right)}{\cos\left(\frac{x}{2}\right)}}$$

2. We integrate S_N'.

$$S_N(x) - S_N(0) = x - \int_0^x \frac{(-1)^N \cos\left(\left(N+\frac{1}{2}\right)\xi\right)}{\cos\left(\frac{\xi}{2}\right)}\,\mathrm{d}\xi$$

$$\boxed{x - S_N = \int_0^x \frac{\sin\left(\left(N+\frac{1}{2}\right)(\xi-\pi)\right)}{\sin\left(\frac{\xi-\pi}{2}\right)}\,\mathrm{d}\xi}$$

3. We find the extrema of the overshoot $E = x - S_N$ with the first derivative test.

$$E' = \frac{\sin\left(\left(N+\frac{1}{2}\right)(x-\pi)\right)}{\sin\left(\frac{x-\pi}{2}\right)} = 0$$

We look for extrema in the range $(-\pi \ldots \pi)$.

$$\left(N + \frac{1}{2}\right)(x - \pi) = -n\pi$$

$$x = \pi\left(1 - \frac{n}{N+1/2}\right), \quad n \in [1 \ldots 2N]$$

The closest of these extrema to $x = \pi$ is

$$x = \pi\left(1 - \frac{1}{N+1/2}\right)$$

Let E_0 be the overshoot at this point. We approximate E_0 for large N.

$$E_0 = \int_0^{\pi(1-1/(N+1/2))} \frac{\sin\left(\left(N+\frac{1}{2}\right)(\xi-\pi)\right)}{\sin\left(\frac{\xi-\pi}{2}\right)}\,\mathrm{d}\xi$$

We shift the limits of integration.

$$E_0 = \int_{\pi/(N+1/2)}^{\pi} \frac{\sin\left(\left(N+\frac{1}{2}\right)\xi\right)}{\sin\left(\frac{\xi}{2}\right)}\,\mathrm{d}\xi$$

We add and subtract an integral over $[0 \ldots \pi/(N+1/2)]$.

$$E_0 = \int_0^\pi \frac{\sin\left(\left(N+\frac{1}{2}\right)\xi\right)}{\sin\left(\frac{\xi}{2}\right)}\,\mathrm{d}\xi - \int_0^{\pi/(N+1/2)} \frac{\sin\left(\left(N+\frac{1}{2}\right)\xi\right)}{\sin\left(\frac{\xi}{2}\right)}\,\mathrm{d}\xi$$

We can evaluate the first integral with contour integration on the unit circle C.

$$\begin{aligned}
\int_0^\pi \frac{\sin\left(\left(N+\frac{1}{2}\right)\xi\right)}{\sin\left(\frac{\xi}{2}\right)}\,\mathrm{d}\xi &= \int_0^\pi \frac{\sin\left((2N+1)\,\xi\right)}{\sin\left(\xi\right)}\,\mathrm{d}\xi \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \frac{\sin\left((2N+1)\,\xi\right)}{\sin\left(\xi\right)}\,\mathrm{d}\xi \\
&= \frac{1}{2}\oint_C \frac{\Im\left(z^{2N+1}\right)}{(z-1/z)/(\mathrm{i}2)}\,\frac{\mathrm{d}z}{\mathrm{i}z} \\
&= \Im\left(\oint_C \frac{z^{2N+1}}{(z^2-1)}\,\mathrm{d}z\right) \\
&= \Im\left(\mathrm{i}\pi\,\mathrm{Res}\left(\frac{z^{2N+1}}{(z+1)(z-1)}, 1\right) + \mathrm{i}\pi\,\mathrm{Res}\left(\frac{z^{2N+1}}{(z+1)(z-1)}, -1\right)\right) \\
&= \pi\Re\left(\frac{1^{2N+1}}{2} + \frac{(-1)^{2N+1}}{-2}\right) \\
&= \pi
\end{aligned}$$

We approximate the second integral.

$$
\begin{aligned}
\int_0^{\pi/(N+1/2)} \frac{\sin\left(\left(N+\frac{1}{2}\right)\xi\right)}{\sin\left(\frac{\xi}{2}\right)} \,\mathrm{d}\xi &= \frac{2}{2N+1} \int_0^{\pi} \frac{\sin(x)}{\sin\left(\frac{x}{2N+1}\right)} \,\mathrm{d}x \\
&\approx 2 \int_0^{\pi} \frac{\sin(x)}{x} \,\mathrm{d}x \\
&= 2 \int_0^{\pi} \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \,\mathrm{d}x \\
&= 2 \sum_{n=0}^{\infty} \int_0^{\pi} \frac{(-1)^n x^{2n}}{(2n+1)!} \,\mathrm{d}x \\
&= 2 \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)(2n+1)!} \,\mathrm{d}x \\
&\approx 3.70387
\end{aligned}
$$

In the limit as $N \to \infty$, the overshoot is

$$|\pi - 3.70387| \approx 0.56.$$

Solution 30.3

1. The eigenfunctions of the self-adjoint problem

$$-y'' = \lambda y, \quad y(0) = y(1) = 0,$$

are

$$\phi_n = \sin(n\pi x), \quad n \in \mathbb{Z}^+$$

We find the series expansion of the inhomogeneity $f(x) = 1$.

$$1 = \sum_{n=1}^{\infty} f_n \sin(n\pi x)$$

$$f_n = 2\int_0^1 \sin(n\pi x)\,\mathrm{d}x$$

$$f_n = 2\left[-\frac{\cos(n\pi x)}{n\pi}\right]_0^1$$

$$f_n = \frac{2}{n\pi}(1 - (-1)^n)$$

$$f_n = \begin{cases} \frac{4}{n\pi} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

We expand the solution in a series of the eigenfunctions.

$$y = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

We substitute the series into the differential equation.

$$y'' + 2y = 1$$

$$-\sum_{n=1}^{\infty} a_n \pi^2 n^2 \sin(n\pi x) + 2\sum_{n=1}^{\infty} a_n \sin(n\pi x) = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{4}{n\pi} \sin(n\pi x)$$

$$a_n = \begin{cases} \frac{4}{n\pi(2-\pi^2 n^2)} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

$$\boxed{y = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{4}{n\pi(2 - \pi^2 n^2)} \sin(n\pi x)}$$

2. Now we solve the boundary value problem directly.

$$y'' + 2y = 1 \quad y(0) = y(1) = 0$$

The general solution of the differential equation is

$$y = c_1 \cos\left(\sqrt{2}x\right) + c_2 \sin\left(\sqrt{2}x\right) + \frac{1}{2}.$$

We apply the boundary conditions to find the solution.

$$c_1 + \frac{1}{2} = 0, \quad c_1 \cos\left(\sqrt{2}\right) + c_2 \sin\left(\sqrt{2}\right) + \frac{1}{2} = 0$$

$$c_1 = -\frac{1}{2}, \quad c_2 = \frac{\cos\left(\sqrt{2}\right) - 1}{2\sin\left(\sqrt{2}\right)}$$

$$\boxed{y = \frac{1}{2}\left(1 - \cos\left(\sqrt{2}x\right) + \frac{\cos\left(\sqrt{2}\right) - 1}{\sin\left(\sqrt{2}\right)} \sin\left(\sqrt{2}x\right)\right)}$$

We find the Fourier sine series of the solution.

$$y = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

$$a_n = 2\int_0^1 y(x)\sin(n\pi x)\,\mathrm{d}x$$

$$a_n = \int_0^1 \left(1 - \cos\left(\sqrt{2}x\right) + \frac{\cos\left(\sqrt{2}\right) - 1}{\sin\left(\sqrt{2}\right)} \sin\left(\sqrt{2}x\right)\right)\sin(n\pi x)\,\mathrm{d}x$$

$$a_n = \frac{2(1 - (-1)^2}{n\pi(2 - \pi^2 n^2)}$$

$$a_n = \begin{cases} \frac{4}{n\pi(2-\pi^2 n^2)} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

We obtain the same series as in the first part.

Solution 30.4

1. The eigenfunctions of the self-adjoint problem

$$-y'' = \lambda y, \quad y'(0) = y'(\pi) = 0,$$

are

$$\phi_0 = \frac{1}{2}, \quad \phi_n = \cos(nx), \quad n \in \mathbb{Z}^+$$

We find the series expansion of the inhomogeneity $f(x) = \sin(x)$.

$$f(x) = \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n \cos(nx)$$

$$f_0 = \frac{2}{\pi} \int_0^{\pi} \sin(x) \, \mathrm{d}x$$

$$f_0 = \frac{4}{\pi}$$

$$f_n = \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(nx) \, \mathrm{d}x$$

$$f_n = \frac{2(1 + (-1)^n)}{\pi(1 - n^2)}$$

$$f_n = \begin{cases} \frac{4}{\pi(1-n^2)} & \text{for even } n \\ 0 & \text{for odd } n \end{cases}$$

We expand the solution in a series of the eigenfunctions.

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

We substitute the series into the differential equation.

$$y'' + 2y = \sin(x)$$

$$-\sum_{n=1}^{\infty} a_n n^2 \cos(nx) + a_0 + 2\sum_{n=1}^{\infty} a_n \cos(nx) = \frac{2}{\pi} + \sum_{\substack{n=2\\ \text{even } n}}^{\infty} \frac{4}{\pi(1-n^2)} \cos(nx)$$

$$\boxed{y = \frac{1}{\pi} + \sum_{\substack{n=2\\ \text{even } n}}^{\infty} \frac{4}{\pi(1-n^2)(2-n^2)} \cos(nx)}$$

2. We expand the solution in a series of the eigenfunctions.

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

We substitute the series into the differential equation.

$$y'' + 4y = \sin(x)$$

$$-\sum_{n=1}^{\infty} a_n n^2 \cos(nx) + 2a_0 + 4\sum_{n=1}^{\infty} a_n \cos(nx) = \frac{2}{\pi} + \sum_{\substack{n=2\\ \text{even } n}}^{\infty} \frac{4}{\pi(1-n^2)} \cos(nx)$$

It is not possible to solve for the a_2 coefficient. That equation is

$$(0)a_2 = -\frac{4}{3\pi}.$$

This problem is to be expected, as this boundary value problem does not have a solution. The solution of the differential equation is

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{3}\sin(x)$$

The boundary conditions give us an inconsistent set of constraints.

$$y'(0) = 0, \quad y'(\pi) = 0$$
$$c_2 + \frac{1}{3} = 0, \quad c_2 - \frac{1}{3} = 0$$

Thus the problem has no solution.

Solution 30.5
Cosine Series. The coefficients in the cosine series are

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi x^2 \,\mathrm{d}x \\ &= \frac{2\pi^2}{3} \\ a_n &= \frac{2}{\pi} \int_0^\pi x^2 \cos(nx) \,\mathrm{d}x \\ &= \frac{4(-1)^n}{n^2}. \end{aligned}$$

Thus the Fourier cosine series is

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx).$$

In Figure 30.10 the even periodic extension of $f(x)$ is plotted in a dashed line and the sum of the first five terms in the Fourier series is plotted in a solid line. Since the even periodic extension is continuous, the cosine series is differentiable.

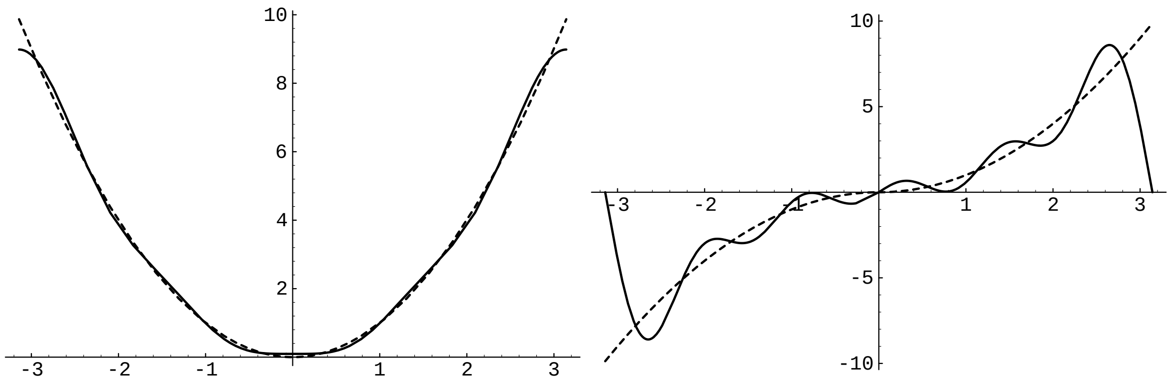

Figure 30.10: The Fourier Cosine and Sine Series of $f(x) = x^2$.

Sine Series. The coefficients in the sine series are

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi x^2 \sin(nx) \, \mathrm{d}x \\ &= -\frac{2(-1)^n \pi}{n} - \frac{4(1-(-1)^n)}{\pi n^3} \\ &= \begin{cases} -\frac{2(-1)^n \pi}{n} & \text{for even } n \\ -\frac{2(-1)^n \pi}{n} - \frac{8}{\pi n^3} & \text{for odd } n. \end{cases} \end{aligned}$$

Thus the Fourier sine series is

$$f(x) \sim -\sum_{n=1}^{\infty} \left(\frac{2(-1)^n \pi}{n} + \frac{4(1-(-1)^n)}{\pi n^3} \right) \sin(nx).$$

In Figure 30.10 the odd periodic extension of $f(x)$ and the sum of the first five terms in the sine series are plotted. Since the odd periodic extension of $f(x)$ is not continuous, the series is not differentiable.

Solution 30.6
We could find the expansion by integrating to find the Fourier coefficients, but it is easier to expand $\cos^n(x)$ directly.

$$\begin{aligned}
\cos^n(x) &= \left[\frac{1}{2}(\mathrm{e}^{ix} + \mathrm{e}^{-ix}) \right]^n \\
&= \frac{1}{2^n} \left[\binom{n}{0} \mathrm{e}^{inx} + \binom{n}{1} \mathrm{e}^{i(n-2)x} + \cdots + \binom{n}{n-1} \mathrm{e}^{-i(n-2)x} + \binom{n}{n} \mathrm{e}^{-inx} \right]
\end{aligned}$$

If n is odd,

$$\begin{aligned}\cos^n(x) &= \frac{1}{2^n}\left[\binom{n}{0}(\mathrm{e}^{inx} + \mathrm{e}^{-inx}) + \binom{n}{1}(\mathrm{e}^{i(n-2)x} + \mathrm{e}^{-i(n-2)x}) + \cdots \right.\\ &\qquad \left. + \binom{n}{(n-1)/2}(\mathrm{e}^{ix} + \mathrm{e}^{-ix})\right]\\ &= \frac{1}{2^n}\left[\binom{n}{0}2\cos(nx) + \binom{n}{1}2\cos((n-2)x) + \cdots + \binom{n}{(n-1)/2}2\cos(x)\right]\\ &= \frac{1}{2^{n-1}}\sum_{m=0}^{(n-1)/2}\binom{n}{m}\cos((n-2m)x)\\ &= \frac{1}{2^{n-1}}\sum_{\substack{k=1\\ \text{odd } k}}^{n}\binom{n}{(n-k)/2}\cos(kx).\end{aligned}$$

If n is even,

$$\cos^n(x) = \frac{1}{2^n}\left[\binom{n}{0}(\,\mathrm{e}^{inx} + \mathrm{e}^{-inx}) + \binom{n}{1}(\,\mathrm{e}^{i(n-2)x} + \mathrm{e}^{-i(n-2)x}) + \cdots \right.$$
$$\left. + \binom{n}{n/2-1}(\,\mathrm{e}^{i2x} + \mathrm{e}^{-i2x}) + \binom{n}{n/2}\right]$$
$$= \frac{1}{2^n}\left[\binom{n}{0}2\cos(nx) + \binom{n}{1}2\cos((n-2)x) + \cdots + \binom{n}{n/2-1}2\cos(2x) + \binom{n}{n/2}\right]$$
$$= \frac{1}{2^n}\binom{n}{n/2} + \frac{1}{2^{n-1}}\sum_{m=0}^{(n-2)/2}\binom{n}{m}\cos((n-2m)x)$$
$$= \frac{1}{2^n}\binom{n}{n/2} + \frac{1}{2^{n-1}}\sum_{\substack{k=2\\ \text{even } k}}^{n}\binom{n}{(n-k)/2}\cos(kx).$$

We may denote,

$$\boxed{\cos^n(x) = \frac{a_0}{2}\sum_{k=1}^{n} a_k\cos(kx),}$$

where

$$\boxed{a_k = \frac{1+(-1)^{n-k}}{2}\frac{1}{2^{n-1}}\binom{n}{(n-k)/2}.}$$

Solution 30.7

We expand $f(x)$ in a cosine series. The coefficients in the cosine series are

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi x^2 \, \mathrm{d}x \\ &= \frac{2\pi^2}{3} \\ a_n &= \frac{2}{\pi} \int_0^\pi x^2 \cos(nx) \, \mathrm{d}x \\ &= \frac{4(-1)^n}{n^2}. \end{aligned}$$

Thus the Fourier cosine series is

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

The Fourier series converges to the even periodic extension of

$$f(x) = x^2 \quad \text{for } 0 < x < \pi,$$

which is

$$\boxed{\hat{f}(x) = \left(x - 2\pi \left(\left\lfloor \frac{x+\pi}{2\pi} \right\rfloor \right) \right)^2.}$$

($\lfloor \cdot \rfloor$ denotes the floor or greatest integer function.) This periodic extension is a continuous function. Since x^2 is an even function, we have

$$\boxed{\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx = x^2 \quad \text{for } -\pi \leq x \leq \pi.}$$

We substitute $x = \pi$ into the Fourier series.

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi) = \pi^2$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

We substitute $x = 0$ into the Fourier series.

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 0$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}}$$

Solution 30.8

1. The Fourier sine coefficients are

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi} \left(\cos x - 1 + \frac{2x}{\pi}\right)\sin(nx)\,\mathrm{d}x \end{aligned}$$

$$\boxed{a_n = \frac{2(1+(-1)^n)}{\pi(n^3 - n)}}$$

2. From our work in the previous part, we see that the Fourier coefficients decay as $1/n^3$. The Fourier sine series converges to the odd periodic extension of $f(x)$. We can determine the rate of decay of the Fourier

coefficients from the smoothness of $\hat{f}(x)$. For $-\pi < x < \pi$, the odd periodic extension of $f(x)$ is defined

$$\hat{f}(x) = \begin{cases} f_+(x) = \cos(x) - 1 + \frac{2x}{\pi} & 0 < x < \pi, \\ f_-(x) = -f_+(-x) = -\cos(x) + 1 + \frac{2x}{\pi} & -\pi < x < 0. \end{cases}$$

Since

$$f_+(0) = f_-(0) = 0 \quad \text{and} \quad f_+(\pi) = f_-(-\pi) = 0$$

$\hat{f}(x)$ is continuous, C^0. Since

$$f_+'(0) = f_-'(0) = \frac{2}{\pi} \quad \text{and} \quad f_+'(\pi) = f_-'(-\pi) = \frac{2}{\pi}$$

$\hat{f}(x)$ is continuously differentiable, C^1. However, since

$$f_+''(0) = -1, \quad f_-''(0) = 1$$

$\hat{f}(x)$ is not C^2. Since $\hat{f}(x)$ is C^1 we know that the Fourier coefficients decay as $1/n^3$.

Solution 30.9

Cosine Series. The even periodic extension of $f(x)$ is a C^0, continuous, function (See Figure 30.11. Thus the coefficients in the cosine series will decay as $1/n^2$. The Fourier cosine coefficients are

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi x \sin x \, \mathrm{d}x \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{2}{\pi} \int_0^\pi x \sin x \cos x \, \mathrm{d}x \\ &= -\frac{1}{2} \end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^\pi x \sin x \cos(nx) \, \mathrm{d}x \\
&= \frac{2(-1)^{n+1}}{n^2 - 1}, \quad \text{for } n \geq 2
\end{aligned}
$$

The Fourier cosine series is

$$
\boxed{\hat{f}(x) = 1 - \frac{1}{2} \cos x - 2 \sum_{n=2}^{\infty} \frac{2(-1)^n}{n^2 - 1} \cos(nx).}
$$

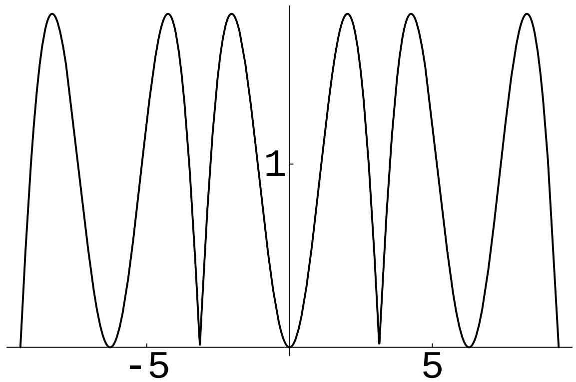

Figure 30.11: The even periodic extension of $x \sin x$.

Sine Series. The odd periodic extension of $f(x)$ is a C^1, continuously differentiable, function (See Figure 30.12. Thus the coefficients in the cosine series will decay as $1/n^3$. The Fourier sine coefficients are

$$
\begin{aligned}
a_1 &= \frac{1}{\pi} \int_0^\pi x \sin x \sin x \, \mathrm{d}x \\
&= \frac{\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^\pi x \sin x \sin(nx)\, \mathrm{d}x \\
&= -\frac{4(1 + (-1)^n)n}{\pi(n^2 - 1)^2}, \quad \text{for } n \geq 2
\end{aligned}
$$

The Fourier sine series is

$$
\boxed{\hat{f}(x) = \frac{\pi}{2} \sin x - \frac{4}{\pi} \sum_{n=2}^{\infty} \frac{(1 + (-1)^n)n}{(n^2 - 1)^2} \cos(nx).}
$$

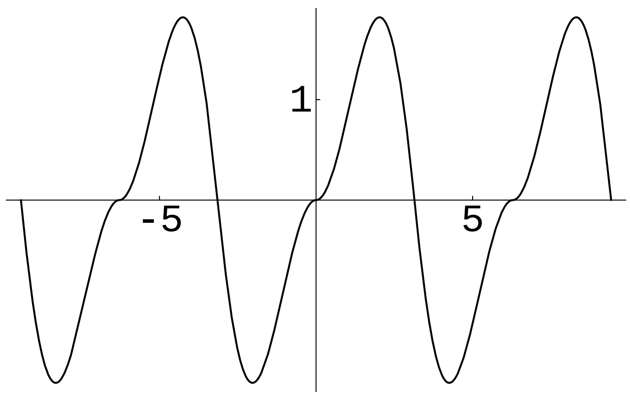

Figure 30.12: The odd periodic extension of $x \sin x$.

Solution 30.10

If $\nu = n$ is an integer, then the Fourier cosine series is $\cos(|n|x)$.

We note that for $\nu \neq n$, the even periodic extension of $cos(\nu x)$ is C^0 so that the series converges to $\cos(\nu x)$

for $-\pi \leq x \leq \pi$ and the coefficients decay as $1/n^2$. If ν is not an integer, then then Fourier cosine coefficients are

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi \cos(\nu x) \, \mathrm{d}x \\ &= \frac{2}{\pi \nu} \sin(\pi \nu) \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi \cos(\nu x) \cos(n x) \, \mathrm{d}x \\ &= (-1)^n \left(\frac{1}{\nu - n} + \frac{1}{\nu + n} \right) \sin(\pi \nu) \end{aligned}$$

The Fourier cosine series is

$$\boxed{\cos(\nu x) = \frac{1}{\pi \nu} \sin(\pi \nu) + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\nu - n} + \frac{1}{\nu + n} \right) \sin(\pi \nu) \cos(n x).}$$

For $\nu \neq n$ we substitute $x = 0$ into the Fourier cosine series.

$$1 = \frac{1}{\pi \nu} \sin(\pi \nu) + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\nu - n} + \frac{1}{\nu + n} \right) \sin(\pi \nu)$$

$$\boxed{\frac{\pi}{\sin \pi \nu} = \frac{1}{\nu} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\nu - n} + \frac{1}{\nu + n} \right)}$$

For $\nu \neq n$ we substitute $x = \pi$ into the Fourier cosine series.

$$\cos(\nu \pi) = \frac{1}{\pi \nu} \sin(\pi \nu) + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\nu - n} + \frac{1}{\nu + n} \right) \sin(\pi \nu) (-1)^n$$

$$\boxed{\pi \cot \pi \nu = \frac{1}{\nu} + \sum_{n=1}^{\infty} \left(\frac{1}{\nu - n} + \frac{1}{\nu + n} \right)}$$

We write the last formula as

$$\pi \cot \pi\nu - \frac{1}{\nu} = \sum_{n=1}^{\infty} \left(\frac{1}{\nu - n} + \frac{1}{\nu + n} \right)$$

We integrate from $\nu = 0$ to $\nu = \theta < 1$.

$$\left[\log \left(\frac{\sin(\pi\nu)}{\nu} \right) \right]_0^\theta = \sum_{n=1}^{\infty} \left([\log(n - \nu)]_0^\theta + [\log(n + \nu)]_0^\theta \right)$$

$$\log \left(\frac{\sin(\pi\theta)}{\theta} \right) - \log(\pi) = \sum_{n=1}^{\infty} \left(\log \left(\frac{n - \theta}{n} \right) + \log \left(\frac{n + \theta}{n} \right) \right)$$

$$\log \left(\frac{\sin(\pi\theta)}{\pi\theta} \right) = \sum_{n=1}^{\infty} \log \left(1 - \frac{\theta^2}{n^2} \right)$$

$$\log \left(\frac{\sin(\pi\theta)}{\pi\theta} \right) = \log \left(\prod_{n=1}^{\infty} \left(1 - \frac{\theta^2}{n^2} \right) \right)$$

$$\boxed{\frac{\sin(\pi\theta)}{\pi\theta} = \prod_{n=1}^{\infty} \left(1 - \frac{\theta^2}{n^2} \right)}$$

Solution 30.11

1. We will consider the principal branch of the logarithm, $-\pi < \arg(\log z) \leq \pi$. For $-\pi < x < \pi$, $\cos(x/2)$ is positive so that $\log(\cos(x/2))$ is real-valued. At $x = \pm\pi$, $\log(\cos(x/2))$ is singular. However, the function is integrable so it has a Fourier series which converges except at $x = (2k + 1)\pi$, $k \in \mathbb{Z}$.

$$\begin{aligned} \log \left(\cos \frac{x}{2} \right) &= \log \left(\frac{e^{ix/2} + e^{-ix/2}}{2} \right) \\ &= -\log 2 + \log \left(e^{-ix/2} \left(1 + e^{ix} \right) \right) \\ &= -\log 2 - i\frac{x}{2} + \log \left(1 + e^{ix} \right) \end{aligned}$$

Since $|\mathrm{e}^{ix}| \leq 1$ and $\mathrm{e}^{ix} \neq -1$ for $\Im(x) \geq 0$, $x \neq (2k+1)\pi$, we can expand the last term in a Taylor series in that domain.

$$= -\log 2 - i\frac{x}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\mathrm{e}^{ix}\right)^n$$

$$= -\log 2 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos nx - i\left(\frac{x}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx\right)$$

For $-\pi < x < \pi$, $\log(\cos(x/2))$ is real-valued. We equate the real parts of the equation on this domain to obtain the Fourier series of $\log(\cos(x/2))$.

$$\boxed{\log\left(\cos\frac{x}{2}\right) = -\log 2 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos(nx), \quad -\pi < x < \pi.}$$

The domain of convergence for this series is $\Im(x) = 0$, $x \neq (2k+1)\pi$. The Fourier series converges to the periodic extension of the function.

$$\boxed{\log\left|\cos\frac{x}{2}\right| = -\log 2 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos(nx), \quad x \neq (2k+1)\pi, \ k \in \mathbb{Z}}$$

2. Now we integrate the function from 0 to π.

$$\int_0^\pi \log\left(\cos\frac{x}{2}\right) \mathrm{d}x = \int_0^\pi \left(-\log 2 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos(nx)\right) \mathrm{d}x$$

$$= -\pi \log 2 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^\pi \cos(nx)\,\mathrm{d}x$$

$$= -\pi \log 2 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[\frac{\sin(nx)}{n}\right]_0^\pi \mathrm{d}x$$

$$\boxed{\int_0^\pi \log\left(\cos\frac{x}{2}\right)\,\mathrm{d}x = -\pi\log 2}$$

3.

$$\frac{1}{2}\log\left|\frac{\sin((x+\xi)/2)}{\sin((x-\xi)/2)}\right| = \frac{1}{2}\log|\sin((x+\xi)/2)| - \frac{1}{2}\log|\sin((x-\xi)/2)|$$

Consider the function $\log|\sin(y/2)|$. Since $\sin(x) = \cos(x-\pi/2)$, we can use the result of part (a) to obtain,

$$\begin{aligned}
\log\left|\sin\frac{y}{2}\right| &= \log\left|\cos\frac{y-\pi}{2}\right| \\
&= -\log 2 - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(n(y-\pi)) \\
&= -\log 2 - \sum_{n=1}^{\infty}\frac{1}{n}\cos(ny), \quad \text{for } y \neq 2\pi k,\ k\in\mathbb{Z}.
\end{aligned}$$

We return to the original function:

$$\frac{1}{2}\log\left|\frac{\sin((x+\xi)/2)}{\sin((x-\xi)/2)}\right| = \frac{1}{2}\left(-\log 2 - \sum_{n=1}^{\infty}\frac{1}{n}\cos(n(x+\xi)) + \log 2 + \sum_{n=1}^{\infty}\frac{1}{n}\cos(n(x-\xi))\right),$$

for $x \pm \xi \neq 2\pi k$, $k \in \mathbb{Z}$.

$$\boxed{\frac{1}{2}\log\left|\frac{\sin((x+\xi)/2)}{\sin((x-\xi)/2)}\right| = \sum_{n=1}^{\infty}\frac{\sin nx \sin n\xi}{n}, \quad x \neq \pm\xi + 2k\pi.}$$

Solution 30.12

The eigenfunction problem associated with this problem is

$$\phi'' + \lambda^2 \phi = 0, \quad \phi(a) = \phi(b) = 0,$$

which has the solutions,

$$\lambda_n = \frac{n\pi}{b-a}, \quad \phi_n = \sin\left(\frac{n\pi(x-a)}{b-a}\right), \quad n \in \mathbb{N}.$$

We expand the solution and the inhomogeneity in the eigenfunctions.

$$y(x) = \sum_{n=1}^{\infty} y_n \sin\left(\frac{n\pi(x-a)}{b-a}\right)$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi(x-a)}{b-a}\right), \quad f_n = \frac{2}{b-a}\int_a^b f(x) \sin\left(\frac{n\pi(x-a)}{b-a}\right) \mathrm{d}x$$

Since the solution $y(x)$ satisfies the same homogeneous boundary conditions as the eigenfunctions, we can differentiate the series. We substitute the series expansions into the differential equation.

$$y'' + \alpha y = f(x)$$

$$\sum_{n=1}^{\infty} y_n \left(-\lambda_n^2 + \alpha\right) \sin\left(\lambda_n x\right) = \sum_{n=1}^{\infty} f_n \sin\left(\lambda_n x\right)$$

$$y_n = \frac{f_n}{\alpha - \lambda_n^2}$$

Thus the solution of the problem has the series representation,

$$\boxed{y(x) = \sum_{n=1}^{\infty} \left(\alpha - \lambda_n^2\right) \sin\left(\frac{n\pi(x-a)}{b-a}\right).}$$

Solution 30.13
The eigenfunction problem associated with this problem is

$$\phi'' + \lambda^2 \phi = 0, \quad \phi(a) = \phi(b) = 0,$$

which has the solutions,

$$\lambda_n = \frac{n\pi}{b-a}, \quad \phi_n = \sin\left(\frac{n\pi(x-a)}{b-a}\right), \quad n \in \mathbb{N}.$$

We expand the solution and the inhomogeneity in the eigenfunctions.

$$y(x) = \sum_{n=1}^{\infty} y_n \sin\left(\frac{n\pi(x-a)}{b-a}\right)$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi(x-a)}{b-a}\right), \quad f_n = \frac{2}{b-a}\int_a^b f(x) \sin\left(\frac{n\pi(x-a)}{b-a}\right) \mathrm{d}x$$

Since the solution $y(x)$ does not satisfy the same homogeneous boundary conditions as the eigenfunctions, we can differentiate the series. We multiply the differential equation by an eigenfunction and integrate from a to b. We use integration by parts to move derivatives from y to the eigenfunction.

$$y'' + \alpha y = f(x)$$

$$\int_a^b y''(x)\sin(\lambda_m x)\,\mathrm{d}x + \alpha \int_a^b y(x)\sin(\lambda_m x)\,\mathrm{d}x = \int_a^b f(x)\sin(\lambda_m x)\,\mathrm{d}x$$

$$[y'\sin(\lambda_m x)]_a^b - \int_a^b y'\lambda_m \cos(\lambda_m x)\,\mathrm{d}x + \alpha\frac{b-a}{2} y_m = \frac{b-a}{2} f_m$$

$$-[y\lambda_m \cos(\lambda_m x)]_a^b - \int_a^b y\lambda_m^2 \sin(\lambda_m x)\,\mathrm{d}x + \alpha\frac{b-a}{2} y_m = \frac{b-a}{2} f_m$$

$$-B\lambda_m(-1)^m + A\lambda_m(-1)^{m+1} - \lambda_m^2 y_m + \alpha\frac{b-a}{2} y_m = \frac{b-a}{2} f_m$$

$$y_m = \frac{f_m + (-1)^m \lambda_m (A+B)}{\alpha - \lambda_m^2}$$

Thus the solution of the problem has the series representation,

$$\boxed{y(x) = \sum_{n=1}^{\infty} \frac{f_m + (-1)^m \lambda_m (A+B)}{\alpha - \lambda_m^2} \sin\left(\frac{n\pi(x-a)}{b-a}\right).}$$

Solution 30.14

1.

$$\begin{aligned} A + iB &= \frac{1}{1-z^2} \\ &= \sum_{n=0}^{\infty} z^{2n} \\ &= \sum_{n=0}^{\infty} r^{2n} \mathrm{e}^{i2nx} \\ &= \sum_{n=0}^{\infty} r^{2n} \cos(2nx) + i \sum_{n=1}^{\infty} r^{2n} \sin(2nx) \end{aligned}$$

$$\boxed{A = \sum_{n=0}^{\infty} r^{2n} \cos(2nx), \qquad B = \sum_{n=1}^{\infty} r^{2n} \sin(2nx)}$$

$$
\begin{aligned}
A + iB &= \frac{1}{1 - z^2} \\
&= \frac{1}{1 - r^2 \mathrm{e}^{2ix}} \\
&= \frac{1}{1 - r^2 \cos(2x) - ir^2 \sin(2x)} \\
&= \frac{1 - r^2 \cos(2x) + ir^2 \sin(2x)}{(1 - r^2 \cos(2x))^2 + (r^2 \sin(2x))^2}
\end{aligned}
$$

$$
\boxed{A = \frac{1 - r^2 \cos(2x)}{1 - 2r^2 \cos(2x) + r^4}, \qquad B = \frac{r^2 \sin(2x)}{1 - 2r^2 \cos(2x) + r^4}}
$$

2. We consider the principal branch of the logarithm.

$$
\begin{aligned}
A + iB &= \log(1 + z) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} r^n \mathrm{e}^{inx} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} r^n \big(\cos(nx) + i \sin(nx) \big)
\end{aligned}
$$

$$
\boxed{A = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} r^n \cos(nx), \qquad B = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} r^n \sin(nx)}
$$

$$
\begin{aligned}
A + iB &= \log(1+z) \\
&= \log\left(1 + r\,\mathrm{e}^{ix}\right) \\
&= \log\left(1 + r\cos x + ir\sin x\right) \\
&= \log\left|1 + r\cos x + ir\sin x\right| + i\arg\left(1 + r\cos x + ir\sin x\right) \\
&= \log\sqrt{(1 + r\cos x)^2 + (r\sin x)^2} + i\arctan\left(1 + r\cos x, r\sin x\right)
\end{aligned}
$$

$$
\boxed{A = \frac{1}{2}\log\left(1 + 2r\cos x + r^2\right), \qquad B = \arctan\left(1 + r\cos x, r\sin x\right)}
$$

3.

$$
\begin{aligned}
A_n + iB_n &= \sum_{k=1}^{n} z^k \\
&= \frac{1 - z^{n+1}}{1 - z} \\
&= \frac{1 - r^{n+1}\,\mathrm{e}^{i(n+1)x}}{1 - r\,\mathrm{e}^{ix}} \\
&= \frac{1 - r\,\mathrm{e}^{-ix} - r^{n+1}\,\mathrm{e}^{i(n+1)x} + r^{n+2}\,\mathrm{e}^{inx}}{1 - 2r\cos x + r^2}
\end{aligned}
$$

$$
\boxed{A_n = \frac{1 - r\cos x - r^{n+1}\cos((n+1)x) + r^{n+2}\cos(nx)}{1 - 2r\cos x + r^2}}
$$

$$
\boxed{B_n = \frac{r\sin x - r^{n+1}\sin((n+1)x) + r^{n+2}\sin(nx)}{1 - 2r\cos x + r^2}}
$$

$$
\begin{aligned}
A_n + iB_n &= \sum_{k=1}^{n} z^k \\
&= \sum_{k=1}^{n} r^k \, \mathrm{e}^{ikx}
\end{aligned}
$$

$$
\boxed{A_n = \sum_{k=1}^{n} r^k \cos(kx), \qquad B_n = \sum_{k=1}^{n} r^k \sin(kx)}
$$

Solution 30.15

1.

$$
\int_0^{\pi} 1 \cdot \sin x \, \mathrm{d}x = [-\cos x]_0^{\pi} = 2
$$

Thus the system is not orthogonal on the interval $[0, \pi]$. Consider the interval $[a, a + \pi]$.

$$
\begin{aligned}
\int_a^{a+\pi} 1 \cdot \sin x \, \mathrm{d}x &= [-\cos x]_a^{a+\pi} = 2\cos a \\
\int_a^{a+\pi} 1 \cdot \cos x \, \mathrm{d}x &= [\sin x]_a^{a+\pi} = -2\sin a
\end{aligned}
$$

Since there is no value of a for which both $\cos a$ and $\sin a$ vanish, the system is not orthogonal for any interval of length π.

2. First note that

$$
\int_0^{\pi} \cos nx \, \mathrm{d}x = 0 \text{ for } n \in \mathbb{N}.
$$

If $n \neq m$, $n \geq 1$ and $m \geq 0$ then

$$\int_0^\pi \cos nx \cos mx \, \mathrm{d}x = \frac{1}{2} \int_0^\pi \big(\cos((n-m)x) + \cos((n+m)x) \big) \, \mathrm{d}x = 0$$

Thus the set $\{1, \cos x, \cos 2x, \ldots\}$ is orthogonal on $[0, \pi]$. Since

$$\int_0^\pi \mathrm{d}x = \pi$$

$$\int_0^\pi \cos^2(nx) \, \mathrm{d}x = \frac{\pi}{2},$$

the set

$$\boxed{\left\{ \sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}} \cos x, \sqrt{\frac{2}{\pi}} \cos 2x, \ldots \right\}}$$

is orthonormal on $[0, \pi]$.

If $n \neq m$, $n \geq 1$ and $m \geq 1$ then

$$\int_0^\pi \sin nx \sin mx \, \mathrm{d}x = \frac{1}{2} \int_0^\pi \big(\cos((n-m)x) - \cos((n+m)x) \big) \, \mathrm{d}x = 0$$

Thus the set $\{\sin x, \sin 2x, \ldots\}$ is orthogonal on $[0, \pi]$. Since

$$\int_0^\pi \sin^2(nx) \, \mathrm{d}x = \frac{\pi}{2},$$

the set

$$\boxed{\left\{ \sqrt{\frac{2}{\pi}} \sin x, \sqrt{\frac{2}{\pi}} \sin 2x, \ldots \right\}}$$

is orthonormal on $[0, \pi]$.

Solution 30.16

Since the periodic extension of $|x|$ in $[-\pi, \pi]$ is an even function its Fourier series is a cosine series. Because of the anti-symmetry about $x = \pi/2$ we see that except for the constant term, there will only be odd cosine terms. Since the periodic extension is a continuous function, but has a discontinuous first derivative, the Fourier coefficients will decay as $1/n^2$.

$$|x| = \sum_{n=0}^{\infty} a_n \cos(nx), \quad \text{for } x \in [-\pi, \pi]$$

$$a_0 = \frac{1}{\pi} \int_0^\pi x \, \mathrm{d}x = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^\pi = \frac{\pi}{2}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^\pi x \cos(nx) \, \mathrm{d}x \\
&= \frac{2}{\pi} \left[x \frac{\sin(nx)}{n} \right]_0^\pi - \frac{2}{\pi} \int_0^\pi \frac{\sin(nx)}{n} \, \mathrm{d}x \\
&= -\frac{2}{\pi} \left[\frac{\cos(nx)}{n^2} \right]_0^\pi \\
&= -\frac{2}{\pi n^2} (\cos(n\pi) - 1) \\
&= \frac{2(1 - (-1)^n)}{\pi n^2}
\end{aligned}$$

$$|x| = \frac{\pi}{2} + \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n^2} \cos(nx) \quad \text{for } x \in [-\pi, \pi]$$

Define $R_N(x) = f(x) - S_N(x)$. We seek an upper bound on $|R_N(x)|$.

$$|R_N(x)| = \left| \frac{4}{\pi} \sum_{\substack{n=N+1 \\ \text{odd } n}}^{\infty} \frac{1}{n^2} \cos(nx) \right|$$

$$\leq \frac{4}{\pi} \sum_{\substack{n=N+1 \\ \text{odd } n}}^{\infty} \frac{1}{n^2}$$

$$= \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n^2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{N} \frac{1}{n^2}$$

Since

$$\sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

We can bound the error with,

$$|R_N(x)| \leq \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{N} \frac{1}{n^2}.$$

$N = 7$ is the smallest number for which our error bound is less than 10^{-1}. $N \geq 7$ is sufficient to make the error less that 0.1.

$$|R_7(x)| \leq \frac{\pi}{2} - \frac{4}{\pi} \left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} \right) \approx 0.079$$

$N \geq 7$ is also necessary because.

$$|R_N(0)| = \frac{4}{\pi} \sum_{\substack{n=N+1 \\ \text{odd } n}}^{\infty} \frac{1}{n^2}.$$

Solution 30.17

1.

$$1 \sim \sum_{n=1}^{\infty} a_n \sin(nx), \quad 0 \leq x \leq \pi$$

Since the odd periodic extension of the function is discontinuous, the Fourier coefficients will decay as $1/n$. Because of the symmetry about $x = \pi/2$, there will be only odd sine terms.

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin(nx) \, dx \\ &= \frac{2}{n\pi} (-\cos(n\pi) + \cos(0)) \\ &= \frac{2}{n\pi} (1 - (-1)^n) \end{aligned}$$

$$\boxed{1 \sim \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{\sin(nx)}{n}}$$

2. It's always OK to integrate a Fourier series term by term. We integrate the series in part (a).

$$\int_a^x 1 \, dx \sim \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \int_a^x \frac{\sin(n\xi)}{n} \, dx$$

$$x - a \sim \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{\cos(na) - \cos(nx)}{n^2}$$

Since the series converges uniformly, we can replace the $\sim$ with $=$.

$$x - a = \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{\cos(na)}{n^2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{\cos(nx)}{n^2}$$

Now we have a Fourier cosine series. The first sum on the right is the constant term. If we choose $a = \pi/2$ this sum vanishes since $\cos(n\pi/2) = 0$ for odd integer n.

$$\boxed{x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{\cos(nx)}{n^2}}$$

3. If $f(x)$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

then Parseval's theorem states that

$$\int_{-\pi}^{\pi} f^2(x) \, \mathrm{d}x = \frac{\pi}{2} a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

We apply this to the Fourier sine series from part (a).

$$\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x = \pi \sum_{\substack{n=1\\ \text{odd } n}}^{\infty} \left(\frac{4}{\pi n}\right)^2$$

$$\int_{-\pi}^{0} (-1)^2\,\mathrm{d}x + \int_{0}^{\pi} (1)^2\,\mathrm{d}x = \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}}$$

We substitute $x = \pi$ in the series from part (b) to corroborate the result.

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}$$

$$\pi = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi)}{(2n-1)^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Solution 30.18

1.

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

Since the periodic extension of the function is discontinuous, the Fourier coefficients will decay like $1/n$. Because of the anti-symmetry about $x = \pi/2$, there will be only odd cosine terms.

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2}$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^\pi f(x) \cos(nx) \, \mathrm{d}x \\
&= \frac{2}{\pi} \int_0^{\pi/2} \cos(nx) \, \mathrm{d}x \\
&= \frac{2}{\pi n} \sin(n\pi/2) \\
&= \begin{cases} \frac{2}{\pi n} (-1)^{(n-1)/2}, & \text{for odd } n \\ 0 & \text{for even } n \end{cases}
\end{aligned}
$$

The Fourier cosine series of $f(x)$ is

$$
\boxed{f(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos((2n+1)x).}
$$

2. The N^{th} partial sum is

$$
S_N(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{N} \frac{(-1)^n}{2n+1} \cos((2n+1)x).
$$

We wish to evaluate the sum from part (a). First we make the change of variables $y = x - \pi/2$ to get rid of the $(-1)^n$ factor.

$$
\begin{aligned}
\sum_{n=0}^{\infty} & \frac{(-1)^n}{2n+1} \cos((2n+1)x) \\
&= \sum_{n=0}^{N} \frac{(-1)^n}{2n+1} \cos((2n+1)(y+\pi/2)) \\
&= \sum_{n=0}^{N} \frac{(-1)^n}{2n+1} (-1)^{n+1} \sin((2n+1)y) \\
&= -\sum_{n=0}^{N} \frac{1}{2n+1} \sin((2n+1)y)
\end{aligned}
$$

We write the summand as an integral and interchange the order of summation and integration to get rid of the $1/(2n+1)$ factor.

$$
\begin{aligned}
&= -\sum_{n=0}^{N} \int_0^y \cos((2n+1)t)\,dt \\
&= -\int_0^y \sum_{n=0}^{N} \cos((2n+1)t)\,dt \\
&= -\int_0^y \left(\sum_{n=1}^{2N+1} \cos(nt) - \sum_{n=1}^{N} \cos(2nt) \right) dt \\
&= -\int_0^y \Re \left(\sum_{n=1}^{2N+1} \mathrm{e}^{int} - \sum_{n=1}^{N} \mathrm{e}^{i2nt} \right) dt \\
&= -\int_0^y \Re \left(\frac{\mathrm{e}^{it} - \mathrm{e}^{i(2N+2)t}}{1 - \mathrm{e}^{it}} - \frac{\mathrm{e}^{i2t} - \mathrm{e}^{i2(N+1)t}}{1 - \mathrm{e}^{i2t}} \right) dt \\
&= -\int_0^y \Re \left(\frac{(\mathrm{e}^{it} - \mathrm{e}^{i2(N+1)t})(1 - \mathrm{e}^{i2t}) - (\mathrm{e}^{i2t} - \mathrm{e}^{i2(N+1)t})(1 - \mathrm{e}^{it})}{(1 - \mathrm{e}^{it})(1 - \mathrm{e}^{i2t})} \right) dt \\
&= -\int_0^y \Re \left(\frac{\mathrm{e}^{it} - \mathrm{e}^{i2t} + \mathrm{e}^{i(2N+4)t} - \mathrm{e}^{i(2N+3)t}}{(1 - \mathrm{e}^{it})(1 - \mathrm{e}^{i2t})} \right) dt \\
&= -\int_0^y \Re \left(\frac{\mathrm{e}^{it} - \mathrm{e}^{i(2N+3)t}}{1 - \mathrm{e}^{i2t}} \right) dt \\
&= -\int_0^y \Re \left(\frac{\mathrm{e}^{i(2N+2)t} - 1}{\mathrm{e}^{it} - \mathrm{e}^{-it}} \right) dt \\
&= -\int_0^y \Re \left(\frac{-i\,\mathrm{e}^{i2(N+1)t} + i}{2\sin t} \right) dt \\
&= -\frac{1}{2} \int_0^y \frac{\sin(2(N+1)t)}{\sin t}\,dt \\
&= -\frac{1}{2} \int_0^{x-\pi/2} \frac{\sin(2(N+1)t)}{\sin t}\,dt
\end{aligned}
$$

Now we have a tidy representation of the partial sum.

$$\boxed{S_N(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^{x-\pi/2} \frac{\sin(2(N+1)t)}{\sin t}\,dt}$$

3. We solve $\frac{\mathrm{d}S_N(x)}{\mathrm{d}x} = 0$ to find the relative extrema of $S_N(x)$.

$$\begin{gathered}
S_N'(x) = 0 \\
-\frac{1}{\pi}\frac{\sin(2(N+1)(x-\pi/2))}{\sin(x-\pi/2)} = 0 \\
\frac{(-1)^{N+1}\sin(2(N+1)x)}{-\cos(x)} = 0 \\
\frac{\sin(2(N+1)x)}{\cos(x)} = 0 \\
\boxed{x = x_n = \frac{n\pi}{2(N+1)}, \quad n = 0, 1, \ldots, N, N+2, \ldots, 2N+2}
\end{gathered}$$

Note that $x_{N+1} = \pi/2$ is not a solution as the denominator vanishes there. The function has a removable singularity at $x = \pi/2$ with limiting value $(-1)^N$.

4.

$$S_N(x_N) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\frac{\pi N}{2(N+1)}-\pi/2} \frac{\sin(2(N+1)t)}{\sin t}\,dt$$

We note that the integrand is even.

$$\int_0^{\frac{\pi N}{2(N+1)}-\pi/2} = \int_0^{-\frac{\pi}{2(N+1)}} = -\int_0^{\frac{\pi}{2(N+1)}}$$

$$\boxed{S_N(x_N) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\frac{\pi}{2(N+1)}} \frac{\sin(2(N+1)t)}{\sin t}\,dt}$$

5. We make the change of variables $2(N+1)t \to t$.

$$S_N(x_N) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin(t)}{2(N+1)\sin(t/(2(N+1)))}\,dt$$

Note that

$$\lim_{\epsilon\to 0} \frac{\sin(\epsilon t)}{\epsilon} = \lim_{\epsilon\to 0} \frac{t\cos(\epsilon t)}{1} = t$$

$$\boxed{S_N(x_N) \to \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin(t)}{t}\,dt \approx 1.0895 \quad \text{as } N \to \infty}$$

This is not equal to the limiting value of $f(x)$, $f(\pi/2 - 0) = 1$.

Solution 30.19

With the parametrization in t, $x(t)$ and $y(t)$ are continuous functions on the range $[0, 2\pi]$. Since the curve is closed, we have $x(0) = x(2\pi)$ and $y(0) = y(2\pi)$. This means that the periodic extensions of $x(t)$ and $y(t)$ are continuous functions. Thus we can differentiate their Fourier series. First we define formal Fourier series for $x(t)$

and $y(t)$.

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nt) + b_n \sin(nt)\big)$$
$$y(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} \big(c_n \cos(nt) + d_n \sin(nt)\big)$$
$$x'(t) = \sum_{n=1}^{\infty} \big(nb_n \cos(nt) - na_n \sin(nt)\big)$$
$$y'(t) = \sum_{n=1}^{\infty} \big(nd_n \cos(nt) - nc_n \sin(nt)\big)$$

In this problem we will be dealing with integrals on $[0, 2\pi]$ of products of Fourier series. We derive a general formula for later use.

$$\begin{aligned}
\int_0^{2\pi} xy\, dt &= \int_0^{2\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nt) + b_n \sin(nt)\big)\right) \left(\frac{c_0}{2} + \sum_{n=1}^{\infty} \big(c_n \cos(nt) + d_n \sin(nt)\big)\right) dt \\
&= \int_0^{2\pi} \left(\frac{a_0 c_0}{4} + \sum_{n=1}^{\infty} \big(a_n c_n \cos^2(nt) + b_n d_n \sin^2(nt)\big)\right) dt \\
&= \pi \left(\frac{1}{2} a_0 c_0 + \sum_{n=1}^{\infty} (a_n c_n + b_n d_n)\right)
\end{aligned}$$

In the arclength parametrization we have

$$\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 = 1.$$

In terms of $t = 2\pi s/L$ this is

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \left(\frac{L}{2\pi}\right)^2.$$

We integrate this identity on $[0, 2\pi]$.

$$\begin{aligned}
\frac{L^2}{2\pi} &= \int_0^{2\pi} \left(\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right) dt \\
&= \pi \left(\sum_{n=1}^{\infty} \left((nb_n)^2 + (-na_n)^2\right) + \sum_{n=1}^{\infty} \left((nd_n)^2 + (-nc_n)^2\right)\right) \\
&= \pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2 + c_n^2 + d_n^2)
\end{aligned}$$

$$L^2 = 2\pi^2 \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2 + c_n^2 + d_n^2)$$

We assume that the curve is parametrized so that the area is positive. (Reversing the orientation changes the sign of the area as defined above.) The area is

$$\begin{aligned}
A &= \int_0^{2\pi} x \frac{\mathrm{d}y}{\mathrm{d}t}\, dt \\
&= \int_0^{2\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt)\right)\right) \left(\sum_{n=1}^{\infty} \left(nd_n \cos(nt) - nc_n \sin(nt)\right)\right) dt \\
&= \pi \sum_{n=1}^{\infty} n (a_n d_n - b_n c_n)
\end{aligned}$$

Now we find an upper bound on the area. We will use the inequality $|ab| \leq \frac{1}{2}|a^2 + b^2|$, which follows from expanding $(a-b)^2 \geq 0$.

$$\begin{aligned} A &\leq \frac{\pi}{2} \sum_{n=1}^{\infty} n \left(a_n^2 + b_n^2 + c_n^2 + d_n^2\right) \\ &\leq \frac{\pi}{2} \sum_{n=1}^{\infty} n^2 \left(a_n^2 + b_n^2 + c_n^2 + d_n^2\right) \end{aligned}$$

We can express this in terms of the perimeter.

$$= \frac{L^2}{4\pi}$$

$$\boxed{L^2 \geq 4\pi A}$$

Now we determine the curves for which $L^2 = 4\pi A$. To do this we find conditions for which A is equal to the upper bound we obtained for it above. First note that

$$\sum_{n=1}^{\infty} n \left(a_n^2 + b_n^2 + c_n^2 + d_n^2\right) = \sum_{n=1}^{\infty} n^2 \left(a_n^2 + b_n^2 + c_n^2 + d_n^2\right)$$

implies that all the coefficients except a_0, c_0, a_1, b_1, c_1 and d_1 are zero. The constraint,

$$\pi \sum_{n=1}^{\infty} n(a_n d_n - b_n c_n) = \frac{\pi}{2} \sum_{n=1}^{\infty} n \left(a_n^2 + b_n^2 + c_n^2 + d_n^2\right)$$

then becomes

$$a_1 d_1 - b_1 c_1 = a_1^2 + b_1^2 + c_1^2 + d_1^2.$$

This implies that $d_1 = a_1$ and $c_1 = -b_1$. a_0 and c_0 are arbitrary. Thus curves for which $L^2 = 4\pi A$ have the parametrization

$$x(t) = \frac{a_0}{2} + a_1 \cos t + b_1 \sin t, \qquad y(t) = \frac{c_0}{2} - b_1 \cos t + a_1 \sin t.$$

Note that

$$\left(x(t) - \frac{a_0}{2}\right)^2 + \left(y(t) - \frac{c_0}{2}\right)^2 = a_1^2 + b_1^2.$$

The curve is a circle of radius $\sqrt{a_1^2 + b_1^2}$ and center $(a_0/2, c_0/2)$.

Solution 30.20

1. The Fourier sine series has the form

$$x(1-x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x).$$

   The norm of the eigenfunctions is

$$\int_0^1 \sin^2(n\pi x)\,\mathrm{d}x = \frac{1}{2}.$$

   The coefficients in the expansion are

$$\begin{aligned} a_n &= 2\int_0^1 x(1-x)\sin(n\pi x)\,\mathrm{d}x \\ &= \frac{2}{\pi^3 n^3}(2 - 2\cos(n\pi) - n\pi\sin(n\pi)) \\ &= \frac{4}{\pi^3 n^3}(1 - (-1)^n). \end{aligned}$$

Thus the Fourier sine series is

$$\boxed{x(1-x) = \frac{8}{\pi^3} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{\sin(n\pi x)}{n^3} = \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{(2n-1)^3}.}$$

The Fourier cosine series has the form

$$x(1-x) = \sum_{n=0}^{\infty} a_n \cos(n\pi x).$$

The norm of the eigenfunctions is

$$\int_0^1 1^2 \, \mathrm{d}x = 1, \quad \int_0^1 \cos^2(n\pi x) \, \mathrm{d}x = \frac{1}{2}.$$

The coefficients in the expansion are

$$a_0 = \int_0^1 x(1-x) \, \mathrm{d}x = \frac{1}{6},$$

$$\begin{aligned}
a_n &= 2 \int_0^1 x(1-x) \cos(n\pi x) \, \mathrm{d}x \\
&= -\frac{2}{\pi^2 n^2} + \frac{4 \sin(n\pi) - n\pi \cos(n\pi)}{\pi^3 n^3} \\
&= -\frac{2}{\pi^2 n^2}(1 + (-1)^n)
\end{aligned}$$

Thus the Fourier cosine series is

$$\boxed{x(1-x) = \frac{1}{6} - \frac{4}{\pi^2} \sum_{\substack{n=1 \\ \text{even } n}}^{\infty} \frac{\cos(n\pi x)}{n^2} = \frac{1}{6} - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n\pi x)}{n^2}.}$$

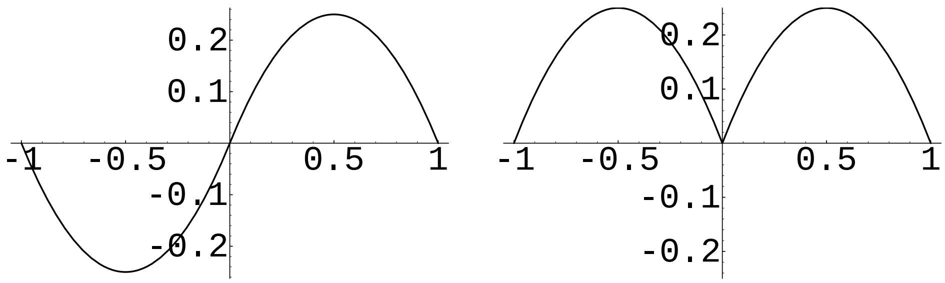

Figure 30.13: The odd and even periodic extension of $x(1-x)$, $0 \le x \le 1$.

The Fourier sine series converges to the odd periodic extension of the function. Since this function is C^1, continuously differentiable, we know that the Fourier coefficients must decay as $1/n^3$. The Fourier cosine series converges to the even periodic extension of the function. Since this function is only C^0, continuous, the Fourier coefficients must decay as $1/n^2$. The odd and even periodic extensions are shown in Figure 30.13. The sine series is better because of the faster convergence of the series.

2. (a) We substitute $x = 0$ into the cosine series.

$$0 = \frac{1}{6} - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

(b) We substitute $x = 1/2$ into the cosine series.

$$\frac{1}{4} = \frac{1}{6} - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}}$$

(c) We substitute $x = 1/2$ into the sine series.

$$\frac{1}{4} = \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi/2)}{(2n-1)^3}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} = -\frac{\pi^3}{32}}$$

Chapter 31

Regular Sturm-Liouville Problems

I learned there are troubles
Of more than one kind.
Some come from ahead
And some come from behind.

But I've bought a big bat.
I'm all ready, you see.
Now my troubles are going
To have troubles with *me*!

-*I Had Trouble in Getting to Solla Sollew*
-Theodor S. Geisel, (Dr. Suess)

31.1 Derivation of the Sturm-Liouville Form

Consider the eigenvalue problem on the finite interval $[a, b]$

$$p_2(x)y'' + p_1(x)y' + p_0(x)y = \mu y,$$

subject to the homogeneous unmixed boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \qquad \beta_1 y(b) + \beta_2 y'(b) = 0.$$

Here the p_j's are real and continuous and $p_2 > 0$ on the interval $[a, b]$. The α_j's and β_j's are real. (Note that if p_2 were negative we could multiply the equation by (-1) and replace μ by $-\mu$.)

We would like to write this problem in a form that can be used to obtain qualitative information about the problem. First we will write the operator in self-adjoint form. Since p_2 is positive on the interval,

$$y'' + \frac{p_1}{p_2} y' + \frac{p_0}{p_2} y = \frac{\mu}{p_2} y.$$

Multiplying by the factor

$$\exp\left(\int^x \frac{p_1}{p_2} \, d\xi \right) = \mathrm{e}^{P(x)}$$

yields

$$\mathrm{e}^{P(x)} \left(y'' + \frac{p_1}{p_2} y' + \frac{p_0}{p_2} y \right) = \mathrm{e}^{P(x)} \frac{\mu}{p_2} y$$
$$\left(\mathrm{e}^{P(x)} y' \right)' + \mathrm{e}^{P(x)} \frac{p_0}{p_2} y = \mathrm{e}^{P(x)} \frac{\mu}{p_2} y.$$

We define the following functions

$$p = \mathrm{e}^{P(x)}, \quad q = \mathrm{e}^{P(x)} \frac{p_0}{p_2}, \quad \sigma = \mathrm{e}^{P(x)} \frac{1}{p_2}, \quad \lambda = -\mu.$$

Since the p_j's are continuous and p_2 is positive, p, q, and σ are continuous. p and σ are positive functions. The problem now has the form

$$(py')' + qy + \lambda \sigma y = 0,$$

subject to the boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \qquad \beta_1 y(b) + \beta_2 y'(b) = 0.$$

This is known as a **Regular Sturm-Liouville** problem. We will devote much of this chapter to studying the properties of this problem. We will encounter many results that are analogous to the properties of self-adjoint eigenvalue problems.

Example 31.1.1

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\log x \frac{\mathrm{d}y}{\mathrm{d}x}\right) + \lambda x y = 0, \qquad y(1) = y(2) = 0$$

is not a regular Sturm-Liouville problem since $\log x$ vanishes at $x = 1$.

Result 31.1.1 Any eigenvalue problem of the form

$$p_2 y'' + p_1 y' + p_0 y = \mu y, \quad \text{for } a \leq x \leq b,$$
$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0,$$

where the p_j's are real and continuous, $p_2 > 0$ on $[a, b]$, and the α_j's and β_j's are real can be written in the form of a regular Sturm-Liouville problem

$$(py')' + qy + \lambda \sigma y = 0, \quad \text{on } a \leq x \leq b,$$
$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0.$$

31.2 Properties of Regular Sturm-Liouville Problems

Self-Adjoint. Writing the problem in the form

$$L[y] = (py')' + qy = -\lambda\sigma y,$$

we see that the operator is formally self-adjoint. Now to see if the problem is self-adjoint.

$$\begin{aligned}
\langle v|L[u]\rangle - \langle L[v]|u\rangle &= \langle v|(pu')' + qu\rangle - \langle (pv')' + qv|u\rangle \\
&= [\overline{v}pu']_a^b - \langle \overline{v}'|pu'\rangle + \langle \overline{v}|qu\rangle - [p\overline{v}'u]_a^b + \langle p\overline{v}'|u'\rangle - \langle q\overline{v}|u\rangle \\
&= [\overline{v}pu']_a^b - [p\overline{v}'u]_a^b \\
&= p(b)\big(\overline{v}(b)u'(b) - \overline{v}'(b)u(b)\big) + p(a)\big(\overline{v}(a)u'(a) - \overline{v}'(a)u(a)\big) \\
&= p(b)\left(\overline{v}(b)\left(-\frac{\beta_1}{\beta_2}\right)u(b) - \left(-\frac{\beta_1}{\beta_2}\right)\overline{v}(b)u(b)\right) \\
&\quad + p(a)\left(\overline{v}(a)\left(-\frac{\alpha_1}{\alpha_2}\right)u(a) - \left(-\frac{\alpha_1}{\alpha_2}\right)\overline{v}(a)u(a)\right) \\
&= 0
\end{aligned}$$

Note that α_i and β_i are real so

$$\overline{\left(\frac{\alpha_1}{\alpha_2}\right)} = \left(\frac{\alpha_1}{\alpha_2}\right), \qquad \overline{\left(\frac{\beta_1}{\beta_2}\right)} = \left(\frac{\beta_1}{\beta_2}\right)$$

Thus $L[y]$ subject to the boundary conditions is self-adjoint.

Real Eigenvalues. Let λ be an eigenvalue with the eigenfunction ϕ. Starting with Green's formula,

$$\begin{gathered}
\langle \phi|L[\phi]\rangle - \langle L[\phi]|\phi\rangle = 0 \\
\langle \phi| - \lambda\sigma\phi\rangle - \langle -\lambda\sigma\phi|\phi\rangle = 0 \\
-\lambda\langle \phi|\sigma|\phi\rangle + \overline{\lambda}\langle \phi|\sigma|\phi\rangle = 0 \\
(\overline{\lambda} - \lambda)\langle \phi|\sigma|\phi\rangle = 0.
\end{gathered}$$

Since $\langle\phi|\sigma|\phi\rangle > 0$, $\overline{\lambda} - \lambda = 0$. Thus the eigenvalues are real.

Infinite Number of Eigenvalues. There are an infinite of eigenvalues which have no finite cluster point. This result is analogous to the result that we derived for self-adjoint eigenvalue problems. When we cover the Rayleigh quotient, we will find that there is a least eigenvalue. Since the eigenvalues are distinct and have no finite cluster point, $\lambda_n \to \infty$ as $n \to \infty$. Thus the eigenvalues form an ordered sequence,

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots .$$

Orthogonal Eigenfunctions. Let λ and μ be two distinct eigenvalues with the eigenfunctions ϕ and ψ. Green's formula states

$$\begin{gathered}
\langle\psi|L[\phi]\rangle - \langle L[\psi]|\phi\rangle = 0. \\
\langle\psi| - \lambda\sigma\phi\rangle - \langle -\mu\sigma\psi|\phi\rangle = 0 \\
-\lambda\langle\psi|\sigma|\phi\rangle + \overline{\mu}\langle\psi|\sigma|\phi\rangle = 0 \\
(\mu - \lambda)\langle\psi|\sigma|\phi\rangle = 0
\end{gathered}$$

Since the eigenvalues are distinct, $\langle\psi|\sigma|\phi\rangle = 0$. Thus eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to σ.

Unique Eigenfunctions. Let λ be an eigenvalue. Suppose ϕ and ψ are two independent eigenfunctions corresponding to λ. The eigenfunctions satisfy the equations

$$\begin{gathered}
L[\phi] + \lambda\sigma\phi = 0 \\
L[\psi] + \lambda\sigma\psi = 0.
\end{gathered}$$

Taking the difference of ψ times the first equation and ϕ times the second equation gives us

$$\psi L[\phi] - \phi L[\psi] = 0$$
$$\psi(p\phi')' - \phi(p\psi')' = 0$$
$$(p(\psi\phi' - \psi'\phi))' = 0$$
$$p(\psi\phi' - \psi'\phi) = \text{const.}$$

In order to satisfy the boundary conditions, the constant must be zero.

$$p(\psi\phi' - \psi'\phi) = 0$$

Since $p > 0$,

$$\psi\phi' - \psi'\phi = 0$$
$$\frac{\phi'}{\psi} - \frac{\psi'\phi}{\psi^2} = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi}{\psi}\right) = 0$$
$$\frac{\phi}{\psi} = \text{const.}$$

ϕ and ψ are not independent. Thus each eigenvalue has a unique, (to within a multiplicative constant), eigenfunction.

Real Eigenfunctions. If λ is an eigenvalue with eigenfunction ϕ, then

$$(p\phi')' + q\phi + \lambda\sigma\phi = 0.$$

Taking the complex conjugate of this equation,

$$\left(p\overline{\phi}'\right)' + q\overline{\phi} + \lambda\sigma\overline{\phi} = 0.$$

Thus $\overline{\phi}$ is also an eigenfunction corresponding to λ. Are ϕ and $\overline{\phi}$ independent functions, or do they just differ by a multiplicative constant? (For example, e^{ix} and e^{-ix} are independent functions, but ix and $-ix$ are dependent.) From our argument on unique eigenfunctions, we see that

$$\phi = (\text{const})\overline{\phi}.$$

Since ϕ and $\overline{\phi}$ only differ by a multiplicative constant, the eigenfunctions can be chosen so that they are real-valued functions.

Rayleigh's Quotient. Let λ be an eigenvalue with the eigenfunction ϕ.

$$\langle \phi | L[\phi] \rangle = \langle \phi | -\lambda\sigma\phi \rangle$$
$$\langle \phi | (p\phi')' + q\phi \rangle = -\lambda \langle \phi|\sigma|\phi \rangle$$
$$\left[\overline{\phi} p \phi'\right]_a^b - \langle \phi'|p|\phi' \rangle + \langle \phi|q|\phi \rangle = -\lambda \langle \phi|\sigma|\phi \rangle$$
$$\boxed{\lambda = \frac{-\left[p\overline{\phi}\phi'\right]_a^b + \langle \phi'|p|\phi' \rangle - \langle \phi|q|\phi \rangle}{\langle \phi|\sigma|\phi \rangle}}$$

This is known as **Rayleigh's quotient**. It is useful for obtaining qualitative information about the eigenvalues.

Minimum Property of Rayleigh's Quotient. Note that since p, q, σ and ϕ are bounded functions, the Rayleigh quotient is bounded below. Thus there is a least eigenvalue. If we restrict u to be a real continuous function that satisfies the boundary conditions, then

$$\lambda_1 = \min_u \frac{-[puu']_a^b + \langle u'|p|u' \rangle - \langle u|q|u \rangle}{\langle u|\sigma|u \rangle},$$

where λ_1 is the least eigenvalue. This form allows us to get upper and lower bounds on λ_1.

To derive this formula, we first write it in terms of the operator L.

$$\lambda_1 = \min_u \frac{-\langle u|L[u] \rangle}{\langle u|\sigma|u \rangle}$$

Since u is continuous and satisfies the boundary conditions, we can expand u in a series of the eigenfunctions.

$$
\begin{aligned}
-\frac{\langle u|L[u]\rangle}{\langle u|\sigma|u\rangle} &= -\frac{\left\langle \sum_{n=1}^{\infty} c_n \phi_n \middle| L\left[\sum_{m=1}^{\infty} c_m \phi_m\right]\right\rangle}{\left\langle \sum_{n=1}^{\infty} c_n \phi_n \middle|\sigma\middle| \sum_{m=1}^{\infty} c_m \phi_m\right\rangle} \\
&= -\frac{\left\langle \sum_{n=1}^{\infty} c_n \phi_n \middle| -\sum_{m=1}^{\infty} c_m \lambda_m \sigma \phi_m\right\rangle}{\left\langle \sum_{n=1}^{\infty} c_n \phi_n \middle|\sigma\middle| \sum_{m=1}^{\infty} c_m \phi_m\right\rangle}
\end{aligned}
$$

Assuming that we can interchange summation and integration,

$$
\begin{aligned}
&= \frac{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \overline{c_n} c_m \lambda_n \langle \phi_m|\sigma|\phi_n\rangle}{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \overline{c_n} c_m \langle \phi_m|\sigma|\phi_n\rangle} \\
&= \frac{\sum_{n=1}^{\infty} |c_n|^2 \lambda_n \langle \phi_n|\sigma|\phi_n\rangle}{\sum_{n=1}^{\infty} |c_n|^2 \langle \phi_n|\sigma|\phi_n\rangle} \\
&\leq \lambda_1 \frac{\sum_{n=1}^{\infty} |c_n|^2 \langle \phi_n|\sigma|\phi_n\rangle}{\sum_{n=1}^{\infty} |c_n|^2 \langle \phi_n|\sigma|\phi_n\rangle} \\
&= \lambda_1.
\end{aligned}
$$

We see that the minimum value of Rayleigh's quotient is λ_1. The minimum is attained when $c_n = 0$ for all $n \geq 2$, that is, when $u = c_1 \phi_1$.

Completeness. The set of the eigenfunctions of a regular Sturm-Liouville problem is complete. That is, any piecewise continuous function defined on $[a, b]$ can be expanded in a series of the eigenfunctions

$$
f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),
$$

where the c_n are the generalized Fourier coefficients

$$
c_n = \frac{\langle \phi_n|\sigma|f\rangle}{\langle \phi_n|\sigma|\phi_n\rangle}.
$$

Here the sum is convergent in the mean. For any fixed x, the sum converges to $\frac{1}{2}(f(x^-) + f(x^+))$. If $f(x)$ is continuous and satisfies the boundary conditions, then the convergence is uniform.

Result 31.2.1 Properties of regular Sturm-Liouville problems.

- The eigenvalues λ are real.
- There are an infinite number of eigenvalues
$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots .$$
There is a least eigenvalue λ_1 but there is no greatest eigenvalue, ($\lambda_n \to \infty$ as $n \to \infty$).
- For each eigenvalue, there is one unique, (to within a multiplicative constant), eigenfunction ϕ_n. The eigenfunctions can be chosen to be real-valued. (Assume the ϕ_n following are real-valued.) The eigenfunction ϕ_n has exactly $n-1$ zeros in the open interval $a < x < b$.
- The eigenfunctions are orthogonal with respect to the weighting function $\sigma(x)$.
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if } n \neq m.$$
- The eigenfunctions are complete. Any piecewise continuous function $f(x)$ defined on $a \leq x \leq b$ can be expanded in a series of eigenfunctions
$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$
where
$$c_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx}.$$
The sum converges to $\frac{1}{2}(f(x^-) + f(x^+))$.
- The eigenvalues can be related to the eigenfunctions with a formula known as the Rayleigh quotient.
$$\lambda_n = \frac{-p\phi_n \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\Big|_a^b + \int_a^b \left(p\left(\frac{\mathrm{d}\phi_n}{\mathrm{d}x}\right)^2 - q\phi_n^2\right)dx}{\int_a^b \phi_n^2 \sigma\,dx}$$

Example 31.2.1 A simple example of a Sturm-Liouville problem is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \lambda y = 0, \qquad y(0) = y(\pi) = 0.$$

Bounding The Least Eigenvalue. The Rayleigh quotient for the first eigenvalue is

$$\lambda_1 = \frac{\int_0^\pi (\phi_1')^2 \, dx}{\int_0^\pi \phi_1^2 \, dx}.$$

Immediately we see that the eigenvalues are non-negative. If $\int_0^\pi (\phi_1')^2 \, dx = 0$ then $\phi = \text{(const)}$. The only constant that satisfies the boundary conditions is $\phi = 0$. Since the trivial solution is not an eigenfunction, $\lambda = 0$ is not an eigenvalue. Thus all the eigenvalues are positive.

Now we get an upper bound for the first eigenvalue.

$$\lambda_1 = \min_u \frac{\int_0^\pi (u')^2 \, dx}{\int_0^\pi u^2 \, dx}$$

where u is continuous and satisfies the boundary conditions. We choose $u = x(x - \pi)$ as a trial function.

$$\begin{aligned}
\lambda_1 &\leq \frac{\int_0^\pi (u')^2 \, dx}{\int_0^\pi u^2 \, dx} \\
&= \frac{\int_0^\pi (2x - \pi)^2 \, dx}{\int_0^\pi (x^2 - \pi x)^2 \, dx} \\
&= \frac{\pi^3/3}{\pi^5/30} \\
&= \frac{10}{\pi^2} \\
&\approx 1.013
\end{aligned}$$

Finding the Eigenvalues and Eigenfunctions. We consider the cases of negative, zero, and positive eigenvalues to check our results above.

$\boldsymbol{\lambda < 0}$. The general solution is

$$y = c\,\mathrm{e}^{\sqrt{-\lambda}x} + d\,\mathrm{e}^{-\sqrt{-\lambda}x}.$$

The only solution that satisfies the boundary conditions is the trivial solution, $y = 0$. Thus there are no negative eigenvalues.

$\boldsymbol{\lambda = 0}$. The general solution is

$$y = c + dx.$$

Again only the trivial solution satisfies the boundary conditions, so $\lambda = 0$ is not an eigenvalue.

$\boldsymbol{\lambda > 0}$. The general solution is

$$y = c\cos(\sqrt{\lambda}x) + d\sin(\sqrt{\lambda}x).$$

Applying the boundary conditions,

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad c = 0 \\ y(\pi) = 0 \quad &\Rightarrow \quad d\sin(\sqrt{\lambda}\pi) = 0 \end{aligned}$$

The nontrivial solutions are

$$\sqrt{\lambda} = n = 1, 2, 3, \ldots \qquad y = d\sin(n\pi).$$

Thus the eigenvalues and eigenfunctions are

$$\lambda_n = n^2, \qquad \phi_n = \sin(nx), \qquad \text{for } n = 1, 2, 3, \ldots$$

We can verify that this example satisfies all the properties listed in Result 31.2.1. Note that there are an infinite number of eigenvalues. There is a least eigenvalue $\lambda_1 = 1$ but there is no greatest eigenvalue. For each eigenvalue, there is one eigenfunction. The n^{th} eigenfunction $\sin(nx)$ has $n-1$ zeroes in the interval $0 < x < \pi$.

Since a series of the eigenfunctions is the familiar Fourier sine series, we know that the eigenfunctions are orthogonal and complete. Checking Rayleigh's quotient,

$$
\begin{aligned}
\lambda_n &= \frac{-p\phi_n \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\Big|_0^\pi + \int_0^\pi \left(p \left(\frac{\mathrm{d}\phi_n}{\mathrm{d}x}\right)^2 - q\phi_n^2 \right) dx}{\int_0^\pi \phi_n^2 \sigma \, dx} \\
&= \frac{-\sin(nx) \frac{\mathrm{d}\sin(nx)}{\mathrm{d}x}\Big|_0^\pi + \int_0^\pi \left(\left(\frac{\mathrm{d}\sin(nx)}{\mathrm{d}x}\right)^2 \right) dx}{\int_0^\pi \sin^2(nx) dx} \\
&= \frac{\int_0^\pi n^2 \cos^2(nx) \, dx}{\pi/2} \\
&= n^2.
\end{aligned}
$$

Example 31.2.2 Consider the eigenvalue problem

$$x^2 y'' + x y' + y = \mu y, \qquad y(1) = y(2) = 0.$$

Since $x^2 > 0$ on $[1, 2]$, we can write this problem in terms of a regular Sturm-Liouville eigenvalue problem. Dividing by x^2,

$$y'' + \frac{1}{x} y' + \frac{1}{x^2}(1-\mu) y = 0.$$

We multiply by the factor $\exp(\int^x \frac{1}{\xi} \, d\xi) = \exp(\log x) = x$ and make the substitution, $\lambda = 1 - \mu$ to obtain the Sturm-Liouville form

$$
\begin{gathered}
x y'' + y' + \lambda \frac{1}{x} y = 0 \\
(x y')' + \lambda \frac{1}{x} y = 0.
\end{gathered}
$$

We see that the eigenfunctions will be orthogonal with respect to the weighting function $\sigma = 1/x$.

From the Rayleigh quotient,

$$\begin{aligned}\lambda &= \frac{-\left[p\bar{\phi}\phi'\right]_a^b + \langle \phi' | x | \phi' \rangle}{\langle \phi | \frac{1}{x} | \phi \rangle} \\ &= \frac{\langle \phi' | x | \phi' \rangle}{\langle \phi | \frac{1}{x} | \phi \rangle}.\end{aligned}$$

If $\phi' = 0$, then only the trivial solution, $\phi = 0$, satisfies the boundary conditions. Thus the eigenvalues λ are positive.

Returning to the original problem We see that the eigenvalues, μ, satisfy $\mu < 1$. Since this is an Euler equation, the substitution $y = x^\alpha$ yields

$$\begin{aligned}\alpha(\alpha - 1) + \alpha + 1 - \mu = 0 \\ \alpha^2 + 1 - \mu = 0.\end{aligned}$$

Since $\mu < 1$,

$$\alpha = \pm i\sqrt{1 - \mu}.$$

The general solution is

$$y = c_1 x^{i\sqrt{1-\mu}} + c_2 x^{-i\sqrt{1-\mu}}.$$

We know that the eigenfunctions can be written as real functions. We can rewrite the solution as

$$y = c_1 \,\mathrm{e}^{i\sqrt{1-\mu}\log x} + c_2 \,\mathrm{e}^{-i\sqrt{1-\mu}\log x}.$$

An equivalent form is

$$y = c_1 \cos(\sqrt{1-\mu}\log x) + c_2 \sin(\sqrt{1-\mu}\log x).$$

Applying the boundary conditions,

$$
\begin{aligned}
y(1) = 0 &\quad\Rightarrow\quad c_1 = 0 \\
y(2) = 0 &\quad\Rightarrow\quad \sin(\sqrt{1-\mu}\log 2) = 0 \\
&\quad\Rightarrow\quad \sqrt{1-\mu}\log 2 = n\pi, \quad \text{for } n = 1, 2, \ldots
\end{aligned}
$$

Thus the eigenvalues and eigenfunctions are

$$
\boxed{\mu_n = 1 - \left(\frac{n\pi}{\log 2}\right)^2, \qquad \phi_n = \sin\left(n\pi\frac{\log x}{\log 2}\right) \quad \text{for } n = 1, 2, \ldots}
$$

31.3 Solving Differential Equations With Eigenfunction Expansions

Linear Algebra. Consider the eigenvalue problem,

$$
\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.
$$

If the matrix $\mathbf{A}$ has a complete, orthonormal set of eigenvectors $\{\boldsymbol{\xi}_k\}$ with eigenvalues $\{\lambda_k\}$ then we can represent any vector as a linear combination of the eigenvectors.

$$
\mathbf{y} = \sum_{k=1}^{n} a_k\boldsymbol{\xi}_k, \qquad a_k = \boldsymbol{\xi}_k \cdot \mathbf{y}
$$

$$
\mathbf{y} = \sum_{k=1}^{n} (\boldsymbol{\xi}_k \cdot \mathbf{y})\,\boldsymbol{\xi}_k
$$

This property allows us to solve the inhomogeneous equation

$$
\mathbf{A}\mathbf{x} - \mu\mathbf{x} = \mathbf{b}. \tag{31.1}
$$

Before we try to solve this equation, we should consider the existence/uniqueness of the solution. If μ is not an eigenvalue, then the range of $L \equiv \mathbf{A} - \mu$ is $\mathbb{R}^n$. The problem has a unique solution. If μ is an eigenvalue, then the null space of L is the span of the eigenvectors of μ. That is, if $\mu = \lambda_i$, then $\operatorname{nullspace}(L) = \operatorname{span}(\boldsymbol{\xi}_{i_1}, \boldsymbol{\xi}_{i_2}, \ldots, \boldsymbol{\xi}_{i_m})$. ($\{\boldsymbol{\xi}_{i_1}, \boldsymbol{\xi}_{i_2}, \ldots, \boldsymbol{\xi}_{i_m}\}$ are the eigenvalues of λ_i.) If $\mathbf{b}$ is orthogonal to $\operatorname{nullspace}(L)$ then Equation 31.1 has a solution, but it is not unique. If $\mathbf{y}$ is a solution then we can add any linear combination of $\{\boldsymbol{\xi}_{i_j}\}$ to obtain another solution. Thus the solutions have the form

$$\mathbf{x} = \mathbf{y} + \sum_{j=1}^{m} c_j \boldsymbol{\xi}_{i_j}.$$

If $\mathbf{b}$ is not orthogonal to $\operatorname{nullspace}(L)$ then Equation 31.1 has no solution.

Now we solve Equation 31.1. We assume that μ is not an eigenvalue. We expand the solution $\mathbf{x}$ and the inhomogeneity in the orthonormal eigenvectors.

$$\mathbf{x} = \sum_{k=1}^{n} a_k \boldsymbol{\xi}_k, \qquad \mathbf{b} = \sum_{k=1}^{n} b_k \boldsymbol{\xi}_k$$

We substitute the expansions into Equation 31.1.

$$\begin{aligned}
\mathbf{A} \sum_{k=1}^{n} a_k \boldsymbol{\xi}_k - \mu \sum_{k=1}^{n} a_k \boldsymbol{\xi}_k &= \sum_{k=1}^{n} b_k \boldsymbol{\xi}_k \\
\sum_{k=1}^{n} a_k \lambda_k \boldsymbol{\xi}_k - \mu \sum_{k=1}^{n} a_k \boldsymbol{\xi}_k &= \sum_{k=1}^{n} b_k \boldsymbol{\xi}_k \\
a_k &= \frac{b_k}{\lambda_k - \mu}
\end{aligned}$$

The solution is

$$\mathbf{x} = \sum_{k=1}^{n} \frac{b_k}{\lambda_k - \mu} \boldsymbol{\xi}_k.$$

Inhomogeneous Boundary Value Problems. Consider the self-adjoint eigenvalue problem,

$$Ly = \lambda y, \quad a < x < b,$$
$$B_1[y] = B_2[y] = 0.$$

If the problem has a complete, orthonormal set of eigenfunctions $\{\phi_k\}$ with eigenvalues $\{\lambda_k\}$ then we can represent any square-integrable function as a linear combination of the eigenfunctions.

$$f = \sum_k f_k \phi_k, \qquad f_k = \langle \phi_k | f \rangle = \int_a^b \overline{\phi_k(x)} f(x)\, dx$$
$$f = \sum_k \langle \phi_k | f \rangle \phi_k$$

This property allows us to solve the inhomogeneous differential equation

$$Ly - \mu y = f, \quad a < x < b, \tag{31.2}$$
$$B_1[y] = B_2[y] = 0.$$

Before we try to solve this equation, we should consider the existence/uniqueness of the solution. If μ is not an eigenvalue, then the range of $L - \mu$ is the space of square-integrable functions. The problem has a unique solution. If μ is an eigenvalue, then the null space of L is the span of the eigenfunctions of μ. That is, if $\mu = \lambda_i$, then $\operatorname{nullspace}(L) = \operatorname{span}(\phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_m})$. ($\{\phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_m}\}$ are the eigenvalues of λ_i.) If f is orthogonal to $\operatorname{nullspace}(L - \mu)$ then Equation 31.2 has a solution, but it is not unique. If u is a solution then we can add any linear combination of $\{\phi_{i_j}\}$ to obtain another solution. Thus the solutions have the form

$$y = u + \sum_{j=1}^{m} c_j \phi_{i_j}.$$

If f is not orthogonal to $\operatorname{nullspace}(L - \mu)$ then Equation 31.2 has no solution.

Now we solve Equation 31.2. We assume that μ is not an eigenvalue. We expand the solution y and the inhomogeneity in the orthonormal eigenfunctions.

$$y = \sum_k y_k \phi_k, \qquad f = \sum_k f_k \phi_k$$

It would be handy if we could substitute the expansions into Equation 31.2. However, the expansion of a function is not necessarily differentiable. Thus we demonstrate that since y is $C^2(a \ldots b)$ and satisfies the boundary conditions $B_1[y] = B_2[y] = 0$, we are justified in substituting it into the differential equation. In particular, we will show that

$$L[y] = L\left[\sum_k y_k \phi_k\right] = \sum_k y_k L\left[\phi_k\right] = \sum_k y_k \lambda_k \phi_k.$$

To do this we will use Green's identity. If u and v are $C^2(a \ldots b)$ and satisfy the boundary conditions $B_1[y] = B_2[y] = 0$ then

$$\langle u | L[v] \rangle = \langle L[u] | v \rangle.$$

First we assume that we can differentiate y term-by-term.

$$L[y] = \sum_k y_k \lambda_k \phi_k$$

Now we directly expand $L[y]$ and show that we get the same result.

$$L[y] = \sum_k c_k \phi_k$$

$$\begin{aligned}
c_k &= \langle \phi_k | L[y] \rangle \\
&= \langle L[\phi_k] | y \rangle \\
&= \langle \lambda_k \phi_k | y \rangle \\
&= \lambda_k \langle \phi_k | y \rangle \\
&= \lambda_k y_k
\end{aligned}$$

$$L[y] = \sum_k y_k \lambda \phi_k$$

The series representation of y may *not* be differentiable, but we *are* justified in applying L term-by-term.

Now we substitute the expansions into Equation 31.2.

$$L\left[\sum_k y_k \phi_k\right] - \mu \sum_k y_k \phi_k = \sum_k f_k \phi_k$$

$$\sum_k \lambda_k y_k \phi_k - \mu \sum_k y_k \phi_k = \sum_k f_k \phi_k$$

$$y_k = \frac{f_k}{\lambda_k - \mu}$$

The solution is

$$y = \sum_k \frac{f_k}{\lambda_k - \mu} \phi_k$$

Consider a second order, inhomogeneous problem.

$$L[y] = f(x), \quad B_1[y] = b_1, \quad B_2[y] = b_2$$

We will expand the solution in an orthogonal basis.

$$y = \sum_n a_n \phi_n$$

We would like to substitute the series into the differential equation, but in general we are not allowed to differentiate such series. To get around this, we use integration by parts to move derivatives from the solution y, to the ϕ_n.

Example 31.3.1 Consider the problem,

$$y'' + \alpha y = f(x), \quad y(0) = a, \quad y(\pi) = b,$$

where $\alpha \neq n^2$, $n \in \mathbb{Z}^+$. We expand the solution in a cosine series.

$$y(x) = \frac{y_0}{\sqrt{\pi}} + \sum_{n=1}^{\infty} y_n \sqrt{\frac{2}{\pi}} \cos(nx)$$

We also expand the inhomogeneous term.

$$f(x) = \frac{f_0}{\sqrt{\pi}} + \sum_{n=1}^{\infty} f_n \sqrt{\frac{2}{\pi}} \cos(nx)$$

We multiply the differential equation by the orthonormal functions and integrate over the interval. We neglect the special case $\phi_0 = 1/\sqrt{\pi}$ for now.

$$\int_0^\pi \sqrt{\frac{2}{\pi}} \cos(nx) y'' \, dx + \alpha \int_0^\pi \sqrt{\frac{2}{\pi}} \cos(nx) y \, dx = \int_0^\pi \sqrt{\frac{2}{\pi}} f(x) \, dx$$

$$\left[\sqrt{\frac{2}{\pi}} \cos(nx) y'(x) \right]_0^\pi + \int_0^\pi \sqrt{\frac{2}{\pi}} n \sin(nx) y'(x) \, dx + \alpha y_n = f_n$$

$$\sqrt{\frac{2}{\pi}} \left((-1)^n y'(\pi) - y'(0) \right) + \left[\sqrt{\frac{2}{\pi}} n \sin(nx) y(x) \right]_0^\pi - \int_0^\pi \sqrt{\frac{2}{\pi}} n^2 \cos(nx) y(x) \, dx + \alpha y_n = f_n$$

$$\sqrt{\frac{2}{\pi}} \left((-1)^n y'(\pi) - y'(0) \right) - n^2 y_n + \alpha y_n = f_n$$

Unfortunately we don't know the values of $y'(0)$ and $y'(\pi)$.

CONTINUE HERE

31.4 Exercises

Exercise 31.1
Find the eigenvalues and eigenfunctions of

$$y'' + 2\alpha y' + \lambda y = 0, \quad y(a) = y(b) = 0,$$

where $a < b$.

Write the problem in Sturm Liouville form. Verify that the eigenvalues and eigenfunctions satisfy the properties of regular Sturm-Liouville problems. Find the coefficients in the expansion of an arbitrary function $f(x)$ in a series of the eigenfunctions.
Hint, Solution

Exercise 31.2
Find the eigenvalues and eigenfunctions of the boundary value problem

$$y'' + \frac{\lambda}{(z+1)^2} y = 0$$

on the interval $1 \leq z \leq 2$ with boundary conditions $y(1) = y(2) = 0$. Discuss how the results confirm the concepts presented in class relating to boundary value problems of this type.
Hint, Solution

Exercise 31.3
Find the eigenvalues and eigenfunctions of

$$y'' + \frac{2\alpha + 1}{x} y' + \frac{\lambda}{x^2} y = 0, \quad y(a) = y(b) = 0,$$

where $0 < a < b$. Write the problem in Sturm Liouville form. Verify that the eigenvalues and eigenfunctions satisfy the properties of regular Sturm-Liouville problems. Find the coefficients in the expansion of an arbitrary function $f(x)$ in a series of the eigenfunctions.
Hint, Solution

Exercise 31.4
Find the eigenvalues and eigenfunctions of

$$y'' - y' + \lambda y = 0, \qquad y(0) = y(1) = 0.$$

Find the coefficients in the expansion of an arbitrary, $f(x)$, in a series of the eigenfunctions.
Hint, Solution

Exercise 31.5
Find the eigenvalues and eigenfunctions for,

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0.$$

Show that the transcendental equation for λ has infinitely many roots $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Find the limit of λ_n as $n \to \infty$. How is this approached?
Hint, Solution

Exercise 31.6
Consider

$$y'' + y = f(x) \quad y(0) = 0 \quad y(1) + y'(1) = 0.$$

Find the eigenfunctions for this problem and the equation which the eigenvalues satisfy. Give the general solution in terms of these eigenfunctions.
Hint, Solution

Exercise 31.7
Show that the eigenvalue problem,

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(0) - y(1) = 0,$$

(note the mixed boundary condition), has only one real eigenvalue. Find it and the corresponding eigenfunction. Show that this problem is not self-adjoint. Thus the proof, valid for unmixed, homogeneous boundary conditions, that all eigenvalues are real fails in this case.
Hint, Solution

Exercise 31.8
Determine the Rayleigh quotient, $R[\phi]$ for,

$$y'' + \frac{1}{x}y' + \lambda y = 0, \quad |y(0)| < \infty, \quad y(1) = 0.$$

Use the trial function $\phi = 1 - x$ in $R[\phi]$ to deduce that the smallest zero of $J_0(x)$, the Bessel function of the first kind and order zero, is less than $\sqrt{6}$.
Hint, Solution

Exercise 31.9
Discuss the eigenvalues of the equation

$$y'' + \lambda q(z) y = 0, \qquad y(0) = y(\pi) = 0$$

where

$$q(z) = \begin{cases} a > 0, & 0 \leq z \leq l \\ b > 0, & l < z \leq \pi. \end{cases}$$

This is an example that indicates that the results we obtained in class for eigenfunctions and eigenvalues with $q(z)$ continuous and bounded also hold if $q(z)$ is simply integrable; that is

$$\int_0^\pi |q(z)| \, dz$$

is finite.
Hint, Solution

Exercise 31.10

1. Find conditions on the smooth real functions $p(x)$, $q(x)$, $r(x)$ and $s(x)$ so that the eigenvalues, λ, of:

$$Lv \equiv (p(x)v''(x))'' - (q(x)v'(x))' + r(x)v(x) = \lambda s(x)v(x), \quad a < x < b$$
$$v(a) = v''(a) = 0$$
$$v''(b) = 0, \quad p(b)v'''(b) - q(b)v'(b) = 0$$

are positive. Prove the assertion.

2. Show that for any smooth $p(x)$, $q(x)$, $r(x)$ and $s(x)$ the eigenfunctions belonging to distinct eigenvalues are orthogonal relative to the weight $s(x)$. That is:

$$\int_a^b v_m(x)v_k(x)s(x)\,dx = 0 \text{ if } \lambda_k \neq \lambda_m.$$

3. Find the eigenvalues and eigenfunctions for:

$$\frac{d^4\phi}{dx^4} = \lambda\phi, \quad \begin{cases}\phi(0) = \phi''(0) = 0, \\ \phi(1) = \phi''(1) = 0.\end{cases}$$

Hint, Solution

31.5 Hints

Hint 31.1

Hint 31.2

Hint 31.3

Hint 31.4
Write the problem in Sturm-Liouville form to show that the eigenfunctions are orthogonal with respect to the weighting function $\sigma = e^{-x}$.

Hint 31.5
Note that the solution is a regular Sturm-Liouville problem and thus the eigenvalues are real. Use the Rayleigh quotient to show that there are only positive eigenvalues. Informally show that there are an infinite number of eigenvalues with a graph.

Hint 31.6

Hint 31.7
Find the solution for $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$. A problem is self-adjoint if it satisfies Green's identity.

Hint 31.8

Write the equation in self-adjoint form. The Bessel equation of the first kind and order zero satisfies the problem,

$$y'' + \frac{1}{x}y' + y = 0, \quad |y(0)| < \infty, \quad y(r) = 0,$$

where r is a positive root of $J_0(x)$. Make the change of variables $\xi = x/r$, $u(\xi) = y(x)$.

Hint 31.9

Hint 31.10

31.6 Solutions

Solution 31.1

Recall that constant coefficient equations are shift invariant. If $u(x)$ is a solution, then so is $u(x-c)$.

We substitute $y = e^{\gamma x}$ into the constant coefficient equation.

$$y'' + 2\alpha y' + \lambda y = 0$$
$$\gamma^2 + 2\alpha\gamma + \lambda = 0$$
$$\gamma = -\alpha \pm \sqrt{\alpha^2 - \lambda}$$

First we consider the case $\lambda = \alpha^2$. A set of solutions of the differential equation is

$$\left\{ e^{-\alpha x}, x\, e^{-\alpha x} \right\}$$

The homogeneous solution that satisfies the left boundary condition $y(a) = 0$ is

$$y = c(x-a)\, e^{-\alpha x}.$$

Since only the trivial solution with $c = 0$ satisfies the right boundary condition, $\lambda = \alpha^2$ is not an eigenvalue.

Next we consider the case $\lambda \neq \alpha^2$. We write

$$\gamma = -\alpha \pm i\sqrt{\lambda - \alpha^2}.$$

Note that $\Re(\sqrt{\lambda - \alpha^2}) \geq 0$. A set of solutions of the differential equation is

$$\left\{ e^{(-\alpha \pm i\sqrt{\lambda-\alpha^2})x} \right\}$$

By taking the sum and difference of these solutions we obtain a new set of linearly independent solutions.

$$\left\{ e^{-\alpha x} \cos\left(\sqrt{\lambda - \alpha^2} x\right), e^{-\alpha x} \sin\left(\sqrt{\lambda - \alpha^2} x\right) \right\}$$

The solution which satisfies the left boundary condition is

$$y = c\, e^{-\alpha x} \sin\left(\sqrt{\lambda - \alpha^2}(x-a)\right).$$

For nontrivial solutions, the right boundary condition $y(b) = 0$ imposes the constraint

$$\mathrm{e}^{-\alpha b} \sin\left(\sqrt{\lambda - \alpha^2}(b-a)\right) = 0$$
$$\sqrt{\lambda - \alpha^2}(b-a) = n\pi, \quad n \in \mathbb{Z}$$

We have the eigenvalues

$$\boxed{\lambda_n = \alpha^2 + \left(\frac{n\pi}{b-a}\right)^2, \quad n \in \mathbb{Z}}$$

with the eigenfunctions

$$\boxed{\phi_n = \mathrm{e}^{-\alpha x} \sin\left(n\pi \frac{x-a}{b-a}\right).}$$

To write the problem in Sturm-Liouville form, we multiply by the integrating factor

$$\mathrm{e}^{\int 2\alpha \, dx} = \mathrm{e}^{2\alpha x}.$$

$$\boxed{\left(\mathrm{e}^{2\alpha x} y'\right)' + \lambda \, \mathrm{e}^{2\alpha x} y = 0, \quad y(a) = y(b) = 0}$$

Now we verify that the Sturm-Liouville properties are satisfied.

- The eigenvalues

$$\lambda_n = \alpha^2 + \left(\frac{n\pi}{b-a}\right)^2, \quad n \in \mathbb{Z}$$

  are real.

- There are an infinite number of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots,$$

$$\alpha^2 + \left(\frac{\pi}{b-a}\right)^2 < \alpha^2 + \left(\frac{2\pi}{b-a}\right)^2 < \alpha^2 + \left(\frac{3\pi}{b-a}\right)^2 < \cdots.$$

  There is a least eigenvalue

$$\lambda_1 = \alpha^2 + \left(\frac{\pi}{b-a}\right)^2,$$

  but there is no greatest eigenvalue, ($\lambda_n \to \infty$ as $n \to \infty$).

- For each eigenvalue, we found one unique, (to within a multiplicative constant), eigenfunction ϕ_n. We were able to choose the eigenfunctions to be real-valued. The eigenfunction

$$\phi_n = \mathrm{e}^{-\alpha x} \sin\left(n\pi \frac{x-a}{b-a}\right).$$

  has exactly $n-1$ zeros in the open interval $a < x < b$.

- The eigenfunctions are orthogonal with respect to the weighting function $\sigma(x) = \mathrm{e}^{2\alpha x}$.

$$\begin{aligned}
\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx &= \int_a^b \mathrm{e}^{-\alpha x} \sin\left(n\pi\frac{x-a}{b-a}\right) \mathrm{e}^{-\alpha x} \sin\left(m\pi\frac{x-a}{b-a}\right) \mathrm{e}^{2\alpha x}\,dx \\
&= \int_a^b \sin\left(n\pi\frac{x-a}{b-a}\right) \sin\left(m\pi\frac{x-a}{b-a}\right) dx \\
&= \frac{b-a}{\pi} \int_0^\pi \sin(nx)\sin(mx)\,dx \\
&= \frac{b-a}{2\pi} \int_0^\pi (\cos((n-m)x) - \cos((n+m)x))\,dx \\
&= 0 \quad \text{if } n \neq m
\end{aligned}$$

- The eigenfunctions are complete. Any piecewise continuous function $f(x)$ defined on $a \leq x \leq b$ can be expanded in a series of eigenfunctions

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx}.$$

The sum converges to $\frac{1}{2}(f(x^-) + f(x^+))$. (We do not prove this property.)

- The eigenvalues can be related to the eigenfunctions with the Rayleigh quotient.

$$\begin{aligned}
\lambda_n &= \frac{\left[-p\phi_n \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\right]_a^b + \int_a^b \left(p\left(\frac{\mathrm{d}\phi_n}{\mathrm{d}x}\right)^2 - q\phi_n^2\right) dx}{\int_a^b \phi_n^2 \sigma\,dx} \\
&= \frac{\int_a^b \left(\mathrm{e}^{2\alpha x}\left(\mathrm{e}^{-\alpha x}\left(\frac{n\pi}{b-a}\cos\left(n\pi\frac{x-a}{b-a}\right) - \alpha\sin\left(n\pi\frac{x-a}{b-a}\right)\right)\right)^2\right) dx}{\int_a^b \left(\mathrm{e}^{-\alpha x}\sin\left(n\pi\frac{x-a}{b-a}\right)\right)^2 \mathrm{e}^{2\alpha x}\,dx} \\
&= \frac{\int_a^b \left(\left(\frac{n\pi}{b-a}\right)^2\cos^2\left(n\pi\frac{x-a}{b-a}\right) - 2\alpha\frac{n\pi}{b-a}\cos\left(n\pi\frac{x-a}{b-a}\right)\sin\left(n\pi\frac{x-a}{b-a}\right) + \alpha^2\sin^2\left(n\pi\frac{x-a}{b-a}\right)\right) dx}{\int_a^b \sin^2\left(n\pi\frac{x-a}{b-a}\right) dx} \\
&= \frac{\int_0^\pi \left(\left(\frac{n\pi}{b-a}\right)^2\cos^2(x) - 2\alpha\frac{n\pi}{b-a}\cos(x)\sin(x) + \alpha^2\sin^2(x)\right) dx}{\int_0^\pi \sin^2(x)\,dx} \\
&= \alpha^2 + \left(\frac{n\pi}{b-a}\right)^2
\end{aligned}$$

Now we expand a function $f(x)$ in a series of the eigenfunctions.

$$f(x) \sim \sum_{n=1}^{\infty} c_n \, \mathrm{e}^{-\alpha x} \sin\left(n\pi \frac{x-a}{b-a}\right),$$

where

$$\begin{aligned} c_n &= \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx} \\ &= \frac{2n}{b-a} \int_a^b f(x)\,\mathrm{e}^{\alpha x} \sin\left(n\pi \frac{x-a}{b-a}\right) dx \end{aligned}$$

Solution 31.2

This is an Euler equation. We substitute $y = (z+1)^\alpha$ into the equation.

$$\begin{gathered} y'' + \frac{\lambda}{(z+1)^2} y = 0 \\ \alpha(\alpha-1) + \lambda = 0 \\ \alpha = \frac{1 \pm \sqrt{1-4\lambda}}{2} \end{gathered}$$

First consider the case $\lambda = 1/4$. A set of solutions is

$$\left\{\sqrt{z+1}, \sqrt{z+1}\log(z+1)\right\}.$$

Another set of solutions is

$$\left\{\sqrt{z+1}, \sqrt{z+1}\log\left(\frac{z+1}{2}\right)\right\}.$$

The solution which satisfies the boundary condition $y(1) = 0$ is

$$y = c\sqrt{z+1}\log\left(\frac{z+1}{2}\right).$$

Since only the trivial solution satisfies the $y(2) = 0$, $\lambda = 1/4$ is not an eigenvalue.

Now consider the case $\lambda \neq 1/4$. A set of solutions is

$$\left\{(z+1)^{(1+\sqrt{1-4\lambda})/2}, (z+1)^{(1-\sqrt{1-4\lambda})/2}\right\}.$$

We can write this in terms of the exponential and the logarithm.

$$\left\{\sqrt{z+1}\exp\left(i\frac{\sqrt{4\lambda-1}}{2}\log(z+1)\right), \sqrt{z+1}\exp\left(-i\frac{\sqrt{4\lambda-1}}{2}\log(z+1)\right)\right\}.$$

Note that

$$\left\{\sqrt{z+1}\exp\left(i\frac{\sqrt{4\lambda-1}}{2}\log\left(\frac{z+1}{2}\right)\right), \sqrt{z+1}\exp\left(-i\frac{\sqrt{4\lambda-1}}{2}\log\left(\frac{z+1}{2}\right)\right)\right\}.$$

is also a set of solutions. The new factor of 2 in the logarithm just multiplies the solutions by a constant. We write the solution in terms of the cosine and sine.

$$\left\{\sqrt{z+1}\cos\left(\frac{\sqrt{4\lambda-1}}{2}\log\left(\frac{z+1}{2}\right)\right), \sqrt{z+1}\sin\left(\frac{\sqrt{4\lambda-1}}{2}\log\left(\frac{z+1}{2}\right)\right)\right\}.$$

The solution of the differential equation which satisfies the boundary condition $y(1) = 0$ is

$$y = c\sqrt{z+1}\sin\left(\frac{\sqrt{1-4\lambda}}{2}\log\left(\frac{z+1}{2}\right)\right).$$

Now we use the second boundary condition to find the eigenvalues.

$$
\begin{gathered}
y(2) = 0 \\
\sin\left(\frac{\sqrt{4\lambda - 1}}{2} \log\left(\frac{3}{2}\right)\right) = 0 \\
\frac{\sqrt{4\lambda - 1}}{2} \log\left(\frac{3}{2}\right) = n\pi, \quad n \in \mathbb{Z} \\
\lambda = \frac{1}{4}\left(1 + \left(\frac{2n\pi}{\log(3/2)}\right)^2\right), \quad n \in \mathbb{Z}
\end{gathered}
$$

$n = 0$ gives us a trivial solution, so we discard it. Discarding duplicate solutions, The eigenvalues and eigenfunctions are

$$
\boxed{\lambda_n = \frac{1}{4} + \left(\frac{n\pi}{\log(3/2)}\right)^2, \qquad y_n = \sqrt{z+1}\sin\left(n\pi\frac{\log((z+1)/2)}{\log(3/2)}\right), \quad n \in \mathbb{Z}^+.}
$$

Now we verify that the eigenvalues and eigenfunctions satisfy the properties of regular Sturm-Liouville problems.

- The eigenvalues are real.
- There are an infinite number of eigenvalues

$$
\begin{gathered}
\lambda_1 < \lambda_2 < \lambda_3 < \cdots \\
\frac{1}{4} + \left(\frac{\pi}{\log(3/2)}\right)^2 < \frac{1}{4} + \left(\frac{2\pi}{\log(3/2)}\right)^2 < \frac{1}{4} + \left(\frac{3\pi}{\log(3/2)}\right)^2 < \cdots
\end{gathered}
$$

  There is a least least eigenvalue

$$
\lambda_1 = \frac{1}{4} + \left(\frac{\pi}{\log(3/2)}\right)^2,
$$

  but there is no greatest eigenvalue.

- The eigenfunctions are orthogonal with respect to the weighting function $\sigma(z) = 1/(z+1)^2$. Let $n \neq m$.

$$
\begin{aligned}
&\int_1^2 y_n(z) y_m(z) \sigma(z)\, dz \\
&\qquad = \int_1^2 \sqrt{z+1} \sin\left(n\pi \frac{\log((z+1)/2)}{\log(3/2)}\right) \sqrt{z+1} \sin\left(m\pi \frac{\log((z+1)/2)}{\log(3/2)}\right) \frac{1}{(z+1)^2}\, dz \\
&\qquad = \int_0^\pi \sin(nx) \sin(mx) \frac{\log(3/2)}{\pi}\, dx \\
&\qquad = \frac{\log(3/2)}{2\pi} \int_0^\pi (\cos((n-m)x) - \cos((n+m)x))\, dx \\
&\qquad = 0
\end{aligned}
$$

- The eigenfunctions are complete. A function $f(x)$ defined on $(1 \ldots 2)$ has the series representation

$$
f(x) \sim \sum_{n=1}^\infty c_n y_n(x) = \sum_{n=1}^\infty c_n \sqrt{z+1} \sin\left(n\pi \frac{\log((z+1)/2)}{\log(3/2)}\right),
$$

where

$$
c_n = \frac{\langle y_n | 1/(z+1)^2 | f \rangle}{\langle y_n | 1/(z+1)^2 | y_n \rangle} = \frac{2}{\log(3/2)} \int_1^2 \sin\left(n\pi \frac{\log((z+1)/2)}{\log(3/2)}\right) \frac{1}{(z+1)^{3/2}} f(x)\, dz
$$

Solution 31.3

Recall that Euler equations are scale invariant. If $u(x)$ is a solution, then so is $u(cx)$ for any nonzero constant c.

We substitute $y = x^\gamma$ into the Euler equation.

$$
\begin{gathered}
y'' + \frac{2\alpha+1}{x} y' + \frac{\lambda}{x^2} y = 0 \\
\gamma(\gamma-1) + (2\alpha+1)\gamma + \lambda = 0 \\
\gamma^2 + 2\alpha\gamma + \lambda = 0 \\
\gamma = -\alpha \pm \sqrt{\alpha^2 - \lambda}
\end{gathered}
$$

First we consider the case $\lambda = \alpha^2$. A set of solutions of the differential equation is

$$\{x^{-\alpha}, x^{-\alpha} \log x\}$$

The homogeneous solution that satisfies the left boundary condition $y(a) = 0$ is

$$y = cx^{-\alpha}(\log x - \log a) = cx^{-\alpha} \log\left(\frac{x}{a}\right).$$

Since only the trivial solution with $c = 0$ satisfies the right boundary condition, $\lambda = \alpha^2$ is not an eigenvalue.

Next we consider the case $\lambda \neq \alpha^2$. We write

$$\gamma = -\alpha \pm i\sqrt{\lambda - \alpha^2}.$$

Note that $\Re(\sqrt{\lambda - \alpha^2}) \geq 0$. A set of solutions of the differential equation is

$$\left\{x^{-\alpha \pm i\sqrt{\lambda-\alpha^2}}\right\}$$

$$\left\{x^{-\alpha}\, \mathrm{e}^{\pm i\sqrt{\lambda-\alpha^2}\log x}\right\}.$$

By taking the sum and difference of these solutions we obtain a new set of linearly independent solutions.

$$\left\{x^{-\alpha} \cos\left(\sqrt{\lambda - \alpha^2} \log x\right), x^{-\alpha} \sin\left(\sqrt{\lambda - \alpha^2} \log x\right),\right\}$$

The solution which satisfies the left boundary condition is

$$y = cx^{-\alpha} \sin\left(\sqrt{\lambda - \alpha^2} \log\left(\frac{x}{a}\right)\right).$$

For nontrivial solutions, the right boundary condition $y(b) = 0$ imposes the constraint

$$b^{-\alpha} \sin\left(\sqrt{\lambda - \alpha^2} \log\left(\frac{b}{a}\right)\right)$$

$$\sqrt{\lambda - \alpha^2} \log\left(\frac{b}{a}\right) = n\pi, \quad n \in \mathbb{Z}$$

We have the eigenvalues

$$\boxed{\lambda_n = \alpha^2 + \left(\frac{n\pi}{\log(b/a)}\right)^2, \quad n \in \mathbb{Z}}$$

with the eigenfunctions

$$\boxed{\phi_n = x^{-\alpha} \sin\left(n\pi \frac{\log(x/a)}{\log(b/a)}\right).}$$

To write the problem in Sturm-Liouville form, we multiply by the integrating factor

$$\mathrm{e}^{\int (2\alpha+1)/x \, dx} = \mathrm{e}^{(2\alpha+1)\log x} = x^{2\alpha+1}.$$

$$\boxed{\left(x^{2\alpha+1} y'\right)' + \lambda x^{2\alpha-1} y = 0, \quad y(a) = y(b) = 0}$$

Now we verify that the Sturm-Liouville properties are satisfied.

- The eigenvalues

$$\lambda_n = \alpha^2 + \left(\frac{n\pi}{\log(b/a)}\right)^2, \quad n \in \mathbb{Z}$$

  are real.

- There are an infinite number of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots,$$
$$\alpha^2 + \left(\frac{\pi}{\log(b/a)}\right)^2 < \alpha^2 + \left(\frac{2\pi}{\log(b/a)}\right)^2 < \alpha^2 + \left(\frac{3\pi}{\log(b/a)}\right)^2 < \cdots$$

There is a least eigenvalue

$$\lambda_1 = \alpha^2 + \left(\frac{\pi}{\log(b/a)} \right)^2,$$

but there is no greatest eigenvalue, ($\lambda_n \to \infty$ as $n \to \infty$).

- For each eigenvalue, we found one unique, (to within a multiplicative constant), eigenfunction ϕ_n. We were able to choose the eigenfunctions to be real-valued. The eigenfunction

$$\phi_n = x^{-\alpha} \sin\left(n\pi \frac{\log(x/a)}{\log(b/a)} \right).$$

has exactly $n-1$ zeros in the open interval $a < x < b$.

- The eigenfunctions are orthogonal with respect to the weighting function $\sigma(x) = x^{2\alpha-1}$.

$$\begin{aligned}
\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx &= \int_a^b x^{-\alpha} \sin\left(n\pi \frac{\log(x/a)}{\log(b/a)} \right) x^{-\alpha} \sin\left(m\pi \frac{\log(x/a)}{\log(b/a)} \right) x^{2\alpha-1}\,dx \\
&= \int_a^b \sin\left(n\pi \frac{\log(x/a)}{\log(b/a)} \right) \sin\left(m\pi \frac{\log(x/a)}{\log(b/a)} \right) \frac{1}{x}\,dx \\
&= \frac{\log(b/a)}{\pi} \int_0^\pi \sin(nx)\sin(mx)\,dx \\
&= \frac{\log(b/a)}{2\pi} \int_0^\pi (\cos((n-m)x) - \cos((n+m)x))\,dx \\
&= 0 \quad \text{if } n \neq m
\end{aligned}$$

- The eigenfunctions are complete. Any piecewise continuous function $f(x)$ defined on $a \leq x \leq b$ can be expanded in a series of eigenfunctions

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx}.$$

The sum converges to $\frac{1}{2}(f(x^-) + f(x^+))$. (We do not prove this property.)

- The eigenvalues can be related to the eigenfunctions with the Rayleigh quotient.

$$\begin{aligned}
\lambda_n &= \frac{\left[-p\phi_n \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\right]_a^b + \int_a^b \left(p\left(\frac{\mathrm{d}\phi_n}{\mathrm{d}x}\right)^2 - q\phi_n^2\right)\,dx}{\int_a^b \phi_n^2 \sigma\,dx} \\
&= \frac{\int_a^b \left(x^{2\alpha+1}\left(x^{-\alpha-1}\left(\frac{n\pi}{\log(b/a)}\cos\left(n\pi\frac{\log(x/a)}{\log(b/a)}\right) - \alpha\sin\left(n\pi\frac{\log(x/a)}{\log(b/a)}\right)\right)\right)^2\right)\,dx}{\int_a^b \left(x^{-\alpha}\sin\left(n\pi\frac{\log(x/a)}{\log(b/a)}\right)\right)^2 x^{2\alpha-1}\,dx} \\
&= \frac{\int_a^b \left(\left(\frac{n\pi}{\log(b/a)}\right)^2\cos^2(\cdot) - 2\alpha\frac{n\pi}{\log(b/a)}\cos(\cdot)\sin(\cdot) + \alpha^2\sin^2(\cdot)\right)x^{-1}\,dx}{\int_a^b \sin^2\left(n\pi\frac{\log(x/a)}{\log(b/a)}\right)x^{-1}\,dx} \\
&= \frac{\int_0^\pi \left(\left(\frac{n\pi}{\log(b/a)}\right)^2\cos^2(x) - 2\alpha\frac{n\pi}{\log(b/a)}\cos(x)\sin(x) + \alpha^2\sin^2(x)\right)\,dx}{\int_0^\pi \sin^2(x)\,dx} \\
&= \alpha^2 + \left(\frac{n\pi}{\log(b/a)}\right)^2
\end{aligned}$$

Now we expand a function $f(x)$ in a series of the eigenfunctions.

$$f(x) \sim \sum_{n=1}^\infty c_n x^{-\alpha}\sin\left(n\pi\frac{\log(x/a)}{\log(b/a)}\right),$$

where

$$
\begin{aligned}
c_n &= \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx} \\
&= \frac{2n}{\log(b/a)} \int_a^b f(x) x^{\alpha-1} \sin\left(n\pi \frac{\log(x/a)}{\log(b/a)}\right) dx
\end{aligned}
$$

Solution 31.4

$$
y'' - y' + \lambda y = 0, \qquad y(0) = y(1) = 0.
$$

The factor that will put this equation in Sturm-Liouville form is

$$
F(x) = \exp\left(\int^x -1\,dx\right) = \mathrm{e}^{-x}.
$$

The differential equation becomes

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x} y'\right) + \lambda\,\mathrm{e}^{-x} y = 0.
$$

Thus we see that the eigenfunctions will be orthogonal with respect to the weighting function $\sigma = \mathrm{e}^{-x}$.

Substituting $y = \mathrm{e}^{\alpha x}$ into the differential equation yields

$$
\begin{aligned}
\alpha^2 - \alpha + \lambda &= 0 \\
\alpha &= \frac{1 \pm \sqrt{1 - 4\lambda}}{2} \\
\alpha &= \frac{1}{2} \pm \sqrt{1/4 - \lambda}.
\end{aligned}
$$

If $\lambda < 1/4$ then the solutions to the differential equation are exponential and only the trivial solution satisfies the boundary conditions.

If $\lambda = 1/4$ then the solution is $y = c_1 \mathrm{e}^{x/2} + c_2 x \mathrm{e}^{x/2}$ and again only the trivial solution satisfies the boundary conditions.

Now consider the case that $\lambda > 1/4$.

$$\alpha = \frac{1}{2} \pm i\sqrt{\lambda - 1/4}$$

The solutions are

$$\mathrm{e}^{x/2} \cos(\sqrt{\lambda - 1/4}\, x), \qquad \mathrm{e}^{x/2} \sin(\sqrt{\lambda - 1/4}\, x).$$

The left boundary condition gives us

$$y = c\, \mathrm{e}^{x/2} \sin(\sqrt{\lambda - 1/4}\, x).$$

The right boundary condition demands that

$$\sqrt{\lambda - 1/4} = n\pi, \qquad n = 1, 2, \ldots$$

Thus we see that the eigenvalues and eigenfunctions are

$$\lambda_n = \frac{1}{4} + (n\pi)^2, \qquad y_n = \mathrm{e}^{x/2} \sin(n\pi x).$$

If $f(x)$ is a piecewise continuous function then we can expand it in a series of the eigenfunctions.

$$f(x) = \sum_{n=1}^{\infty} a_n \, \mathrm{e}^{x/2} \sin(n\pi x)$$

The coefficients are

$$
\begin{aligned}
a_n &= \frac{\int_0^1 f(x)\,\mathrm{e}^{-x}\,\mathrm{e}^{x/2}\sin(n\pi x)\,dx}{\int_0^1 \mathrm{e}^{-x}(\,\mathrm{e}^{x/2}\sin(n\pi x))^2\,dx} \\
&= \frac{\int_0^1 f(x)\,\mathrm{e}^{-x/2}\sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx} \\
&= 2\int_0^1 f(x)\,\mathrm{e}^{-x/2}\sin(n\pi x)\,dx.
\end{aligned}
$$

Solution 31.5

Since this is a Sturm-Liouville problem, there are only real eigenvalues. By the Rayleigh quotient, the eigenvalues are

$$
\lambda = \frac{-\phi\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big|_0^1 + \int_0^1\left(\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2\right)\,dx}{\int_0^1 \phi^2\,dx},
$$

$$
\lambda = \frac{\phi^2(1) + \int_0^1\left(\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2\right)\,dx}{\int_0^1 \phi^2\,dx}.
$$

This demonstrates that there are only positive eigenvalues. The general solution of the differential equation for positive, real λ is

$$
y = c_1\cos\left(\sqrt{\lambda}x\right) + c_2\sin\left(\sqrt{\lambda}x\right).
$$

The solution that satisfies the left boundary condition is

$$
y = c\sin\left(\sqrt{\lambda}x\right).
$$

For nontrivial solutions we must have

$$\sin\left(\sqrt{\lambda}\right) + \sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) = 0$$

$$\sqrt{\lambda} = -\tan\left(\sqrt{\lambda}\right).$$

The positive solutions of this equation are eigenvalues with corresponding eigenfunctions $\sin\left(\sqrt{\lambda}x\right)$. In Figure 31.1 we plot the functions x and $-\tan(x)$ and draw vertical lines at $x = (n - 1/2)\pi$, $n \in \mathbb{N}$.

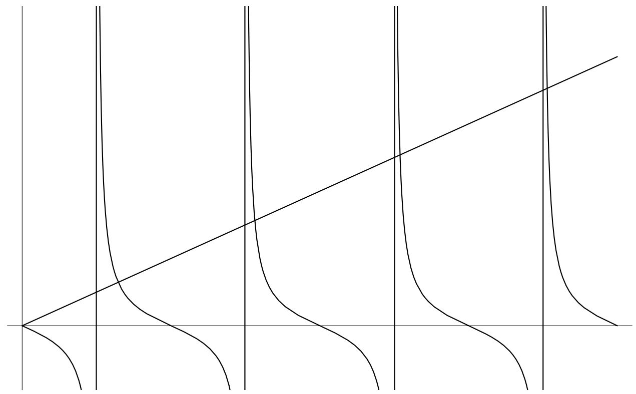

Figure 31.1: x and $-\tan(x)$.

From this we see that there are an infinite number of eigenvalues, $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. In the limit as $n \to \infty$, $\lambda_n \to (n - 1/2)\pi$. The limit is approached from above.

Solution 31.6

Consider the eigenvalue problem

$$y'' + y = \mu y \quad y(0) = 0 \quad y(1) + y'(1) = 0.$$

From Exercise 31.5 we see that the eigenvalues satisfy

$$\sqrt{1-\mu} = -\tan\left(\sqrt{1-\mu}\right)$$

and that there are an infinite number of eigenvalues. For large n, $\mu_n \approx 1 - (n-1/2)\pi$. The eigenfunctions are

$$\phi_n = \sin\left(\sqrt{1-\mu_n} x\right).$$

To solve the inhomogeneous problem, we expand the solution and the inhomogeneity in a series of the eigenfunctions.

$$f = \sum_{n=1}^{\infty} f_n \phi_n, \quad f_n = \frac{\int_0^1 f(x)\phi_n(x)\,\mathrm{d}x}{\int_0^1 \phi_n^2(x)\,\mathrm{d}x}$$

$$y = \sum_{n=1}^{\infty} y_n \phi_n$$

We substitite the expansions into the differential equation to determine the coefficients.

$$y'' + y = f$$

$$\sum_{n=1}^{\infty} \mu_n y_n \phi_n = \sum_{n=1}^{\infty} f_n \phi_n$$

$$\boxed{y = \sum_{n=1}^{\infty} \frac{f_n}{\mu_n} \sin\left(\sqrt{1-\mu_n} x\right)}$$

Solution 31.7

First consider $\lambda = 0$. The general solution is

$$y = c_1 + c_2 x.$$

$y = cx$ satisfies the boundary conditions. Thus $\lambda = 0$ is an eigenvalue.

Now consider negative real λ. The general solution is

$$y = c_1 \cosh\left(\sqrt{-\lambda}x\right) + c_2 \sinh\left(\sqrt{-\lambda}x\right).$$

The solution that satisfies the left boundary condition is

$$y = c \sinh\left(\sqrt{-\lambda}x\right).$$

For nontrivial solutions of the boundary value problem, there must be negative real solutions of

$$\sqrt{-\lambda} - \sinh\left(\sqrt{-\lambda}\right) = 0.$$

Since $x = \sinh x$ has no nonzero real solutions, this equation has no solutions for negative real λ. There are no negative real eigenvalues.

Finally consider positive real λ. The general solution is

$$y = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right).$$

The solution that satisfies the left boundary condition is

$$y = c \sin\left(\sqrt{\lambda}x\right).$$

For nontrivial solutions of the boundary value problem, there must be positive real solutions of

$$\sqrt{\lambda} - \sin\left(\sqrt{\lambda}\right) = 0.$$

Since $x = \sin x$ has no nonzero real solutions, this equation has no solutions for positive real λ. There are no positive real eigenvalues.

There is only one real eigenvalue, $\lambda = 0$, with corresponding eigenfunction $\phi = x$.

The difficulty with the boundary conditions, $y(0) = 0$, $y'(0) - y(1) = 0$ is that the problem is not self-adjoint. We demonstrate this by showing that the problem does not satisfy Green's identity. Let u and v be two functions that satisfy the boundary conditions, but not necessarily the differential equation.

$$\begin{aligned} \langle u, L[v]\rangle - \langle L[u], v\rangle &= \langle u, v''\rangle - \langle u'', v\rangle \\ &= [uv']_0^1 - \langle u', v'\rangle - \langle u', v'\rangle - [u'v]_0^1 + \langle u', v'\rangle - \langle u', v'\rangle \\ &= u(1)v'(1) - u'(1)v(1) \end{aligned}$$

Green's identity is not satisfied,

$$\langle u, L[v]\rangle - \langle L[u], v\rangle \neq 0;$$

The problem is not self-adjoint.

Solution 31.8
First we write the equation in formally self-adjoint form,

$$L[y] \equiv (xy')' = -\lambda xy, \quad |y(0)| < \infty, \quad y(1) = 0.$$

Let λ be an eigenvalue with corresponding eigenfunction ϕ. We derive the Rayleigh quotient for λ.

$$\begin{aligned} \langle \phi, L[\phi]\rangle &= \langle \phi, -\lambda x\phi\rangle \\ \langle \phi, (x\phi')'\rangle &= -\lambda\langle \phi, x\phi\rangle \\ [\phi x\phi']_0^1 - \langle \phi', x\phi'\rangle &= -\lambda\langle \phi, x\phi\rangle \end{aligned}$$

We apply the boundary conditions and solve for λ.

$$\boxed{\lambda = \frac{\langle \phi', x\phi'\rangle}{\langle \phi, x\phi\rangle}}$$

The Bessel equation of the first kind and order zero satisfies the problem,

$$y'' + \frac{1}{x}y' + y = 0, \quad |y(0)| < \infty, \quad y(r) = 0,$$

where r is a positive root of $J_0(x)$. We make the change of variables $\xi = x/r$, $u(\xi) = y(x)$ to obtain the problem

$$\frac{1}{r^2}u'' + \frac{1}{r\xi}\frac{1}{r}u' + u = 0, \quad |u(0)| < \infty, \quad u(1) = 0,$$

$$u'' + \frac{1}{\xi}u' + r^2 u = 0, \quad |u(0)| < \infty, \quad u(1) = 0.$$

Now r^2 is the eigenvalue of the problem for $u(\xi)$. From the Rayleigh quotient, the minimum eigenvalue obeys the inequality

$$r^2 \leq \frac{\langle \phi', x\phi' \rangle}{\langle \phi, x\phi \rangle},$$

where ϕ is any test function that satisfies the boundary conditions. Taking $\phi = 1 - x$ we obtain,

$$r^2 \leq \frac{\int_0^1 (-1)x(-1)\, dx}{\int_0^1 (1-x)x(1-x)\, dx} = 6,$$

$$\boxed{r \leq \sqrt{6}}$$

Thus the smallest zero of $J_0(x)$ is less than or equal to $\sqrt{6} \approx 2.4494$. (The smallest zero of $J_0(x)$ is approximately 2.40483.)

Solution 31.9

We assume that $0 < l < \pi$.

Recall that the solution of a second order differential equation with piecewise continuous coefficient functions is piecewise C^2. This means that the solution is C^2 except for a finite number of points where it is C^1.

First consider the case $\lambda = 0$. A set of linearly independent solutions of the differential equation is $\{1, z\}$. The solution which satisfies $y(0) = 0$ is $y_1 = c_1 z$. The solution which satisfies $y(\pi) = 0$ is $y_2 = c_2(\pi - z)$. There is a solution for the problem if there are there are values of c_1 and c_2 such that y_1 and y_2 have the same position and slope at $z = l$.

$$y_1(l) = y_2(l), \quad y_1'(l) = y_2'(l)$$
$$c_1 l = c_2(\pi - l), \quad c_1 = -c_2$$

Since there is only the trivial solution, $c_1 = c_2 = 0$, $\lambda = 0$ is not an eigenvalue.

Now consider $\lambda \neq 0$. For $0 \leq z \leq l$ a set of linearly independent solutions is

$$\left\{\cos(\sqrt{a\lambda}z), \sin(\sqrt{a\lambda}z)\right\}.$$

The solution which satisfies $y(0) = 0$ is

$$y_1 = c_1 \sin(\sqrt{a\lambda}z).$$

For $l < z \leq \pi$ a set of linearly independent solutions is

$$\left\{\cos(\sqrt{b\lambda}z), \sin(\sqrt{b\lambda}z)\right\}.$$

The solution which satisfies $y(\pi) = 0$ is

$$y_2 = c_2 \sin(\sqrt{b\lambda}(\pi - z)).$$

$\lambda \neq 0$ is an eigenvalue if there are nontrivial solutions of

$$y_1(l) = y_2(l), \quad y_1'(l) = y_2'(l)$$
$$c_1 \sin(\sqrt{a\lambda}l) = c_2 \sin(\sqrt{b\lambda}(\pi - l)), \quad c_1\sqrt{a\lambda}\cos(\sqrt{a\lambda}l) = -c_2\sqrt{b\lambda}\cos(\sqrt{b\lambda}(\pi - l))$$

We divide the second equation by $\sqrt(\lambda)$ since $\lambda \neq 0$ and write this as a linear algebra problem.

$$\begin{pmatrix} \sin(\sqrt{a\lambda}l) & -\sin(\sqrt{b\lambda}(\pi - l)) \\ \sqrt{a}\cos(\sqrt{a\lambda}l) & \sqrt{b}\sin(\sqrt{b\lambda}(\pi - l)) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This system of equations has nontrivial solutions if and only if the determinant of the matrix is zero.

$$\sqrt{b}\sin(\sqrt{a\lambda}l)\sin(\sqrt{b\lambda}(\pi - l)) + \sqrt{a}\cos(\sqrt{a\lambda}l)\sin(\sqrt{b\lambda}(\pi - l)) = 0$$

We can use trigonometric identities to write this equation as

$$\boxed{(\sqrt{b} - \sqrt{a})\sin\left(\sqrt{\lambda}(l\sqrt{a} - (\pi - l)\sqrt{b})\right) + (\sqrt{b} + \sqrt{a})\sin\left(\sqrt{\lambda}(l\sqrt{a} + (\pi - l)\sqrt{b})\right) = 0}$$

Clearly this equation has an infinite number of solutions for real, positive λ. However, it is not clear that this equation does not have non-real solutions. In order to prove that, we will show that the problem is self-adjoint. Before going on to that we note that the eigenfunctions have the form

$$\boxed{\phi_n(z) = \begin{cases} \sin\left(\sqrt{a\lambda_n}z\right) & 0 \leq z \leq l \\ \sin\left(\sqrt{b\lambda_n}(\pi - z)\right) & l < z \leq \pi. \end{cases}}$$

Now we prove that the problem is self-adjoint. We consider the class of functions which are C^2 in $(0 \ldots \pi)$ except at the interior point $x = l$ where they are C^1 and which satisfy the boundary conditions $y(0) = y(\pi) = 0$. Note that the differential operator is not defined at the point $x = l$. Thus Green's identity,

$$\langle u|q|Lv\rangle = \langle Lu|q|v\rangle$$

is not well-defined. To remedy this we must define a new inner product. We choose

$$\langle u|v\rangle \equiv \int_0^l \overline{u}v\,dx + \int_l^\pi \overline{u}v\,dx.$$

This new inner product does not require differentiability at the point $x = l$.

The problem is self-adjoint if Green's indentity is satisfied. Let u and v be elements of our class of functions. In addition to the boundary conditions, we will use the fact that u and v satisfy $y(l^-) = y(l^+)$ and $y'(l^-) = y'(l^+)$.

$$
\begin{aligned}
\langle v|Lu\rangle &= \int_0^l \overline{v}u''\,dx + \int_l^\pi \overline{v}u''\,dx \\
&= [\overline{v}u']_0^l - \int_0^l \overline{v}'u'\,dx + [\overline{v}u']_l^\pi - \int_l^\pi \overline{v}'u'\,dx \\
&= \overline{v}(l)u'(l) - \int_0^l \overline{v}'u'\,dx - \overline{v}(l)u'(l) - \int_l^\pi \overline{v}'u'\,dx \\
&= -\int_0^l \overline{v}'u'\,dx - \int_l^\pi \overline{v}'u'\,dx \\
&= -[\overline{v}'u]_0^l + \int_0^l \overline{v}''u\,dx - [\overline{v}'u]_l^\pi + \int_l^\pi \overline{v}''u\,dx \\
&= -\overline{v}'(l)u(l) + \int_0^l \overline{v}''u\,dx + \overline{v}'(l)u(l) + \int_l^\pi \overline{v}''u\,dx \\
&= \int_0^l \overline{v}''u\,dx + \int_l^\pi \overline{v}''u\,dx \\
&= \langle Lv|Lu\rangle
\end{aligned}
$$

The problem is self-adjoint. Hence the eigenvalues are real. There are an infinite number of positive, real eigenvalues λ_n.

Solution 31.10

1. Let v be an eigenfunction with the eigenvalue λ. We start with the differential equation and then take the

inner product with v.

$$(pv'')'' - (qv')' + rv = \lambda sv$$
$$\langle v, (pv'')'' - (qv')' + rv\rangle = \langle v, \lambda sv\rangle$$

We use integration by parts and utilize the homogeneous boundary conditions.

$$[v(pv'')']_a^b - \langle v', (pv'')'\rangle - [vqv']_a^b + \langle v', qv'\rangle + \langle v, rv\rangle = \lambda\langle v, sv\rangle$$
$$-[v'pv'']_a^b + \langle v'', pv''\rangle + \langle v', qv'\rangle + \langle v, rv\rangle = \lambda\langle v, sv\rangle$$
$$\lambda = \frac{\langle v'', pv''\rangle + \langle v', qv'\rangle + \langle v, rv\rangle}{\langle v, sv\rangle}$$

We see that if $p, q, r, s \geq 0$ then the eigenvalues will be positive. (Of course we assume that p and s are not identically zero.)

2. First we prove that this problem is self-adjoint. Let u and v be functions that satisfy the boundary conditions, but do not necessarily satsify the differential equation.

$$\langle v, L[u]\rangle - \langle L[v], u\rangle = \langle v, (pu'')'' - (qu')' + ru\rangle - \langle (pv'')'' - (qv')' + rv, u\rangle$$

Following our work in part (a) we use integration by parts to move the derivatives.

$$= (\langle v'', pu''\rangle + \langle v', qu'\rangle + \langle v, ru\rangle) - (\langle pv'', u''\rangle + \langle qv', u'\rangle + \langle rv, u\rangle)$$
$$= 0$$

This problem satisfies Green's identity,

$$\langle v, L[u]\rangle - \langle L[v], u\rangle = 0,$$

and is thus self-adjoint.

Let v_k and v_m be eigenfunctions corresponding to the distinct eigenvalues λ_k and λ_m. We start with Green's identity.

$$\begin{aligned}
\langle v_k, L[v_m]\rangle - \langle L[v_k], v_m\rangle &= 0 \\
\langle v_k, \lambda_m s v_m\rangle - \langle \lambda_k s v_k, v_m\rangle &= 0 \\
(\lambda_m - \lambda_k)\langle v_k, s v_m\rangle &= 0 \\
\langle v_k, s v_m\rangle &= 0
\end{aligned}$$

The eigenfunctions are orthogonal with respect to the weighting function s.

3. From part (a) we know that there are only positive eigenvalues. The general solution of the differential equation is

$$\phi = c_1 \cos(\lambda^{1/4} x) + c_2 \cosh(\lambda^{1/4} x) + c_3 \sin(\lambda^{1/4} x) + c_4 \sinh(\lambda^{1/4} x).$$

Applying the condition $\phi(0) = 0$ we obtain

$$\phi = c_1(\cos(\lambda^{1/4} x) - \cosh(\lambda^{1/4} x)) + c_2 \sin(\lambda^{1/4} x) + c_3 \sinh(\lambda^{1/4} x).$$

The condition $\phi''(0) = 0$ reduces this to

$$\phi = c_1 \sin(\lambda^{1/4} x) + c_2 \sinh(\lambda^{1/4} x).$$

We substitute the solution into the two right boundary conditions.

$$\begin{aligned}
c_1 \sin(\lambda^{1/4}) + c_2 \sinh(\lambda^{1/4}) &= 0 \\
-c_1 \lambda^{1/2} \sin(\lambda^{1/4}) + c_2 \lambda^{1/2} \sinh(\lambda^{1/4}) &= 0
\end{aligned}$$

We see that $\sin(\lambda^{1/4}) = 0$. The eigenvalues and eigenfunctions are

$$\lambda_n = (n\pi)^4, \quad \phi_n = \sin(n\pi x), \quad n \in \mathbb{N}.$$

Chapter 32

Integrals and Convergence

Never try to teach a pig to sing. It wastes your time and annoys the pig.

-?

32.1 Uniform Convergence of Integrals

Consider the improper integral

$$\int_c^\infty f(x,t)\,dt.$$

The integral is convergent to $S(x)$ if, given any $\epsilon > 0$, there exists $T(x,\epsilon)$ such that

$$\left| \int_c^\tau f(x,t)\,dt - S(x) \right| < \epsilon \quad \text{for all } \tau > T(x,\epsilon).$$

The sum is uniformly convergent if T is independent of x.

Similar to the Weierstrass M-test for infinite sums we have a uniform convergence test for integrals. If there exists a continuous function $M(t)$ such that $|f(x,t)| \leq M(t)$ and $\int_c^\infty M(t)\,dt$ is convergent, then $\int_c^\infty f(x,t)\,dt$ is uniformly convergent.

If $\int_c^\infty f(x,t)\,dt$ is uniformly convergent, we have the following properties:

- If $f(x,t)$ is continuous for $x \in [a,b]$ and $t \in [c,\infty)$ then for $a < x_0 < b$,

$$\lim_{x\to x_0} \int_c^\infty f(x,t)\,dt = \int_c^\infty \left(\lim_{x\to x_0} f(x,t)\right)\,dt.$$

- If $a \leq x_1 < x_2 \leq b$ then we can interchange the order of integration.

$$\int_{x_1}^{x_2} \left(\int_c^\infty f(x,t)\,dt\right)\,dx = \int_c^\infty \left(\int_{x_1}^{x_2} f(x,t)\,dx\right)\,dt$$

- If $\frac{\partial f}{\partial x}$ is continuous, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_c^\infty f(x,t)\,dt = \int_c^\infty \frac{\partial}{\partial x} f(x,t)\,dt.$$

32.2 The Riemann-Lebesgue Lemma

Result 32.2.1 If $\int_a^b |f(x)|\,dx$ exists, then

$$\int_a^b f(x)\sin(\lambda x)\,dx \to 0 \text{ as } \lambda \to \infty.$$

Before we try to justify the Riemann-Lebesgue lemma, we will need a preliminary result. Let λ be a positive constant.

$$\left| \int_a^b \sin(\lambda x)\, dx \right| = \left| \left[-\frac{1}{\lambda} \cos(\lambda x) \right]_a^b \right|$$
$$\leq \frac{2}{\lambda}.$$

We will prove the Riemann-Lebesgue lemma for the case when $f(x)$ has limited total fluctuation on the interval (a, b). We can express $f(x)$ as the difference of two functions

$$f(x) = \psi_+(x) - \psi_-(x),$$

where ψ_+ and ψ_- are positive, increasing, bounded functions.

From the mean value theorem for positive, increasing functions, there exists an x_0, $a \leq x_0 \leq b$, such that

$$\left| \int_a^b \psi_+(x) \sin(\lambda x)\, dx \right| = \left| \psi_+(b) \int_{x_0}^b \sin(\lambda x)\, dx \right|$$
$$\leq |\psi_+(b)| \frac{2}{\lambda}.$$

Similarly,

$$\left| \int_a^b \psi_-(x) \sin(\lambda x)\, dx \right| \leq |\psi_-(b)| \frac{2}{\lambda}.$$

Thus

$$\left| \int_a^b f(x) \sin(\lambda x)\, dx \right| \leq \frac{2}{\lambda} (|\psi_+(b)| + |\psi_-(b)|)$$
$$\to 0 \quad \text{as } \lambda \to \infty.$$

32.3 Cauchy Principal Value

32.3.1 Integrals on an Infinite Domain

The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty} \int_a^0 f(x)\,dx + \lim_{b\to\infty} \int_0^b f(x)\,dx,$$

when these limits exist. The Cauchy principal value of the integral is defined

$$\mathrm{PV} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to\infty} \int_{-a}^{a} f(x)\,dx.$$

The principal value may exist when the integral diverges.

Example 32.3.1 $\int_{-\infty}^{\infty} x\,dx$ diverges, but

$$\mathrm{PV} \int_{-\infty}^{\infty} x\,dx = \lim_{a\to\infty} \int_{-a}^{a} x\,dx = \lim_{a\to\infty} (0) = 0.$$

If the improper integral converges, then the Cauchy principal value exists and is equal to the value of the integral. The principal value of the integral of an odd function is zero. If the principal value of the integral of an even function exists, then the integral converges.

32.3.2 Singular Functions

Let $f(x)$ have a singularity at $x = 0$. Let a and b satisfy $a < 0 < b$. The integral of $f(x)$ is defined

$$\int_a^b f(x)\,dx = \lim_{\epsilon_1\to0^-} \int_a^{\epsilon_1} f(x)\,dx + \lim_{\epsilon_2\to0^+} \int_{\epsilon_2}^{b} f(x)\,dx,$$

when the limits exist. The Cauchy principal value of the integral is defined

$$\mathrm{PV} \int_a^b f(x)\, dx = \lim_{\epsilon \to 0^+} \left(\int_a^{-\epsilon} f(x)\, dx + \int_\epsilon^b f(x)\, dx \right),$$

when the limit exists.

Example 32.3.2 The integral

$$\int_{-1}^2 \frac{1}{x}\, dx$$

diverges, but the principal value exists.

$$\begin{aligned}
\mathrm{PV} \int_{-1}^2 \frac{1}{x}\, dx &= \lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} \frac{1}{x}\, dx + \int_\epsilon^2 \frac{1}{x}\, dx \right) \\
&= \lim_{\epsilon \to 0^+} \left(-\int_\epsilon^1 \frac{1}{x}\, dx + \int_\epsilon^2 \frac{1}{x}\, dx \right) \\
&= \int_1^2 \frac{1}{x}\, dx \\
&= \log 2
\end{aligned}$$

Chapter 33

The Laplace Transform

33.1 The Laplace Transform

The Laplace transform of the function $f(t)$ is defined

$$\mathcal{L}[f(t)] = \int_0^\infty \mathrm{e}^{-st} f(t) \, \mathrm{d}t,$$

for all values of s for which the integral exists. The Laplace transform of $f(t)$ is a function of s which we will denote $\hat{f}(s)$. [1]

A function $f(t)$ is of exponential order α if there exist constants t_0 and M such that

$$|f(t)| < M \, \mathrm{e}^{\alpha t}, \quad \text{for all } t > t_0.$$

If $\int_0^{t_0} f(t) \, \mathrm{d}t$ exists and $f(t)$ is of exponential order α then the Laplace transform $F(s)$ exists for $\Re(s) > \alpha$. Here are a few examples of these concepts.

- $\sin t$ is of exponential order 0.

[1] Denoting the Laplace transform of $f(t)$ as $F(s)$ is also common.

- $t\,\mathrm{e}^{2t}$ is of exponential order α for any $\alpha > 2$.
- e^{t^2} is not of exponential order α for any α.
- t^n is of exponential order α for any $\alpha > 0$.
- t^{-2} does not have a Laplace transform as the integral diverges.

Example 33.1.1 Consider the Laplace transform of $f(t) = 1$. Since $f(t) = 1$ is of exponential order α for any $\alpha > 0$, the Laplace transform integral converges for $\Re(s) > 0$.

$$\begin{aligned} \hat{f}(s) &= \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t \\ &= \left[-\frac{1}{s}\,\mathrm{e}^{-st}\right]_0^\infty \\ &= \frac{1}{s} \end{aligned}$$

Example 33.1.2 The function $f(t) = t\,\mathrm{e}^t$ is of exponential order α for any $\alpha > 1$. We compute the Laplace transform of this function.

$$\begin{aligned} \hat{f}(s) &= \int_0^\infty \mathrm{e}^{-st} t\,\mathrm{e}^t\,\mathrm{d}t \\ &= \int_0^\infty t\,\mathrm{e}^{(1-s)t}\,\mathrm{d}t \\ &= \left[\frac{1}{1-s} t\,\mathrm{e}^{(1-s)t}\right]_0^\infty - \int_0^\infty \frac{1}{1-s}\,\mathrm{e}^{(1-s)t}\,\mathrm{d}t \\ &= -\left[\frac{1}{(1-s)^2}\,\mathrm{e}^{(1-s)t}\right]_0^\infty \\ &= \frac{1}{(1-s)^2} \quad \text{for } \Re(s) > 1. \end{aligned}$$

Example 33.1.3 Consider the Laplace transform of the Heaviside function,

$$H(t-c) = \begin{cases} 0 & \text{for } t < c \\ 1 & \text{for } t > c, \end{cases}$$

where $c > 0$.

$$\begin{aligned}
\mathcal{L}[H(t-c)] &= \int_0^\infty \mathrm{e}^{-st} H(t-c)\,\mathrm{d}t \\
&= \int_c^\infty \mathrm{e}^{-st}\,\mathrm{d}t \\
&= \left[\frac{\mathrm{e}^{-st}}{-s}\right]_c^\infty \\
&= \frac{\mathrm{e}^{-cs}}{s} \quad \text{for } \Re(s) > 0
\end{aligned}$$

Example 33.1.4 Next consider $H(t-c)f(t-c)$.

$$\begin{aligned}
\mathcal{L}[H(t-c)f(t-c)] &= \int_0^\infty \mathrm{e}^{-st} H(t-c) f(t-c)\,\mathrm{d}t \\
&= \int_c^\infty \mathrm{e}^{-st} f(t-c)\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-s(t+c)} f(t)\,\mathrm{d}t \\
&= \mathrm{e}^{-cs} \hat{f}(s)
\end{aligned}$$

33.2 The Inverse Laplace Transform

The *inverse Laplace transform* in denoted

$$f(t) = \mathcal{L}^{-1}[\hat{f}(s)].$$

We compute the inverse Laplace transform with the *Mellin inversion formula.*

$$f(t) = \frac{1}{i2\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{st} \hat{f}(s) \, ds$$

Here α is a real constant that is to the right of the singularities of $\hat{f}(s)$.

To see why the Mellin inversion formula is correct, we take the Laplace transform of it. Assume that $f(t)$ is of exponential order α. Then α will be to the right of the singularities of $\hat{f}(s)$.

$$\begin{aligned} \mathcal{L}[\mathcal{L}^{-1}[\hat{f}(s)]] &= \mathcal{L}\left[\frac{1}{i2\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{zt} \hat{f}(z) \, dz\right] \\ &= \int_0^\infty e^{-st} \frac{1}{i2\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{zt} \hat{f}(z) \, dz \, dt \end{aligned}$$

We interchange the order of integration.

$$= \frac{1}{i2\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} \hat{f}(z) \int_0^\infty e^{(z-s)t} \, dt \, dz$$

Since $\Re(z) = \alpha$, the integral in t exists for $\Re(s) > \alpha$.

$$= \frac{1}{i2\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{\hat{f}(z)}{s - z} \, dz$$

We would like to evaluate this integral by closing the path of integration with a semi-circle of radius R in the right half plane and applying the residue theorem. However, in order for the integral along the semi-circle to vanish as $R \to \infty$, $\hat{f}(z)$ must vanish as $|z| \to \infty$. If $\hat{f}(z)$ vanishes we can use the maximum modulus bound to show that the integral along the semi-circle vanishes. This we assume that $\hat{f}(z)$ vanishes at infinity.

Consider the integral,

$$\frac{1}{i2\pi} \oint_C \frac{\hat{f}(z)}{s - z} \, dz,$$

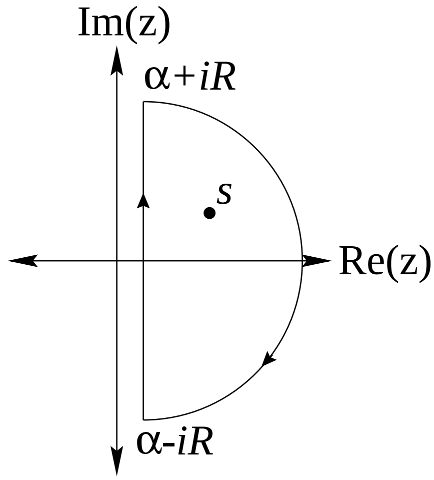

Figure 33.1: The Laplace Transform Pair Contour.

where C is the contour that starts at $\alpha - iR$, goes straight up to $\alpha + iR$, and then follows a semi-circle back down to $\alpha - iR$. This contour is shown in Figure 33.1.

If s is inside the contour then

$$\frac{1}{\mathrm{i}2\pi} \oint_C \frac{\hat{f}(z)}{s - z} \, \mathrm{d}z = \hat{f}(s).$$

Note that the contour is traversed in the negative direction. Since $\hat{f}(z)$ decays as $|z| \to \infty$, the semicircular contribution to the integral will vanish as $R \to \infty$. Thus

$$\frac{1}{\mathrm{i}2\pi} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} \frac{\hat{f}(z)}{s - z} \, \mathrm{d}z = \hat{f}(s).$$

Therefore, we have shown than

$$\mathcal{L}[\mathcal{L}^{-1}[\hat{f}(s)]] = F(s).$$

$f(t)$ and $\hat{f}(s)$ are known as Laplace transform pairs.

33.2.1 F(s) with Poles

Example 33.2.1 Consider the inverse Laplace transform of $1/s^2$. $s = 1$ is to the right of the singularity of $1/s^2$.

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = \frac{1}{\mathrm{i}2\pi}\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} \mathrm{e}^{st}\frac{1}{s^2}\,\mathrm{d}s$$

Let B_R be the contour starting at $1 - iR$ and following a straight line to $1 + iR$; let C_R be the contour starting at $1 + iR$ and following a semicircular path down to $1 - iR$. Let C be the combination of B_R and C_R. This contour is shown in Figure 33.2.

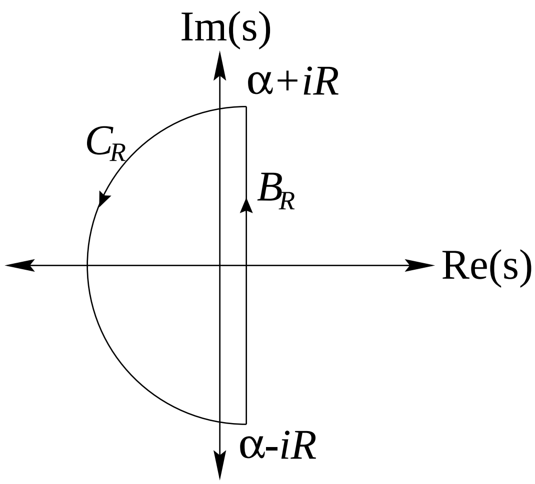

Figure 33.2: The Path of Integration for the Inverse Laplace Transform.

Consider the line integral on C for $R > 1$.

$$
\begin{aligned}
\frac{1}{\mathrm{i}2\pi} \oint_C \mathrm{e}^{st} \frac{1}{s^2} \, \mathrm{d}s &= \operatorname{Res}\left(\mathrm{e}^{st} \frac{1}{s^2}, 0 \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{e}^{st} \bigg|_{s=0} \\
&= t
\end{aligned}
$$

If $t \geq 0$, the integral along C_R vanishes as $R \to \infty$. We parameterize s.

$$
s = 1 + R\,\mathrm{e}^{\mathrm{i}\theta}, \qquad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}
$$

$$
\left| \mathrm{e}^{st} \right| = \left| \mathrm{e}^{t(1+R\,\mathrm{e}^{\mathrm{i}\theta})} \right| = \mathrm{e}^t \, \mathrm{e}^{tR\cos\theta} \leq \mathrm{e}^t
$$

$$
\begin{aligned}
\left| \int_{C_R} \mathrm{e}^{st} \frac{1}{s^2} \, \mathrm{d}s \right| &\leq \int_{C_R} \left| \mathrm{e}^{st} \frac{1}{s^2} \right| \, \mathrm{d}s \\
&\leq \pi R \, \mathrm{e}^t \frac{1}{(R-1)^2} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}
$$

Thus the inverse Laplace transform of $1/s^2$ is

$$
\boxed{\mathcal{L}^{-1}\left[\frac{1}{s^2} \right] = t, \quad \text{for } t \geq 0.}
$$

Let $\hat{f}(s)$ be analytic except for isolated poles at $s_1, s_2, \ldots, s_N$ and let α be to the right of these poles. Also, let $\hat{f}(s) \to 0$ as $|s| \to \infty$. Define B_R to be the straight line from $\alpha - \mathrm{i}R$ to $\alpha + \mathrm{i}R$ and C_R to be the semicircular path from $\alpha + \mathrm{i}R$ to $\alpha - \mathrm{i}R$. If R is large enough to enclose all the poles, then

$$\frac{1}{\mathrm{i}2\pi} \oint_{B_R+C_R} \mathrm{e}^{st} F(s) \,\mathrm{d}s = \sum_{n=1}^{N} \operatorname{Res}\left(\mathrm{e}^{st} F(s), s_n \right)$$

$$\frac{1}{\mathrm{i}2\pi} \int_{B_R} \mathrm{e}^{st} F(s) \,\mathrm{d}s = \sum_{n=1}^{N} \operatorname{Res}\left(\mathrm{e}^{st} F(s), s_n \right) - \frac{1}{\mathrm{i}2\pi} \int_{C_R} \mathrm{e}^{st} F(s) \,\mathrm{d}s.$$

Now let's examine the integral along C_R. Let the maximum of $|\hat{f}(s)|$ on C_R be M_R. We can parameterize the contour with $s = \alpha + R\,\mathrm{e}^{\mathrm{i}\theta}$, $\pi/2 < \theta < 3\pi/2$.

$$\begin{aligned}
\left| \int_{C_R} \mathrm{e}^{st} F(s) \,\mathrm{d}s \right| &= \left| \int_{\pi/2}^{3\pi/2} \mathrm{e}^{t(\alpha + R\,\mathrm{e}^{i\theta})} \hat{f}(\alpha + R\,\mathrm{e}^{\mathrm{i}\theta}) R\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta} \,\mathrm{d}\theta \right| \\
&\leq \int_{\pi/2}^{3\pi/2} \mathrm{e}^{\alpha t}\, \mathrm{e}^{tR\cos\theta} R M_R \,\mathrm{d}\theta \\
&= R M_R\, \mathrm{e}^{\alpha t} \int_0^{\pi} \mathrm{e}^{-tR\sin\theta} \,\mathrm{d}\theta
\end{aligned}$$

If $t \geq 0$ we can use Jordan's Lemma to obtain,

$$\begin{aligned}
&< R M_R\, \mathrm{e}^{\alpha t} \frac{\pi}{tR}. \\
&= M_R\, \mathrm{e}^{\alpha t} \frac{\pi}{t}
\end{aligned}$$

We use that $M_R \to 0$ as $R \to \infty$.

$$\to 0 \quad \text{as } R \to \infty$$

Thus we have an expression for the inverse Laplace transform of $\hat{f}(s)$.

$$\frac{1}{\mathrm{i}2\pi}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} \mathrm{e}^{st}\hat{f}(s)\,\mathrm{d}s = \sum_{n=1}^{N} \operatorname{Res}\left(\mathrm{e}^{st}\hat{f}(s), s_n\right)$$
$$\mathcal{L}^{-1}[\hat{f}(s)] = \sum_{n=1}^{N} \operatorname{Res}\left(\mathrm{e}^{st}\hat{f}(s), s_n\right)$$

Result 33.2.1 If $\hat{f}(s)$ is analytic except for poles at $s_1, s_2, \ldots, s_N$ and $\hat{f}(s) \to 0$ as $|s| \to \infty$ then the inverse Laplace transform of $\hat{f}(s)$ is

$$f(t) = \mathcal{L}^{-1}[\hat{f}(s)] = \sum_{n=1}^{N} \operatorname{Res}\left(\mathrm{e}^{st}\hat{f}(s), s_n\right), \quad \text{for } t > 0.$$

Example 33.2.2 Consider the inverse Laplace transform of $\frac{1}{s^3-s^2}$.

First we factor the denominator.

$$\frac{1}{s^3 - s^2} = \frac{1}{s^2}\frac{1}{s-1}.$$

Taking the inverse Laplace transform,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^3 - s^3}\right] &= \operatorname{Res}\left(\mathrm{e}^{st}\frac{1}{s^2}\frac{1}{s-1}, 0\right) + \operatorname{Res}\left(\mathrm{e}^{st}\frac{1}{s^2}\frac{1}{s-1}, 1\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{e}^{st}}{s-1}\bigg|_{s=0} + \mathrm{e}^t \\
&= \frac{-1}{(-1)^2} + \frac{t}{-1} + \mathrm{e}^t
\end{aligned}$$

Thus we have that

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s^3 - s^2}\right] = \mathrm{e}^t - t - 1, \quad \text{for } t > 0.}$$

Example 33.2.3 Consider the inverse Laplace transform of

$$\frac{s^2 + s - 1}{s^3 - 2s^2 + s - 2}.$$

We factor the denominator.

$$\frac{s^2 + s - 1}{(s - 2)(s - \mathrm{i})(s + \mathrm{i})}.$$

Then we take the inverse Laplace transform.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s^2 + s - 1}{s^3 - 2s^2 + s - 2}\right] &= \operatorname{Res}\left(\mathrm{e}^{st}\frac{s^2 + s - 1}{(s - 2)(s - \mathrm{i})(s + \mathrm{i})}, 2\right) + \operatorname{Res}\left(\mathrm{e}^{st}\frac{s^2 + s - 1}{(s - 2)(s - \mathrm{i})(s + \mathrm{i})}, \mathrm{i}\right) \\
&\quad + \operatorname{Res}\left(\mathrm{e}^{st}\frac{s^2 + s - 1}{(s - 2)(s - \mathrm{i})(s + \mathrm{i})}, -\mathrm{i}\right) \\
&= \mathrm{e}^{2t} + \mathrm{e}^{\mathrm{i}t}\frac{1}{\mathrm{i}2} + \mathrm{e}^{-\mathrm{i}t}\frac{-1}{\mathrm{i}2}
\end{aligned}$$

Thus we have

$$\boxed{\mathcal{L}^{-1}\left[\frac{s^2 + s - 1}{s^3 - 2s^2 + s - 2}\right] = \sin t + \mathrm{e}^{2t}, \quad \text{for } t > 0.}$$

33.2.2 $\hat{f}(s)$ with Branch Points

Example 33.2.4 Consider the inverse Laplace transform of $\frac{1}{\sqrt{s}}$. $\sqrt{s}$ denotes the principal branch of $s^{1/2}$. There is a branch cut from $s = 0$ to $s = -\infty$ and

$$\frac{1}{\sqrt{s}} = \frac{\mathrm{e}^{-\mathrm{i}\theta/2}}{\sqrt{r}}, \quad \text{for } -\pi < \theta < \pi.$$

Let α be any positive number. The inverse Laplace transform of $\frac{1}{\sqrt{s}}$ is

$$f(t) = \frac{1}{\mathrm{i}2\pi} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} \mathrm{e}^{st} \frac{1}{\sqrt{s}} \, \mathrm{d}s.$$

We will evaluate the integral by deforming it to wrap around the branch cut. Consider the integral on the contour shown in Figure 33.3. C_R^+ and C_R^- are circular arcs of radius R. B is the vertical line at $\Re(s) = \alpha$ joining the two arcs. C_ϵ is a semi-circle in the right half plane joining $\mathrm{i}\epsilon$ and $-\mathrm{i}\epsilon$. L^+ and L^- are lines joining the circular arcs at $\Im(s) = \pm\epsilon$.

Since there are no residues inside the contour, we have

$$\frac{1}{\mathrm{i}2\pi} \left(\int_B + \int_{C_R^+} + \int_{L^+} + \int_{C_\epsilon} + \int_{L^-} + \int_{C_R^-} \right) \mathrm{e}^{st} \frac{1}{\sqrt{s}} \, \mathrm{d}s = 0.$$

We will evaluate the inverse Laplace transform for $t > 0$.

First we will show that the integral along C_R^+ vanishes as $R \to \infty$. As $\epsilon \to 0$, we have

$$\int_{C_R^+} \cdots \, \mathrm{d}s = \int_{\pi/2-\delta}^{\pi/2} \cdots \, \mathrm{d}\theta + \int_{\pi/2}^{\pi} \cdots \, \mathrm{d}\theta.$$

The first integral vanishes by the maximum modulus bound. Note that the length of the path of integration is

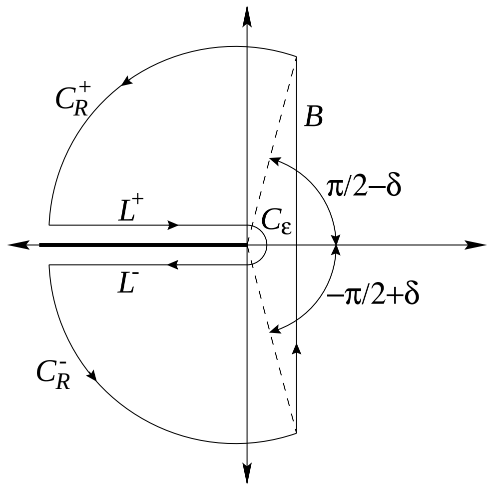

Figure 33.3: Path of Integration for $1/\sqrt{s}$

less than 2α.

$$
\begin{aligned}
\left| \int_{\pi/2-\delta}^{\pi/2} \cdots \, \mathrm{d}\theta \right| &\leq \left(\max_{s \in C_R^+} \left| \mathrm{e}^{st} \frac{1}{\sqrt{s}} \right| \right) (2\alpha) \\
&= \mathrm{e}^{\alpha t} \frac{1}{\sqrt{R}} (2\alpha) \\
&\to 0 \text{ as } R \to \infty
\end{aligned}
$$

The second integral vanishes by Jordan's Lemma. A parameterization of C_R^+ is $s = R\,\mathrm{e}^{\mathrm{i}\theta}$.

$$\begin{aligned}
\left| \int_{\pi/2}^{\pi} \mathrm{e}^{R\,\mathrm{e}^{\mathrm{i}\theta} t} \frac{1}{\sqrt{R\,\mathrm{e}^{\mathrm{i}\theta}}}\,\mathrm{d}\theta \right| &\leq \int_{\pi/2}^{\pi} \left| \mathrm{e}^{R\,\mathrm{e}^{\mathrm{i}\theta} t} \frac{1}{\sqrt{R\,\mathrm{e}^{\mathrm{i}\theta}}} \right| \mathrm{d}\theta \\
&\leq \frac{1}{\sqrt{R}} \int_{\pi/2}^{\pi} \mathrm{e}^{R\cos(\theta)t}\,\mathrm{d}\theta \\
&\leq \frac{1}{\sqrt{R}} \int_{0}^{\pi/2} \mathrm{e}^{-Rt\sin(\phi)}\,\mathrm{d}\phi \\
&< \frac{1}{\sqrt{R}} \frac{\pi}{2Rt} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

We could show that the integral along C_R^- vanishes by the same method. Now we have

$$\frac{1}{\mathrm{i}2\pi} \left(\int_B + \int_{L^+} + \int_{C_\epsilon} + \int_{L^-} \right) \mathrm{e}^{st} \frac{1}{\sqrt{s}}\,\mathrm{d}s = 0.$$

We can show that the integral along C_ϵ vanishes as $\epsilon \to 0$ with the maximum modulus bound.

$$\begin{aligned}
\left| \int_{C_\epsilon} \mathrm{e}^{st} \frac{1}{\sqrt{s}}\,\mathrm{d}s \right| &\leq \left(\max_{s \in C_\epsilon} \left| \mathrm{e}^{st} \frac{1}{\sqrt{s}} \right| \right) (\pi\epsilon) \\
&< \mathrm{e}^{\epsilon t} \frac{1}{\sqrt{\epsilon}} \pi\epsilon \\
&\to 0 \quad \text{as } \epsilon \to 0
\end{aligned}$$

Now we can express the inverse Laplace transform in terms of the integrals along L^+ and L^-.

$$f(t) \equiv \frac{1}{\mathrm{i}2\pi} \int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} \mathrm{e}^{st} \frac{1}{\sqrt{s}}\,\mathrm{d}s = -\frac{1}{\mathrm{i}2\pi} \int_{L^+} \mathrm{e}^{st} \frac{1}{\sqrt{s}}\,\mathrm{d}s - \frac{1}{\mathrm{i}2\pi} \int_{L^-} \mathrm{e}^{st} \frac{1}{\sqrt{s}}\,\mathrm{d}s$$

On L^+, $s = r\,\mathrm{e}^{\mathrm{i}\pi}$, $\mathrm{d}s = \mathrm{e}^{\mathrm{i}\pi}\mathrm{d}r = -\mathrm{d}r$; on L^-, $s = r\,\mathrm{e}^{-\mathrm{i}\pi}$, $\mathrm{d}s = \mathrm{e}^{-\mathrm{i}\pi}\mathrm{d}r = -\mathrm{d}r$. We can combine the integrals along the top and bottom of the branch cut.

$$\begin{aligned} f(t) &= -\frac{1}{\mathrm{i}2\pi}\int_\infty^0 \mathrm{e}^{-rt}\frac{-\mathrm{i}}{\sqrt{r}}(-1)\,\mathrm{d}r - \frac{1}{\mathrm{i}2\pi}\int_0^\infty \mathrm{e}^{-rt}\frac{\mathrm{i}}{\sqrt{r}}(-1)\,\mathrm{d}r \\ &= \frac{1}{\mathrm{i}2\pi}\int_0^\infty \mathrm{e}^{-rt}\frac{\mathrm{i}2}{\sqrt{r}}\,\mathrm{d}r \end{aligned}$$

We make the change of variables $x = rt$.

$$= \frac{1}{\pi\sqrt{t}}\int_0^\infty \mathrm{e}^{-x}\frac{1}{\sqrt{x}}\,\mathrm{d}x$$

We recognize this integral as $\Gamma(1/2)$.

$$\begin{aligned} &= \frac{1}{\pi\sqrt{t}}\Gamma(1/2) \\ &= \frac{1}{\sqrt{\pi t}} \end{aligned}$$

Thus the inverse Laplace transform of $\frac{1}{\sqrt{s}}$ is

$$\boxed{f(t) = \frac{1}{\sqrt{\pi t}}, \quad \text{for } t > 0.}$$

33.2.3 Asymptotic Behavior of F(s)

Consider the behavior of

$$\hat{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

as $s \to +\infty$. Assume that $f(t)$ is analytic in a neighborhood of $t = 0$. Only the behavior of the integrand near $t = 0$ will make a significant contribution to the value of the integral. As you move away from $t = 0$, the e^{-st} term dominates. Thus we could approximate the value of $\hat{f}(s)$ by replacing $f(t)$ with the first few terms in its Taylor series expansion about the origin.

$$\hat{f}(s) \sim \int_0^\infty \mathrm{e}^{-st} \left[f(0) + t f'(0) + \frac{t^2}{2} f''(0) + \cdots \right] \mathrm{d}t \quad \text{as } s \to +\infty$$

Using

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}}$$

we obtain

$$\boxed{\hat{f}(s) \sim \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \cdots \quad \text{as } s \to +\infty.}$$

Example 33.2.5 The Taylor series expansion of $\sin t$ about the origin is

$$\sin t = t - \frac{t^3}{6} + \mathcal{O}(t^5).$$

Thus the Laplace transform of $\sin t$ has the behavior

$$\mathcal{L}[\sin t] \sim \frac{1}{s^2} - \frac{1}{s^4} + \mathcal{O}(s^{-6}) \quad \text{as } s \to +\infty.$$

We corroborate this by expanding $\mathcal{L}[\sin t]$.

$$\begin{aligned}
\mathcal{L}[\sin t] &= \frac{1}{s^2+1} \\
&= \frac{s^{-2}}{1+s^{-2}} \\
&= s^{-2} \sum_{n=0}^{\infty} (-1)^n s^{-2n} \\
&= \frac{1}{s^2} - \frac{1}{s^4} + \mathcal{O}(s^{-6})
\end{aligned}$$

33.3 Properties of the Laplace Transform

In this section we will list several useful properties of the Laplace transform. If a result is not derived, it is shown in the Problems section. Unless otherwise stated, assume that $f(t)$ and $g(t)$ are piecewise continuous and of exponential order α.

- $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$
- $\mathcal{L}[\mathrm{e}^{ct} f(t)] = F(s-c)$ for $s > c + \alpha$
- $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\hat{f}(s)] \quad \text{for } n = 1, 2, \ldots$
- If $\int_0^\beta \frac{f(t)}{t}\,\mathrm{d}t$ exists for positive β then

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\sigma)\,\mathrm{d}\sigma.$$

- $\mathcal{L}\left[\int_0^t f(\tau)\,\mathrm{d}\tau\right] = \frac{\hat{f}(s)}{s}$

- $\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right] = s\hat{f}(s) - f(0)$

  $\mathcal{L}\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\right] = s^2\hat{f}(s) - sf(0) - f'(0)$

  To derive these formulas,

$$\begin{aligned}\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right] &= \int_0^\infty \mathrm{e}^{-st} f'(t)\,\mathrm{d}t \\ &= \left[\mathrm{e}^{-st} f(t)\right]_0^\infty - \int_0^\infty -s\,\mathrm{e}^{-st} f(t)\,\mathrm{d}t \\ &= -f(0) + s\hat{f}(s)\end{aligned}$$

$$\begin{aligned}\mathcal{L}\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\right] &= s\mathcal{L}[f'(t)] - f'(0) \\ &= s^2\hat{f}(s) - sf(0) - f'(0)\end{aligned}$$

- Let $f(t)$ and $g(t)$ be continuous. The convolution of $f(t)$ and $g(t)$ is defined

$$h(t) = (f * g) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$$

  The **convolution theorem** states

$$\hat{h}(s) = \hat{f}(s)\hat{g}(s).$$

To show this,

$$\begin{aligned}
\hat{h}(s) &= \int_0^\infty \mathrm{e}^{-st} \int_0^t f(\tau) g(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_0^\infty \int_\tau^\infty \mathrm{e}^{-st} f(\tau) g(t-\tau)\,\mathrm{d}t\,\mathrm{d}\tau \\
&= \int_0^\infty \mathrm{e}^{-s\tau} f(\tau) \int_\tau^\infty \mathrm{e}^{-s(t-\tau)} g(t-\tau)\,\mathrm{d}t\,\mathrm{d}\tau \\
&= \int_0^\infty \mathrm{e}^{-s\tau} f(\tau)\,\mathrm{d}\tau \int_0^\infty \mathrm{e}^{-s\eta} g(\eta)\,\mathrm{d}\eta \\
&= \hat{f}(s)\hat{g}(s)
\end{aligned}$$

- If $f(t)$ is periodic with period T then

$$\mathcal{L}[f(t)] = \frac{\int_0^T \mathrm{e}^{-st} f(t)\,\mathrm{d}t}{1 - \mathrm{e}^{-sT}}.$$

Example 33.3.1 Consider the inverse Laplace transform of $\frac{1}{s^3-s^2}$. First we factor the denominator.

$$\frac{1}{s^3-s^2} = \frac{1}{s^2}\frac{1}{s-1}$$

We know the inverse Laplace transforms of each term.

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t, \qquad \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = \mathrm{e}^t$$

We apply the convolution theorem.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{1}{s-1}\right] &= \int_0^t \tau\,\mathrm{e}^{t-\tau}\,\mathrm{d}\tau \\
&= \mathrm{e}^t\left[-\tau\,\mathrm{e}^{-\tau}\right]_0^t - \mathrm{e}^t\int_0^t -\mathrm{e}^{-\tau}\,\mathrm{d}\tau \\
&= -t - 1 + \mathrm{e}^t
\end{aligned}$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{1}{s-1}\right] = \mathrm{e}^t - t - 1.}$$

Example 33.3.2 We can find the inverse Laplace transform of

$$\frac{s^2+s-1}{s^3-2s^2+s-2}$$

with the aid of a table of Laplace transform pairs. We factor the denominator.

$$\frac{s^2+s-1}{(s-2)(s-\mathrm{i})(s+\mathrm{i})}$$

We expand the function in partial fractions and then invert each term.

$$\begin{gathered}
\frac{s^2+s-1}{(s-2)(s-\mathrm{i})(s+\mathrm{i})} = \frac{1}{s-2} - \frac{\mathrm{i}/2}{s-\mathrm{i}} + \frac{\mathrm{i}/2}{s+\mathrm{i}} \\
\frac{s^2+s-1}{(s-2)(s-\mathrm{i})(s+\mathrm{i})} = \frac{1}{s-2} + \frac{1}{s^2+1} \\
\boxed{\mathcal{L}^{-1}\left[\frac{1}{s-2} + \frac{1}{s^2+1}\right] = \mathrm{e}^{2t} + \sin t}
\end{gathered}$$

33.4 Constant Coefficient Differential Equations

Example 33.4.1 Consider the differential equation

$$y' + y = \cos t, \quad \text{for } t > 0, \quad y(0) = 1.$$

We take the Laplace transform of this equation.

$$s\hat{y}(s) - y(0) + \hat{y}(s) = \frac{s}{s^2 + 1}$$
$$\hat{y}(s) = \frac{s}{(s+1)(s^2+1)} + \frac{1}{s+1}$$
$$\hat{y}(s) = \frac{1/2}{s+1} + \frac{1}{2}\frac{s+1}{s^2+1}$$

Now we invert $\hat{y}(s)$.

$$\boxed{y(t) = \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t, \quad \text{for } t > 0}$$

Notice that the initial condition was included when we took the Laplace transform.

One can see from this example that taking the Laplace transform of a constant coefficient differential equation reduces the differential equation for $y(t)$ to an algebraic equation for $\hat{y}(s)$.

Example 33.4.2 Consider the differential equation

$$y'' + y = \cos(2t), \qquad \text{for } t > 0, \qquad y(0) = 1,\ y'(0) = 0.$$

We take the Laplace transform of this equation.

$$s^2\hat{y}(s) - sy(0) - y'(0) + \hat{y}(s) = \frac{s}{s^2 + 4}$$
$$\hat{y}(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{s}{s^2+1}$$

From the table of Laplace transform pairs we know

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \cos t, \qquad \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin(2t).$$

We use the convolution theorem to find the inverse Laplace transform of $\hat{y}(s)$.

$$\begin{aligned}
y(t) &= \int_0^t \frac{1}{2}\sin(2\tau)\cos(t-\tau)\,\mathrm{d}\tau + \cos t \\
&= \frac{1}{4}\int_0^t \sin(t+\tau) + \sin(3\tau - t)\,\mathrm{d}\tau + \cos t \\
&= \frac{1}{4}\left[-\cos(t+\tau) - \frac{1}{3}\cos(3\tau - t)\right]_0^t + \cos t \\
&= \frac{1}{4}\left(-\cos(2t) + \cos t - \frac{1}{3}\cos(2t) + \frac{1}{3}\cos(t)\right) + \cos t \\
&= -\frac{1}{3}\cos(2t) + \frac{4}{3}\cos(t)
\end{aligned}$$

Alternatively, we can find the inverse Laplace transform of $\hat{y}(s)$ by first finding its partial fraction expansion.

$$\begin{aligned}
\hat{y}(s) &= \frac{s/3}{s^2+1} - \frac{s/3}{s^2+4} + \frac{s}{s^2+1} \\
&= -\frac{s/3}{s^2+4} + \frac{4s/3}{s^2+1}
\end{aligned}$$

$$y(t) = -\frac{1}{3}\cos(2t) + \frac{4}{3}\cos(t)$$

Example 33.4.3 Consider the initial value problem

$$y'' + 5y' + 2y = 0, \qquad y(0) = 1, \quad y'(0) = 2.$$

Without taking a Laplace transform, we know that since

$$y(t) = 1 + 2t + \mathcal{O}(t^2)$$

the Laplace transform has the behavior

$$\hat{y}(s) \sim \frac{1}{s} + \frac{2}{s^2} + \mathcal{O}(s^{-3}), \quad \text{as } s \to +\infty.$$

33.5 Systems of Constant Coefficient Differential Equations

The Laplace transform can be used to transform a system of constant coefficient differential equations into a system of algebraic equations. This should not be surprising, as a system of differential equations can be written as a single differential equation, and vice versa.

Example 33.5.1 Consider the set of differential equations

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= -y_3 - y_2 - y_1 + t^3 \end{aligned}$$

with the initial conditions

$$y_1(0) = y_2(0) = y_3(0) = 0.$$

We take the Laplace transform of this system.

$$\begin{aligned} s\hat{y}_1 - y_1(0) &= \hat{y}_2 \\ s\hat{y}_2 - y_2(0) &= \hat{y}_3 \\ s\hat{y}_3 - y_3(0) &= -\hat{y}_3 - \hat{y}_2 - \hat{y}_1 + \frac{6}{s^4} \end{aligned}$$

The first two equations can be written as

$$\hat{y}_1 = \frac{\hat{y}_3}{s^2}$$
$$\hat{y}_2 = \frac{\hat{y}_3}{s}.$$

We substitute this into the third equation.

$$s\hat{y}_3 = -\hat{y}_3 - \frac{\hat{y}_3}{s} - \frac{\hat{y}_3}{s^2} + \frac{6}{s^4}$$
$$(s^3 + s^2 + s + 1)\hat{y}_3 = \frac{6}{s^2}$$
$$\hat{y}_3 = \frac{6}{s^2(s^3 + s^2 + s + 1)}.$$

We solve for $\hat{y}_1$.

$$\hat{y}_1 = \frac{6}{s^4(s^3 + s^2 + s + 1)}$$
$$\hat{y}_1 = \frac{1}{s^4} - \frac{1}{s^3} + \frac{1}{2(s+1)} + \frac{1-s}{2(s^2+1)}$$

We then take the inverse Laplace transform of $\hat{y}_1$.

$$y_1 = \frac{t^3}{6} - \frac{t^2}{2} + \frac{1}{2}\,\mathrm{e}^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t.$$

We can find y_2 and y_3 by differentiating the expression for y_1.

$$y_2 = \frac{t^2}{2} - t - \frac{1}{2}\,\mathrm{e}^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t$$
$$y_3 = t - 1 + \frac{1}{2}\,\mathrm{e}^{-t} - \frac{1}{2}\sin t + \frac{1}{2}\cos t$$

33.6 Exercises

Exercise 33.1
Find the Laplace transform of the following functions:

1. $f(t) = \mathrm{e}^{at}$
2. $f(t) = \sin(at)$
3. $f(t) = \cos(at)$
4. $f(t) = \sinh(at)$
5. $f(t) = \cosh(at)$
6. $f(t) = \dfrac{\sin(at)}{t}$
7. $f(t) = \displaystyle\int_0^t \frac{\sin(au)}{u}\,\mathrm{d}u$
8. $f(t) = \begin{cases} 1, & 0 \leq t < \pi \\ 0, & \pi \leq t < 2\pi \end{cases}$
   and $f(t + 2\pi) = f(t)$ for $t > 0$. That is, $f(t)$ is periodic for $t > 0$.

Hint, Solution

Exercise 33.2
Show that $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$.
Hint, Solution

Exercise 33.3
Show that if $f(t)$ is of exponential order α,

$$\mathcal{L}[\,\mathrm{e}^{ct} f(t)] = F(s-c) \text{ for } s > c + \alpha.$$

Hint, Solution

Exercise 33.4
Show that

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\hat{f}(s)] \quad \text{for } n = 1, 2, \ldots$$

Hint, Solution

Exercise 33.5
Show that if $\int_0^\beta \frac{f(t)}{t}\,\mathrm{d}t$ exists for positive β then

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\sigma)\,\mathrm{d}\sigma.$$

Hint, Solution

Exercise 33.6
Show that

$$\mathcal{L}\left[\int_0^t f(\tau)\,\mathrm{d}\tau\right] = \frac{\hat{f}(s)}{s}.$$

Hint, Solution

Exercise 33.7
Show that if $f(t)$ is periodic with period T then

$$\mathcal{L}[f(t)] = \frac{\int_0^T \mathrm{e}^{-st} f(t)\, \mathrm{d}t}{1 - \mathrm{e}^{-sT}}.$$

Hint, Solution

Exercise 33.8
The function $f(t)$ $t \geq 0$, is periodic with period $2T$; i.e. $f(t + 2T) \equiv f(t)$, and is also odd with period T; i.e. $f(t + T) = -f(t)$. Further,

$$\int_0^T f(t)\, \mathrm{e}^{-st}\, \mathrm{d}t = \hat{g}(s).$$

Show that the Laplace transform of $f(t)$ is $\hat{f}(s) = \hat{g}(s)/(1 + \mathrm{e}^{-sT})$. Find $f(t)$ such that $\hat{f}(s) = s^{-1} \tanh(sT/2)$.
Hint, Solution

Exercise 33.9
Find the Laplace transform of t^ν, $\nu > -1$ by two methods.

1. Assume that s is complex-valued. Make the change of variables $z = st$ and use integration in the complex plane.

2. Show that the Laplace transform of t^ν is an analytic function for $\Re(s) > 0$. Assume that s is real-valued. Make the change of variables $x = st$ and evaluate the integral. Then use analytic continuation to extend the result to complex-valued s.

Hint, Solution

Exercise 33.10 (mathematica/ode/laplace/laplace.nb)
Show that the Laplace transform of $f(t) = \ln t$ is

$$\hat{f}(s) = -\frac{\operatorname{Log} s}{s} - \frac{\gamma}{s}, \quad \text{where} \quad \gamma = -\int_0^\infty \mathrm{e}^{-t} \ln t \, \mathrm{d}t.$$

[$\gamma = 0.5772\ldots$ is known as Euler's constant.]
Hint, Solution

Exercise 33.11
Find the Laplace transform of $t^\nu \ln t$. Write the answer in terms of the digamma function, $\psi(\nu) = \Gamma'(\nu)/\Gamma(\nu)$. What is the answer for $\nu = 0$?
Hint, Solution

Exercise 33.12
Find the inverse Laplace transform of

$$\hat{f}(s) = \frac{1}{s^3 - 2s^2 + s - 2}$$

with the following methods.

1. Expand $\hat{f}(s)$ using partial fractions and then use the table of Laplace transforms.

2. Factor the denominator into $(s-2)(s^2+1)$ and then use the convolution theorem.

3. Use Result 33.2.1.

Hint, Solution

Exercise 33.13
Solve the differential equation

$$y'' + \epsilon y' + y = \sin t, \qquad y(0) = y'(0) = 0, \qquad 0 < \epsilon \ll 1$$

using the Laplace transform. This equation represents a weakly damped, driven, linear oscillator.
Hint, Solution

Exercise 33.14
Solve the problem,

$$y'' - ty' + y = 0, \quad y(0) = 0, \; y'(0) = 1,$$

with the Laplace transform.
Hint, Solution

Exercise 33.15
Prove the following relation between the inverse Laplace transform and the inverse Fourier transform,

$$\mathcal{L}^{-1}[\hat{f}(s)] = \frac{1}{2\pi} \mathrm{e}^{ct} \mathcal{F}^{-1}[\hat{f}(c + \mathrm{i}\omega)],$$

where c is to the right of the singularities of $\hat{f}(s)$.
Hint, Solution

Exercise 33.16 (mathematica/ode/laplace/laplace.nb)
Show by evaluating the Laplace inversion integral that if

$$\hat{f}(s) = \left(\frac{\pi}{s}\right)^{1/2} \mathrm{e}^{-2(as)^{1/2}}, \quad s^{1/2} = \sqrt{s} \text{ for } s > 0,$$

then $f(t) = \mathrm{e}^{-a/t}/\sqrt{t}$. Hint: cut the s-plane along the negative real axis and deform the contour onto the cut. Remember that $\int_0^\infty \mathrm{e}^{-ax^2} \cos(bx)\,\mathrm{d}x = \sqrt{\pi/4a}\,\mathrm{e}^{-b^2/4a}$.
Hint, Solution

Exercise 33.17 (mathematica/ode/laplace/laplace.nb)
Use Laplace transforms to solve the initial value problem

$$\frac{\mathrm{d}^4 y}{\mathrm{d}t^4} - y = t, \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Hint, Solution

Exercise 33.18 (mathematica/ode/laplace/laplace.nb)
Solve, by Laplace transforms,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sin t + \int_0^t y(\tau)\cos(t-\tau)\,\mathrm{d}\tau, \quad y(0) = 0.$$

Hint, Solution

Exercise 33.19 (mathematica/ode/laplace/laplace.nb)
Suppose $u(t)$ satisfies the difference-differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + u(t) - u(t-1) = 0, \quad t \geq 0,$$

and the 'initial condition' $u(t) = u_0(t)$, $-1 \leq t \leq 0$, where $u_0(t)$ is given. Show that the Laplace transform $\hat{u}(s)$ of $u(t)$ satisfies

$$\hat{u}(s) = \frac{u_0(0)}{1 + s - \mathrm{e}^{-s}} + \frac{\mathrm{e}^{-s}}{1 + s - \mathrm{e}^{-s}} \int_{-1}^0 \mathrm{e}^{-st} u_0(t)\,\mathrm{d}t.$$

Find $u(t)$, $t \geq 0$, when $u_0(t) = 1$. Check the result.
Hint, Solution

Exercise 33.20
Let the function $f(t)$ be defined by

$$f(t) = \begin{cases} 1 & 0 \leq t < \pi \\ 0 & \pi \leq t < 2\pi, \end{cases}$$

and for all positive values of t so that $f(t+2\pi) = f(t)$. That is, $f(t)$ is periodic with period 2π. Find the solution of the intial value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - y = f(t); \qquad y(0) = 1, \quad y'(0) = 0.$$

Examine the continuity of the solution at $t = n\pi$, where n is a positive integer, and verify that the solution is continuous and has a continuous derivative at these points.
Hint, Solution

Exercise 33.21
Use Laplace transforms to solve

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \int_0^t y(\tau)\,\mathrm{d}\tau = \mathrm{e}^{-t}, \quad y(0) = 1.$$

Hint, Solution

Exercise 33.22
An electric circuit gives rise to the system

$$L\frac{\mathrm{d}i_1}{\mathrm{d}t} + Ri_1 + q/C = E_0$$
$$L\frac{\mathrm{d}i_2}{\mathrm{d}t} + Ri_2 - q/C = 0$$
$$\frac{\mathrm{d}q}{\mathrm{d}t} = i_1 - i_2$$

with initial conditions

$$i_1(0) = i_2(0) = \frac{E_0}{2R}, \quad q(0) = 0.$$

Solve the system by Laplace transform methods and show that

$$i_1 = \frac{E_0}{2R} + \frac{E_0}{2\omega L}\,\mathrm{e}^{-\alpha t}\sin(\omega t)$$

where

$$\alpha = \frac{R}{2L} \quad \text{and} \quad \omega^2 = \frac{2}{LC} - \alpha^2.$$

Hint, Solution

33.7 Hints

Hint 33.1
Use the differentiation and integration properties of the Laplace transform where appropriate.

Hint 33.2

Hint 33.3

Hint 33.4
If the integral is uniformly convergent and $\frac{\partial g}{\partial s}$ is continuous then

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_a^b g(s,t)\,\mathrm{d}t = \int_a^b \frac{\partial}{\partial s} g(s,t)\,\mathrm{d}t$$

Hint 33.5

$$\int_s^\infty \mathrm{e}^{-tx}\,\mathrm{d}t = \frac{1}{x}\,\mathrm{e}^{-sx}$$

Hint 33.6
Use integration by parts.

Hint 33.7

$$\int_0^\infty e^{-st} f(t)\,dt = \int_{n=0}^\infty \sum_{nT}^{(n+1)T} e^{-st} f(t)\,dt$$

The sum can be put in the form of a geometric series.

$$\sum_{n=0}^\infty \alpha^n = \frac{1}{1-\alpha}, \quad \text{for } |\alpha| < 1$$

Hint 33.8

Hint 33.9
Write the answer in terms of the Gamma function.

Hint 33.10

Hint 33.11

Hint 33.12

Hint 33.13

Hint 33.14

Hint 33.15

Hint 33.16

Hint 33.17

Hint 33.18

Hint 33.19

Hint 33.20

Hint 33.21

Hint 33.22

33.8 Solutions

Solution 33.1

1.

$$
\begin{aligned}
\mathcal{L}\left[\mathrm{e}^{at}\right] &= \int_0^\infty \mathrm{e}^{-st}\,\mathrm{e}^{at}\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-(s-a)t}\,\mathrm{d}t \\
&= \left[-\frac{\mathrm{e}^{-(s-a)t}}{s-a}\right]_0^\infty, \quad \text{for } \Re(s) > \Re(a)
\end{aligned}
$$

$$
\boxed{\mathcal{L}\left[\mathrm{e}^{at}\right] = \frac{1}{s-a}}
$$

2.

$$
\begin{aligned}
\mathcal{L}[\sin(at)] &= \int_0^\infty \mathrm{e}^{-st}\sin(at)\,\mathrm{d}t \\
&= \frac{1}{2\mathrm{i}}\int_0^\infty \left(\mathrm{e}^{(-s+\mathrm{i}a)t} - \mathrm{e}^{(-s-\mathrm{i}a)t}\right)\mathrm{d}t \\
&= \frac{1}{2\mathrm{i}}\left[\frac{-\,\mathrm{e}^{(-s+\mathrm{i}a)t}}{s-\mathrm{i}a} + \frac{\mathrm{e}^{(-s-\mathrm{i}a)t}}{s+\mathrm{i}a}\right]_0^\infty \\
&= \frac{1}{2\mathrm{i}}\left(\frac{1}{s-\mathrm{i}a} - \frac{1}{s+\mathrm{i}a}\right)
\end{aligned}
$$

$$
\boxed{\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}.}
$$

3.

$$
\begin{aligned}
\mathcal{L}[\cos(at)] &= \mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\sin(at)}{a}\right] \\
&= s\mathcal{L}\left[\frac{\sin(at)}{a}\right] - \sin(0)
\end{aligned}
$$

$$
\boxed{\mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2}}
$$

4.

$$
\begin{aligned}
\mathcal{L}[\sinh(at)] &= \int_0^\infty \mathrm{e}^{-st}\sinh(at)\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^\infty \left(\mathrm{e}^{(-s+a)t} - \mathrm{e}^{(-s-a)t}\right)\,\mathrm{d}t \\
&= \frac{1}{2}\left[\frac{-\mathrm{e}^{(-s+a)t}}{s-a} + \frac{\mathrm{e}^{(-s-a)t}}{s+a}\right]_0^\infty, \quad \text{for } \Re(s) > |\Re(a)| \\
&= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)
\end{aligned}
$$

$$
\boxed{\mathcal{L}[\sinh(at)] = \frac{a}{s^2 - a^2}, \quad \text{for } \Re(s) > |\Re(a)|}
$$

5.

$$
\begin{aligned}
\mathcal{L}[\cosh(at)] &= \mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\sinh(at)}{a}\right] \\
&= s\mathcal{L}\left[\frac{\sinh(at)}{a}\right] - \sinh(0)
\end{aligned}
$$

$$\boxed{\mathcal{L}[\cosh(at)] = \frac{s}{s^2 - a^2}}$$

6. First note that

$$\mathcal{L}\left[\frac{\sin(at)}{t}\right](s) = \int_s^\infty \mathcal{L}[\sin(at)](\sigma)\,\mathrm{d}\sigma.$$

Now we use the Laplace transform of $\sin(at)$ to compute the Laplace transform of $\sin(at)/t$.

$$\begin{aligned}
\mathcal{L}\left[\frac{\sin(at)}{t}\right] &= \int_s^\infty \frac{a}{\sigma^2 + a^2}\,\mathrm{d}\sigma \\
&= \int_s^\infty \frac{1}{(\sigma/a)^2 + 1}\,\frac{d\sigma}{a} \\
&= \left[\arctan\left(\frac{\sigma}{a}\right)\right]_s^\infty \\
&= \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right)
\end{aligned}$$

$$\boxed{\mathcal{L}\left[\frac{\sin(at)}{t}\right] = \arctan\left(\frac{a}{s}\right)}$$

7.

$$\mathcal{L}\left[\int_0^t \frac{\sin(a\tau)}{\tau}\,\mathrm{d}\tau\right] = \frac{1}{s}\mathcal{L}\left[\frac{\sin(at)}{t}\right]$$

$$\boxed{\mathcal{L}\left[\int_0^t \frac{\sin(a\tau)}{\tau}\,\mathrm{d}\tau\right] = \frac{1}{s}\arctan\left(\frac{a}{s}\right)}$$

8.

$$\begin{aligned}
\mathcal{L}[f(t)] &= \frac{\int_0^{2\pi} \mathrm{e}^{-st} f(t)\,\mathrm{d}t}{1 - \mathrm{e}^{-2\pi s}} \\
&= \frac{\int_0^{\pi} \mathrm{e}^{-st}\,\mathrm{d}t}{1 - \mathrm{e}^{-2\pi s}} \\
&= \frac{1 - \mathrm{e}^{-\pi s}}{s(1 - \mathrm{e}^{-2\pi s})}
\end{aligned}$$

$$\boxed{\mathcal{L}[f(t)] = \frac{1}{s(1 + \mathrm{e}^{-\pi s})}}$$

Solution 33.2

$$\begin{aligned}
\mathcal{L}[af(t) + bg(t)] &= \int_0^\infty \mathrm{e}^{-st}\big(af(t) + bg(t)\big)\,\mathrm{d}t \\
&= a\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t + b\int_0^\infty \mathrm{e}^{-st} g(t)\,\mathrm{d}t \\
&= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]
\end{aligned}$$

Solution 33.3

If $f(t)$ is of exponential order α, then $\mathrm{e}^{ct} f(t)$ is of exponential order $c + \alpha$.

$$\begin{aligned}
\mathcal{L}[\mathrm{e}^{ct} f(t)] &= \int_0^\infty \mathrm{e}^{-st}\,\mathrm{e}^{ct} f(t)\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-(s-c)t} f(t)\,\mathrm{d}t \\
&= \hat{f}(s - c) \text{ for } s > c + \alpha
\end{aligned}$$

Solution 33.4
First consider the Laplace transform of $t^0 f(t)$.

$$\mathcal{L}[t^0 f(t)] = \hat{f}(s)$$

Now consider the Laplace transform of $t^n f(t)$ for $n \geq 1$.

$$\begin{aligned}
\mathcal{L}[t^n f(t)] &= \int_0^\infty \mathrm{e}^{-st} t^n f(t)\,\mathrm{d}t \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{e}^{-st} t^{n-1} f(t)\,\mathrm{d}t \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}[t^{n-1} f(t)]
\end{aligned}$$

Thus we have a difference equation for the Laplace transform of $t^n f(t)$ with the solution

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\mathcal{L}[t^0 f(t)] \text{ for } n \in \mathbb{Z}^{0+},$$

$$\boxed{\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\hat{f}(s) \text{ for } n \in \mathbb{Z}^{0+}.}$$

Solution 33.5
If $\int_0^\beta \frac{f(t)}{t}\,\mathrm{d}t$ exists for positive β and $f(t)$ is of exponential order α then the Laplace transform of $f(t)/t$ is defined

for $s > \alpha$.

$$
\begin{aligned}
\mathcal{L}\left[\frac{f(t)}{t}\right] &= \int_0^\infty \mathrm{e}^{-st}\frac{1}{t}f(t)\,\mathrm{d}t \\
&= \int_0^\infty \int_s^\infty \mathrm{e}^{-\sigma t}\,\mathrm{d}\sigma\, f(t)\,\mathrm{d}t \\
&= \int_s^\infty \int_0^\infty e^{-\sigma t} f(t)\,\mathrm{d}t\,\mathrm{d}\sigma \\
&= \int_s^\infty \hat{f}(\sigma)\,\mathrm{d}\sigma
\end{aligned}
$$

Solution 33.6

$$
\begin{aligned}
\mathcal{L}\left[\int_0^t f(\tau)\,\mathrm{d}\tau\right] &= \int_0^\infty \mathrm{e}^{-st}\int_0^t f(\tau)\,\mathrm{d}\tau\,\mathrm{d}x \\
&= \left[-\frac{\mathrm{e}^{-st}}{s}\int_0^t f(\tau)\,\mathrm{d}\tau\right]_0^\infty - \int_0^\infty -\frac{\mathrm{e}^{-st}}{s}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t f(\tau)\,\mathrm{d}\tau\right]\,\mathrm{d}t \\
&= \frac{1}{s}\int_0^\infty \mathrm{e}^{-st}f(t)\,\mathrm{d}t \\
&= \frac{1}{s}\hat{f}(s)
\end{aligned}
$$

Solution 33.7
$f(t)$ is periodic with period T.

$$\begin{aligned}
\mathcal{L}[f(t)] &= \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^T \mathrm{e}^{-st} f(t)\,\mathrm{d}t + \int_T^{2T} \mathrm{e}^{-st} f(t)\,\mathrm{d}t + \cdots \\
&= \sum_{n=0}^\infty \int_{nT}^{(n+1)T} \mathrm{e}^{-st} f(t)\,\mathrm{d}t \\
&= \sum_{n=0}^\infty \int_0^T \mathrm{e}^{-s(t+nT)} f(t+nT)\,\mathrm{d}t \\
&= \sum_{n=0}^\infty \mathrm{e}^{-snT} \int_0^T \mathrm{e}^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^T \mathrm{e}^{-st} f(t)\,\mathrm{d}t \sum_{n=0}^\infty \mathrm{e}^{-snT} \\
&= \frac{\int_0^T \mathrm{e}^{-st} f(t)\,\mathrm{d}t}{1 - \mathrm{e}^{-sT}}
\end{aligned}$$

Solution 33.8

$$
\begin{aligned}
\hat{f}(s) &= \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \\
&= \sum_0^n \int_{nT}^{(n+1)T} \mathrm{e}^{-st} f(t)\,\mathrm{d}t \\
&= \sum_0^n \int_0^T \mathrm{e}^{-s(t+nT)} f(t+nT)\,\mathrm{d}t \\
&= \sum_0^n \mathrm{e}^{-snT} \int_0^T \mathrm{e}^{-st} (-1)^n f(t)\,\mathrm{d}t \\
&= \int_0^T \mathrm{e}^{-st} f(t)\,\mathrm{d}t \sum_0^n (-1)^n \left(\mathrm{e}^{-sT}\right)^n
\end{aligned}
$$

$$
\boxed{\hat{f}(s) = \frac{\hat{g}(s)}{1+\mathrm{e}^{-sT}}, \quad \text{for } \Re(s) > 0}
$$

Consider $\hat{f}(s) = s^{-1}\tanh(sT/2)$.

$$
\begin{aligned}
s^{-1}\tanh(sT/2) &= s^{-1}\frac{\mathrm{e}^{sT/2} - \mathrm{e}^{-sT/2}}{\mathrm{e}^{sT/2} + \mathrm{e}^{-sT/2}} \\
&= s^{-1}\frac{1-\mathrm{e}^{-sT}}{1+\mathrm{e}^{-sT}}
\end{aligned}
$$

We have

$$
\hat{g}(s) \equiv \int_0^T f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t = \frac{1-\mathrm{e}^{-st}}{s}.
$$

By inspection we see that this is satisfied for $f(t) = 1$ for $0 < t < T$. We conclude:

$$\boxed{f(t) = \begin{cases} 1 & \text{for } t \in [2nT \ldots (2n+1)T), \\ -1 & \text{for } t \in [(2n+1)T \ldots (2n+2)T), \end{cases}}$$

where $n \in \mathbb{Z}$.

Solution 33.9
The Laplace transform of t^ν, $\nu > -1$ is

$$\hat{f}(s) = \int_0^\infty \mathrm{e}^{-st} t^\nu \, \mathrm{d}t.$$

Assume s is complex-valued. The integral converges for $\Re(s) > 0$ and $\nu > -1$.

Method 1. We make the change of variables $z = st$.

$$\begin{aligned} \hat{f}(s) &= \int_C \mathrm{e}^{-z} \left(\frac{z}{s}\right)^\nu \frac{1}{s} \, \mathrm{d}z \\ &= s^{-(\nu+1)} \int_C \mathrm{e}^{-z} z^\nu \, \mathrm{d}z \end{aligned}$$

C is the path from 0 to ∞ along $\arg(z) = \arg(s)$. (Shown in Figure 33.4).

Since the integrand is analytic in the domain $\epsilon < r < R$, $0 < \theta < \arg(s)$, the integral along the boundary of this domain vanishes.

$$\left(\int_\epsilon^R + \int_R^{R\,\mathrm{e}^{\mathrm{i}\arg(s)}} + \int_{R\,\mathrm{e}^{\mathrm{i}\arg(s)}}^{\epsilon\,\mathrm{e}^{\mathrm{i}\arg(s)}} + \int_{\epsilon\,\mathrm{e}^{\mathrm{i}\arg(s)}}^{\epsilon} \right) \mathrm{e}^{-z} z^\nu \, \mathrm{d}z = 0$$

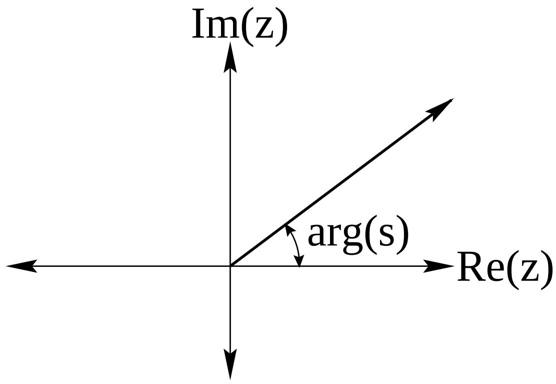

Figure 33.4: The Path of Integration.

We show that the integral along C_R, the circular arc of radius R, vanishes as $R \to \infty$ with the maximum modulus integral bound.

$$\begin{aligned}\left| \int_{C_R} \mathrm{e}^{-z} z^\nu \,\mathrm{d}z \right| &\leq R |\arg(s)| \max_{z \in C_R} \left| \mathrm{e}^{-z} z^\nu \right| \\ &= R |\arg(s)| \,\mathrm{e}^{-R\cos(\arg(s))} R^\nu \\ &\to 0 \quad \text{as } R \to \infty.\end{aligned}$$

The integral along C_ϵ, the circular arc of radius ϵ, vanishes as $\epsilon \to 0$. We demonstrate this with the maximum modulus integral bound.

$$\begin{aligned}\left| \int_{C_\epsilon} \mathrm{e}^{-z} z^\nu \,\mathrm{d}z \right| &\leq \epsilon |\arg(s)| \max_{z \in C_\epsilon} \left| \mathrm{e}^{-z} z^\nu \right| \\ &= \epsilon |\arg(s)| \,\mathrm{e}^{-\epsilon\cos(\arg(s))} \epsilon^\nu \\ &\to 0 \quad \text{as } \epsilon \to 0.\end{aligned}$$

Taking the limit as $\epsilon \to 0$ and $R \to \infty$, we see that the integral along C is equal to the integral along the real axis.

$$\int_C \mathrm{e}^{-z} z^\nu \, \mathrm{d}z = \int_0^\infty \mathrm{e}^{-z} z^\nu \, dz$$

We can evaluate the Laplace transform of t^ν in terms of this integral.

$$\mathcal{L}\left[t^\nu\right] = s^{-(\nu+1)} \int_0^\infty \mathrm{e}^{-t} t^\nu \, \mathrm{d}t$$

$$\boxed{\mathcal{L}\left[t^\nu\right] = \frac{\Gamma(\nu+1)}{s^{-(\nu+1)}}}$$

In the case that ν is a non-negative integer $\nu = n > -1$ we can write this in terms of the factorial.

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{-(n+1)}}$$

Method 2. First note that the integral

$$\hat{f}(s) = \int_0^\infty \mathrm{e}^{-st} t^\nu \, \mathrm{d}t$$

exists for $\Re(s) > 0$. It converges uniformly for $\Re(s) \geq c > 0$. On this domain of uniform convergence we can interchange differentiation and integration.

$$\begin{aligned} \frac{\mathrm{d}\hat{f}}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty \mathrm{e}^{-st} t^\nu \, \mathrm{d}t \\ &= \int_0^\infty \frac{\partial}{\partial s} \left(\mathrm{e}^{-st} t^\nu \right) \mathrm{d}t \\ &= \int_0^\infty -t \, \mathrm{e}^{-st} t^\nu \, \mathrm{d}t \\ &= -\int_0^\infty \mathrm{e}^{-st} t^{\nu+1} \, \mathrm{d}t \end{aligned}$$

Since $\hat{f}'(s)$ is defined for $\Re(s) > 0$, $\hat{f}(s)$ is analytic for $\Re(s) > 0$.

Let σ be real and positive. We make the change of variables $x = \sigma t$.

$$\begin{aligned}
\hat{f}(\sigma) &= \int_0^\infty \mathrm{e}^{-x} \left(\frac{x}{\sigma}\right)^\nu \frac{1}{\sigma} \, \mathrm{d}x \\
&= \sigma^{-(\nu+1)} \int_0^\infty \mathrm{e}^{-x} x^\nu \, \mathrm{d}x \\
&= \frac{\Gamma(\nu+1)}{\sigma^{\nu+1}}
\end{aligned}$$

Note that the function

$$\hat{f}(s) = \frac{\Gamma(\nu+1)}{s^{\nu+1}}$$

is the analytic continuation of $\hat{f}(\sigma)$. Thus we can define the Laplace transform for all complex s in the right half plane.

$$\boxed{\hat{f}(s) = \frac{\Gamma(\nu+1)}{s^{\nu+1}}}$$

Solution 33.10

Note that $\hat{f}(s)$ is an analytic function for $\Re(s) > 0$. Consider real-valued $s > 0$. By definition, $\hat{f}(s)$ is

$$\hat{f}(s) = \int_0^\infty \mathrm{e}^{-st} \ln t \, \mathrm{d}t.$$

We make the change of variables $x = st$.

$$\begin{aligned}
\hat{f}(s) &= \int_0^\infty \mathrm{e}^{-x} \ln\left(\frac{x}{s}\right) \frac{\mathrm{d}x}{s} \\
&= \frac{1}{s} \int_0^\infty \mathrm{e}^{-x} (\ln x - \ln s) \,\mathrm{d}x \\
&= -\frac{\ln|s|}{s} \int_0^\infty \mathrm{e}^{-x} \,\mathrm{d}x + \frac{1}{s} \int_0^\infty \mathrm{e}^{-x} \ln x \,\mathrm{d}x \\
&= -\frac{\ln s}{s} - \frac{\gamma}{s}, \quad \text{for real } s > 0
\end{aligned}$$

The analytic continuation of $\hat{f}(s)$ into the right half-plane is

$$\boxed{\hat{f}(s) = -\frac{\operatorname{Log} s}{s} - \frac{\gamma}{s}.}$$

Solution 33.11
Define

$$\hat{f}(s) = \mathcal{L}[t^\nu \ln t] = \int_0^\infty \mathrm{e}^{-st} t^\nu \ln t \,\mathrm{d}t.$$

This integral defines $\hat{f}(s)$ for $\Re(s) > 0$. Note that the integral converges uniformly for $\Re(s) \geq c > 0$. On this domain we can interchange differentiation and integration.

$$\hat{f}'(s) = \int_0^\infty \frac{\partial}{\partial s} \left(\mathrm{e}^{-st} t^\nu \ln t \right) \mathrm{d}t = -\int_0^\infty t\,\mathrm{e}^{-st} t^\nu \operatorname{Log} t \,\mathrm{d}t$$

Since $\hat{f}'(s)$ also exists for $\Re(s) > 0$, $\hat{f}(s)$ is analytic in that domain.

Let σ be real and positive. We make the change of variables $x = \sigma t$.

$$\begin{aligned}
\hat{f}(\sigma) &= \mathcal{L}\left[t^\nu \ln t\right] \\
&= \int_0^\infty \mathrm{e}^{-\sigma t} t^\nu \ln t \,\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-x} \left(\frac{x}{\sigma}\right)^\nu \ln \frac{x}{\sigma} \frac{1}{\sigma} \,\mathrm{d}x \\
&= \frac{1}{\sigma^{\nu+1}} \int_0^\infty \mathrm{e}^{-x} x^\nu (\ln x - \ln \sigma) \,\mathrm{d}x \\
&= \frac{1}{\sigma^{\nu+1}} \left(\int_0^\infty \mathrm{e}^{-x} x^\nu \ln x \,\mathrm{d}x - \ln \sigma \int_0^\infty \mathrm{e}^{-x} x^\nu \,\mathrm{d}x \right) \\
&= \frac{1}{\sigma^{\nu+1}} \left(\int_0^\infty \frac{\partial}{\partial \nu} \left(\mathrm{e}^{-x} x^\nu\right) \,\mathrm{d}x - \ln \sigma \Gamma(\nu+1) \right) \\
&= \frac{1}{\sigma^{\nu+1}} \left(\frac{\mathrm{d}}{\mathrm{d}\nu} \int_0^\infty \mathrm{e}^{-x} x^\nu \,\mathrm{d}x - \ln \sigma \Gamma(\nu+1) \right) \\
&= \frac{1}{\sigma^{\nu+1}} \left(\frac{\mathrm{d}}{\mathrm{d}\nu} \Gamma(\nu+1) - \ln \sigma \Gamma(\nu+1) \right) \\
&= \frac{1}{\sigma^{\nu+1}} \Gamma(\nu+1) \left(\frac{\Gamma'(\nu+1)}{\Gamma(\nu+1)} - \ln \sigma \right) \\
&= \frac{1}{\sigma^{\nu+1}} \Gamma(\nu+1) \left(\psi(\nu+1) - \ln \sigma \right)
\end{aligned}$$

Note that the function

$$\hat{f}(s) = \frac{1}{s^{\nu+1}} \Gamma(\nu+1) \left(\psi(\nu+1) - \ln s \right)$$

is an analytic continuation of $\hat{f}(\sigma)$. Thus we can define the Laplace transform for all s in the right half plane.

$$\boxed{\mathcal{L}[t^\nu \ln t] = \frac{1}{s^{\nu+1}} \Gamma(\nu+1) \left(\psi(\nu+1) - \ln s \right) \quad \text{for } \Re(s) > 0.}$$

For the case $\nu = 0$, we have

$$\mathcal{L}[\ln t] = \frac{1}{s^1}\Gamma(1)\left(\psi(1) - \ln s\right)$$
$$\boxed{\mathcal{L}[\ln t] = \frac{-\gamma - \ln s}{s},}$$

where γ is Euler's constant

$$\gamma = \int_0^\infty \mathrm{e}^{-x} \ln x \, \mathrm{d}x = 0.5772156629\ldots$$

Solution 33.12
Method 1. We factor the denominator.

$$\hat{f}(s) = \frac{1}{(s-2)(s^2+1)} = \frac{1}{(s-2)(s-\mathrm{i})(s+\mathrm{i})}$$

We expand the function in partial fractions and simplify the result.

$$\frac{1}{(s-2)(s-\mathrm{i})(s+\mathrm{i})} = \frac{1/5}{s-2} - \frac{(1-\mathrm{i}2)/10}{s-\mathrm{i}} - \frac{(1+\mathrm{i}2)/10}{s+\mathrm{i}}$$
$$\hat{f}(s) = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{s+2}{s^2+1}$$

We use a table of Laplace transforms to do the inversion.

$$\mathcal{L}[\mathrm{e}^{2t}] = \frac{1}{s-2}, \quad \mathcal{L}[\cos t] = \frac{s}{s^2+1}, \quad \mathcal{L}[\sin t] = \frac{1}{s^2+1}$$
$$\boxed{f(t) = \frac{1}{5}\left(\mathrm{e}^{2t} - \cos t - 2\sin t\right)}$$

Method 2. We factor the denominator.

$$\hat{f}(s) = \frac{1}{s-2}\frac{1}{s^2+1}$$

From a table of Laplace transforms we note

$$\mathcal{L}[\,\mathrm{e}^{2t}] = \frac{1}{s-2}, \qquad \mathcal{L}[\sin t] = \frac{1}{s^2+1}.$$

We apply the convolution theorem.

$$f(t) = \int_0^t \sin\tau \;\; \mathrm{e}^{2(t-\tau)}\,\mathrm{d}\tau$$

$$\boxed{f(t) = \frac{1}{5}\left(\mathrm{e}^{2t} - \cos t - 2\sin t\right)}$$

Method 3. We factor the denominator.

$$\hat{f}(s) = \frac{1}{(s-2)(s-\mathrm{i})(s+\mathrm{i})}$$

$\hat{f}(s)$ is analytic except for poles and vanishes at infinity.

$$\begin{aligned}
f(t) &= \sum_{s_n=2,\mathrm{i},-\mathrm{i}} \operatorname{Res}\left(\frac{\mathrm{e}^{st}}{(s-2)(s-\mathrm{i})(s+\mathrm{i})}, s_n\right) \\
&= \frac{\mathrm{e}^{2t}}{(2-\mathrm{i})(2+\mathrm{i})} + \frac{\mathrm{e}^{\mathrm{i}t}}{(\mathrm{i}-2)(\mathrm{i}2)} + \frac{\mathrm{e}^{-\mathrm{i}t}}{(-\mathrm{i}-2)(-\mathrm{i}2)} \\
&= \frac{\mathrm{e}^{2t}}{5} + \frac{(-1+\mathrm{i}2)\,\mathrm{e}^{\mathrm{i}t}}{10} + \frac{(-1-\mathrm{i}2)\,\mathrm{e}^{-\mathrm{i}t}}{10} \\
&= \frac{\mathrm{e}^{2t}}{5} + -\frac{\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}}{10} + \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}}{5}
\end{aligned}$$

$$\boxed{f(t) = \frac{1}{5}\left(e^{2t} - \cos t - 2\sin t\right)}$$

Solution 33.13

$$y'' + \epsilon y' + y = \sin t, \qquad y(0) = y'(0) = 0, \qquad 0 < \epsilon \ll 1$$

We take the Laplace transform of this equation.

$$\begin{gathered}
(s^2\hat{y}(s) - sy(0) - y'(0)) + \epsilon(s\hat{y}(s) - y(0)) + \hat{y}(s) = \mathcal{L}[\sin(t)] \\
(s^2 + \epsilon s + 1)\hat{y}(s) = \mathcal{L}[\sin(t)] \\
\hat{y}(s) = \frac{1}{s^2 + \epsilon s + 1}\mathcal{L}[\sin(t)] \\
\hat{y}(s) = \frac{1}{(s + \frac{\epsilon}{2})^2 + 1 - \frac{\epsilon^2}{4}}\mathcal{L}[\sin(t)]
\end{gathered}$$

We use a table of Laplace transforms to find the inverse Laplace transform of the first term.

$$\mathcal{L}^{-1}\left[\frac{1}{(s + \frac{\epsilon}{2})^2 + 1 - \frac{\epsilon^2}{4}}\right] = \frac{1}{\sqrt{1 - \frac{\epsilon^2}{4}}}\, e^{-\epsilon t/2} \sin\left(\sqrt{1 - \frac{\epsilon^2}{4}}\, t\right)$$

We define

$$\alpha = \sqrt{1 - \frac{\epsilon^2}{4}}$$

to get rid of some clutter. Now we apply the convolution theorem to invert [2] $\hat{y}s$.

$$y(t) = \int_0^t \frac{1}{\alpha} \mathrm{e}^{-\epsilon\tau/2} \sin(\alpha\tau) \sin(t-\tau) \, \mathrm{d}\tau$$

$$\boxed{y(t) = \mathrm{e}^{-\epsilon t/2} \left(\frac{1}{\epsilon} \cos(\alpha t) + \frac{1}{2\alpha} \sin(\alpha t) \right) - \frac{1}{\epsilon} \cos t}$$

The solution is plotted in Figure 33.5 for $\epsilon = 0.05$.

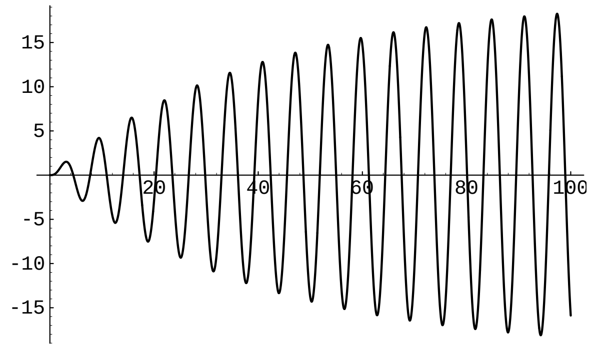

Figure 33.5: The Weakly Damped, Driven Oscillator

[2] Evaluate the convolution integral by inspection.

Solution 33.14
We consider the solutions of

$$y'' - ty' + y = 0, \quad y(0) = 0,\ y'(0) = 1$$

which are of exponential order α for any $\alpha > 0$. We take the Laplace transform of the differential equation.

$$s^2\hat{y} - 1 + \frac{\mathrm{d}}{\mathrm{d}s}(s\hat{y}) + \hat{y} = 0$$
$$\hat{y}' + \left(s + \frac{2}{s}\right)\hat{y} = \frac{1}{s}$$
$$\hat{y}(s) = \frac{1}{s^2} + c\frac{\mathrm{e}^{-s^2/2}}{s^2}$$

We use that

$$\hat{y}(s) \sim \frac{y(0)}{s} + \frac{y'(0)}{s^2} + \cdots$$

to conclude that $c = 0$.

$$\hat{y}(s) = \frac{1}{s^2}$$
$$\boxed{y(t) = t}$$

Solution 33.15

$$\mathcal{L}^{-1}[\hat{f}(s)] = \frac{1}{\mathrm{i}2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{st}\hat{f}(s)\,\mathrm{d}s$$

First we make the change of variable $s = c + \sigma$.

$$\mathcal{L}^{-1}[\hat{f}(s)] = \frac{1}{\mathrm{i}2\pi}\mathrm{e}^{ct}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{\sigma t}F(c+\sigma)\,\mathrm{d}\sigma$$

Then we make the change of variable $\sigma = \mathrm{i}\omega$.

$$\mathcal{L}^{-1}[\hat{f}(s)] = \frac{1}{2\pi}\,\mathrm{e}^{ct}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega t}F(c+\mathrm{i}\omega)\,\mathrm{d}\omega$$
$$\mathcal{L}^{-1}[\hat{f}(s)] = \frac{1}{2\pi}\,\mathrm{e}^{ct}\mathcal{F}^{-1}[F(c+\mathrm{i}\omega)]$$

Solution 33.16

We assume that $\Re(a) \geq 0$. We are considering the principal branch of the square root: $s^{1/2} = \sqrt{s}$. There is a branch cut on the negative real axis. $\hat{f}(s)$ is singular at $s = 0$ and along the negative real axis. Let α be any positive number. The inverse Laplace transform of $\left(\frac{\pi}{s}\right)^{1/2}\mathrm{e}^{-2(as)^{1/2}}$ is

$$f(t) = \frac{1}{\mathrm{i}2\pi}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty}\mathrm{e}^{st}\left(\frac{\pi}{s}\right)^{1/2}\mathrm{e}^{-2(as)^{1/2}}\,\mathrm{d}s.$$

We will evaluate the integral by deforming it to wrap around the branch cut. Consider the integral on the contour shown in Figure 33.6. C_R^+ and C_R^- are circular arcs of radius R. B is the vertical line at $\Re(s) = \alpha$ joining the two arcs. C_ϵ is a semi-circle in the right half plane joining $i\epsilon$ and $-i\epsilon$. L^+ and L^- are lines joining the circular arcs at $\Im(s) = \pm\epsilon$.

Since there are no residues inside the contour, we have

$$\frac{1}{\mathrm{i}2\pi}\left(\int_B + \int_{C_R^+} + \int_{L^+} + \int_{C_\epsilon} + \int_{L^-} + \int_{C_R^-}\right)\mathrm{e}^{st}\left(\frac{\pi}{s}\right)^{1/2}\mathrm{e}^{-2(as)^{1/2}}\,\mathrm{d}s = 0.$$

We will evaluate the inverse Laplace transform for $t > 0$.

First we will show that the integral along C_R^+ vanishes as $R \to \infty$. We parametrize the path of integration with $s = R\,\mathrm{e}^{\mathrm{i}\theta}$ and write the integral along C_R^+ as the sum of two integrals.

$$\int_{C_R^+}\cdots\,\mathrm{d}s = \int_{\pi/2-\delta}^{\pi/2}\cdots\,\mathrm{d}\theta + \int_{\pi/2}^{\pi}\cdots\,\mathrm{d}\theta$$

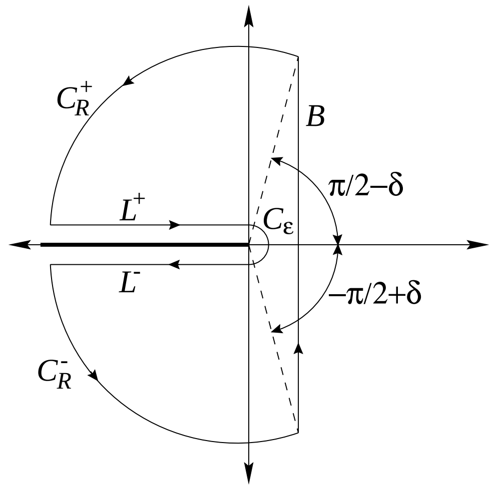

Figure 33.6: Path of Integration

The first integral vanishes by the maximum modulus bound. Note that the length of the path of integration is less than 2α.

$$
\begin{aligned}
\left| \int_{\pi/2-\delta}^{\pi/2} \cdots \, \mathrm{d}\theta \right| &\leq \left(\max_{\theta \in [\pi/2-\delta \ldots \pi/2]} \left| \mathrm{e}^{st} \left(\frac{\pi}{s} \right)^{1/2} \mathrm{e}^{-2(as)^{1/2}} \right| \right) (2\alpha) \\
&= \mathrm{e}^{\alpha t} \frac{\sqrt{\pi}}{\sqrt{R}} (2\alpha) \\
&\to 0 \text{ as } R \to \infty
\end{aligned}
$$

The second integral vanishes by Jordan's Lemma.

$$
\begin{aligned}
\left| \int_{\pi/2}^{\pi} \mathrm{e}^{R\,\mathrm{e}^{\mathrm{i}\theta} t} \frac{\sqrt{\pi}}{\sqrt{R\,\mathrm{e}^{\mathrm{i}\theta}}}\, \mathrm{e}^{-2\sqrt{aR\,\mathrm{e}^{\mathrm{i}\theta}}} \,\mathrm{d}\theta \right| &\leq \int_{\pi/2}^{\pi} \left| \mathrm{e}^{R\,\mathrm{e}^{\mathrm{i}\theta} t} \frac{\sqrt{\pi}}{\sqrt{R\,\mathrm{e}^{\mathrm{i}\theta}}}\, \mathrm{e}^{-2\sqrt{a}\sqrt{R}\,\mathrm{e}^{\mathrm{i}\theta/2}} \right| \mathrm{d}\theta \\
&\leq \frac{\sqrt{\pi}}{\sqrt{R}} \int_{\pi/2}^{\pi} \mathrm{e}^{R\cos(\theta) t} \,\mathrm{d}\theta \\
&\leq \frac{\sqrt{\pi}}{\sqrt{R}} \int_{0}^{\pi/2} \mathrm{e}^{-Rt\sin(\phi)} \,\mathrm{d}\phi \\
&< \frac{\sqrt{\pi}}{\sqrt{R}} \frac{\pi}{2Rt} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}
$$

We could show that the integral along C_R^- vanishes by the same method.

Now we have

$$
\frac{1}{\mathrm{i}2\pi} \left(\int_B + \int_{L^+} + \int_{C_\epsilon} + \int_{L^-} \right) \mathrm{e}^{st} \left(\frac{\pi}{s} \right)^{1/2} \mathrm{e}^{-2(as)^{1/2}} \,\mathrm{d}s = 0.
$$

We show that the integral along C_ϵ vanishes as $\epsilon \to 0$ with the maximum modulus bound.

$$
\begin{aligned}
\left| \int_{C_\epsilon} \mathrm{e}^{st} \left(\frac{\pi}{s} \right)^{1/2} \mathrm{e}^{-2(as)^{1/2}} \,\mathrm{d}s \right| &\leq \left(\max_{s \in C_\epsilon} \left| \mathrm{e}^{st} \left(\frac{\pi}{s} \right)^{1/2} \mathrm{e}^{-2(as)^{1/2}} \right| \right) (\pi\epsilon) \\
&\leq \mathrm{e}^{\epsilon t} \frac{\sqrt{\pi}}{\sqrt{\epsilon}} \pi\epsilon \\
&\to 0 \quad \text{as } \epsilon \to 0.
\end{aligned}
$$

Now we can express the inverse Laplace transform in terms of the integrals along L^+ and L^-

$$
\begin{aligned}
f(t) &\equiv \frac{1}{\mathrm{i}2\pi} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} \mathrm{e}^{st} \left(\frac{\pi}{s} \right)^{1/2} \mathrm{e}^{-2(as)^{1/2}} \,\mathrm{d}s \\
&= -\frac{1}{\mathrm{i}2\pi} \int_{L^+} \mathrm{e}^{st} \left(\frac{\pi}{s} \right)^{1/2} \mathrm{e}^{-2(as)^{1/2}} \,\mathrm{d}s - \frac{1}{\mathrm{i}2\pi} \int_{L^-} \mathrm{e}^{st} \left(\frac{\pi}{s} \right)^{1/2} \mathrm{e}^{-2(as)^{1/2}} \,\mathrm{d}s.
\end{aligned}
$$

On L^+, $s = r\,\mathrm{e}^{\mathrm{i}\pi}$, $\mathrm{d}s = \mathrm{e}^{\mathrm{i}\pi}\mathrm{d}r = -\mathrm{d}r$; on L^-, $s = r\,\mathrm{e}^{-\mathrm{i}\pi}$, $\mathrm{d}s = \mathrm{e}^{-\mathrm{i}\pi}\mathrm{d}r = -\mathrm{d}r$. We can combine the integrals along the top and bottom of the branch cut.

$$\begin{aligned}
f(t) &= -\frac{1}{\mathrm{i}2\pi}\int_\infty^0 \mathrm{e}^{-rt}\frac{\sqrt{\pi}}{i\sqrt{r}}\,\mathrm{e}^{-\mathrm{i}2\sqrt{a}\sqrt{r}}(-\,\mathrm{d}r) - \frac{1}{\mathrm{i}2\pi}\int_0^\infty \mathrm{e}^{-rt}\frac{\sqrt{\pi}}{-\mathrm{i}\sqrt{r}}\,\mathrm{e}^{\mathrm{i}2\sqrt{a}\sqrt{r}}(-\,\mathrm{d}r)\\
&= \frac{1}{2\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-rt}\frac{1}{\sqrt{r}}\left(\mathrm{e}^{-\mathrm{i}2\sqrt{a}\sqrt{r}} + \mathrm{e}^{\mathrm{i}2\sqrt{a}\sqrt{r}}\right)\,\mathrm{d}r\\
&= \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{r}}\,\mathrm{e}^{-rt}2\cos\left(2\sqrt{a}\sqrt{r}\right)\,\mathrm{d}r
\end{aligned}$$

We make the change of variables $x = \sqrt{r}$.

$$\begin{aligned}
&= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{x}\,\mathrm{e}^{-tx^2}\cos\left(2\sqrt{a}x\right)2x\,\mathrm{d}x\\
&= \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-tx^2}\cos\left(2\sqrt{a}x\right)\,\mathrm{d}x\\
&= \frac{2}{\sqrt{\pi}}\sqrt{\frac{\pi}{4t}}\,\mathrm{e}^{-4a/(4t)}\\
&= \frac{\mathrm{e}^{-a/t}}{\sqrt{t}}
\end{aligned}$$

Thus the inverse Laplace transform is

$$\boxed{f(t) = \frac{\mathrm{e}^{-a/t}}{\sqrt{t}}}$$

Solution 33.17

We consider the problem

$$\frac{\mathrm{d}^4y}{\mathrm{d}t^4} - y = t, \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

We take the Laplace transform of the differential equation.

$$
\begin{aligned}
s^4 \hat{y}(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - \hat{y}(s) &= \frac{1}{s^2} \\
s^4 \hat{y}(s) - \hat{y}(s) &= \frac{1}{s^2} \\
\hat{y}(s) &= \frac{1}{s^2(s^4 - 1)}
\end{aligned}
$$

There are several ways in which we could carry out the inverse Laplace transform to find $y(t)$. We could expand the right side in partial fractions and then use a table of Laplace transforms. Since the function is analytic except for isolated singularities and vanishes as $s \to \infty$ we could use the result,

$$
\mathcal{L}^{-1}[\hat{f}(s)] = \sum_{n=1}^{N} \operatorname{Res}\left(\mathrm{e}^{st} \hat{f}(s), s_n\right),
$$

where $\{s_k\}_{k=1}^{n}$ are the singularities of $\hat{f}(s)$. Since we can write the function as a product of simpler terms we could also apply the convolution theorem.

We will first do the inverse Laplace transform by expanding the function in partial fractions to obtain simpler rational functions.

$$
\begin{aligned}
\frac{1}{s^2(s^4-1)} &= \frac{1}{s^2(s-1)(s+1)(s-\mathrm{i})(s+\mathrm{i})} \\
&= \frac{a}{s^2} + \frac{b}{s} + \frac{c}{s-1} + \frac{d}{s+1} + \frac{e}{s-\mathrm{i}} + \frac{f}{s+\mathrm{i}}
\end{aligned}
$$

$$a = \left[\frac{1}{s^4 - 1}\right]_{s=0} = -1$$
$$b = \left[\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^4 - 1}\right]_{s=0} = 0$$
$$c = \left[\frac{1}{s^2(s+1)(s-\mathrm{i})(s+\mathrm{i})}\right]_{s=1} = \frac{1}{4}$$
$$d = \left[\frac{1}{s^2(s-1)(s-\mathrm{i})(s+\mathrm{i})}\right]_{s=-1} = -\frac{1}{4}$$
$$e = \left[\frac{1}{s^2(s-1)(s+1)(s+\mathrm{i})}\right]_{s=\mathrm{i}} = -\mathrm{i}\frac{1}{4}$$
$$f = \left[\frac{1}{s^2(s-1)(s+1)(s-\mathrm{i})}\right]_{s=-\mathrm{i}} = \mathrm{i}\frac{1}{4}$$

Now we have simple functions that we can look up in a table.

$$\hat{y}(s) = -\frac{1}{s^2} + \frac{1/4}{s-1} - \frac{1/4}{s+1} + \frac{1/2}{s^2+1}$$
$$y(t) = \left(-t + \frac{1}{4}\,\mathrm{e}^t - \frac{1}{4}\,\mathrm{e}^{-t} + \frac{1}{2}\sin t\right) H(t)$$
$$\boxed{y(t) = \left(-t + \frac{1}{2}\left(\sinh t + \sin t\right)\right) H(t)}$$

We can also do the inversion with the convolution theorem.

$$\frac{1}{s^2(s^4-1)} = \frac{1}{s^2}\frac{1}{s^2+1}\frac{1}{s^2-1}$$

From a table of Laplace transforms we know,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t,$$
$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t,$$
$$\mathcal{L}^{-1}\left[\frac{1}{s^2-1}\right] = \sinh t.$$

Now we use the convolution theorem to find the solution for $t > 0$.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^4-1}\right] &= \int_0^t \sinh(\tau)\sin(t-\tau)\,\mathrm{d}\tau \\
&= \frac{1}{2}(\sinh t - \sin t)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^2(s^4-1)}\right] &= \int_0^t \frac{1}{2}(\sinh\tau - \sin\tau)(t-\tau)\,\mathrm{d}\tau \\
&= -t + \frac{1}{2}(\sinh t + \sin t)
\end{aligned}$$

Solution 33.18

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \sin t + \int_0^t y(\tau)\cos(t-\tau)\,\mathrm{d}\tau \\
s\hat{y}(s) - y(0) &= \frac{1}{s^2+1} + \hat{y}(s)\frac{s}{s^2+1} \\
(s^3+s)\hat{y}(s) - s\hat{y}(s) &= 1 \\
\hat{y}(s) &= \frac{1}{s^3} \\
\Aboxed{y(t) &= \frac{t^2}{2}}
\end{aligned}
$$

Solution 33.19
The Laplace transform of $u(t-1)$ is

$$
\begin{aligned}
\mathcal{L}[u(t-1)] &= \int_0^\infty \mathrm{e}^{-st} u(t-1)\,\mathrm{d}t \\
&= \int_{-1}^\infty \mathrm{e}^{-s(t+1)} u(t)\,\mathrm{d}t \\
&= \mathrm{e}^{-s} \int_{-1}^0 \mathrm{e}^{-st} u(t)\,\mathrm{d}t + \mathrm{e}^{-s} \int_0^\infty \mathrm{e}^{-st} u(t)\,\mathrm{d}t \\
&= \mathrm{e}^{-s} \int_{-1}^0 \mathrm{e}^{-st} u_0(t)\,\mathrm{d}t + \mathrm{e}^{-s}\hat{u}(s).
\end{aligned}
$$

We take the Laplace transform of the difference-differential equation.

$$s\hat{u}(s) - u(0) + \hat{u}(s) - \mathrm{e}^{-s} \int_{-1}^{0} \mathrm{e}^{-st} u_0(t) \, \mathrm{d}t + \mathrm{e}^{-s} \hat{u}(s) = 0$$

$$(1 + s - \mathrm{e}^{-s})\hat{u}(s) = u_0(0) + \mathrm{e}^{-s} \int_{-1}^{0} \mathrm{e}^{-st} u_0(t) \, \mathrm{d}t$$

$$\boxed{\hat{u}(s) = \frac{u_0(0)}{1 + s - \mathrm{e}^{-s}} + \frac{\mathrm{e}^{-s}}{1 + s - \mathrm{e}^{-s}} \int_{-1}^{0} \mathrm{e}^{-st} u_0(t) \, \mathrm{d}t}$$

Consider the case $u_0(t) = 1$.

$$\hat{u}(s) = \frac{1}{1 + s - \mathrm{e}^{-s}} + \frac{\mathrm{e}^{-s}}{1 + s - \mathrm{e}^{-s}} \int_{-1}^{0} \mathrm{e}^{-st} \, \mathrm{d}t$$

$$\hat{u}(s) = \frac{1}{1 + s - \mathrm{e}^{-s}} + \frac{\mathrm{e}^{-s}}{1 + s - \mathrm{e}^{-s}} \left(-\frac{1}{s} + \frac{1}{s} \mathrm{e}^{s} \right)$$

$$\hat{u}(s) = \frac{1/s + 1 - \mathrm{e}^{-s}/s}{1 + s - \mathrm{e}^{-s}}$$

$$\hat{u}(s) = \frac{1}{s}$$

$$\boxed{u(t) = 1}$$

Clearly this solution satisfies the difference-differential equation.

Solution 33.20
We consider the problem,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - y = f(t), \quad y(0) = 1, \quad y'(0) = 0,$$

where $f(t)$ is periodic with period 2π and is defined by,

$$f(t) = \begin{cases} 1 & 0 \le t < \pi, \\ 0 & \pi \le t < 2\pi. \end{cases}$$

We take the Laplace transform of the differential equation.

$$s^2 \hat{y}(s) - s y(0) - y'(0) - \hat{y}(s) = \hat{f}(s)$$
$$s^2 \hat{y}(s) - s - \hat{y}(s) = \hat{f}(s)$$
$$\hat{y}(s) = \frac{s}{s^2 - 1} + \frac{\hat{f}(s)}{s^2 - 1}$$

By inspection, (of a table of Laplace transforms), we see that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2 - 1}\right] = \cosh(t) H(t),$$
$$\mathcal{L}^{-1}\left[\frac{1}{s^2 - 1}\right] = \sinh(t) H(t).$$

Now we use the convolution theorem.

$$\mathcal{L}^{-1}\left[\frac{\hat{f}(s)}{s^2 - 1}\right] = \int_0^t f(\tau) \sinh(t - \tau) \, \mathrm{d}\tau$$

The solution for positive t is

$$\boxed{y(t) = \cosh(t) + \int_0^t f(\tau) \sinh(t - \tau) \, \mathrm{d}\tau.}$$

Clearly the solution is continuous because the integral of a bounded function is continuous. The first derivative of the solution is

$$y'(t) = \sinh t + f(t)\sinh(0) + \int_0^t f(\tau)\cosh(t-\tau)\,\mathrm{d}\tau$$
$$y'(t) = \sinh t + \int_0^t f(\tau)\cosh(t-\tau)\,\mathrm{d}\tau$$

We see that the first derivative is also continuous.

Solution 33.21
We consider the problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \int_0^t y(\tau)\,\mathrm{d}\tau = \mathrm{e}^{-t}, \quad y(0) = 1.$$

We take the Laplace transform of the equation and solve for $\hat{y}$.

$$s\hat{y} - y(0) + \frac{\hat{y}}{s} = \frac{1}{s+1}$$
$$\hat{y} = \frac{s(s+2)}{(s+1)(s^2+1)}$$

We expand the right side in partial fractions.

$$\hat{y} = -\frac{1}{2(s+1)} + \frac{1+3s}{2(s^2+1)}$$

We use a table of Laplace transforms to do the inversion.

$$\boxed{y = -\frac{1}{2}\,\mathrm{e}^{-t} + \frac{1}{2}(\sin(t) + 3\cos(t))}$$

Solution 33.22
We consider the problem

$$L \frac{\mathrm{d}i_1}{\mathrm{d}t} + R i_1 + q/C = E_0$$
$$L \frac{\mathrm{d}i_2}{\mathrm{d}t} + R i_2 - q/C = 0$$
$$\frac{\mathrm{d}q}{\mathrm{d}t} = i_1 - i_2$$
$$i_1(0) = i_2(0) = \frac{E_0}{2R}, \quad q(0) = 0.$$

We take the Laplace transform of the system of differential equations.

$$L\left(s\hat{i}_1 - \frac{E_0}{2R}\right) + R\hat{i}_1 + \frac{\hat{q}}{C} = \frac{E_0}{s}$$
$$L\left(s\hat{i}_2 - \frac{E_0}{2R}\right) + R\hat{i}_2 - \frac{\hat{q}}{C} = 0$$
$$s\hat{q} = \hat{i}_1 - \hat{i}_2$$

We solve for $\hat{i}_1$, $\hat{i}_2$ and $\hat{q}$.

$$\hat{i}_1 = \frac{E_0}{2}\left(\frac{1}{Rs} + \frac{1/L}{s^2 + Rs/L + 2/(CL)}\right)$$
$$\hat{i}_2 = \frac{E_0}{2}\left(\frac{1}{Rs} - \frac{1/L}{s^2 + Rs/L + 2/(CL)}\right)$$
$$\hat{q} = \frac{CE_0}{2}\left(\frac{1}{s} - \frac{s + R/L}{s^2 + Rs/L + 2/(CL)}\right)$$

We factor the polynomials in the denominators.

$$\hat{i}_1 = \frac{E_0}{2}\left(\frac{1}{Rs} + \frac{1/L}{(s+\alpha-\mathrm{i}\omega)(s+\alpha+\mathrm{i}\omega)}\right)$$
$$\hat{i}_2 = \frac{E_0}{2}\left(\frac{1}{Rs} - \frac{1/L}{(s+\alpha-\mathrm{i}\omega)(s+\alpha+\mathrm{i}\omega)}\right)$$
$$\hat{q} = \frac{CE_0}{2}\left(\frac{1}{s} - \frac{s+2\alpha}{(s+\alpha-\mathrm{i}\omega)(s+\alpha+\mathrm{i}\omega)}\right)$$

Here we have defined

$$\alpha = \frac{R}{2L} \quad \text{and} \quad \omega^2 = \frac{2}{LC} - \alpha^2.$$

We expand the functions in partial fractions.

$$\hat{i}_1 = \frac{E_0}{2}\left(\frac{1}{Rs} + \frac{\mathrm{i}}{2\omega L}\left(\frac{1}{s+\alpha+\mathrm{i}\omega} - \frac{1}{s+\alpha-\mathrm{i}\omega}\right)\right)$$
$$\hat{i}_2 = \frac{E_0}{2}\left(\frac{1}{Rs} - \frac{\mathrm{i}}{2\omega L}\left(\frac{1}{s+\alpha+\mathrm{i}\omega} - \frac{1}{s+\alpha-\mathrm{i}\omega}\right)\right)$$
$$\hat{q} = \frac{CE_0}{2}\left(\frac{1}{s} + \frac{\mathrm{i}}{2\omega}\left(\frac{\alpha+\mathrm{i}\omega}{s+\alpha-\mathrm{i}\omega} - \frac{\alpha-\mathrm{i}\omega}{s+\alpha+\mathrm{i}\omega}\right)\right)$$

Now we can do the inversion with a table of Laplace transforms.

$$i_1 = \frac{E_0}{2}\left(\frac{1}{R} + \frac{\mathrm{i}}{2\omega L}\left(\mathrm{e}^{(-\alpha-\mathrm{i}\omega)t} - \mathrm{e}^{(-\alpha+\mathrm{i}\omega)t}\right)\right)$$
$$i_2 = \frac{E_0}{2}\left(\frac{1}{R} - \frac{\mathrm{i}}{2\omega L}\left(\mathrm{e}^{(-\alpha-\mathrm{i}\omega)t} - \mathrm{e}^{(-\alpha+\mathrm{i}\omega)t}\right)\right)$$
$$q = \frac{CE_0}{2}\left(1 + \frac{\mathrm{i}}{2\omega}\left((\alpha+\mathrm{i}\omega)\,\mathrm{e}^{(-\alpha+\mathrm{i}\omega)t} - (\alpha-\mathrm{i}\omega)\,\mathrm{e}^{(-\alpha-\mathrm{i}\omega)t}\right)\right)$$

We simplify the expressions to obtain the solutions.

$$i_1 = \frac{E_0}{2} \left(\frac{1}{R} + \frac{1}{\omega L} \mathrm{e}^{-\alpha t} \sin(\omega t) \right)$$
$$i_2 = \frac{E_0}{2} \left(\frac{1}{R} - \frac{1}{\omega L} \mathrm{e}^{-\alpha t} \sin(\omega t) \right)$$
$$q = \frac{C E_0}{2} \left(1 - \mathrm{e}^{-\alpha t} \left(\cos(\omega t) + \frac{\alpha}{\omega} \sin(\omega t) \right) \right)$$

Chapter 34

The Fourier Transform

34.1 Derivation from a Fourier Series

Consider the eigenvalue problem

$$y'' + \lambda y = 0, \qquad y(-L) = y(L), \quad y'(-L) = y'(L).$$

The eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for } n \in \mathbb{Z}^{0+}$$
$$\phi_n = \frac{\pi}{L} \mathrm{e}^{\mathrm{i}n\pi x/L}, \quad \text{for } n \in \mathbb{Z}$$

The eigenfunctions form an orthogonal set. A piecewise continuous function defined on $[-L \ldots L]$ can be expanded in a series of the eigenfunctions.

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \frac{\pi}{L} \mathrm{e}^{\mathrm{i}n\pi x/L}$$

The Fourier coefficients are

$$c_n = \frac{\left\langle \frac{\pi}{L} \,\mathrm{e}^{\mathrm{i}n\pi x/L} \middle| f(x) \right\rangle}{\left\langle \frac{\pi}{L} \,\mathrm{e}^{\mathrm{i}n\pi x/L} \middle| \frac{\pi}{L} \,\mathrm{e}^{\mathrm{i}n\pi x/L} \right\rangle}$$
$$= \frac{1}{2\pi} \int_{-L}^{L} \mathrm{e}^{-\mathrm{i}n\pi x/L} f(x) \,\mathrm{d}x.$$

We substitute the expression for c_n into the series for $f(x)$.

$$f(x) \sim \sum_{n=-\infty}^{\infty} \left[\frac{1}{2L} \int_{-L}^{L} \mathrm{e}^{-\mathrm{i}n\pi\xi/L} f(\xi) \,\mathrm{d}\xi \right] \mathrm{e}^{\mathrm{i}n\pi x/L}.$$

We let $\omega_n = n\pi/L$ and $\Delta\omega = \pi/L$.

$$f(x) \sim \sum_{\omega_n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-L}^{L} \mathrm{e}^{-\mathrm{i}\omega_n\xi} f(\xi) \,\mathrm{d}\xi \right] \mathrm{e}^{\mathrm{i}\omega_n x} \Delta\omega.$$

In the limit as $L \to \infty$, (and thus $\Delta\omega \to 0$), the sum becomes an integral.

$$f(x) \sim \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega\xi} f(\xi) \,\mathrm{d}\xi \right] \mathrm{e}^{\mathrm{i}\omega x} \,\mathrm{d}\omega.$$

Thus the expansion of $f(x)$ for finite L

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \frac{\pi}{L} \,\mathrm{e}^{\mathrm{i}n\pi x/L}$$
$$c_n = \frac{1}{2\pi} \int_{-L}^{L} \mathrm{e}^{-\mathrm{i}n\pi x/L} f(x) \,\mathrm{d}x$$

in the limit as $L \to \infty$ becomes

$$f(x) \sim \int_{-\infty}^{\infty} \hat{f}(\omega)\, \mathrm{e}^{\mathrm{i}\omega x}\, \mathrm{d}\omega$$
$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}\omega x}\, \mathrm{d}x.$$

Of course this derivation is only heuristic. In the next section we will explore these formulas more carefully.

34.2 The Fourier Transform

Let $f(x)$ be piecewise continuous and let $\int_{-\infty}^{\infty} |f(x)|\, \mathrm{d}x$ exist. We define the function $I(x, L)$.

$$I(x, L) = \frac{1}{2\pi} \int_{-L}^{L} \left(\int_{-\infty}^{\infty} f(\xi)\, \mathrm{e}^{\mathrm{i}\omega\xi}\, \mathrm{d}\xi \right) \mathrm{e}^{-\mathrm{i}\omega x}\, \mathrm{d}\omega.$$

Since the integral in parentheses is uniformly convergent, we can interchange the order of integration.

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-L}^{L} f(\xi)\, \mathrm{e}^{\mathrm{i}\omega(\xi - x)}\, \mathrm{d}\omega \right) \mathrm{d}\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[f(\xi) \frac{\mathrm{e}^{\mathrm{i}\omega(\xi - x)}}{\mathrm{i}(\xi - x)} \right]_{-L}^{L} \mathrm{d}\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \frac{1}{\mathrm{i}(\xi - x)} \left(\mathrm{e}^{\mathrm{i}L(\xi - x)} - \mathrm{e}^{-\mathrm{i}L(\xi - x)} \right) \mathrm{d}\xi \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{\sin(L(\xi - x))}{\xi - x}\, \mathrm{d}\xi \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi + x) \frac{\sin(L\xi)}{\xi}\, \mathrm{d}\xi.
\end{aligned}$$

In Example 34.3.3 we will show that

$$\int_0^\infty \frac{\sin(L\xi)}{\xi}\,\mathrm{d}\xi = \frac{\pi}{2}.$$

Continuous Functions. Suppose that $f(x)$ is continuous.

$$\begin{aligned}
f(x) &= \frac{1}{\pi}\int_{-\infty}^\infty f(x)\frac{\sin(L\xi)}{\xi}\,\mathrm{d}\xi \\
I(x,L) - f(x) &= \frac{1}{\pi}\int_{-\infty}^\infty \frac{f(x+\xi)-f(x)}{\xi}\sin(L\xi)\,\mathrm{d}\xi.
\end{aligned}$$

If $f(x)$ has a left and right derivative at x then $\frac{f(x+\xi)-f(x)}{\xi}$ is bounded and $\int_{-\infty}^\infty \left|\frac{f(x+\xi)-f(x)}{\xi}\right|\,\mathrm{d}\xi < \infty$. We use the Riemann-Lebesgue lemma to show that the integral vanishes as $L \to \infty$.

$$\frac{1}{\pi}\int_{-\infty}^\infty \frac{f(x+\xi)-f(x)}{\xi}\sin(L\xi)\,\mathrm{d}\xi \to 0 \text{ as } L \to \infty.$$

Now we have an identity for $f(x)$.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(\xi)\,\mathrm{e}^{\mathrm{i}\omega\xi}\,\mathrm{d}\xi\right)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}\omega.$$

Piecewise Continuous Functions. Now consider the case that $f(x)$ is only piecewise continuous.

$$\begin{aligned}
\frac{f(x^+)}{2} &= \frac{1}{\pi}\int_0^\infty f(x^+)\frac{\sin(L\xi)}{\xi}\,\mathrm{d}\xi \\
\frac{f(x^-)}{2} &= \frac{1}{\pi}\int_{-\infty}^0 f(x^-)\frac{\sin(L\xi)}{\xi}\,\mathrm{d}\xi
\end{aligned}$$

$$
\begin{aligned}
I(x,L) - \frac{f(x^+) + f(x^-)}{2} = \int_{-\infty}^{0} &\left(\frac{f(x+\xi) - f(x^-)}{\xi} \right) \sin(L\xi)\, \mathrm{d}\xi \\
&- \int_{0}^{\infty} \left(\frac{f(x+\xi) - f(x^+)}{\xi} \right) \sin(L\xi)\, \mathrm{d}\xi
\end{aligned}
$$

If $f(x)$ has a left and right derivative at x, then

$$
\begin{aligned}
&\frac{f(x+\xi) - f(x^-)}{\xi} \text{ is bounded for } \xi \leq 0, \text{ and} \\
&\frac{f(x+\xi) - f(x^+)}{\xi} \text{ is bounded for } \xi \geq 0.
\end{aligned}
$$

Again using the Riemann-Lebesgue lemma we see that

$$
\frac{f(x^+) + f(x^-)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\xi)\, \mathrm{e}^{\mathrm{i}\omega\xi}\, \mathrm{d}\xi \right) \mathrm{e}^{-\mathrm{i}\omega x}\, \mathrm{d}\omega.
$$

Result 34.2.1 Let $f(x)$ be piecewise continuous with $\int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x < \infty$. The Fourier transform of $f(x)$ is defined

$$\hat{f}(\omega) = \mathcal{F}[f(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \, \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x.$$

We see that the integral is uniformly convergent. The inverse Fourier transform is defined

$$\frac{f(x^+) + f(x^-)}{2} = \mathcal{F}^{-1}[\hat{f}(\omega)] = \int_{-\infty}^{\infty} \hat{f}(\omega) \, \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}\omega.$$

If $f(x)$ is continuous then this reduces to

$$f(x) = \mathcal{F}^{-1}[\hat{f}(\omega)] = \int_{-\infty}^{\infty} \hat{f}(\omega) \, \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}\omega.$$

34.2.1 A Word of Caution

Other texts may define the Fourier transform differently. The important relation is

$$f(x) = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \, \mathrm{e}^{\mp \mathrm{i}\omega\xi} \, d\xi \right) \mathrm{e}^{\pm \mathrm{i}\omega x} \, \mathrm{d}\omega.$$

Multiplying the right side of this equation by $1 = \frac{1}{\alpha}\alpha$ yields

$$f(x) = \frac{1}{\alpha} \int_{-\infty}^{\infty} \left(\frac{\alpha}{2\pi} \int_{-\infty}^{\infty} f(\xi) \, \mathrm{e}^{\mp \mathrm{i}\omega\xi} \, \mathrm{d}\xi \right) \mathrm{e}^{\pm \mathrm{i}\omega x} \, \mathrm{d}\omega.$$

Setting $\alpha = \sqrt{2\pi}$ and choosing sign in the exponentials gives us the Fourier transform pair

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) \, \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}\omega.$$

Other equally valid pairs are

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) \, \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x$$
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \, \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}\omega,$$

and

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) \, \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}x$$
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \, \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}\omega.$$

Be aware of the different definitions when reading other texts or consulting tables of Fourier transforms.

34.3 Evaluating Fourier Integrals

34.3.1 Integrals that Converge

If the Fourier integral

$$\mathcal{F}[f(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \, \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x,$$

converges for real ω, then finding the transform of a function is just a matter of direct integration. We will consider several examples of such garden variety functions in this subsection. Later on we will consider the more interesting cases when the integral does not converge for real ω.

Example 34.3.1 Consider the Fourier transform of $\mathrm{e}^{-\alpha|x|}$, where $\alpha > 0$. Since the integral of $\mathrm{e}^{-\alpha|x|}$ is absolutely convergent, we know that the Fourier transform integral converges for real ω. We write out the integral.

$$\begin{aligned}
\mathcal{F}\left[\mathrm{e}^{-\alpha|x|}\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|x|}\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} \mathrm{e}^{\alpha x-\mathrm{i}\omega x}\,\mathrm{d}x + \frac{1}{2\pi}\int_{0}^{\infty} \mathrm{e}^{-\alpha x-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} \mathrm{e}^{(\alpha-\mathrm{i}\Re(\omega)+\Im(\omega))x}\,\mathrm{d}x + \frac{1}{2\pi}\int_{0}^{\infty} \mathrm{e}^{(-\alpha-\mathrm{i}\Re(\omega)+\Im(\omega))x}\,\mathrm{d}x
\end{aligned}$$

The integral converges for $|\Im(\omega)| < \alpha$. This domain is shown in Figure 34.1.

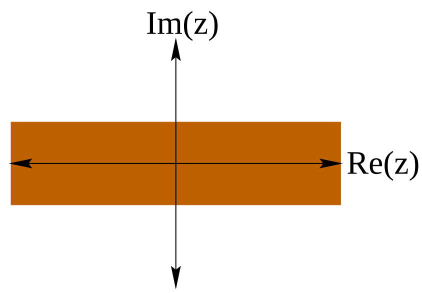

Figure 34.1: The Domain of Convergence

Now We do the integration.

$$
\begin{aligned}
\mathcal{F}\left[\mathrm{e}^{-\alpha|x|}\right] &= \frac{1}{2\pi}\int_{-\infty}^{0} \mathrm{e}^{(\alpha-\mathrm{i}\omega)x}\,\mathrm{d}x + \frac{1}{2\pi}\int_{0}^{\infty} \mathrm{e}^{-(\alpha+\mathrm{i}\omega)x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\frac{\mathrm{e}^{(\alpha-\mathrm{i}\omega)x}}{\alpha-\mathrm{i}\omega}\right]_{-\infty}^{0} + \frac{1}{2\pi}\left[-\frac{\mathrm{e}^{-(\alpha+\mathrm{i}\omega)x}}{\alpha+\mathrm{i}\omega}\right]_{0}^{\infty} \\
&= \frac{1}{2\pi}\left(\frac{1}{\alpha-\mathrm{i}\omega} + \frac{1}{\alpha+\mathrm{i}\omega}\right) \\
&= \frac{1}{\pi}\frac{\alpha}{\pi(\omega^2+\alpha^2)}, \quad \text{for } |\Im(\omega)| < \alpha
\end{aligned}
$$

We can extend the domain of the Fourier transform with analytic continuation.

$$
\mathcal{F}\left[\mathrm{e}^{-\alpha|x|}\right] = \frac{\alpha}{\pi(\omega^2+\alpha^2)}, \quad \text{for } \omega \neq \pm\mathrm{i}\alpha
$$

Example 34.3.2 Consider the Fourier transform of $f(x) = \frac{1}{x-\mathrm{i}\alpha}$, $\alpha > 0$.

$$
\mathcal{F}\left[\frac{1}{x-\mathrm{i}\alpha}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{x-\mathrm{i}\alpha}\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x
$$

The integral converges for $\Im(\omega) = 0$. We will evaluate the integral for positive and negative real values of ω.

For $\omega > 0$, we will close the path of integration in the lower half-plane. Let C_R be the contour from $x = R$ to $x = -R$ following a semicircular path in the lower half-plane. The integral along C_R vanishes as $R \to \infty$ by Jordan's Lemma.

$$
\int_{C_R} \frac{1}{x-\mathrm{i}\alpha}\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \to 0 \quad \text{as } R \to \infty.
$$

Since the integrand is analytic in the lower half-plane the integral vanishes.

$$
\mathcal{F}\left[\frac{1}{x-\mathrm{i}\alpha}\right] = 0
$$

For $\omega < 0$, we will close the path of integration in the upper half-plane. Let C_R denote the semicircular contour from $x = R$ to $x = -R$ in the upper half-plane. The integral along C_R vanishes as R goes to infinity by Jordan's Lemma. We evaluate the Fourier transform integral with the Residue Theorem.

$$\begin{aligned}\mathcal{F}\left[\frac{1}{x - \mathrm{i}\alpha}\right] &= \frac{1}{2\pi} 2\pi i \operatorname{Res}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega x}}{x - i\alpha}, i\alpha\right) \\ &= \mathrm{i}\,\mathrm{e}^{\alpha\omega}\end{aligned}$$

We combine the results for positive and negative values of ω.

$$\mathcal{F}\left[\frac{1}{x - \mathrm{i}\alpha}\right] = \begin{cases} 0 & \text{for } \omega > 0, \\ \mathrm{i}\,\mathrm{e}^{\alpha\omega} & \text{for } \omega < 0 \end{cases}$$

34.3.2 Cauchy Principal Value and Integrals that are Not Absolutely Convergent.

That the integral of $f(x)$ is absolutely convergent is a sufficient but not a necessary condition that the Fourier transform of $f(x)$ exists. The integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x$ may converge even if $\int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x$ does not. Furthermore, if the Fourier transform integral diverges, its principal value may exist. We will say that the Fourier transform of $f(x)$ exists if the principal value of the integral exists.

$$\mathcal{F}[f(x)] = ⨍_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x$$

Example 34.3.3 Consider the Fourier transform of $f(x) = 1/x$.

$$\hat{f}(\omega) = \frac{1}{2\pi} ⨍_{-\infty}^{\infty} \frac{1}{x}\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x$$

If $\omega > 0$, we can close the contour in the lower half-plane. The integral along the semi-circle vanishes due to Jordan's Lemma.

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{x}\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x = 0$$

We can evaluate the Fourier transform with the Residue Theorem.

$$\hat{f}(\omega) = \frac{1}{2\pi}\left(\frac{-1}{2}\right)(2\pi i)\,\mathrm{Res}\left(\frac{1}{x}\,\mathrm{e}^{-\mathrm{i}\omega x}, 0\right)$$

$$\hat{f}(\omega) = -\frac{\mathrm{i}}{2}, \quad \text{for } \omega > 0.$$

The factor of $-1/2$ in the above derivation arises because the path of integration is in the negative, (clockwise), direction and the path of integration crosses through the first order pole at $x = 0$. The path of integration is shown in Figure 34.2.

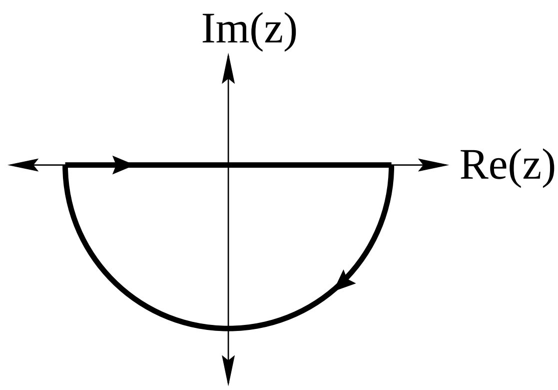

Figure 34.2: The Path of Integration

If $\omega < 0$, we can close the contour in the upper half plane to obtain

$$\hat{f}(\omega) = \frac{\mathrm{i}}{2}, \quad \text{for } \omega < 0.$$

For $\omega = 0$ the integral vanishes because $\frac{1}{x}$ is an odd function.

$$\hat{f}(0) = \frac{1}{2\pi} = \int_{-\infty}^{\infty} \frac{1}{x}\,\mathrm{d}x = 0$$

We collect the results in one formula.

$$\boxed{\hat{f}(\omega) = -\frac{\mathrm{i}}{2}\operatorname{sign}(\omega)}$$

We write the integrand for $\omega > 0$ as the sum of an odd and and even function.

$$\frac{1}{2\pi} ⨍_{-\infty}^{\infty} \frac{1}{x} \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x = -\frac{\mathrm{i}}{2}$$
$$⨍_{-\infty}^{\infty} \frac{1}{x} \cos(\omega x) \, \mathrm{d}x + ⨍_{-\infty}^{\infty} \frac{-\mathrm{i}}{x} \sin(\omega x) \, \mathrm{d}x = -\mathrm{i}\pi$$

The principal value of the integral of any odd function is zero.

$$⨍_{-\infty}^{\infty} \frac{1}{x} \sin(\omega x) \, \mathrm{d}x = \pi$$

If the principal value of the integral of an even function exists, then the integral converges.

$$\int_{-\infty}^{\infty} \frac{1}{x} \sin(\omega x) \, \mathrm{d}x = \pi$$
$$\boxed{\int_{0}^{\infty} \frac{1}{x} \sin(\omega x) \, \mathrm{d}x = \frac{\pi}{2}}$$

Thus we have evaluated an integral that we used in deriving the Fourier transform.

34.3.3 Analytic Continuation

Consider the Fourier transform of $f(x) = 1$. The Fourier integral is not convergent, and its principal value does not exist. Thus we will have to be a little creative in order to define the Fourier transform. Define the two functions

$$f_+(x) = \begin{cases} 1 & \text{for } x > 0 \\ 1/2 & \text{for } x = 0 \\ 0 & \text{for } x < 0 \end{cases}, \qquad f_-(x) = \begin{cases} 0 & \text{for } x > 0 \\ 1/2 & \text{for } x = 0 \\ 1 & \text{for } x < 0 \end{cases}.$$

Note that $1 = f_-(x) + f_+(x)$.

The Fourier transform of $f_+(x)$ converges for $\Im(\omega) < 0$.

$$\begin{aligned}
\mathcal{F}[f_+(x)] &= \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{(-\mathrm{i}\Re(\omega)+\Im(\omega))x}\,\mathrm{d}x. \\
&= \frac{1}{2\pi}\left[\frac{\mathrm{e}^{-\mathrm{i}\omega x}}{-\mathrm{i}\omega}\right]_0^\infty \\
&= -\frac{\mathrm{i}}{2\pi\omega} \quad \text{for } \Im(\omega) < 0
\end{aligned}$$

Using analytic continuation, we can define the Fourier transform of $f_+(x)$ for all ω except the point $\omega = 0$.

$$\mathcal{F}[f_+(x)] = -\frac{\mathrm{i}}{2\pi\omega}$$

We follow the same procedure for $f_-(x)$. The integral converges for $\Im(\omega) > 0$.

$$\begin{aligned}\mathcal{F}[f_-(x)] &= \frac{1}{2\pi}\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} \mathrm{e}^{(-\mathrm{i}\Re(\omega)+\Im(\omega))x}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\left[\frac{\mathrm{e}^{-\mathrm{i}\omega x}}{-\mathrm{i}\omega}\right]_{-\infty}^{0} \\ &= \frac{\mathrm{i}}{2\pi\omega}.\end{aligned}$$

Using analytic continuation we can define the transform for all nonzero ω.

$$\mathcal{F}[f_-(x)] = \frac{\mathrm{i}}{2\pi\omega}$$

Now we are prepared to define the Fourier transform of $f(x) = 1$.

$$\begin{aligned}\mathcal{F}[1] &= \mathcal{F}[f_-(x)] + \mathcal{F}[f_+(x)] \\ &= -\frac{\mathrm{i}}{2\pi\omega} + \frac{\mathrm{i}}{2\pi\omega} \\ &= 0, \quad \text{for } \omega \neq 0\end{aligned}$$

When $\omega = 0$ the integral diverges. When we consider the closure relation for the Fourier transform we will see that

$$\boxed{\mathcal{F}[1] = \delta(\omega).}$$

34.4 Properties of the Fourier Transform

In this section we will explore various properties of the Fourier Transform. I would like to avoid stating assumptions on various functions at the beginning of each subsection. Unless otherwise indicated, assume that the integrals converge.

34.4.1 Closure Relation.

Recall the closure relation for an orthonormal set of functions $\{\phi_1, \phi_2, \dots\}$,

$$\sum_{n=1}^{\infty} \phi_n(x)\overline{\phi_n(\xi)} \sim \delta(x - \xi).$$

There is a similar closure relation for Fourier integrals. We compute the Fourier transform of $\delta(x - \xi)$.

$$\begin{aligned}\mathcal{F}[\delta(x-\xi)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x-\xi)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega\xi}\end{aligned}$$

Next we take the inverse Fourier transform.

$$\delta(x-\xi) \sim \int_{-\infty}^{\infty} \frac{1}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega\xi}\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega$$

$$\boxed{\delta(x-\xi) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega(x-\xi)}\,\mathrm{d}\omega.}$$

Note that the integral is divergent, but it would be impossible to represent $\delta(x - \xi)$ with a convergent integral.

34.4.2 Fourier Transform of a Derivative.

Consider the Fourier transform of $y'(x)$.

$$\begin{aligned}
\mathcal{F}[y'(x)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} y'(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \left[\frac{1}{2\pi}y(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\right]_{-\infty}^{\infty} - \frac{1}{2\pi}\int_{-\infty}^{\infty}(-\mathrm{i}\omega)y(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \mathrm{i}\omega\frac{1}{2\pi}\int_{-\infty}^{\infty} y(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \mathrm{i}\omega\mathcal{F}[y(x)]
\end{aligned}$$

Next consider $y''(x)$.

$$\begin{aligned}
\mathcal{F}[y''(x)] &= \mathcal{F}\left[\frac{\mathrm{d}}{\mathrm{d}x}(y'(x))\right] \\
&= \mathrm{i}\omega\mathcal{F}[y'(x)] \\
&= (\mathrm{i}\omega)^2\mathcal{F}[y(x)] \\
&= -\omega^2\mathcal{F}[y(x)]
\end{aligned}$$

In general,

$$\boxed{\mathcal{F}\left[y^{(n)}(x)\right] = (\mathrm{i}\omega)^n\mathcal{F}[y(x)].}$$

Example 34.4.1 The Dirac delta function can be expressed as the derivative of the Heaviside function.

$$H(x-c) = \begin{cases} 0 & \text{for } x < c, \\ 1 & \text{for } x > c \end{cases}$$

Thus we can express the Fourier transform of $H(x-c)$ in terms of the Fourier transform of the delta function.

$$\mathcal{F}[\delta(x-c)] = \mathrm{i}\omega\mathcal{F}[H(x-c)]$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x-c)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x = \mathrm{i}\omega\mathcal{F}[H(x-c)]$$
$$\frac{1}{2\pi}\,\mathrm{e}^{-\mathrm{i}c\omega} = \mathrm{i}\omega\mathcal{F}[H(x-c)]$$

$$\boxed{\mathcal{F}[H(x-c)] = \frac{1}{2\pi\mathrm{i}\omega}\,\mathrm{e}^{-\mathrm{i}c\omega}}$$

34.4.3 Fourier Convolution Theorem.

Consider the Fourier transform of a product of two functions.

$$\begin{aligned}
\mathcal{F}[f(x)g(x)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)g(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\hat{f}(\eta)\,\mathrm{e}^{\mathrm{i}\eta x}\,\mathrm{d}\eta\right) g(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\hat{f}(\eta)g(x)\,\mathrm{e}^{\mathrm{i}(\eta-\omega)x}\,\mathrm{d}x\right)\mathrm{d}\eta \\
&= \int_{-\infty}^{\infty}\hat{f}(\eta)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\,\mathrm{e}^{-\mathrm{i}(\omega-\eta)x}\,\mathrm{d}x\right)\mathrm{d}\eta \\
&= \int_{-\infty}^{\infty}\hat{f}(\eta)G(\omega-\eta)\,\mathrm{d}\eta
\end{aligned}$$

The convolution of two functions is defined

$$f * g(x) = \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,\mathrm{d}\xi.$$

Thus

$$\boxed{\mathcal{F}[f(x)g(x)] = \hat{f} * \hat{g}(\omega) = \int_{-\infty}^{\infty} \hat{f}(\eta)\hat{g}(\omega - \eta)\,\mathrm{d}\eta.}$$

Now consider the inverse Fourier Transform of a product of two functions.

$$\begin{aligned}
\mathcal{F}^{-1}[\hat{f}(\omega)\hat{g}(\omega)] &= \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\,\mathrm{e}^{-\mathrm{i}\omega\xi}\,\mathrm{d}\xi \right) \hat{g}(\omega)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\xi)\hat{g}(\omega)\,\mathrm{e}^{\mathrm{i}\omega(x-\xi)}\,\mathrm{d}\omega \right) \mathrm{d}\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \left(\int_{-\infty}^{\infty} \hat{g}(\omega)\,\mathrm{e}^{\mathrm{i}\omega(x-\xi)}\,\mathrm{d}\omega \right) \mathrm{d}\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,\mathrm{d}\xi
\end{aligned}$$

Thus

$$\boxed{\mathcal{F}^{-1}[\hat{f}(\omega)\hat{g}(\omega)] = \frac{1}{2\pi} f * g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,\mathrm{d}\xi,}$$

$$\boxed{\mathcal{F}[f * g(x)] = 2\pi \hat{f}(\omega)\hat{g}(\omega).}$$

These relations are known as the Fourier convolution theorem.

Example 34.4.2 Using the convolution theorem and the table of Fourier transform pairs in the appendix, we can find the Fourier transform of

$$f(x) = \frac{1}{x^4 + 5x^2 + 4}.$$

We factor the fraction.

$$f(x) = \frac{1}{(x^2+1)(x^2+4)}$$

From the table, we know that

$$\mathcal{F}\left[\frac{2c}{x^2+c^2}\right] = \mathrm{e}^{-c|\omega|} \quad \text{for } c > 0.$$

We apply the convolution theorem.

$$\begin{aligned}
\mathcal{F}[f(x)] &= \mathcal{F}\left[\frac{1}{8}\frac{2}{x^2+1}\frac{4}{x^2+4}\right] \\
&= \frac{1}{8}\left(\int_{-\infty}^{\infty} \mathrm{e}^{-|\eta|}\,\mathrm{e}^{-2|\omega-\eta|}\,\mathrm{d}\eta\right) \\
&= \frac{1}{8}\left(\int_{-\infty}^{0} \mathrm{e}^{\eta}\,\mathrm{e}^{-2|\omega-\eta|}\,\mathrm{d}\eta + \int_{0}^{\infty} \mathrm{e}^{-\eta}\,\mathrm{e}^{-2|\omega-\eta|}\,\mathrm{d}\eta\right)
\end{aligned}$$

First consider the case $\omega > 0$.

$$\begin{aligned}
\mathcal{F}[f(x)] &= \frac{1}{8}\left(\int_{-\infty}^{0} \mathrm{e}^{-2\omega+3\eta}\,\mathrm{d}\eta + \int_{0}^{\omega} \mathrm{e}^{-2\omega+\eta}\,\mathrm{d}\eta + \int_{\omega}^{\infty} \mathrm{e}^{2\omega-3\eta}\,\mathrm{d}\eta\right) \\
&= \frac{1}{8}\left(\frac{1}{3}\,\mathrm{e}^{-2\omega} + \mathrm{e}^{-\omega} - \mathrm{e}^{-2\omega} + \frac{1}{3}\,\mathrm{e}^{-\omega}\right) \\
&= \frac{1}{6}\,\mathrm{e}^{-\omega} - \frac{1}{12}\,\mathrm{e}^{-2\omega}
\end{aligned}$$

Now consider the case $\omega < 0$.

$$
\begin{aligned}
\mathcal{F}[f(x)] &= \frac{1}{8}\left(\int_{-\infty}^{\omega} \mathrm{e}^{-2\omega+3\eta}\,\mathrm{d}\eta + \int_{\omega}^{0} \mathrm{e}^{2\omega-\eta}\,\mathrm{d}\eta + \int_{0}^{\infty} \mathrm{e}^{2\omega-3\eta}\,\mathrm{d}\eta\right) \\
&= \frac{1}{8}\left(\frac{1}{3}\,\mathrm{e}^{\omega} - \mathrm{e}^{2\omega} + \mathrm{e}^{\omega} + \frac{1}{3}\,\mathrm{e}^{2\omega}\right) \\
&= \frac{1}{6}\,\mathrm{e}^{\omega} - \frac{1}{12}\,\mathrm{e}^{2\omega}
\end{aligned}
$$

We collect the result for positive and negative ω.

$$
\boxed{\mathcal{F}[f(x)] = \frac{1}{6}\,\mathrm{e}^{-|\omega|} - \frac{1}{12}\,\mathrm{e}^{-2|\omega|}}
$$

A better way to find the Fourier transform of

$$
f(x) = \frac{1}{x^4 + 5x^2 + 4}
$$

is to first expand the function in partial fractions.

$$
f(x) = \frac{1/3}{x^2+1} - \frac{1/3}{x^2+4}
$$

$$
\begin{aligned}
\mathcal{F}[f(x)] &= \frac{1}{6}\mathcal{F}\left[\frac{2}{x^2+1}\right] - \frac{1}{12}\mathcal{F}\left[\frac{4}{x^2+4}\right] \\
&= \frac{1}{6}\,\mathrm{e}^{-|\omega|} - \frac{1}{12}\,\mathrm{e}^{-2|\omega|}
\end{aligned}
$$

34.4.4 Parseval's Theorem.

Recall Parseval's theorem for Fourier series. If $f(x)$ is a complex valued function with the Fourier series $\sum_{n=-\infty}^{\infty} c_n \, \mathrm{e}^{\mathrm{i}nx}$ then

$$2\pi \sum_{n=-\infty}^{\infty} |c_n|^2 = \int_{-\pi}^{\pi} |f(x)|^2 \, \mathrm{d}x.$$

Analogous to this result is Parseval's theorem for Fourier transforms.

Let $f(x)$ be a complex valued function that is both absolutely integrable and square integrable.

$$\int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x < \infty$$

The Fourier transform of $\overline{f(-x)}$ is $\overline{\hat{f}(\omega)}$.

$$\begin{aligned}
\mathcal{F}\left[\overline{f(-x)}\right] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{f(-x)} \, \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x \\
&= -\frac{1}{2\pi} \int_{\infty}^{-\infty} \overline{f(x)} \, \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{f(x) \, \mathrm{e}^{-\mathrm{i}\omega x}} \, \mathrm{d}x \\
&= \overline{\hat{f}(\omega)}
\end{aligned}$$

We apply the convolution theorem.

$$\mathcal{F}^{-1}[2\pi \hat{f}(\omega) \overline{\hat{f}(\omega)}] = \int_{-\infty}^{\infty} f(\xi) \overline{f(-(x-\xi))} \, \mathrm{d}\xi$$

$$\int_{-\infty}^{\infty} 2\pi \hat{f}(\omega) \overline{\hat{f}(\omega)} \, \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}\omega = \int_{-\infty}^{\infty} f(\xi) \overline{f(\xi - x)} \, \mathrm{d}\xi$$

We set $x = 0$.

$$2\pi \int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{f}(\omega)}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} f(\xi)\overline{f(\xi)}\,\mathrm{d}\xi$$

$$\boxed{2\pi \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,\mathrm{d}\omega = \int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x}$$

This is known as **Parseval's theorem**.

34.4.5 Shift Property.

The Fourier transform of $f(x+c)$ is

$$\begin{aligned}\mathcal{F}[f(x+c)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x+c)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}\omega(x-c)}\,\mathrm{d}x\end{aligned}$$

$$\boxed{\mathcal{F}[f(x+c)] = \mathrm{e}^{\mathrm{i}\omega c}\hat{f}(\omega)}$$

The inverse Fourier transform of $\hat{f}(\omega+c)$ is

$$\begin{aligned}\mathcal{F}^{-1}[\hat{f}(\omega+c)] &= \int_{-\infty}^{\infty} \hat{f}(\omega+c)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} \hat{f}(\omega)\,\mathrm{e}^{\mathrm{i}(\omega-c)x}\,\mathrm{d}\omega\end{aligned}$$

$$\boxed{\mathcal{F}^{-1}[\hat{f}(\omega+c)] = \mathrm{e}^{-\mathrm{i}cx}f(x)}$$

34.4.6 Fourier Transform of x f(x).

The Fourier transform of $xf(x)$ is

$$\begin{aligned}
\mathcal{F}[xf(x)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} x f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{i} f(x)\frac{\partial}{\partial\omega}(\,\mathrm{e}^{-\mathrm{i}\omega x})\,\mathrm{d}x \\
&= \mathrm{i}\frac{\partial}{\partial\omega}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x\right)
\end{aligned}$$

$$\boxed{\mathcal{F}[xf(x)] = \mathrm{i}\frac{\partial \hat{f}}{\partial\omega}.}$$

Similarly, you can show that

$$\boxed{\mathcal{F}[x^n f(x)] = (i)^n \frac{\partial^n \hat{f}}{\partial\omega^n}.}$$

34.5 Solving Differential Equations with the Fourier Transform

The Fourier transform is useful in solving some differential equations on the domain $(-\infty \ldots \infty)$ with homogeneous boundary conditions at infinity. We take the Fourier transform of the differential equation $L[y] = f$ and solve for $\hat{y}$. We take the inverse transform to determine the solution y. Note that this process is only applicable if the Fourier transform of y exists. Hence the requirement for homogeneous boundary conditions at infinity.

We will use the table of Fourier transforms in the appendix in solving the examples in this section.

Example 34.5.1 Consider the problem

$$y'' - y = \mathrm{e}^{-\alpha|x|}, \qquad y(\pm\infty) = 0, \qquad \alpha > 0, \alpha \neq 1.$$

We take the Fourier transform of this equation.

$$-\omega^2 \hat{y}(\omega) - \hat{y}(\omega) = \frac{\alpha/\pi}{\omega^2 + \alpha^2}$$

We take the inverse Fourier transform to determine the solution.

$$\begin{aligned}
\hat{y}(\omega) &= \frac{-\alpha/\pi}{(\omega^2 + \alpha^2)(\omega^2 + 1)} \\
&= \frac{-\alpha}{\pi} \frac{1}{\alpha^2 - 1} \left(\frac{1}{\omega^2 + 1} - \frac{1}{\omega^2 + \alpha^2} \right) \\
&= \frac{1}{\alpha^2 - 1} \left(\frac{\alpha/\pi}{\omega^2 + \alpha^2} - \alpha \frac{1/\pi}{\omega^2 + 1} \right)
\end{aligned}$$

$$\boxed{y(x) = \frac{\mathrm{e}^{-\alpha|x|} - \alpha\,\mathrm{e}^{-|x|}}{\alpha^2 - 1}}$$

Example 34.5.2 Consider the Green function problem

$$G'' - G = \delta(x - \xi), \qquad y(\pm\infty) = 0.$$

We take the Fourier transform of this equation.

$$\begin{aligned}
-\omega^2 \hat{G} - \hat{G} &= \mathcal{F}[\delta(x - \xi)] \\
\hat{G} &= -\frac{1}{\omega^2 + 1} \mathcal{F}[\delta(x - \xi)]
\end{aligned}$$

We use the Table of Fourier transforms.

$$\hat{G} = -\pi \mathcal{F}\left[\mathrm{e}^{-|x|}\right] \mathcal{F}[\delta(x - \xi)]$$

We use the convolution theorem to do the inversion.

$$
\begin{gathered}
G=-\pi \frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathrm{e}^{-|x-\eta|} \delta(\eta-\xi) \mathrm{d} \eta \\
G(x \mid \xi)=-\frac{1}{2} \mathrm{e}^{|x-\xi|}
\end{gathered}
$$

The inhomogeneous differential equation

$$
y^{\prime \prime}-y=f(x), \qquad y(\pm \infty)=0,
$$

has the solution

$$
\boxed{y=-\frac{1}{2} \int_{-\infty}^{\infty} f(\xi) \mathrm{e}^{-|x-\xi|} \mathrm{d} \xi .}
$$

When solving the differential equation $L[y]=f$ with the Fourier transform, it is quite common to use the convolution theorem. With this approach we have no need to compute the Fourier transform of the right side. We merely *denote* it as $\mathcal{F}[f]$ until we use f in the convolution integral.

34.6 The Fourier Cosine and Sine Transform

34.6.1 The Fourier Cosine Transform

Suppose $f(x)$ is an even function. In this case the Fourier transform of $f(x)$ coincides with the *Fourier cosine transform* of $f(x)$.

$$\begin{aligned}
\mathcal{F}[f(x)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)(\cos(\omega x) - \mathrm{i}\sin(\omega x))\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{0}^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x
\end{aligned}$$

The Fourier cosine transform is defined:

$$\mathcal{F}_c[f(x)] = \hat{f}_c(\omega) = \frac{1}{\pi}\int_0^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x.$$

Note that $\hat{f}_c(\omega)$ is an even function. The inverse Fourier cosine transform is

$$\begin{aligned}
\mathcal{F}_c^{-1}[\hat{f}_c(\omega)] &= \int_{-\infty}^{\infty} \hat{f}_c(\omega)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} \hat{f}_c(\omega)(\cos(\omega x) + \mathrm{i}\sin(\omega x))\,\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} \hat{f}_c(\omega)\cos(\omega x)\,\mathrm{d}\omega \\
&= 2\int_{0}^{\infty} \hat{f}_c(\omega)\cos(\omega x)\,\mathrm{d}\omega.
\end{aligned}$$

Thus we have the Fourier cosine transform pair

$$f(x) = \mathcal{F}_c^{-1}[\hat{f}_c(\omega)] = 2\int_0^\infty \hat{f}_c(\omega)\cos(\omega x)\,\mathrm{d}\omega, \qquad \hat{f}_c(\omega) = \mathcal{F}_c[f(x)] = \frac{1}{\pi}\int_0^\infty f(x)\cos(\omega x)\,\mathrm{d}x.$$

34.6.2 The Fourier Sine Transform

Suppose $f(x)$ is an odd function. In this case the Fourier transform of $f(x)$ coincides with the *Fourier sine transform* of $f(x)$.

$$\begin{aligned}
\mathcal{F}[f(x)] &= \frac{1}{2\pi}\int_{-\infty}^\infty f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty f(x)(\cos(\omega x) - \mathrm{i}\sin(\omega x))\,\mathrm{d}x \\
&= -\frac{\mathrm{i}}{\pi}\int_0^\infty f(x)\sin(\omega x)\,\mathrm{d}x
\end{aligned}$$

Note that $\hat{f}(\omega) = \mathcal{F}[f(x)]$ is an odd function of ω. The inverse Fourier transform of $\hat{f}(\omega)$ is

$$\begin{aligned}
\mathcal{F}^{-1}[\hat{f}(\omega)] &= \int_{-\infty}^\infty \hat{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega \\
&= 2\mathrm{i}\int_0^\infty \hat{f}(\omega)\sin(\omega x)\,\mathrm{d}\omega.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
f(x) &= 2\mathrm{i}\int_0^\infty \left(-\frac{\mathrm{i}}{\pi}\int_0^\infty f(x)\sin(\omega x)\,\mathrm{d}x\right)\sin(\omega x)\,\mathrm{d}\omega \\
&= 2\int_0^\infty \left(\frac{1}{\pi}\int_0^\infty f(x)\sin(\omega x)\,\mathrm{d}x\right)\sin(\omega x)\,\mathrm{d}\omega.
\end{aligned}$$

This gives us the Fourier sine transform pair

$$f(x) = \mathcal{F}_s^{-1}[\hat{f}_s(\omega)] = 2\int_0^\infty \hat{f}_s(\omega)\sin(\omega x)\,\mathrm{d}\omega, \qquad \hat{f}_s(\omega) = \mathcal{F}_s[f(x)] = \frac{1}{\pi}\int_0^\infty f(x)\sin(\omega x)\,\mathrm{d}x.$$

Result 34.6.1 The Fourier cosine transform pair is defined:

$$f(x) = \mathcal{F}_c^{-1}[\hat{f}_c(\omega)] = 2\int_0^\infty \hat{f}_c(\omega)\cos(\omega x)\,\mathrm{d}\omega$$
$$\hat{f}_c(\omega) = \mathcal{F}_c[f(x)] = \frac{1}{\pi}\int_0^\infty f(x)\cos(\omega x)\,\mathrm{d}x$$

The Fourier sine transform pair is defined:

$$f(x) = \mathcal{F}_s^{-1}[\hat{f}_s(\omega)] = 2\int_0^\infty \hat{f}_s(\omega)\sin(\omega x)\,\mathrm{d}\omega$$
$$\hat{f}_s(\omega) = \mathcal{F}_s[f(x)] = \frac{1}{\pi}\int_0^\infty f(x)\sin(\omega x)\,\mathrm{d}x$$

34.7 Properties of the Fourier Cosine and Sine Transform

34.7.1 Transforms of Derivatives

Cosine Transform. Using integration by parts we can find the Fourier cosine transform of derivatives. Let y be a function for which the Fourier cosine transform of y and its first and second derivatives exists. Further assume that y and y' vanish at infinity. We calculate the transforms of the first and second derivatives.

$$
\begin{aligned}
\mathcal{F}_c[y'] &= \frac{1}{\pi}\int_0^\infty y' \cos(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[y\cos(\omega x)\right]_0^\infty + \frac{\omega}{\pi}\int_0^\infty y\sin(\omega x)\,\mathrm{d}x \\
&= \omega \hat{y}_c(\omega) - \frac{1}{\pi} y(0) \\
\mathcal{F}_c[y''] &= \frac{1}{\pi}\int_0^\infty y'' \cos(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[y'\cos(\omega x)\right]_0^\infty + \frac{\omega}{\pi}\int_0^\infty y'\sin(\omega x)\,\mathrm{d}x \\
&= -\frac{1}{\pi}y'(0) + \frac{\omega}{\pi}\left[y\sin(\omega x)\right]_0^\infty - \frac{\omega^2}{\pi}\int_0^\infty y\cos(\omega x)\,\mathrm{d}x \\
&= -\omega^2 \hat{f}_c(\omega) - \frac{1}{\pi}y'(0)
\end{aligned}
$$

Sine Transform. You can show, (see Exercise 34.3), that the Fourier sine transform of the first and second derivatives are

$$
\begin{aligned}
\mathcal{F}_s[y'] &= -\omega \hat{f}_c(\omega) \\
\mathcal{F}_s[y''] &= -\omega^2 \hat{y}_c(\omega) + \frac{\omega}{\pi} y(0).
\end{aligned}
$$

34.7.2 Convolution Theorems

Cosine Transform of a Product. Consider the Fourier cosine transform of a product of functions. Let $f(x)$ and $g(x)$ be two functions defined for $x \geq 0$. Let $\mathcal{F}_c[f(x)] = \hat{f}_c(\omega)$, and $\mathcal{F}_c[g(x)] = \hat{g}_c(\omega)$.

$$\begin{aligned}
\mathcal{F}_c[f(x)g(x)] &= \frac{1}{\pi}\int_0^\infty f(x)g(x)\cos(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\infty \left(2\int_0^\infty \hat{f}_c(\eta)\cos(\eta x)\,\mathrm{d}\eta\right) g(x)\cos(\omega x)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\infty \int_0^\infty \hat{f}_c(\eta) g(x)\cos(\eta x)\cos(\omega x)\,\mathrm{d}x\,\mathrm{d}\eta
\end{aligned}$$

We use the identity $\cos a \cos b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$.

$$\begin{aligned}
&= \frac{1}{\pi}\int_0^\infty \int_0^\infty \hat{f}_c(\eta) g(x)\big(\cos((\omega-\eta)x) + \cos((\omega+\eta)x)\big)\,\mathrm{d}x\,\mathrm{d}\eta \\
&= \int_0^\infty \hat{f}_c(\eta)\left[\frac{1}{\pi}\int_0^\infty g(x)\cos((\omega-\eta)x)\,\mathrm{d}x + \frac{1}{\pi}\int_0^\infty g(x)\cos((\omega+\eta)x)\,\mathrm{d}x\right]\mathrm{d}\eta \\
&= \int_0^\infty \hat{f}_c(\eta)\big(\hat{g}_c(\omega-\eta) + \hat{g}_c(\omega+\eta)\big)\,\mathrm{d}\eta
\end{aligned}$$

$\hat{g}_c(\omega)$ is an even function. If we have only defined $\hat{g}_c(\omega)$ for positive argument, then $\hat{g}_c(\omega) = \hat{g}_c(|\omega|)$.

$$= \int_0^\infty \hat{f}_c(\eta)\big(\hat{g}_c(|\omega-\eta|) + \hat{g}_c(\omega+\eta)\big)\,\mathrm{d}\eta$$

Inverse Cosine Transform of a Product. Now consider the inverse Fourier cosine transform of a product of functions. Let $\mathcal{F}_c[f(x)] = \hat{f}_c(\omega)$, and $\mathcal{F}_c[g(x)] = \hat{g}_c(\omega)$.

$$\begin{aligned}
\mathcal{F}_c^{-1}[\hat{f}_c(\omega)\hat{g}_c(\omega)] &= 2\int_0^\infty \hat{f}_c(\omega)\hat{g}_c(\omega)\cos(\omega x)\,\mathrm{d}\omega \\
&= 2\int_0^\infty \left(\frac{1}{\pi}\int_0^\infty f(\xi)\cos(\omega\xi)\,\mathrm{d}\xi\right)\hat{g}_c(\omega)\cos(\omega x)\,\mathrm{d}\omega \\
&= \frac{2}{\pi}\int_0^\infty \int_0^\infty f(\xi)\hat{g}_c(\omega)\cos(\omega\xi)\cos(\omega x)\,\mathrm{d}\omega\,\mathrm{d}\xi \\
&= \frac{1}{\pi}\int_0^\infty \int_0^\infty f(\xi)\hat{g}_c(\omega)\big(\cos(\omega(x-\xi)) + \cos(\omega(x+\xi))\big)\,\mathrm{d}\omega\,\mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_0^\infty f(\xi)\left(2\int_0^\infty \hat{g}_c(\omega)\cos(\omega(x-\xi))\,\mathrm{d}\omega + 2\int_0^\infty \hat{g}_c(\omega)\cos(\omega(x+\xi))\,\mathrm{d}\omega\right)\mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_0^\infty f(\xi)\big(g(|x-\xi|) + g(x+\xi)\big)\,\mathrm{d}\xi
\end{aligned}$$

Sine Transform of a Product. You can show, (see Exercise 34.5), that the Fourier sine transform of a product of functions is

$$\mathcal{F}_s[f(x)g(x)] = \int_0^\infty \hat{f}_s(\eta)\big(\hat{g}_c(|\omega-\eta|) - \hat{g}_c(\omega+\eta)\big)\,\mathrm{d}\eta.$$

Inverse Sine Transform of a Product. You can also show, (see Exercise 34.6), that the inverse Fourier sine transform of a product of functions is

$$\mathcal{F}_s^{-1}[\hat{f}_s(\omega)\hat{g}_c(\omega)] = \frac{1}{2\pi}\int_0^\infty f(\xi)\big(g(|x-\xi|) - g(x+\xi)\big)\,\mathrm{d}\xi.$$

Result 34.7.1 The Fourier cosine and sine transform convolution theorems are

$$\mathcal{F}_c[f(x)g(x)] = \int_0^\infty \hat{f}_c(\eta)\left[\hat{g}_c(|\omega-\eta|) + \hat{g}_c(\omega+\eta)\right] \mathrm{d}\eta$$
$$\mathcal{F}_c^{-1}[\hat{f}_c(\omega)\hat{g}_c(\omega)] = \frac{1}{2\pi}\int_0^\infty f(\xi)\big(g(|x-\xi|) + g(x+\xi)\big) \mathrm{d}\xi$$
$$\mathcal{F}_s[f(x)g(x)] = \int_0^\infty \hat{f}_s(\eta)\big(\hat{g}_c(|\omega-\eta|) - \hat{g}_c(\omega+\eta)\big) \mathrm{d}\eta$$
$$\mathcal{F}_s^{-1}[\hat{f}_s(\omega)\hat{g}_c(\omega)] = \frac{1}{2\pi}\int_0^\infty f(\xi)\big(g(|x-\xi|) - g(x+\xi)\big) \mathrm{d}\xi$$

34.7.3 Cosine and Sine Transform in Terms of the Fourier Transform

We can express the Fourier cosine and sine transform in terms of the Fourier transform. First consider the Fourier cosine transform. Let $f(x)$ be an even function.

$$\mathcal{F}_c[f(x)] = \frac{1}{\pi}\int_0^\infty f(x)\cos(\omega x)\,\mathrm{d}x$$

We extend the domain integration because the integrand is even.

$$= \frac{1}{2\pi}\int_{-\infty}^\infty f(x)\cos(\omega x)\,\mathrm{d}x$$

Note that $\int_{-\infty}^\infty f(x)\sin(\omega x)\,\mathrm{d}x = 0$ because the integrand is odd.

$$= \frac{1}{2\pi}\int_{-\infty}^\infty f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x$$
$$= \mathcal{F}[f(x)]$$

$$\mathcal{F}_c[f(x)] = \mathcal{F}[f(x)], \quad \text{for even } f(x).$$

For general $f(x)$, use the even extension, $f(|x|)$ to write the result.

$$\mathcal{F}_c[f(x)] = \mathcal{F}[f(|x|)]$$

There is an analogous result for the inverse Fourier cosine transform.

$$\mathcal{F}_c^{-1}\left[\hat{f}(\omega)\right] = \mathcal{F}^{-1}\left[\hat{f}(|\omega|)\right]$$

For the sine series, we have

$$\mathcal{F}_s[f(x)] = \mathrm{i}\mathcal{F}\left[\operatorname{sign}(x) f(|x|)\right] \quad \mathcal{F}_s^{-1}\left[\hat{f}(\omega)\right] = -\mathrm{i}\mathcal{F}^{-1}\left[\operatorname{sign}(\omega)\hat{f}(|\omega|)\right]$$

Result 34.7.2 The results:

$$\mathcal{F}_c[f(x)] = \mathcal{F}[f(|x|)] \quad \mathcal{F}_c^{-1}\left[\hat{f}(\omega)\right] = \mathcal{F}^{-1}\left[\hat{f}(|\omega|)\right]$$

$$\mathcal{F}_s[f(x)] = \mathrm{i}\mathcal{F}[\operatorname{sign}(x) f(|x|)] \quad \mathcal{F}_s^{-1}\left[\hat{f}(\omega)\right] = -\mathrm{i}\mathcal{F}^{-1}\left[\operatorname{sign}(\omega)\hat{f}(|\omega|)\right]$$

allow us to evaluate Fourier cosine and sine transforms in terms of the Fourier transform. This enables us to use contour integration methods to do the integrals.

34.8 Solving Differential Equations with the Fourier Cosine and Sine Transforms

Example 34.8.1 Consider the problem

$$y'' - y = 0, \quad y(0) = 1, \quad y(\infty) = 0.$$

Since the initial condition is $y(0) = 1$ and the sine transform of y'' is $-\omega^2 \hat{y}_c(\omega) + \frac{\omega}{\pi} y(0)$ we take the Fourier sine transform of both sides of the differential equation.

$$-\omega^2 \hat{y}_c(\omega) + \frac{\omega}{\pi} y(0) - \hat{y}_c(\omega) = 0$$
$$-(\omega^2 + 1)\hat{y}_c(\omega) = -\frac{\omega}{\pi}$$
$$\hat{y}_c(\omega) = \frac{\omega}{\pi(\omega^2 + 1)}$$

We use the table of Fourier Sine transforms.

$$\boxed{y = \mathrm{e}^{-x}}$$

Example 34.8.2 Consider the problem

$$y'' - y = \mathrm{e}^{-2x}, \quad y'(0) = 0, \quad y(\infty) = 0.$$

Since the initial condition is $y'(0) = 0$, we take the Fourier cosine transform of the differential equation. From the table of cosine transforms, $\mathcal{F}_c[\mathrm{e}^{-2x}] = 2/(\pi(\omega^2 + 4))$.

$$-\omega^2 \hat{y}_c(\omega) - \frac{1}{\pi} y'(0) - \hat{y}_c(\omega) = \frac{2}{\pi(\omega^2 + 4)}$$

$$\begin{aligned}
\hat{y}_c(\omega) &= -\frac{2}{\pi(\omega^2 + 4)(\omega^2 + 1)} \\
&= \frac{-2}{\pi}\left(\frac{1/3}{\omega^2 + 1} - \frac{1/3}{\omega^2 + 4}\right) \\
&= \frac{1}{3}\frac{2/\pi}{\omega^2 + 4} - \frac{2}{3}\frac{1/\pi}{\omega^2 + 1}
\end{aligned}$$

$$\boxed{y = \frac{1}{3}\mathrm{e}^{-2x} - \frac{2}{3}\mathrm{e}^{-x}}$$

34.9 Exercises

Exercise 34.1
Show that

$$H(x+c) - H(x-c) = \frac{\sin(c\omega)}{\pi\omega}.$$

Exercise 34.2
Using contour integration, find the Fourier transform of

$$f(x) = \frac{1}{x^2 + c^2},$$

where $\Re(c) \neq 0$

Exercise 34.3
Find the Fourier sine transforms of $y'(x)$ and $y''(x)$.

Exercise 34.4
Prove the following identities.

1. $\mathcal{F}[f(x-a)] = \mathrm{e}^{-\mathrm{i}\omega a} \hat{f}(\omega)$
2. $\mathcal{F}[f(ax)] = \frac{1}{|a|} \hat{f}\left(\frac{\omega}{a}\right)$

Exercise 34.5
Show that

$$\mathcal{F}_s[f(x)g(x)] = \int_0^\infty \hat{f}_s(\eta) \big(\hat{g}_c(|\omega - \eta|) - \hat{g}_c(\omega + \eta)\big) \, \mathrm{d}\eta.$$

Exercise 34.6
Show that

$$\mathcal{F}_s^{-1}[\hat{f}_s(\omega)\hat{g}_c(\omega)] = \frac{1}{2\pi}\int_0^\infty f(\xi)\big(g(|x-\xi|) - g(x+\xi)\big)\,\mathrm{d}\xi.$$

Exercise 34.7
Let $\hat{f}_c(\omega) = \mathcal{F}_c[f(x)]$, $\hat{f}_c(\omega) = \mathcal{F}_s[f(x)]$, and assume the cosine and sine transforms of $xf(x)$ exist. Express $\mathcal{F}_c[xf(x)]$ and $\mathcal{F}_s[xf(x)]$ in terms of $\hat{f}_c(\omega)$ and $\hat{f}_c(\omega)$.

Exercise 34.8
Solve the problem

$$y'' - y = \mathrm{e}^{-2x}, \qquad y(0) = 1, \qquad y(\infty) = 0,$$

using the Fourier sine transform.

Exercise 34.9
Show that

$$\mathcal{F}_s[f(x)] = \mathrm{i}\mathcal{F}[\operatorname{sign}(x) f(|x|)] \quad \mathcal{F}_s^{-1}\left[\hat{f}(\omega)\right] = -\mathrm{i}\mathcal{F}^{-1}\left[\operatorname{sign}(\omega)\hat{f}(|\omega|)\right]$$

Exercise 34.10
Let $\hat{f}_c(\omega) = \mathcal{F}_c[f(x)]$ and $\hat{f}_c(\omega) = \mathcal{F}_s[f(x)]$. Show that

1. $\mathcal{F}_c[xf(x)] = \frac{\partial}{\partial\omega}\hat{f}_c(\omega)$
2. $\mathcal{F}_s[xf(x)] = -\frac{\partial}{\partial\omega}\hat{f}_c(\omega)$

3. $\mathcal{F}_c[f(cx)] = \frac{1}{c}\hat{f}_c\left(\frac{\omega}{c}\right)$ for $c > 0$

4. $\mathcal{F}_s[f(cx)] = \frac{1}{c}\hat{f}_c\left(\frac{\omega}{c}\right)$ for $c > 0$.

Exercise 34.11
Solve the integral equation,

$$\int_{-\infty}^{\infty} u(\xi)\,\mathrm{e}^{-a(x-\xi)^2}\,\mathrm{d}\xi = \mathrm{e}^{-bx^2},$$

where $a, b > 0$, $a \neq b$, with the Fourier transform.

Exercise 34.12
Evaluate

$$\frac{1}{\pi}\int_0^{\infty} \frac{1}{x}\,\mathrm{e}^{-cx}\sin(\omega x)\,\mathrm{d}x,$$

where ω is a positive, real number and $\Re(c) > 0$.

Exercise 34.13
Use the Fourier transform to solve the equation

$$y'' - a^2 y = \mathrm{e}^{-a|x|}$$

on the domain $-\infty < x < \infty$ with boundary conditions $y(\pm\infty) = 0$.

Exercise 34.14

1. Use the cosine transform to solve

$$y'' - a^2 y = 0 \text{ on } x \geq 0 \text{ with } y'(0) = b,\ y(\infty) = 0.$$

2. Use the cosine transform to show that the Green function for the above with $b = 0$ is

$$G(x, \xi) = -\frac{1}{2a} \,\mathrm{e}^{-a|x-\xi|} - \frac{1}{2a} \,\mathrm{e}^{-a(x-\xi)}.$$

Exercise 34.15

1. Use the sine transform to solve

$$y'' - a^2 y = 0 \text{ on } x \geq 0 \text{ with } y(0) = b, \; y(\infty) = 0.$$

2. Try using the Laplace transform on this problem. Why isn't it as convenient as the Fourier transform?

3. Use the sine transform to show that the Green function for the above with $b = 0$ is

$$g(x; \xi) = \frac{1}{2a} \left(\mathrm{e}^{-a(x-\xi)} - \mathrm{e}^{-a|x+\xi|} \right)$$

Exercise 34.16

1. Find the Green function which solves the equation

$$y'' + 2\mu y' + (\beta^2 + \mu^2) y = \delta(x - \xi), \qquad \mu > 0, \beta > 0,$$

   in the range $-\infty < x < \infty$ with boundary conditions $y(-\infty) = y(\infty) = 0$.

2. Use this Green's function to show that the solution of

$$y'' + 2\mu y' + (\beta^2 + \mu^2) y = g(x), \quad \mu > 0, \beta > 0, \qquad y(-\infty) = y(\infty) = 0,$$

   with $g(\pm\infty) = 0$ in the limit as $\mu \to 0$ is

$$y = \frac{1}{\beta} \int_{-\infty}^{x} g(\xi) \sin[\beta(x - \xi)] d\xi.$$

   You may assume that the interchange of limits is permitted.

Exercise 34.17
Using Fourier transforms, find the solution $u(x)$ to the integral equation

$$\int_{-\infty}^{\infty} \frac{u(\xi)}{[(x-\xi)^2 + a^2]} \, \mathrm{d}\xi = \frac{1}{x^2 + b^2} \qquad 0 < a < b.$$

Exercise 34.18
The Fourer cosine transform is defined by

$$F_c(\omega) = \frac{1}{\pi} \int_0^\infty f(x) \cos(\omega x) \, \mathrm{d}x.$$

1. From the Fourier theorem show that the inverse cosine transform is given by

$$f(x) = 2 \int_0^\infty F_c(\omega) \cos(\omega x) \, \mathrm{d}\omega.$$

2. Show that the cosine transform of $f''(x)$ is

$$-\omega^2 F_c(\omega) - \frac{f'(0)}{\pi}.$$

3. Use the cosine transform to solve

$$y'' - a^2 y = 0 \quad \text{on } x > 0 \quad \text{with } y'(0) = b, \ y(\infty) = 0.$$

Exercise 34.19
The Fourier sine transform is defined by

$$F_s(\omega) = \frac{1}{\pi} \int_0^\infty f(x) \sin(\omega x) \, \mathrm{d}x.$$

1. Show that the inverse sine transform is given by

$$f(x) = 2\int_0^\infty F_s(\omega)\sin(\omega x)\,\mathrm{d}\omega.$$

2. Show that the sine transform of $f''(x)$ is

$$\frac{\omega}{\pi}f(0) - \omega^2 F_s(\omega).$$

3. Use this property to solve the equation

$$y'' - a^2 y = 0 \quad \text{on } x > 0 \quad \text{with } y(0) = b,\ y(\infty) = 0.$$

4. Try using the Laplace transform on this problem. Why isn't it as convenient as the Fourier transform?

Exercise 34.20
Show that

$$\mathcal{F}[f(x)] = \frac{1}{2}\left(\mathcal{F}_c[f(x) + f(-x)] - \mathrm{i}\mathcal{F}_s[f(x) - f(-x)]\right)$$

where $\mathcal{F}$, $\mathcal{F}_c$ and $\mathcal{F}_s$ are respectively the Fourier transform, Fourier cosine transform and Fourier sine transform.

Exercise 34.21
Find $u(x)$ as the solution to the integral equation:

$$\int_{-\infty}^{\infty} \frac{u(\xi)}{(x-\xi)^2 + a^2}\,\mathrm{d}\xi = \frac{1}{x^2 + b^2}, \qquad 0 < a < b.$$

Use Fourier transforms and the inverse transform. Justify the choice of any contours used in the complex plane.

34.10 Hints

Hint 34.1

$$H(x+c) - H(x-c) = \begin{cases} 1 & \text{for } |x| < c, \\ 0 & \text{for } |x| > c \end{cases}$$

Hint 34.2
Consider the two cases $\Re(\omega) < 0$ and $\Re(\omega) > 0$, closing the path of integration with a semi-circle in the lower or upper half plane.

Hint 34.3

Hint 34.4

Hint 34.5

Hint 34.6

Hint 34.7

Hint 34.8

Hint 34.9

Hint 34.10

Hint 34.11
The left side is the convolution of $u(x)$ and e^{-ax^2}.

Hint 34.12

Hint 34.13

Hint 34.14

Hint 34.15

Hint 34.16

Hint 34.17

Hint 34.18

Hint 34.19

Hint 34.20

Hint 34.21

34.11 Solutions

Solution 34.1

$$
\begin{aligned}
\mathcal{F}[H(x+c) - H(x-c)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (H(x+c) - H(x-c))\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-c}^{c} \mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\frac{\mathrm{e}^{-\mathrm{i}\omega x}}{-\mathrm{i}\omega}\right]_{-c}^{c} \\
&= \frac{1}{2\pi}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega c}}{-\mathrm{i}\omega} - \frac{\mathrm{e}^{\mathrm{i}\omega c}}{-\mathrm{i}\omega}\right)
\end{aligned}
$$

$$
\boxed{\mathcal{F}[H(x+c) - H(x-c)] = \frac{\sin(c\omega)}{\pi\omega}}
$$

Solution 34.2

$$
\begin{aligned}
\mathcal{F}\left[\frac{1}{x^2+c^2}\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+c^2}\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega x}}{(x-\mathrm{i}c)(x+\mathrm{i}c)}\,\mathrm{d}x
\end{aligned}
$$

If $\Re(\omega) < 0$ then we close the path of integration with a semi-circle in the upper half plane.

$$
\mathcal{F}\left[\frac{1}{x^2+c^2}\right] = \frac{1}{2\pi} 2\pi i \operatorname{Res}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega x}}{(x-\mathrm{i}c)(x+\mathrm{i}c)}, x = \mathrm{i}c\right) = \frac{1}{2c}\,\mathrm{e}^{c\omega}
$$

If $\omega > 0$ then we close the path of integration in the lower half plane.

$$\mathcal{F}\left[\frac{1}{x^2+c^2}\right] = -\frac{1}{2\pi} 2\pi i \operatorname{Res}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega x}}{(x-\mathrm{i}c)(x+\mathrm{i}c)}, -\mathrm{i}c\right) = \frac{1}{2c}\,\mathrm{e}^{-c\omega}$$

Thus we have that

$$\boxed{\mathcal{F}\left[\frac{1}{x^2+c^2}\right] = \frac{1}{2c}\,\mathrm{e}^{-c|\omega|}, \quad \text{for } \Re(c) \neq 0.}$$

Solution 34.3

$$\begin{aligned}
\mathcal{F}_s[y'] &= \frac{1}{\pi}\int_0^\infty y' \sin(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\Big[y\sin(\omega x)\Big]_0^\infty - \frac{\omega}{\pi}\int_0^\infty y\cos(\omega x)\,\mathrm{d}x \\
&= -\omega \hat{y}_c(\omega) \\
\mathcal{F}_s[y''] &= \frac{1}{\pi}\int_0^\infty y'' \sin(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\Big[y'\sin(\omega x)\Big]_0^\infty - \frac{\omega}{\pi}\int_0^\infty y'\cos(\omega x)\,\mathrm{d}x \\
&= -\frac{\omega}{\pi}\Big[y\cos(\omega x)\Big]_0^\infty - \frac{\omega^2}{\pi}\int_0^\infty y\sin(\omega x)\,\mathrm{d}x \\
&= -\omega^2 \hat{y}_s(\omega) + \frac{\omega}{\pi} y(0).
\end{aligned}$$

Solution 34.4

1.

$$\begin{aligned}
\mathcal{F}[f(x-a)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-a)\,\mathrm{e}^{-\imath\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\imath\omega(x+a)}\,\mathrm{d}x \\
&= \mathrm{e}^{-\imath\omega a}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\imath\omega x}\,\mathrm{d}x
\end{aligned}$$

$$\boxed{\mathcal{F}[f(x-a)] = \mathrm{e}^{-\imath\omega a}\hat{f}(\omega)}$$

2. If $a > 0$, then

$$\begin{aligned}
\mathcal{F}[f(ax)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(ax)\,\mathrm{e}^{-\imath\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\,\mathrm{e}^{-\imath\omega\xi/a}\frac{1}{a}\,\mathrm{d}\xi \\
&= \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right).
\end{aligned}$$

If $a < 0$, then

$$\begin{aligned}
\mathcal{F}[f(ax)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(ax)\,\mathrm{e}^{-\imath\omega x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{\infty}^{-\infty} \mathrm{e}^{-\imath\omega\xi/a}\frac{1}{a}\,\mathrm{d}\xi \\
&= -\frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right).
\end{aligned}$$

Thus

$$\boxed{\mathcal{F}[f(ax)] = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right).}$$

Solution 34.5

$$\begin{aligned}
\mathcal{F}_s[f(x)g(x)] &= \frac{1}{\pi}\int_0^\infty f(x)g(x)\sin(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\infty \left(2\int_0^\infty \hat{f}_s(\eta)\sin(\eta x)\,\mathrm{d}\eta\right) g(x)\sin(\omega x)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\infty\int_0^\infty \hat{f}_s(\eta)g(x)\sin(\eta x)\sin(\omega x)\,\mathrm{d}x\,\mathrm{d}\eta
\end{aligned}$$

Use the identity, $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$.

$$\begin{aligned}
&= \frac{1}{\pi}\int_0^\infty\int_0^\infty \hat{f}_s(\eta)g(x)\Big[\cos((\omega-\eta)x) - \cos((\omega+\eta)x)\Big]\,\mathrm{d}x\,\mathrm{d}\eta \\
&= \int_0^\infty \hat{f}_s(\eta)\left[\frac{1}{\pi}\int_0^\infty g(x)\cos((\omega-\eta)x)\,\mathrm{d}x - \frac{1}{\pi}\int_0^\infty g(x)\cos((\omega+\eta)x)\,\mathrm{d}x\right]\mathrm{d}\eta
\end{aligned}$$

$$\boxed{\mathcal{F}_s[f(x)g(x)] = \int_0^\infty \hat{f}_s(\eta)\big[G_c(|\omega-\eta|) - G_c(\omega+\eta)\big]\,\mathrm{d}\eta}$$

Solution 34.6

$$
\begin{aligned}
\mathcal{F}_s^{-1}[\hat{f}_s(\omega)G_c(\omega)] &= 2\int_0^\infty \hat{f}_s(\omega)G_c(\omega)\sin(\omega x)\,\mathrm{d}\omega \\
&= 2\int_0^\infty \left(\frac{1}{\pi}\int_0^\infty f(\xi)\sin(\omega\xi)\,\mathrm{d}\xi\right) G_c(\omega)\sin(\omega x)\,\mathrm{d}\omega \\
&= \frac{2}{\pi}\int_0^\infty\int_0^\infty f(\xi)G_c(\omega)\sin(\omega\xi)\sin(\omega x)\,\mathrm{d}\omega\,\mathrm{d}\xi \\
&= \frac{1}{\pi}\int_0^\infty\int_0^\infty f(\xi)G_c(\omega)\Big[\cos(\omega(x-\xi)) - \cos(\omega(x+\xi))\Big]\,\mathrm{d}\omega\,\mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_0^\infty f(\xi)\left[2\int_0^\infty G_c(\omega)\cos(\omega(x-\xi))\,\mathrm{d}\omega - 2\int_0^\infty G_c(\omega)\cos(\omega(x+\xi))\,\mathrm{d}\omega)\right]\,\mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_0^\infty f(\xi)[g(x-\xi) - g(x+\xi)]\,\mathrm{d}\xi
\end{aligned}
$$

$$
\boxed{\mathcal{F}_s^{-1}[\hat{f}_s(\omega)G_c(\omega)] = \frac{1}{2\pi}\int_0^\infty f(\xi)\big[g(|x-\xi|) - g(x+\xi)\big]\,\mathrm{d}\xi}
$$

Solution 34.7

$$\begin{aligned}
\mathcal{F}_c[xf(x)] &= \frac{1}{\pi}\int_0^\infty xf(x)\cos(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\infty f(x)\frac{\partial}{\partial\omega}(\sin(\omega x))\,\mathrm{d}x \\
&= \frac{\partial}{\partial\omega}\frac{1}{\pi}\int_0^\infty f(x)\sin(\omega x)\,\mathrm{d}x \\
&= \frac{\partial}{\partial\omega}\hat{f}_s(\omega)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_s[xf(x)] &= \frac{1}{\pi}\int_0^\infty xf(x)\sin(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\infty f(x)\frac{\partial}{\partial\omega}(-\cos(\omega x))\,\mathrm{d}x \\
&= -\frac{\partial}{\partial\omega}\frac{1}{\pi}\int_0^\infty f(x)\cos(\omega x)\,\mathrm{d}x \\
&= -\frac{\partial}{\partial\omega}\hat{f}_c(\omega)
\end{aligned}$$

Solution 34.8

$$y'' - y = \mathrm{e}^{-2x}, \qquad y(0) = 1, \qquad y(\infty) = 0$$

We take the Fourier sine transform of the differential equation.

$$-\omega^2\hat{y}_s(\omega) + \frac{\omega}{\pi}y(0) - \hat{y}_s(\omega) = \frac{2\omega/\pi}{\omega^2+4}$$

$$\begin{aligned}
\hat{y}_s(\omega) &= -\frac{\omega/\pi}{(\omega^2+4)(\omega^2+1)} + \frac{\omega/\pi}{(\omega^2+1)} \\
&= \frac{\omega/(3\pi)}{\omega^2+4} - \frac{\omega/(3\pi)}{\omega^2+1} + \frac{\omega/\pi}{\omega^2+1} \\
&= \frac{2}{3}\frac{\omega/\pi}{\omega^2+1} + \frac{1}{3}\frac{\omega/\pi}{\omega^2+4}
\end{aligned}$$

$$\boxed{y = \frac{2}{3}\,\mathrm{e}^{-x} + \frac{1}{3}\,\mathrm{e}^{-2x}}$$

Solution 34.9

Consider the Fourier sine transform. Let $f(x)$ be an odd function.

$$\mathcal{F}_s[f(x)] = \frac{1}{\pi}\int_0^\infty f(x)\sin(\omega x)\,\mathrm{d}x$$

Extend the integration because the integrand is even.

$$= \frac{1}{2\pi}\int_{-\infty}^\infty f(x)\sin(\omega x)\,\mathrm{d}x$$

Note that $\int_{-\infty}^\infty f(x)\cos(\omega x)\,\mathrm{d}x = 0$ as the integrand is odd.

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^\infty f(x)\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \mathrm{i}\mathcal{F}[f(x)]
\end{aligned}$$

$$\mathcal{F}_s[f(x)] = \mathrm{i}\mathcal{F}[f(x)], \quad \text{for odd } f(x).$$

For general $f(x)$, use the odd extension, $\operatorname{sign}(x) f(|x|)$ to write the result.

$$\mathcal{F}_s[f(x)] = \mathrm{i}\mathcal{F}[\operatorname{sign}(x) f(|x|)]$$

Now consider the inverse Fourier sine transform. Let $\hat{f}(\omega)$ be an odd function.

$$\mathcal{F}_s^{-1}\left[\hat{f}(\omega)\right] = 2\int_0^\infty \hat{f}(\omega)\sin(\omega x)\,\mathrm{d}\omega$$

Extend the integration because the integrand is even.

$$= \int_{-\infty}^\infty \hat{f}(\omega)\sin(\omega x)\,\mathrm{d}\omega$$

Note that $\int_{-\infty}^\infty \hat{f}(\omega)\cos(\omega x)\,\mathrm{d}\omega = 0$ as the integrand is odd.

$$= \int_{-\infty}^\infty \hat{f}(\omega)(-i)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega$$

$$= -\mathrm{i}\mathcal{F}^{-1}\left[\hat{f}(\omega)\right]$$

$$\mathcal{F}_s^{-1}\left[\hat{f}(\omega)\right] = -\mathrm{i}\mathcal{F}^{-1}\left[\hat{f}(\omega)\right], \quad \text{for odd } \hat{f}(\omega).$$

For general $\hat{f}(\omega)$, use the odd extension, $\operatorname{sign}(\omega)\hat{f}(|\omega|)$ to write the result.

$$\mathcal{F}_s^{-1}\left[\hat{f}(\omega)\right] = -\mathrm{i}\mathcal{F}^{-1}\left[\operatorname{sign}(\omega)\hat{f}(|\omega|)\right]$$

Solution 34.10

$$\begin{aligned}
\mathcal{F}_c[xf(x)] &= \frac{1}{\pi}\int_0^\infty xf(x)\cos(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\infty f(x)\frac{\partial}{\partial\omega}\sin(\omega x)\,\mathrm{d}x \\
&= \frac{\partial}{\partial\omega}\frac{1}{\pi}\int_0^\infty f(x)\sin(\omega x)\,\mathrm{d}x \\
&= \frac{\partial}{\partial\omega}\hat{f}_s(\omega)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_s[xf(x)] &= \frac{1}{\pi}\int_0^\infty xf(x)\sin(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\infty f(x)\frac{\partial}{\partial\omega}(-\cos(\omega x))\,\mathrm{d}x \\
&= -\frac{\partial}{\partial\omega}\frac{1}{\pi}\int_0^\infty f(x)\cos(\omega x)\,\mathrm{d}x \\
&= -\frac{\partial}{\partial\omega}\hat{f}_c(\omega)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_c[f(cx)] &= \frac{1}{\pi}\int_0^\infty f(cx)\cos(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\infty f(\xi)\cos\left(\frac{\omega}{c}\xi\right)\frac{d\xi}{c} \\
&= \frac{1}{c}\hat{f}_c\left(\frac{\omega}{c}\right)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{F}_s[f(cx)] &= \frac{1}{\pi}\int_0^\infty f(cx)\sin(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\infty f(\xi)\sin\left(\frac{\omega}{c}\xi\right)\frac{d\xi}{c} \\
&= \frac{1}{c}\hat{f}_s\left(\frac{\omega}{c}\right)
\end{aligned}
$$

Solution 34.11

$$
\int_{-\infty}^{\infty} u(\xi)\,\mathrm{e}^{-a(x-\xi)^2}\,\mathrm{d}\xi = \mathrm{e}^{-bx^2}
$$

We take the Fourier transform and solve for $U(\omega)$.

$$
\begin{aligned}
2\pi U(\omega)\mathcal{F}\left[\mathrm{e}^{-ax^2}\right] &= \mathcal{F}\left[\mathrm{e}^{-bx^2}\right] \\
2\pi U(\omega)\frac{1}{\sqrt{4\pi a}}\,\mathrm{e}^{-\omega^2/(4a)} &= \frac{1}{\sqrt{4\pi b}}\,\mathrm{e}^{-\omega^2/(4b)} \\
U(\omega) &= \frac{1}{2\pi}\sqrt{\frac{a}{b}}\,\mathrm{e}^{-\omega^2(a-b)/(4ab)}
\end{aligned}
$$

Now we take the inverse Fourier transform.

$$
U(\omega) = \frac{1}{2\pi}\sqrt{\frac{a}{b}}\frac{\sqrt{4\pi ab/(a-b)}}{\sqrt{4\pi ab/(a-b)}}\,\mathrm{e}^{-\omega^2(a-b)/(4ab)}
$$

$$
\boxed{u(x) = \frac{a}{\sqrt{\pi(a-b)}}\,\mathrm{e}^{-abx^2/(a-b)}}
$$

Solution 34.12

$$
\begin{aligned}
I &= \frac{1}{\pi} \int_0^\infty \frac{1}{x} \mathrm{e}^{-cx} \sin(\omega x) \,\mathrm{d}x \\
&= \frac{1}{\pi} \int_0^\infty \left(\int_c^\infty \mathrm{e}^{-zx} \,\mathrm{d}z \right) \sin(\omega x) \,\mathrm{d}x \\
&= \frac{1}{\pi} \int_c^\infty \int_0^\infty \mathrm{e}^{-zx} \sin(\omega x) \,\mathrm{d}x \,\mathrm{d}z \\
&= \frac{1}{\pi} \int_c^\infty \frac{\omega}{z^2 + \omega^2} \,\mathrm{d}z \\
&= \frac{1}{\pi} \left[\arctan\left(\frac{z}{\omega} \right) \right]_c^\infty \\
&= \frac{1}{\pi} \left(\frac{\pi}{2} - \arctan\left(\frac{c}{\omega} \right) \right) \\
&= \frac{1}{\pi} \arctan\left(\frac{\omega}{c} \right)
\end{aligned}
$$

Solution 34.13
We consider the differential equation

$$
y'' - a^2 y = \mathrm{e}^{-a|x|}
$$

on the domain $-\infty < x < \infty$ with boundary conditions $y(\pm\infty) = 0$. We take the Fourier transform of the differential equation.

$$
-\omega^2 \hat{y} - a^2 \hat{y} = \frac{a}{\pi(\omega^2 + a^2)}
$$

We solve for $\hat{y}(\omega)$.

$$
\hat{y}(\omega) = -\frac{a}{\pi(\omega^2 + a^2)^2}
$$

We take the inverse Fourier transform to find the solution of the differential equation.

$$y(x) = \int_{-\infty}^{\infty} -\frac{a}{\pi(\omega^2 + a^2)^2} \, \mathrm{e}^{\mathrm{i}x\omega} \, \mathrm{d}\omega$$

Note that since $\hat{y}(\omega)$ is a real-valued, even function, $y(x)$ is a real-valued, even function. Thus we only need to evaluate the integral for positive x. If we replace x by $|x|$ in this expression we will have the solution that is valid for all x.

For $x > 0$, we evaluate the integral by closing the path of integration in the upper half plane and using the Residue Theorem and Jordan's Lemma.

$$\begin{aligned}
y(x) &= -\frac{a}{\pi} \int_{-\infty}^{\infty} \frac{1}{(\omega - \mathrm{i}a)^2 (\omega + \mathrm{i}a)^2} \, \mathrm{e}^{\mathrm{i}x\omega} \, \mathrm{d}\omega \\
&= -\mathrm{i}2\pi \frac{a}{\pi} \operatorname{Res}\left(\frac{1}{(\omega - \mathrm{i}a)^2 (\omega + \mathrm{i}a)^2} \, \mathrm{e}^{\mathrm{i}x\omega}, \omega = \mathrm{i}a \right) \\
&= -\mathrm{i}2a \lim_{\omega \to \mathrm{i}a} \frac{\mathrm{d}}{\mathrm{d}\omega} \left(\frac{\mathrm{e}^{\mathrm{i}x\omega}}{(\omega + \mathrm{i}a)^2} \right) \\
&= -\mathrm{i}2a \lim_{\omega \to \mathrm{i}a} \left(\frac{\mathrm{i}x \, \mathrm{e}^{\mathrm{i}x\omega}}{(\omega + \mathrm{i}a)^2} - \frac{2 \, \mathrm{e}^{\mathrm{i}x\omega}}{(\omega + \mathrm{i}a)^3} \right) \\
&= -\mathrm{i}2a \left(\frac{\mathrm{i}x \, \mathrm{e}^{-ax}}{-4a^2} - \frac{2 \, \mathrm{e}^{-ax}}{-\mathrm{i}8a^3} \right) \\
&= -\frac{(1 + ax) \, \mathrm{e}^{-ax}}{2a^2}, \quad \text{for } x \geq 0
\end{aligned}$$

The solution of the differential equation is

$$\boxed{y(x) = -\frac{1}{2a^2}(1 + a|x|) \, \mathrm{e}^{-a|x|}.}$$

Solution 34.14

1. We take the Fourier cosine transform of the differential equation.

$$-\omega^2 \hat{y}(\omega) - \frac{b}{\pi} - a^2 \hat{y}(\omega) = 0$$
$$\hat{y}(\omega) = -\frac{b}{\pi(\omega^2 + a^2)}$$

Now we take the inverse Fourier cosine transform. We use the fact that $\hat{y}(\omega)$ is an even function.

$$\begin{aligned} y(x) &= \mathcal{F}_c^{-1}\left[-\frac{b}{\pi(\omega^2 + a^2)}\right] \\ &= \mathcal{F}^{-1}\left[-\frac{b}{\pi(\omega^2 + a^2)}\right] \\ &= -\frac{b}{\pi} \mathrm{i}2\pi \operatorname{Res}\left(\frac{1}{\omega^2 + a^2} \mathrm{e}^{\mathrm{i}\omega x}, \omega = \mathrm{i}a\right) \\ &= -\mathrm{i}2b \lim_{\omega \to \mathrm{i}a}\left(\frac{\mathrm{e}^{\mathrm{i}\omega x}}{\omega + \mathrm{i}a}\right), \quad \text{for } x \geq 0 \end{aligned}$$

$$\boxed{y(x) = -\frac{b}{a} \mathrm{e}^{-ax}}$$

2. The Green function problem is

$$g'' - a^2 g = \delta(x - \xi) \text{ on } x, \xi > 0, \quad g'(0; \xi) = 0, \quad g(\infty; \xi) = 0.$$

We take the Fourier cosine transform and solve for $\hat{g}(\omega; \xi)$.

$$-\omega^2 \hat{g} - a^2 \hat{g} = \mathcal{F}_c[\delta(x - \xi)]$$
$$\hat{g}(\omega; \xi) = -\frac{1}{\omega^2 + a^2} \mathcal{F}_c[\delta(x - \xi)]$$

We express the right side as a product of Fourier cosine transforms.

$$\hat{g}(\omega;\xi) = -\frac{\pi}{a}\mathcal{F}_c[\mathrm{e}^{-ax}]\mathcal{F}_c[\delta(x-\xi)]$$

Now we can apply the Fourier cosine convolution theorem,

$$\mathcal{F}_c^{-1}\left[\mathcal{F}_c[f(x)]\mathcal{F}_c[g(x)]\right] = \frac{1}{2\pi}\int_0^\infty f(t)\big(g(|x-t|) + g(x+t)\big)\,\mathrm{d}t,$$

to obtain

$$g(x;\xi) = -\frac{\pi}{a}\frac{1}{2\pi}\int_0^\infty \delta(t-\xi)\left(\mathrm{e}^{-a|x-t|} + \mathrm{e}^{-a(x+t)}\right)\mathrm{d}t$$

$$\boxed{g(x;\xi) = -\frac{1}{2a}\left(\mathrm{e}^{-a|x-\xi|} + \mathrm{e}^{-a(x+\xi)}\right)}$$

Solution 34.15

1. We take the Fourier sine transform of the differential equation.

$$-\omega^2\hat{y}(\omega) + \frac{b\omega}{\pi} - a^2\hat{y}(\omega) = 0$$

$$\hat{y}(\omega) = \frac{b\omega}{\pi(\omega^2 + a^2)}$$

Now we take the inverse Fourier sine transform. We use the fact that $\hat{y}(\omega)$ is an odd function.

$$\begin{aligned}
y(x) &= \mathcal{F}_s^{-1}\left[\frac{b\omega}{\pi(\omega^2+a^2)}\right] \\
&= -\mathrm{i}\mathcal{F}^{-1}\left[\frac{b\omega}{\pi(\omega^2+a^2)}\right] \\
&= -\mathrm{i}\frac{b}{\pi}\mathrm{i}2\pi \operatorname{Res}\left(\frac{\omega}{\omega^2+a^2}\,\mathrm{e}^{\mathrm{i}\omega x}, \omega = \mathrm{i}a\right) \\
&= 2b \lim_{\omega\to \mathrm{i}a}\left(\frac{\omega\,\mathrm{e}^{\mathrm{i}\omega x}}{\omega+\mathrm{i}a}\right) \\
&= b\,\mathrm{e}^{-ax} \quad \text{for } x \geq 0
\end{aligned}$$

$$\boxed{y(x) = b\,\mathrm{e}^{-ax}}$$

2. Now we solve the differential equation with the Laplace transform.

$$y'' - a^2 y = 0$$
$$s^2\hat{y}(s) - sy(0) - y'(0) - a^2\hat{y}(s) = 0$$

We don't know the value of $y'(0)$, so we treat it as an unknown constant.

$$\hat{y}(s) = \frac{bs + y'(0)}{s^2 - a^2}$$
$$y(x) = b\cosh(ax) + \frac{y'(0)}{a}\sinh(ax)$$

In order to satisfy the boundary condition at infinity we must choose $y'(0) = -ab$.

$$\boxed{y(x) = b\,\mathrm{e}^{-ax}}$$

We see that solving the differential equation with the Laplace transform is not as convenient, because the boundary condition at infinity is not automatically satisfied. We had to find a value of $y'(0)$ so that $y(\infty) = 0$.

3. The Green function problem is

$$g'' - a^2 g = \delta(x - \xi) \text{ on } x, \xi > 0, \quad g(0; \xi) = 0, \quad g(\infty; \xi) = 0.$$

We take the Fourier sine transform and solve for $\hat{g}(\omega; \xi)$.

$$-\omega^2 \hat{g} - a^2 \hat{g} = \mathcal{F}_s[\delta(x - \xi)]$$
$$\hat{g}(\omega; \xi) = -\frac{1}{\omega^2 + a^2} \mathcal{F}_s[\delta(x - \xi)]$$

We write the right side as a product of Fourier cosine transforms and sine transforms.

$$\hat{g}(\omega; \xi) = -\frac{\pi}{a} \mathcal{F}_c[\mathrm{e}^{-ax}] \mathcal{F}_s[\delta(x - \xi)]$$

Now we can apply the Fourier sine convolution theorem,

$$\mathcal{F}_s^{-1}\left[\mathcal{F}_s[f(x)]\mathcal{F}_c[g(x)]\right] = \frac{1}{2\pi} \int_0^\infty f(t) \big(g(|x - t|) - g(x + t)\big) \, \mathrm{d}t,$$

to obtain

$$g(x; \xi) = -\frac{\pi}{a} \frac{1}{2\pi} \int_0^\infty \delta(t - \xi) \left(\mathrm{e}^{-a|x-t|} - \mathrm{e}^{-a(x+t)} \right) \mathrm{d}t$$
$$\boxed{g(x; \xi) = \frac{1}{2a} \left(\mathrm{e}^{-a(x-\xi)} - \mathrm{e}^{-a|x+\xi|} \right)}$$

Solution 34.16

1. We take the Fourier transform of the differential equation, solve for $\hat{G}$ and then invert.

$$G'' + 2\mu G' + (\beta^2 + \mu^2)G = \delta(x - \xi)$$
$$-\omega^2 \hat{G} + \mathrm{i}2\mu\omega\hat{G} + (\beta^2 + \mu^2)\hat{G} = \frac{\mathrm{e}^{-\mathrm{i}\omega\xi}}{2\pi}$$
$$\hat{G} = -\frac{\mathrm{e}^{-\mathrm{i}\omega\xi}}{2\pi(\omega^2 - \mathrm{i}2\mu\omega - \beta^2 - \mu^2)}$$
$$G = \int_{-\infty}^{\infty} -\frac{\mathrm{e}^{-\mathrm{i}\omega\xi}\,\mathrm{e}^{\mathrm{i}\omega x}}{2\pi(\omega^2 - \mathrm{i}2\mu\omega - \beta^2 - \mu^2)}\,\mathrm{d}\omega$$
$$G = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega(x-\xi)}}{(\omega + \beta - \mathrm{i}\mu)(\omega - \beta - \mathrm{i}\mu)}\,\mathrm{d}\omega$$

For $x > \xi$ we close the path of integration in the upper half plane and use the Residue theorem. There are two simple poles in the upper half plane. For $x < \xi$ we close the path of integration in the lower half plane. Since the integrand is analytic there, the integral is zero. $G(x;\xi) = 0$ for $x < \xi$. For $x > \xi$ we have

$$G(x;\xi) = -\frac{1}{2\pi}\mathrm{i}2\pi\left(\operatorname{Res}\left(\frac{\mathrm{e}^{\mathrm{i}\omega(x-\xi)}}{(\omega + \beta - \mathrm{i}\mu)(\omega - \beta - \mathrm{i}\mu)}, \omega = -\beta + \mathrm{i}\mu\right)\right.$$
$$\left. + \operatorname{Res}\left(\frac{\mathrm{e}^{\mathrm{i}\omega(x-\xi)}}{(\omega + \beta - \mathrm{i}\mu)(\omega - \beta - \mathrm{i}\mu)}, \omega = -\beta - \mathrm{i}\mu\right)\right)$$

$$G(x;\xi) = -\mathrm{i}\left(\frac{\mathrm{e}^{\mathrm{i}(-\beta+\mathrm{i}\mu)(x-\xi)}}{-2\beta}\frac{\mathrm{e}^{\mathrm{i}(\beta+\mathrm{i}\mu)(x-\xi)}}{2\beta}\right)$$
$$G(x;\xi) = \frac{1}{\beta}\,\mathrm{e}^{-\mu(x-\xi)}\sin(\beta(x - \xi)).$$

Thus the Green function is

$$\boxed{G(x;\xi) = \frac{1}{\beta}\,\mathrm{e}^{-\mu(x-\xi)}\sin(\beta(x - \xi))H(x - \xi).}$$

2. The solution of the inhomogeneous equation

$$y'' + 2\mu y' + (\beta^2 + \mu^2)y = g(x), \quad y(-\infty) = y(\infty) = 0,$$

is

$$y(x) = \int_{-\infty}^{\infty} g(\xi) G(x;\xi) \, \mathrm{d}\xi$$
$$y(x) = \int_{-\infty}^{\infty} g(\xi) \frac{1}{\beta} \, \mathrm{e}^{-\mu(x-\xi)} \sin(\beta(x-\xi)) \, \mathrm{d}\xi$$
$$y(x) = \frac{1}{\beta} \int_{-\infty}^{x} g(\xi) \, \mathrm{e}^{-\mu(x-\xi)} \sin(\beta(x-\xi)) \, \mathrm{d}\xi.$$

Taking the limit $\mu \to 0$ we have

$$\boxed{y = \frac{1}{\beta} \int_{-\infty}^{x} g(\xi) \sin(\beta(x-\xi)) d\xi.}$$

Solution 34.17

First we consider the Fourier transform of $f(x) = 1/(x^2 + c^2)$ where $\Re(c) > 0$.

$$\begin{aligned}
\hat{f}(\omega) &= \mathcal{F}\left[\frac{1}{x^2 + c^2}\right] \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{x^2 + c^2} \, \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega x}}{(x - \mathrm{i}c)(x + \mathrm{i}c)} \, \mathrm{d}x
\end{aligned}$$

If $\omega < 0$ then we close the path of integration with a semi-circle in the upper half plane.

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{2\pi} 2\pi i \operatorname{Res}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega x}}{(x - \mathrm{i}c)(x + \mathrm{i}c)}, x = \mathrm{i}c\right) \\
&= \frac{\mathrm{e}^{c\omega}}{2c}, \quad \text{for } \omega < 0
\end{aligned}$$

Note that $f(x) = 1/(x^2 + c^2)$ is an even function of x so that $\hat{f}(\omega)$ is an even function of ω. If $\hat{f}(\omega) = g(\omega)$ for $\omega < 0$ then $f(\omega) = g(-|\omega|)$ for all ω. Thus

$$\boxed{\mathcal{F}\left[\frac{1}{x^2 + c^2}\right] = \frac{1}{2c}\,\mathrm{e}^{-c|\omega|}.}$$

Now we consider the integral equation

$$\int_{-\infty}^{\infty} \frac{u(\xi)}{[(x-\xi)^2 + a^2]}\,\mathrm{d}\xi = \frac{1}{x^2 + b^2} \qquad 0 < a < b.$$

We take the Fourier transform, utilizing the convolution theorem.

$$2\pi \hat{u}(\omega) \frac{\mathrm{e}^{-a|\omega|}}{2a} = \frac{\mathrm{e}^{-b|\omega|}}{2b}$$

$$\hat{u}(\omega) = \frac{a\,\mathrm{e}^{-(b-a)|\omega|}}{2\pi b}$$

$$u(x) = \frac{a}{2\pi b} 2(b-a) \frac{1}{x^2 + (b-a)^2}$$

$$\boxed{u(x) = \frac{a(b-a)}{\pi b(x^2 + (b-a)^2)}}$$

Solution 34.18

1. Note that $F_c(\omega)$ is an even function. The inverse Fourier cosine transform is

$$\begin{aligned}
f(x) &= \mathcal{F}_c^{-1}[F_c(\omega)] \\
&= \int_{-\infty}^{\infty} F_c(\omega)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} F_c(\omega)(\cos(\omega x) + \mathrm{i}\sin(\omega x))\,\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} F_c(\omega)\cos(\omega x)\,\mathrm{d}\omega \\
&= 2\int_{0}^{\infty} F_c(\omega)\cos(\omega x)\,\mathrm{d}\omega.
\end{aligned}$$

2.

$$\begin{aligned}
\mathcal{F}_c[y''] &= \frac{1}{\pi}\int_0^\infty y''\cos(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[y'\cos(\omega x)\right]_0^\infty + \frac{\omega}{\pi}\int_0^\infty y'\sin(\omega x)\,\mathrm{d}x \\
&= -\frac{1}{\pi}y'(0) + \frac{\omega}{\pi}\left[y\sin(\omega x)\right]_0^\infty - \frac{\omega^2}{\pi}\int_0^\infty y\cos(\omega x)\,\mathrm{d}x
\end{aligned}$$

$$\boxed{\mathcal{F}_c[y''] = -\omega^2 F_c(\omega) - \frac{y'(0)}{\pi}}$$

3. We take the Fourier cosine transform of the differential equation.

$$\begin{aligned}
-\omega^2\hat{y}(\omega) - \frac{b}{\pi} - a^2\hat{y}(\omega) &= 0 \\
\hat{y}(\omega) &= -\frac{b}{\pi(\omega^2 + a^2)}
\end{aligned}$$

Now we take the inverse Fourier cosine transform. We use the fact that $\hat{y}(\omega)$ is an even function.

$$\begin{aligned} y(x) &= \mathcal{F}_c^{-1}\left[-\frac{b}{\pi(\omega^2+a^2)}\right] \\ &= \mathcal{F}^{-1}\left[-\frac{b}{\pi(\omega^2+a^2)}\right] \\ &= -\frac{b}{\pi}\mathrm{i}2\pi \operatorname{Res}\left(\frac{1}{\omega^2+a^2}\,\mathrm{e}^{\mathrm{i}\omega x}, \omega = \mathrm{i}a\right) \\ &= -\mathrm{i}2b \lim_{\omega\to \mathrm{i}a}\left(\frac{\mathrm{e}^{\mathrm{i}\omega x}}{\omega+\mathrm{i}a}\right), \quad \text{for } x \geq 0 \end{aligned}$$

$$\boxed{y(x) = -\frac{b}{a}\,\mathrm{e}^{-ax}}$$

Solution 34.19

1. Suppose $f(x)$ is an odd function. The Fourier transform of $f(x)$ is

$$\begin{aligned} \mathcal{F}[f(x)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)(\cos(\omega x) - \mathrm{i}\sin(\omega x))\,\mathrm{d}x \\ &= -\frac{\mathrm{i}}{\pi}\int_0^{\infty} f(x)\sin(\omega x)\,\mathrm{d}x. \end{aligned}$$

Note that $F(\omega) = \mathcal{F}[f(x)]$ is an odd function of ω. The inverse Fourier transform of $F(\omega)$ is

$$\begin{aligned} \mathcal{F}^{-1}[F(\omega)] &= \int_{-\infty}^{\infty} F(\omega)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega \\ &= 2\mathrm{i}\int_0^{\infty} F(\omega)\sin(\omega x)\,\mathrm{d}\omega. \end{aligned}$$

Thus we have that

$$
\begin{aligned}
f(x) &= 2\mathrm{i} \int_0^\infty \left(-\frac{\mathrm{i}}{\pi} \int_0^\infty f(x) \sin(\omega x) \,\mathrm{d}x \right) \sin(\omega x) \,\mathrm{d}\omega \\
&= 2 \int_0^\infty \left(\frac{1}{\pi} \int_0^\infty f(x) \sin(\omega x) \,\mathrm{d}x \right) \sin(\omega x) \,\mathrm{d}\omega.
\end{aligned}
$$

This gives us the Fourier sine transform pair

$$
\boxed{f(x) = 2 \int_0^\infty F_s(\omega) \sin(\omega x) \,\mathrm{d}\omega, \qquad F_s(\omega) = \frac{1}{\pi} \int_0^\infty f(x) \sin(\omega x) \,\mathrm{d}x.}
$$

2.

$$
\begin{aligned}
\mathcal{F}_s[y''] &= \frac{1}{\pi} \int_0^\infty y'' \sin(\omega x) \,\mathrm{d}x \\
&= \frac{1}{\pi} \Big[y' \sin(\omega x) \Big]_0^\infty - \frac{\omega}{\pi} \int_0^\infty y' \cos(\omega x) \,\mathrm{d}x \\
&= -\frac{\omega}{\pi} \Big[y \cos(\omega x) \Big]_0^\infty - \frac{\omega^2}{\pi} \int_0^\infty y \sin(\omega x) \,\mathrm{d}x
\end{aligned}
$$

$$
\boxed{\mathcal{F}_s[y''] = -\omega^2 F_s(\omega) + \frac{\omega}{\pi} y(0)}
$$

3. We take the Fourier sine transform of the differential equation.

$$
\begin{aligned}
-\omega^2 \hat{y}(\omega) + \frac{b\omega}{\pi} - a^2 \hat{y}(\omega) &= 0 \\
\hat{y}(\omega) &= \frac{b\omega}{\pi(\omega^2 + a^2)}
\end{aligned}
$$

Now we take the inverse Fourier sine transform. We use the fact that $\hat{y}(\omega)$ is an odd function.

$$\begin{aligned}
y(x) &= \mathcal{F}_s^{-1}\left[\frac{b\omega}{\pi(\omega^2+a^2)}\right] \\
&= -\mathrm{i}\mathcal{F}^{-1}\left[\frac{b\omega}{\pi(\omega^2+a^2)}\right] \\
&= -\mathrm{i}\frac{b}{\pi}\mathrm{i}2\pi \operatorname{Res}\left(\frac{\omega}{\omega^2+a^2}\,\mathrm{e}^{\mathrm{i}\omega x}, \omega = \mathrm{i}a\right) \\
&= 2b \lim_{\omega\to \mathrm{i}a}\left(\frac{\omega\,\mathrm{e}^{\mathrm{i}\omega x}}{\omega + \mathrm{i}a}\right) \\
&= b\,\mathrm{e}^{-ax} \quad \text{for } x \geq 0
\end{aligned}$$

$$\boxed{y(x) = b\,\mathrm{e}^{-ax}}$$

4. Now we solve the differential equation with the Laplace transform.

$$y'' - a^2 y = 0$$
$$s^2\hat{y}(s) - sy(0) - y'(0) - a^2\hat{y}(s) = 0$$

We don't know the value of $y'(0)$, so we treat it as an unknown constant.

$$\hat{y}(s) = \frac{bs + y'(0)}{s^2 - a^2}$$
$$y(x) = b\cosh(ax) + \frac{y'(0)}{a}\sinh(ax)$$

In order to satisfy the boundary condition at infinity we must choose $y'(0) = -ab$.

$$\boxed{y(x) = b\,\mathrm{e}^{-ax}}$$

We see that solving the differential equation with the Laplace transform is not as convenient, because the boundary condition at infinity is not automatically satisfied. We had to find a value of $y'(0)$ so that $y(\infty) = 0$.

Solution 34.20

The Fourier, Fourier cosine and Fourier sine transforms are defined:

$$\mathcal{F}[f(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}\omega x}\, \mathrm{d}x,$$
$$\mathcal{F}[f(x)]_c = \frac{1}{\pi} \int_{0}^{\infty} f(x) \cos(\omega x)\, \mathrm{d}x,$$
$$\mathcal{F}[f(x)]_s = \frac{1}{\pi} \int_{0}^{\infty} f(x) \sin(\omega x)\, \mathrm{d}x.$$

We start with the right side of the identity and apply the usual tricks of integral calculus to reduce the expression to the left side.

$$\frac{1}{2}\left(\mathcal{F}_c[f(x) + f(-x)] - \mathrm{i}\mathcal{F}_s[f(x) - f(-x)]\right)$$

$$\frac{1}{2\pi}\left(\int_0^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x + \int_0^{\infty} f(-x)\cos(\omega x)\,\mathrm{d}x - \mathrm{i}\int_0^{\infty} f(x)\sin(\omega x)\,\mathrm{d}x + \mathrm{i}\int_0^{\infty} f(-x)\sin(\omega x)\,\mathrm{d}x\right)$$
$$\frac{1}{2\pi}\left(\int_0^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x - \int_0^{-\infty} f(x)\cos(-\omega x)\,\mathrm{d}x - \mathrm{i}\int_0^{\infty} f(x)\sin(\omega x)\,\mathrm{d}x - \mathrm{i}\int_0^{-\infty} f(x)\sin(-\omega x)\,\mathrm{d}x\right)$$
$$\frac{1}{2\pi}\left(\int_0^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x + \int_{-\infty}^{0} f(x)\cos(\omega x)\,\mathrm{d}x - \mathrm{i}\int_0^{\infty} f(x)\sin(\omega x)\,\mathrm{d}x - \mathrm{i}\int_{-\infty}^{0} f(x)\sin(\omega x)\,\mathrm{d}x\right)$$

$$\frac{1}{2\pi}\left(\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x - \mathrm{i}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,\mathrm{d}x\right)$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x$$
$$\mathcal{F}[f(x)]$$

Solution 34.21

We take the Fourier transform of the integral equation, noting that the left side is the convolution of $u(x)$ and $\frac{1}{x^2+a^2}$.

$$2\pi\hat{u}(\omega)\mathcal{F}\left[\frac{1}{x^2+a^2}\right] = \mathcal{F}\left[\frac{1}{x^2+b^2}\right]$$

We find the Fourier transform of $f(x) = \frac{1}{x^2+c^2}$. Note that since $f(x)$ is an even, real-valued function, $\hat{f}(\omega)$ is an even, real-valued function.

$$\mathcal{F}\left[\frac{1}{x^2+c^2}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{x^2+c^2}\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x$$

For $x > 0$ we close the path of integration in the upper half plane and apply Jordan's Lemma to evaluate the integral in terms of the residues.

$$\begin{aligned}
&= \frac{1}{2\pi}\mathrm{i}2\pi\operatorname{Res}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega x}}{(x-\mathrm{i}c)(x+\mathrm{i}c)}, x = \mathrm{i}c\right) \\
&= \mathrm{i}\frac{\mathrm{e}^{-\mathrm{i}\omega\mathrm{i}c}}{2\mathrm{i}c} \\
&= \frac{1}{2c}\,\mathrm{e}^{-c\omega}
\end{aligned}$$

Since $\hat{f}(\omega)$ is an even function, we have

$$\mathcal{F}\left[\frac{1}{x^2+c^2}\right] = \frac{1}{2c}\,\mathrm{e}^{-c|\omega|}.$$

Our equation for $\hat{u}(\omega)$ becomes,

$$2\pi\hat{u}(\omega)\frac{1}{2a}\,\mathrm{e}^{-a|\omega|} = \frac{1}{2b}\,\mathrm{e}^{-b|\omega|}$$
$$\hat{u}(\omega) = \frac{a}{2\pi b}\,\mathrm{e}^{-(b-a)|omega|}.$$

We take the inverse Fourier transform using the transform pair we derived above.

$$u(x) = \frac{a}{2\pi b}\frac{2(b-a)}{x^2+(b-a)^2}$$
$$\boxed{u(x) = \frac{a(b-a)}{\pi b(x^2+(b-a)^2)}}$$

Chapter 35

The Gamma Function

35.1 Euler's Formula

For non-negative, integral n the factorial function is

$$n! = n(n-1)\cdots(1), \quad \text{with} \quad 0! = 1.$$

We would like to extend the factorial function so it is defined for all complex numbers.

Consider the function $\Gamma(z)$ defined by Euler's formula

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1}\, dt.$$

(Here we take the principal value of t^{z-1}.) The integral converges for $\Re(z) > 0$. If $\Re(z) \leq 0$ then the integrand will be at least as singular as $1/t$ at $t = 0$ and thus the integral will diverge.

Difference Equation. Using integration by parts,

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty \mathrm{e}^{-t} t^z \, dt \\ &= \left[-\mathrm{e}^{-t} t^z \right]_0^\infty - \int_0^\infty -\mathrm{e}^{-t} z t^{z-1} \, dt.\end{aligned}$$

Since $\Re(z) > 0$ the first term vanishes.

$$\begin{aligned}&= z \int_0^\infty \mathrm{e}^{-t} t^{z-1} \, dt \\ &= z\Gamma(z)\end{aligned}$$

Thus $\Gamma(z)$ satisfies the difference equation

$$\boxed{\Gamma(z+1) = z\Gamma(z).}$$

For general z it is not possible to express the integral in terms of elementary functions. However, we can evaluate the integral for some z. The value $z = 1$ looks particularly simple to do.

$$\Gamma(1) = \int_0^\infty \mathrm{e}^{-t} \, dt = \left[-\mathrm{e}^{-t} \right]_0^\infty = 1.$$

Using the difference equation we can find the value of $\Gamma(n)$ for any positive, integral n.

$$\begin{aligned}\Gamma(1) &= 1 \\ \Gamma(2) &= 1 \\ \Gamma(3) &= (2)(1) = 2 \\ \Gamma(4) &= (3)(2)(1) = 6 \\ \cdots &= \cdots \\ \Gamma(n+1) &= n!.\end{aligned}$$

Thus the Gamma function, $\Gamma(z)$, extends the factorial function to all complex z in the right half-plane. For non-negative, integral n we have

$$\boxed{\Gamma(n+1) = n!.}$$

Analyticity. The derivative of $\Gamma(z)$ is

$$\Gamma'(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1} \log t \, dt.$$

Since this integral converges for $\Re(z) > 0$, $\Gamma(z)$ is analytic in that domain.

35.2 Hankel's Formula

We would like to find the analytic continuation of the Gamma function into the left half-plane. We accomplish this with Hankel's formula

$$\Gamma(z) = \frac{1}{2i \sin(\pi z)} \int_C \mathrm{e}^t t^{z-1} \, dt.$$

Here C is the contour starting at $-\infty$ below the real axis, enclosing the origin and returning to $-\infty$ above the real axis. A graph of this contour is shown in Figure 35.1. Again we use the principle value of t^{z-1} so there is a branch cut on the negative real axis.

The integral in Hankel's formula converges for all complex z. For non-positive, integral z the integral does not vanish. Thus because of the sine term the Gamma function has simple poles at $z = 0, -1, -2, \ldots$. For positive, integral z, the integrand is entire and thus the integral vanishes. Using L'Hospital's rule you can show that the points, $z = 1, 2, 3, \ldots$ are removable singularities and the Gamma function is analytic at these points. Since the only zeroes of $\sin(\pi z)$ occur for integral z, $\Gamma(z)$ is analytic in the entire plane except for the points, $z = 0, -1, -2, \ldots$.

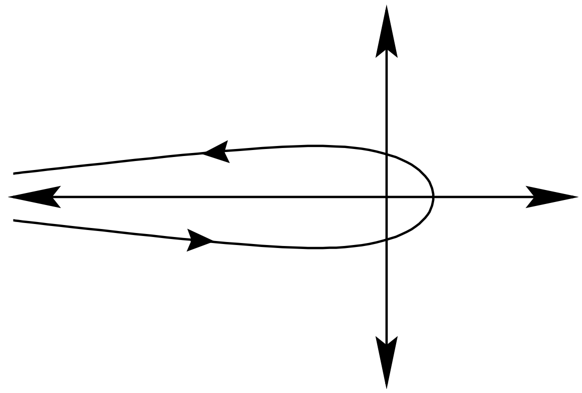

Figure 35.1: The Hankel Contour.

Difference Equation. Using integration by parts we can derive the difference equation from Hankel's formula.

$$
\begin{aligned}
\Gamma(z+1) &= \frac{1}{2i\sin(\pi(z+1))}\int_C \mathrm{e}^t t^z \, dt \\
&= \frac{1}{-2i\sin(\pi z)}\left(\left[\mathrm{e}^t t^z \right]_{-\infty-0i}^{-\infty+0i} - \int_C \mathrm{e}^t z t^{z-1} \, dt \right) \\
&= \frac{1}{2i\sin(\pi z)} z \int_C \mathrm{e}^t t^{z-1} \, dt \\
&= z\Gamma(z).
\end{aligned}
$$

Evaluating $\Gamma(1)$,

$$
\Gamma(1) = \lim_{z \to 1} \frac{\int_C \mathrm{e}^t t^{z-1} \, dt}{2i\sin(\pi z)}.
$$

Both the numerator and denominator vanish. Using L'Hospital's rule,

$$
\begin{aligned}
&= \lim_{z \to 1} \frac{\int_C \mathrm{e}^t t^{z-1} \log t \, dt}{2\pi i \cos(\pi z)} \\
&= \frac{\int_C \mathrm{e}^t \log t \, dt}{2\pi i}
\end{aligned}
$$

Let C_r be the circle of radius r starting at $-\pi$ radians and going to π radians.

$$
\begin{aligned}
&= \frac{1}{2\pi i} \left(\int_{-\infty}^{-r} \mathrm{e}^t [\log(-t) - \pi i] \, dt + \int_{C_r} \mathrm{e}^t \log t \, dt + \int_{-r}^{-\infty} \mathrm{e}^t [\log(-t) + \pi i] \, dt \right) \\
&= \frac{1}{2\pi i} \left(\int_{-r}^{-\infty} \mathrm{e}^t [-\log(-t) + \pi i] \, dt + \int_{-r}^{-\infty} \mathrm{e}^t [\log(-t) + \pi i] \, dt + \int_{C_r} \mathrm{e}^t \log t \, dt \right) \\
&= \frac{1}{2\pi i} \left(\int_{-r}^{-\infty} \mathrm{e}^t 2\pi i \, dt + \int_{C_r} \mathrm{e}^t \log t \, dt \right)
\end{aligned}
$$

The integral on C_r vanishes as $r \to 0$.

$$
\begin{aligned}
&= \frac{1}{2\pi i} 2\pi i \int_0^{-\infty} \mathrm{e}^t \, dt \\
&= 1.
\end{aligned}
$$

Thus we obtain the same value as with Euler's formula. It can be shown that Hankel's formula is the analytic continuation of the Gamma function into the left half-plane.

35.3 Gauss' Formula

Gauss defined the Gamma function as an infinite product. This form is useful in deriving some of its properties. We can obtain the product form from Euler's formula. First recall that

$$
\mathrm{e}^{-t} = \lim_{n \to \infty} \left(1 - \frac{t}{n}\right)^n .
$$

Substituting this into Euler's formula,

$$\begin{aligned}\Gamma(z) &= \int_0^\infty \mathrm{e}^{-t} t^{z-1}\, dt \\ &= \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt.\end{aligned}$$

With the substitution $\tau = t/n$,

$$\begin{aligned}&= \lim_{n\to\infty} \int_0^1 (1-\tau)^n n^{z-1} \tau^{z-1} n\, d\tau \\ &= \lim_{n\to\infty} n^z \int_0^1 (1-\tau)^n \tau^{z-1}\, d\tau.\end{aligned}$$

Let n be an integer. Using integration by parts we can evaluate the integral.

$$\begin{aligned}\int_0^1 (1-\tau)^n \tau^{z-1}\, d\tau &= \left[\frac{(1-\tau)^n \tau^z}{z}\right]_0^1 - \int_0^1 -n(1-\tau)^{n-1} \frac{\tau^z}{z}\, d\tau \\ &= \frac{n}{z} \int_0^1 (1-\tau)^{n-1} \tau^z\, d\tau \\ &= \frac{n(n-1)}{z(z+1)} \int_0^1 (1-\tau)^{n-2} \tau^{z+1}\, d\tau \\ &= \frac{n(n-1)\cdots(1)}{z(z+1)\cdots(z+n-1)} \int_0^1 \tau^{z+n-1}\, d\tau \\ &= \frac{n(n-1)\cdots(1)}{z(z+1)\cdots(z+n-1)} \left[\frac{\tau^{z+n}}{z+n}\right]_0^1 \\ &= \frac{n!}{z(z+1)\cdots(z+n)}\end{aligned}$$

Thus we have that

$$
\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} n^z \frac{n!}{z(z+1)\cdots(z+n)} \\
&= \frac{1}{z}\lim_{n\to\infty} \frac{(1)(2)\cdots(n)}{(z+1)(z+2)\cdots(z+n)} n^z \\
&= \frac{1}{z}\lim_{n\to\infty} \frac{1}{(1+z)(1+z/2)\cdots(1+z/n)} n^z \\
&= \frac{1}{z}\lim_{n\to\infty} \frac{1}{(1+z)(1+z/2)\cdots(1+z/n)} \frac{2^z 3^z \cdots n^z}{1^z 2^z \cdots (n-1)^z}
\end{aligned}
$$

Since $\lim_{n\to\infty} \frac{(n+1)^z}{n^z} = 1$ we can multiply by that factor.

$$
\begin{aligned}
&= \frac{1}{z}\lim_{n\to\infty} \frac{1}{(1+z)(1+z/2)\cdots(1+z/n)} \frac{2^z 3^z \cdots (n+1)^z}{1^z 2^z \cdots n^z} \\
&= \frac{1}{z}\prod_{n=1}^{\infty}\left[\frac{1}{1+z/n}\frac{(n+1)^z}{n^z}\right]
\end{aligned}
$$

Thus we have Gauss' formula for the Gamma function

$$
\boxed{\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}\right].}
$$

We derived this formula from Euler's formula which is valid only in the left half-plane. However, the product formula is valid for all z except $z = 0, -1, -2, \ldots$.

35.4 Weierstrass' Formula

The Euler-Mascheroni Constant. Before deriving Weierstrass' product formula for the Gamma function we will need to define the Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\log n\right] = 0.5772\cdots.$$

In deriving the Euler product formula, we had the equation

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty}\left[n^z\frac{n!}{z(z+1)\cdots(z+n)}\right].\\
&= \lim_{n\to\infty}\left[z^{-1}\left(1+\frac{z}{1}\right)^{-1}\left(1+\frac{z}{2}\right)^{-1}\cdots\left(1+\frac{z}{n}\right)^{-1}n^z\right]\\
\frac{1}{\Gamma(z)} &= \lim_{n\to\infty}\left[z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{n}\right)\mathrm{e}^{-z\log n}\right]\\
&= \lim_{n\to\infty}\left[z\left(1+\frac{z}{1}\right)\mathrm{e}^{-z}\left(1+\frac{z}{2}\right)\mathrm{e}^{-z/2}\cdots\left(1+\frac{z}{n}\right)\mathrm{e}^{-z/n}\exp\left(\left[1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right]z\right)\right]
\end{aligned}$$

Weierstrass' formula for the Gamma function is then

$$\boxed{\frac{1}{\Gamma(z)} = z\,\mathrm{e}^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)\mathrm{e}^{-z/n}\right].}$$

Since the product is uniformly convergent, $1/\Gamma(z)$ is an entire function. Since $1/\Gamma(z)$ has no singularities, we see that $\Gamma(z)$ has no zeros.

Result 35.4.1 Euler's formula for the Gamma function is valid for $\Re(z) > 0$.

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1}\, dt$$

Hankel's formula defines the $\Gamma(z)$ for the entire complex plane except for the points $z = 0, -1, -2, \ldots$.

$$\Gamma(z) = \frac{1}{2i\sin(\pi z)} \int_C \mathrm{e}^{t} t^{z-1}\, dt$$

Gauss' and Weierstrass' product formulas are, respectively

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{1}{n}\right)^z \left(1 + \frac{z}{n}\right)^{-1} \right] \qquad \text{and}$$

$$\frac{1}{\Gamma(z)} = z\,\mathrm{e}^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \mathrm{e}^{-z/n} \right].$$

35.5 Stirling's Approximation

In this section we will try to get an approximation to the Gamma function for large positive argument. Euler's formula is

$$\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1}\, dt.$$

We could first try to approximate the integral by only looking at the domain where the integrand is large. In Figure 35.2 the integrand in the formula for $\Gamma(10)$, $\mathrm{e}^{-t} t^9$, is plotted.

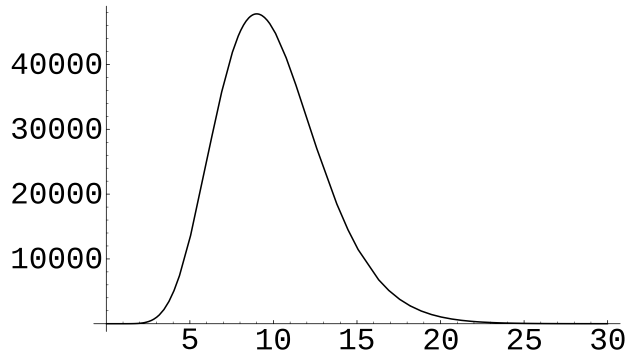

Figure 35.2: Plot of the integrand for $\Gamma(10)$

We see that the "important" part of the integrand is the hump centered around $x = 9$. If we find where the integrand of $\Gamma(x)$ has its maximum

$$\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-t} t^{x-1}\right) = 0 \\
-\mathrm{e}^{-t} t^{x-1} + (x-1)\,\mathrm{e}^{-t} t^{x-2} = 0 \\
(x-1) - t = 0 \\
t = x - 1,
\end{gathered}$$

we see that the maximum varies with x. This could complicate our analysis. To take care of this problem we introduce the change of variables $t = xs$.

$$\begin{aligned}\Gamma(x) &= \int_0^\infty \mathrm{e}^{-xs}(xs)^{x-1} x\, ds \\ &= x^x \int_0^\infty \mathrm{e}^{-xs} s^x s^{-1}\, ds \\ &= x^x \int_0^\infty \mathrm{e}^{-x(s-\log s)} s^{-1}\, ds\end{aligned}$$

The integrands, $(\mathrm{e}^{-x(s-\log s)} s^{-1})$, for $\Gamma(5)$ and $\Gamma(20)$ are plotted in Figure 35.3.

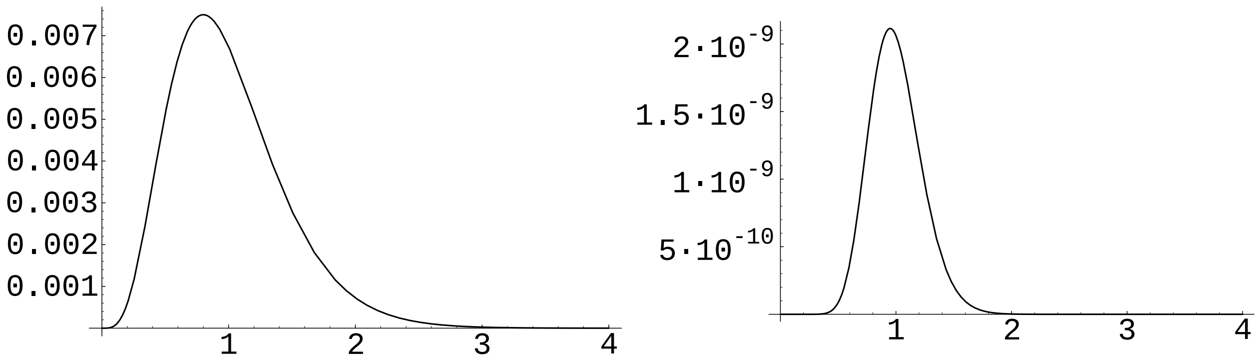

Figure 35.3: Plot of the integrand for $\Gamma(5)$ and $\Gamma(20)$.

We see that the important part of the integrand is the hump that seems to be centered about $s = 1$. Also note that the the hump becomes narrower with increasing x. This makes sense as the $\mathrm{e}^{-x(s-\log s)}$ term is the most rapidly varying term. Instead of integrating from zero to infinity, we could get a good approximation to the integral by just integrating over some small neighborhood centered at $s = 1$. Since $s - \log s$ has a minimum at $s = 1$, $\mathrm{e}^{-x(s-\log s)}$ has a maximum there. Because the important part of the integrand is the small area around

$s = 1$, it makes sense to approximate $s - \log s$ with its Taylor series about that point.

$$s - \log s = 1 + \frac{1}{2}(s-1)^2 + O\big[(s-1)^3\big]$$

Since the hump becomes increasingly narrow with increasing x, we will approximate the $1/s$ term in the integrand with its value at $s = 1$. Substituting these approximations into the integral, we obtain

$$\begin{aligned}\Gamma(x) &\sim x^x \int_{1-\epsilon}^{1+\epsilon} \mathrm{e}^{-x(1+(s-1)^2/2)}\, ds \\ &= x^x\, \mathrm{e}^{-x} \int_{1-\epsilon}^{1+\epsilon} \mathrm{e}^{-x(s-1)^2/2}\, ds\end{aligned}$$

As $x \to \infty$ both of the integrals

$$\int_{-\infty}^{1-\epsilon} \mathrm{e}^{-x(s-1)^2/2}\, ds \qquad \text{and} \qquad \int_{1+\epsilon}^{\infty} \mathrm{e}^{-x(s-1)^2/2}\, ds$$

are exponentially small. Thus instead of integrating from $1-\epsilon$ to $1+\epsilon$ we can integrate from $-\infty$ to ∞.

$$\begin{aligned}\Gamma(x) &\sim x^x\, \mathrm{e}^{-x} \int_{-\infty}^{\infty} \mathrm{e}^{-x(s-1)^2/2}\, ds \\ &= x^x\, \mathrm{e}^{-x} \int_{-\infty}^{\infty} \mathrm{e}^{-xs^2/2}\, ds \\ &= x^x\, \mathrm{e}^{-x} \sqrt{\frac{2\pi}{x}}\end{aligned}$$

$$\boxed{\Gamma(x) \sim \sqrt{2\pi} x^{x-1/2}\, \mathrm{e}^{-x} \quad \text{as x } \to \infty.}$$

This is known as Stirling's approximation to the Gamma function. In the table below, we see that the approximation is pretty good even for relatively small argument.

n	$\Gamma(n)$	$\sqrt{2\pi} x^{x-1/2}\,\mathrm{e}^{-x}$	relative error
5	24	23.6038	0.0165
15	$8.71783 \cdot 10^{10}$	$8.66954 \cdot 10^{10}$	0.0055
25	$6.20448 \cdot 10^{23}$	$6.18384 \cdot 10^{23}$	0.0033
35	$2.95233 \cdot 10^{38}$	$2.94531 \cdot 10^{38}$	0.0024
45	$2.65827 \cdot 10^{54}$	$2.65335 \cdot 10^{54}$	0.0019

In deriving Stirling's approximation to the Gamma function we did a lot of hand waving. However, all of the steps can be justified and better approximations can be obtained by using Laplace's method for finding the asymptotic behavior of integrals.

35.6 Exercises

Exercise 35.1
Given that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\, dx = \sqrt{\pi},$$

deduce the value of $\Gamma(1/2)$. Now find the value of $\Gamma(n + 1/2)$.

Exercise 35.2
Evaluate $\int_0^\infty \mathrm{e}^{-x^3}\, dx$ in terms of the gamma function.

Exercise 35.3
Show that

$$\int_0^\infty \mathrm{e}^{-x} \sin(\log x)\, dx = \frac{\Gamma(i) + \Gamma(-i)}{2}.$$

35.7 Hints

Hint 35.1
Use the change of variables, $\xi = x^2$ in the integral. To find the value of $\Gamma(n + 1/2)$ use the difference relation.

Hint 35.2
Make the change of variable $\xi = x^3$.

35.8 Solutions

Solution 35.1

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,dx = \sqrt{\pi}$$
$$\int_{0}^{\infty} \mathrm{e}^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Make the change of variables $\xi = x^2$.

$$\int_{0}^{\infty} \mathrm{e}^{-\xi}\frac{1}{2}\xi^{-1/2}\,d\xi = \frac{\sqrt{\pi}}{2}$$
$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

Recall the difference relation for the Gamma function $\Gamma(z+1) = z\Gamma(z)$.

$$\begin{aligned}
\Gamma(n+1/2) &= (n-1/2)\Gamma(n-1/2) \\
&= \frac{2n-1}{2}\Gamma(n-1/2) \\
&= \frac{(2n-3)(2n-1)}{2^2}\Gamma(n-3/2) \\
&= \frac{(1)(3)(5)\cdots(2n-1)}{2^n}\Gamma(1/2)
\end{aligned}$$

$$\boxed{\Gamma(n+1/2) = \frac{(1)(3)(5)\cdots(2n-1)}{2^n}\sqrt{\pi}}$$

Solution 35.2
We make the change of variable $\xi = x^3$, $x = \xi^{1/3}$, $dx = \frac{1}{3}\xi^{-2/3}\, d\xi$.

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-x^3}\, dx &= \int_0^\infty \mathrm{e}^{-\xi} \frac{1}{3} \xi^{-2/3}\, d\xi \\
&= \frac{1}{3} \Gamma\left(\frac{1}{3}\right)
\end{aligned}$$

Solution 35.3

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-x} \sin(\log x)\, dx &= \int_0^\infty \mathrm{e}^{-x} \frac{1}{2i} \left(\mathrm{e}^{i \log x} - \mathrm{e}^{-i \log x}\right) dx \\
&= \frac{1}{2i} \int_0^\infty \mathrm{e}^{-x} \left(x^i - x^{-i}\right) dx \\
&= \frac{1}{2i} \left(\Gamma(1+i) - \Gamma(1-i)\right) \\
&= \frac{1}{2i} \left(i\Gamma(i) - (-i)\Gamma(-i)\right) \\
&= \frac{\Gamma(i) + \Gamma(-i)}{2}
\end{aligned}$$

Chapter 36

Bessel Functions

Ideas are angels. Implementations are a bitch.

36.1 Bessel's Equation

A commonly encountered differential equation in applied mathematics is *Bessel's equation*

$$y'' + \frac{1}{z}y' + \left(1 - \frac{\nu^2}{z^2}\right) y = 0.$$

For our purposes, we will consider $\nu \in \mathbb{R}^{0+}$. This equation arises when solving certain partial differential equations with the method of separation of variables in cylindrical coordinates. For this reason, the solutions of this equation are sometimes called *cylindrical functions*.

This equation cannot be solved directly. However, we can find series representations of the solutions. There is a regular singular point at $z = 0$, so the Frobenius method is applicable there. The point at infinity is an irregular singularity, so we will look for asymptotic series about that point. Additionally, we will use Laplace's method to find definite integral representations of the solutions.

Note that Bessel's equation depends only on ν^2 and not ν alone. Thus if we find a solution, (which of course depends on this parameter), $y_\nu(z)$ we know that $y_{-\nu}(z)$ is also a solution. For this reason, we will consider $\nu \in \mathbb{R}^{0+}$. Whether or not $y_\nu(z)$ and $y_{-\nu}(z)$ are linearly independent, (distinct solutions), remains to be seen.

Example 36.1.1 Consider the differential equation

$$y'' + \frac{1}{z} y' + \frac{\nu^2}{z^2} y = 0$$

One solution is $y_\nu(z) = z^\nu$. Since the equation depends only on ν^2, another solution is $y_{-\nu}(z) = z^{-\nu}$. For $\nu \neq 0$, these two solutions are linearly independent.

Now consider the differential equation

$$y'' + \nu^2 y = 0$$

One solution is $y_\nu(z) = \cos(\nu z)$. Therefore, another solution is $y_{-\nu}(z) = \cos(-\nu z) = \cos(\nu z)$. However, these two solutions are not linearly independent.

36.2 Frobeneius Series Solution about z = 0

We note that $z = 0$ is a regular singular point, (the only singular point of Bessel's equation in the finite complex plane.) We will use the Frobenius method at that point to analyze the solutions. We assume that $\nu \geq 0$.

The indicial equation is

$$\alpha(\alpha - 1) + \alpha - \nu^2 = 0$$
$$\alpha = \pm\nu.$$

If $\pm\nu$ do not differ by an integer, (that is if ν is not a half-integer), then there will be two series solutions of the Frobenius form.

$$y_1(z) = z^\nu \sum_{k=0}^{\infty} a_k z^k, \quad y_2(z) = z^{-\nu} \sum_{k=0}^{\infty} b_k z^k$$

If ν is a half-integer, the second solution may or may not be in the Frobenius form. In any case, then will always be at least one solution in the Frobenius form. We will determine that series solution. $y(z)$ and it derivatives are

$$y = \sum_{k=0}^{\infty} a_k z^{k+\nu}, \quad y' = \sum_{k=0}^{\infty} (k+\nu) a_k z^{k+\nu-1}, \quad y'' = \sum_{k=0}^{\infty} (k+\nu)(k+\nu-1) a_k z^{k+\nu-2}.$$

We substitute the Frobenius series into the differential equation.

$$z^2 y'' + z y' + \left(z^2 - \nu^2\right) y = 0$$

$$\sum_{k=0}^{\infty} (k+\nu)(k+\nu-1) a_k z^{k+\nu} + \sum_{k=0}^{\infty} (k+\nu) a_k z^{k+\nu} + \sum_{k=0}^{\infty} a_k z^{k+\nu+2} - \sum_{k=0}^{\infty} \nu^2 a_k z^{k+\nu} = 0$$

$$\sum_{k=0}^{\infty} \left(k^2 + 2k\nu\right) a_k z^k + \sum_{k=2}^{\infty} a_{k-2} z^k = 0$$

We equate powers of z to obtain equations that determine the coefficients. The coefficient of z^0 is the equation $0 \cdot a_0 = 0$. This corroborates that a_0 is arbitrary, (but non-zero). The coefficient of z^1 is the equation

$$(1 + 2\nu) a_1 = 0$$
$$a_1 = 0$$

The coefficient of z^k for $k \geq 2$ gives us

$$\left(k^2 + 2k\nu\right) a_k + a_{k-2} = 0.$$
$$a_k = -\frac{a_{k-2}}{k^2 + 2k\nu} = -\frac{a_{k-2}}{k(k+2\nu)}$$

From the recurrence relation we see that all the odd coefficients are zero, $a_{2k+1} = 0$. The even coefficients are

$$a_{2k} = -\frac{a_{2k-2}}{4k(k+\nu)} = \frac{(-1)^k a_0}{2^{2k} k! \Gamma(k+\nu+1)}$$

Thus we have the series solution

$$y(z) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! \Gamma(k+\nu+1)} z^{2k}.$$

a_0 is arbitrary. We choose $a_0 = 2^{-\nu}$. We call this solution the *Bessel function of the first kind and order* ν and denote it with $J_\nu(z)$.

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu}$$

Recall that the Gamma function is non-zero and finite for all real arguments except non-positive integers. $\Gamma(x)$ has singularities at $x = 0, -1, -2, \ldots$. Therefore, $J_{-\nu}(z)$ is well-defined when ν is not a positive integer. Since $J_{-\nu}(z) \sim z^{-\nu}$ at $z = 0$, $J_{-\nu}(z)$ is clear linearly independent to $J_\nu(z)$ for non-integer ν. In particular we note that there are two solutions of the Frobenius form when ν is a half odd integer.

$$J_{-\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k-\nu+1)} \left(\frac{z}{2}\right)^{2k-\nu}, \quad \text{for } \nu \notin \mathbb{Z}^+$$

Of course for $\nu = 0$, $J_\nu(z)$ and $J_{-\nu}(z)$ are identical. Consider the case that $\nu = n$ is a positive integer. Since $\Gamma(x) \to +\infty$ as $x \to 0, -1, -2, \ldots$ we see the the coefficients in the series for $J_{-nu}(z)$ vanish for $k = 0, \ldots, n-1$.

$$J_{-n}(z) = \sum_{k=n}^{\infty} \frac{(-1)^k}{k! \Gamma(k-n+1)} \left(\frac{z}{2}\right)^{2k-n}$$

$$J_{-n}(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{(k+n)! \Gamma(k+1)} \left(\frac{z}{2}\right)^{2k+n}$$

$$J_{-n}(z) = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{z}{2}\right)^{2k+n}$$

$$J_{-n}(z) = (-1)^n J_n(z)$$

Thus we see that $J_{-n}(z)$ and $J_n(z)$ are not linearly independent for integer n.

36.2.1 Behavior at Infinity

With the change of variables $z = 1/t$, $w(z) = u(t)$ Bessel's equation becomes

$$t^4 u'' + 2t^3 u' + t(-t^2)u' + (1 - \nu^2 t^2)u = 0$$
$$u'' + \frac{1}{t}u' + \left(\frac{1}{t^4} - \frac{\nu^2}{t^2}\right) u = 0.$$

The point $t = 0$ and hence the point $z = \infty$ is an irregular singular point. We will find the leading order asymptotic behavior of the solutions as $z \to +\infty$.

Controlling Factor. Starting with Bessel's equation for real argument

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0,$$

we make the substitution $y = \mathrm{e}^{s(x)}$ to obtain

$$s'' + (s')^2 + \frac{1}{x}s' + 1 - \frac{\nu^2}{x^2} = 0.$$

We know that $\frac{\nu^2}{x^2} \ll 1$ as $x \to \infty$; we will assume that $s'' \ll (s')^2$ as $x \to \infty$. This gives us

$$(s')^2 + \frac{1}{x}s' + 1 \sim 0 \qquad \text{as } x \to \infty.$$

To simplify the equation further, we will try the possible two-term balances.

1. $(s')^2 + \frac{1}{x}s' \sim 0 \quad \Rightarrow \quad s' \sim -\frac{1}{x}$ This balance is not consistent as it violates the assumption that 1 is smaller than the other terms.

2. $(s')^2 + 1 \sim 0 \quad \Rightarrow \quad s' \sim \pm i$ This balance is consistent.

3. $\frac{1}{x}s' + 1 \sim 0 \quad \Rightarrow \quad s' \sim -x$ This balance is inconsistent as $(s')^2$ isn't smaller than the other terms.

Thus the only dominant balance is $s' \sim \pm i$. This balance is consistent with our initial assumption that $s'' \ll (s')^2$. Thus $s \sim \pm ix$ and the controlling factor is $\mathrm{e}^{\pm ix}$.

Leading Order Behavior. In order to find the leading order behavior, we substitute $s = \pm ix + t(x)$ where $t(x) \ll x$ as $x \to \infty$ into the differential equation for s. We first consider the case $s = ix + t(x)$. We assume that $t' \ll 1$ and $t'' \ll 1/x$.

$$t'' + (i + t')^2 + \frac{1}{x}(i + t') + 1 - \frac{\nu^2}{x^2} = 0$$
$$t'' + 2it' + (t')^2 + \frac{i}{x} + \frac{1}{x}t' - \frac{\nu^2}{x^2} = 0$$

Using our assumptions about the behavior of t' and t'',

$$2it' + \frac{i}{x} \sim 0$$
$$t' \sim -\frac{1}{2x}$$
$$t \sim -\frac{1}{2}\log x \quad \text{as } x \to \infty.$$

This asymptotic behavior is consistent with our assumptions.

Substituting $s = -ix + t(x)$ will also yield $t \sim -\frac{1}{2}\log x$. Thus the leading order behavior of the solutions is

$$y \sim c\,\mathrm{e}^{\pm ix - \frac{1}{2}\log x + u(x)} = cx^{-1/2}\,\mathrm{e}^{\pm ix + u(x)} \quad \text{as } x \to \infty,$$

where $u(x) \ll \log x$ as $x \to \infty$.

By substituting $t = -\frac{1}{2}\log x + u(x)$ into the differential equation for t, you could show that $u(x) \to \text{const}$ as $x \to \infty$. Thus the full leading order behavior of the solutions is

$$y \sim cx^{-1/2}\,\mathrm{e}^{\pm ix + u(x)} \quad \text{as } x \to \infty$$

where $u(x) \to 0$ as $x \to \infty$. Writing this in terms of sines and cosines yields

$$y_1 \sim x^{-1/2}\cos(x + u_1(x)), \qquad y_2 \sim x^{-1/2}\sin(x + u_2(x)), \quad \text{as } x \to \infty,$$

where $u_1, u_2 \to 0$ as $x \to \infty$.

Result 36.2.1 Bessel's equation for real argument is

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0.$$

If ν is not an integer then the solutions behave as linear combinations of

$$y_1 = x^\nu, \quad \text{and} \quad y_2 = x^{-\nu}$$

at $x = 0$. If ν is an integer, then the solutions behave as linear combinations of

$$y_1 = x^\nu, \quad \text{and} \quad y_2 = x^{-\nu} + c x^\nu \log x$$

at $x = 0$. The solutions are asymptotic to a linear combination of

$$y_1 = x^{-1/2} \sin(x + u_1(x)), \quad \text{and} \quad y_2 = x^{-1/2} \cos(x + u_2(x))$$

as $x \to +\infty$, where $u_1, u_2 \to 0$ as $x \to \infty$.

36.3 Bessel Functions of the First Kind

Consider the function $\exp(\frac{1}{2}z(t - 1/t))$. We can expand this function in a Laurent series in powers of t,

$$\mathrm{e}^{\frac{1}{2}z(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z) t^n,$$

where the coefficient functions $J_n(z)$ are

$$J_n(z) = \frac{1}{2\pi i} \oint \tau^{-n-1} \, \mathrm{e}^{\frac{1}{2}z(\tau - 1/\tau)} \, d\tau.$$

Here the path of integration is any positive closed path around the origin. $\exp(\frac{1}{2}z(t - 1/t))$ is the **generating function** for Bessel function of the first kind.

36.3.1 The Bessel Function Satisfies Bessel's Equation

We would like to expand $J_n(z)$ in powers of z. The first step in doing this is to make the substitution $\tau = 2t/z$.

$$\begin{aligned} J_n(z) &= \frac{1}{2\pi i} \oint \left(\frac{2t}{z}\right)^{-n-1} \exp\left(\frac{1}{2} z \left(\frac{2t}{z} - \frac{z}{2t}\right)\right) \frac{2}{z} \, dt \\ &= \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \oint t^{-n-1} \, \mathrm{e}^{t - z^2/4t} \, dt \end{aligned}$$

Differentiating the expression for $J_n(z)$,

$$\begin{aligned} J_n'(z) &= \frac{1}{2\pi i} \frac{n z^{n-1}}{2^n} \oint t^{-n-1} \, \mathrm{e}^{t - z^2/4t} \, dt + \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \oint t^{-n-1} \left(\frac{-2z}{4t}\right) \mathrm{e}^{t - z^2/4t} \, dt \\ &= \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \oint \left(\frac{n}{z} - \frac{z}{2t}\right) t^{-n-1} \, \mathrm{e}^{t - z^2/4t} \, dt \end{aligned}$$

$$\begin{aligned} J_n''(z) &= \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \oint \left[\frac{n}{z}\left(\frac{n}{z} - \frac{z}{2t}\right) + \left(-\frac{n}{z^2} - \frac{1}{2t}\right) - \frac{z}{2t}\left(\frac{n}{z} - \frac{z}{2t}\right)\right] t^{-n-1} \, \mathrm{e}^{t - z^2/4t} \, dt \\ &= \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \oint \left[\frac{n^2}{z^2} - \frac{nz}{2zt} - \frac{n}{z^2} - \frac{1}{2t} - \frac{nz}{2zt} + \frac{z^2}{4t^2}\right] t^{-n-1} \, \mathrm{e}^{t - z^2/4t} \, dt \\ &= \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \oint \left[\frac{n(n-1)}{z^2} - \frac{2n+1}{2t} + \frac{z^2}{4t^2}\right] t^{-n-1} \, \mathrm{e}^{t - z^2/4t} \, dt. \end{aligned}$$

Substituting $J_n(z)$ into Bessel's equation,

$$
\begin{aligned}
& J_n'' + \frac{1}{z} J_n' + \left(1 - \frac{n^2}{z^2}\right) J_n \\
&= \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \oint \left[\left(\frac{n(n-1)}{z^2} - \frac{2n+1}{2t} + \frac{z^2}{4t^2} \right) + \left(\frac{n}{z^2} - \frac{1}{2t} \right) + \left(1 - \frac{n^2}{z^2} \right) \right] t^{-n-1} \, \mathrm{e}^{t - z^2/4t} \, dt \\
&= \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \oint \left[1 - \frac{n+1}{t} + \frac{z^2}{4t^2} \right] t^{-n-1} \, \mathrm{e}^{t - z^2/4t} \, dt \\
&= \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \oint \frac{\mathrm{d}}{\mathrm{d}t} \left(t^{-n-1} \, \mathrm{e}^{t - z^2/4t} \right) \, dt
\end{aligned}
$$

Since $t^{-n-1} \, \mathrm{e}^{t - z^2/4t}$ is analytic in $0 < |t| < \infty$ when n is an integer, the integral vanishes.

$$= 0.$$

Thus for integral n, $J_n(z)$ satisfies Bessel's equation.

$J_n(z)$ is called the Bessel function of the first kind. The subscript is the order. Thus $J_1(z)$ is a Bessel function of order 1. $J_0(x)$ and $J_1(x)$ are plotted in the first graph in Figure 36.1. $J_5(x)$ is plotted in the second graph in Figure 36.1. Note that for non-negative, integral n, $J_n(z)$ behaves as z^n at $z = 0$.

36.3.2 Series Expansion of the Bessel Function

Expanding $\exp(-z^2/4t)$ in the integral expression for J_n,

$$
\begin{aligned}
J_n(z) &= \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \oint t^{-n-1} \, \mathrm{e}^{t - z^2/4t} \, dt \\
&= \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \oint t^{-n-1} \, \mathrm{e}^{t} \left(\sum_{m=0}^{\infty} \left(\frac{-z^2}{4t} \right)^m \frac{1}{m!} \right) dt
\end{aligned}
$$

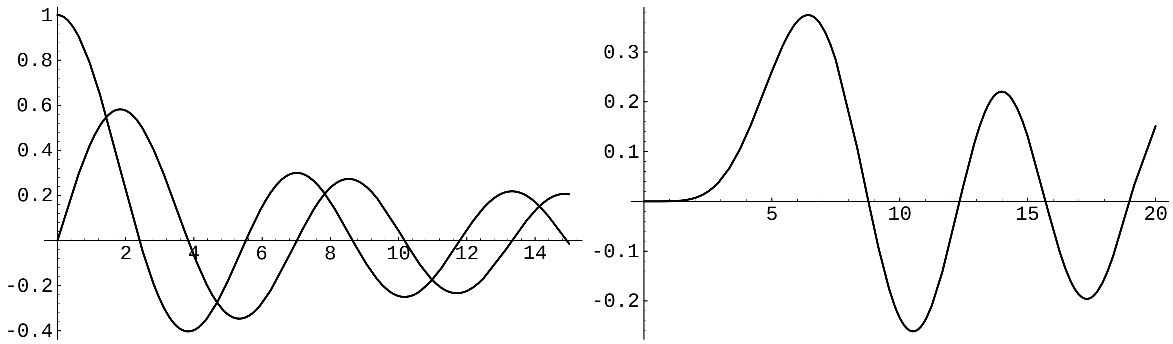

Figure 36.1: Plot of $J_0(x)$, $J_1(x)$ and $J_5(x)$.

For the path of integration, we are free to choose any contour that encloses the origin. Consider the circular path on $|t| = 1$. Since the integral is uniformly convergent, we can interchange the order of integration and summation.

$$J_n(z) = \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{2^{2m} m!} \oint t^{-n-m-1} \, \mathrm{e}^t \, dt$$

If n is a non-negative integer,

$$\frac{1}{2\pi i}\oint t^{-n-m-1}\,\mathrm{e}^t\,dt = \lim_{z\to 0}\left(\frac{1}{(n+m)!}\frac{d^{n+m}}{dz^{n+m}}(\,\mathrm{e}^z)\right)$$
$$= \frac{1}{(n+m)!}.$$

Thus we have the series expansion

$$\boxed{J_n(z) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!(n+m)!}\left(\frac{z}{2}\right)^{n+2m} \qquad \text{for } n \geq 0.}$$

Now consider $J_{-n}(z)$, (n positive).

$$J_{-n}(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^{-n}\sum_{m=1}^{\infty}\frac{(-1)^m z^{2m}}{2^{2m}m!}\oint t^{n-m-1}\,\mathrm{e}^t\,dt$$

For $m \geq n$, the integrand has a pole of order $m-n+1$ at the origin.

$$\frac{1}{2\pi i}\oint t^{n-m-1}\,\mathrm{e}^t\,dt = \begin{cases}\frac{1}{(m-n)!} & \text{for } m \geq n\\ 0 & \text{for } m < n\end{cases}$$

The expression for J_{-n} is then

$$J_{-n}(z) = \sum_{m=n}^{\infty}\frac{(-1)^m}{m!(m-n)!}\left(\frac{z}{2}\right)^{-n+2m}$$
$$= \sum_{m=0}^{\infty}\frac{(-1)^{m+n}}{(m+n)!m!}\left(\frac{z}{2}\right)^{n+2m}$$
$$= (-1)^n J_n(z).$$

Thus we have that

$$\boxed{J_{-n}(z) = (-1)^n J_n(z) \quad \text{for integral } n.}$$

36.3.3 Bessel Functions of Non-Integral Order

The generalization of the factorial function is the Gamma function. For integral values of n, $n! = \Gamma(n+1)$. The Gamma function is defined for all complex-valued arguments. Thus one would guess that if the Bessel function of the first kind were defined for non-integral order, it would have the definition,

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{\nu+2m}.$$

The Integrand for Non-Integral ν. Recall the definition of the Bessel function

$$J_\nu(z) = \frac{1}{2\pi i} \left(\frac{z}{2}\right)^\nu \oint t^{-\nu-1} \mathrm{e}^{t-z^2/4t}\, dt.$$

When ν is an integer, the integrand is single valued. Thus if you start at any point and follow any path around the origin, the integrand will return to its original value. This property was the key to J_n satisfying Bessel's equation. If ν is not an integer, then this property does not hold for arbitrary paths around the origin.

A New Contour. First, since the integrand is multiple-valued, we need to define what branch of the function we are talking about. We will take the principal value of the integrand and introduce a branch cut on the negative real axis. Let C be a contour that starts at $z = -\infty$ below the branch cut, circles the origin, and returns to the point $z = -\infty$ above the branch cut. This contour is shown in Figure 36.2.

Thus we define

$$\boxed{J_\nu(z) = \frac{1}{2\pi i} \left(\frac{z}{2}\right)^\nu \oint_C t^{-\nu-1} \mathrm{e}^{t-z^2/4t}\, dt.}$$

Bessel's Equation. Substituting $J_\nu(z)$ into Bessel's equation yields

$$J_\nu'' + \frac{1}{z} J_\nu' + \left(1 - \frac{\nu^2}{z^2}\right) J_\nu = \frac{1}{2\pi i} \left(\frac{z}{2}\right)^\nu \oint_C \frac{\mathrm{d}}{\mathrm{d}t} \left(t^{-\nu-1} \mathrm{e}^{t-z^2/4t}\right) dt.$$

Since $t^{-\nu-1} \mathrm{e}^{t-z^2/4t}$ is analytic in $0 < |z| < \infty$ and $|\arg(z)| < \pi$, and it vanishes at $z = -\infty$, the integral is zero. Thus the Bessel function of the first kind satisfies Bessel's equation for all complex orders.

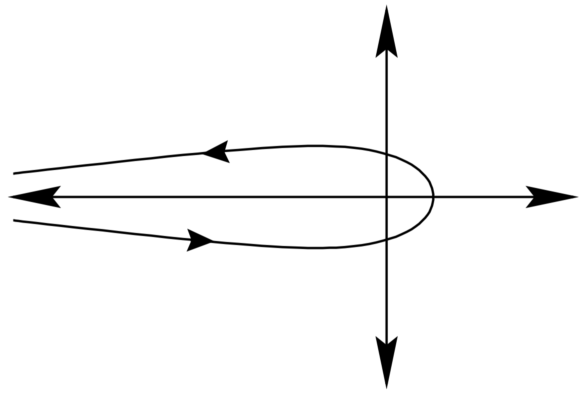

Figure 36.2: The Contour of Integration.

Series Expansion. Because of the e^t factor in the integrand, the integral defining J_ν converges uniformly. Expanding $\mathrm{e}^{-z^2/4t}$ in a Taylor series yields

$$J_\nu(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{2^{2m} m!} \oint_C t^{-\nu-m-1}\,\mathrm{e}^t\,dt$$

Since

$$\frac{1}{\Gamma(\alpha)} = \frac{1}{2\pi i}\oint_C t^{-\alpha-1}\,\mathrm{e}^t\,dt,$$

we have the series expansion of the Bessel function

$$\boxed{J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu+m+1)}\left(\frac{z}{2}\right)^{\nu+2m}.}$$

Linear Independence. The Wronskian of Bessel's equation is

$$W(z) = \exp\left(-\int^z \frac{1}{\zeta}\,d\zeta\right) = \mathrm{e}^{-\log z} = \frac{1}{z}.$$

Thus to within a function of ν, the Wronskian is $1/z$. For any given ν, there are two linearly independent solutions. Note that Bessel's equation is unchanged under the transformation $\nu \to -\nu$. Thus both J_ν and $J_{-\nu}$ satisfy Bessel's equation. Now we must determine if they are linearly independent. We have already shown that for integral values of ν they are not independent. ($J_{-n} = (-1)^n J_n$.) Assume that ν is not an integer. The Wronskian of J_ν and $J_{-\nu}$ is

$$\begin{aligned} W[J_\nu, J_{-\nu}] &= \begin{vmatrix} J_\nu & J_{-\nu} \\ J'_\nu & J'_{-\nu} \end{vmatrix} \\ &= J_\nu J'_{-\nu} - J_{-\nu} J'_\nu \end{aligned}$$

Substituting in the expansion for J_ν,

$$\begin{aligned} &= \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{\nu+2m}\right) \left(\sum_{n=0}^{\infty} \frac{(-1)^n(-\nu+2n)}{n!\Gamma(-\nu+n+1)2} \left(\frac{z}{2}\right)^{-\nu+2n-1}\right) \\ &\quad - \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(-\nu+m+1)} \left(\frac{z}{2}\right)^{-\nu+2m}\right) \left(\sum_{n=0}^{\infty} \frac{(-1)^n(\nu+2n)}{n!\Gamma(\nu+n+1)2} \left(\frac{z}{2}\right)^{\nu+2n-1}\right). \end{aligned}$$

Since the Wronskian is a function of ν times $1/z$ the coefficients of all of the powers of z except $1/z$ must vanish.

$$\begin{aligned} &= \frac{-\nu}{z\Gamma(\nu+1)\Gamma(-\nu+1)} - \frac{\nu}{z\Gamma(-\nu+1)\Gamma(\nu+1)} \\ &= -\frac{2}{z\Gamma(\nu)\Gamma(1-\nu)} \end{aligned}$$

Using an identity for the Gamma function simplifies this expression.

$$= -\frac{2}{\pi z} \sin(\pi \nu)$$

Since the Wronskian is nonzero for non-integral ν, J_ν and $J_{-\nu}$ are independent functions when ν is not an integer. The general solution to the equation is then $aJ_\nu + bJ_{-\nu}$.

36.3.4 Recursion Formulas

In showing that J_ν satisfies Bessel's equation for arbitrary complex ν, we obtained

$$\oint_C \frac{\mathrm{d}}{\mathrm{d}t} \left(t^{-\nu} \, \mathrm{e}^{t - z^2/4t} \right) dt = 0.$$

Expanding the integral,

$$\oint_C \left(t^{-\nu} + \frac{z^2}{4} t^{-\nu-2} - \nu t^{-\nu-1} \right) \mathrm{e}^{t - z^2/4t} \, dt = 0.$$

$$\frac{1}{2\pi i} \left(\frac{z}{2} \right)^\nu \oint_C \left(t^{-\nu} + \frac{z^2}{4} t^{-\nu-2} - \nu t^{-\nu-1} \right) \mathrm{e}^{t - z^2/4t} \, dt = 0.$$

Since $J_\nu(z) = \frac{1}{2\pi i}(z/2)^\nu \oint_C t^{-\nu-1} \, \mathrm{e}^{t - z^2/4t} \, dt$,

$$\left[\left(\frac{2}{z} \right)^{-1} J_{\nu-1} + \left(\frac{2}{z} \right) \frac{z^2}{4} J_{\nu+1} - \nu J_\nu \right] = 0.$$

$$\boxed{J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{z} J_\nu}$$

Differentiating the integral expression for J_ν,

$$J'_\nu(z) = \frac{1}{2\pi i}\frac{\nu z^{\nu-1}}{2^\nu}\oint_C t^{-\nu-1}\,\mathrm{e}^{t-z^2/4t}\,dt + \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \oint_C t^{-\nu-1}\left(-\frac{z}{2t}\right)\,\mathrm{e}^{t-z^2/4t}\,dt$$

$$J'_\nu(z) = \frac{\nu}{z}\frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \oint_C t^{-\nu-1}\,\mathrm{e}^{t-z^2/4t}\,dt - \frac{1}{2\pi i}\left(\frac{z}{2}\right)^{\nu+1} \oint_C t^{-\nu-2}\,\mathrm{e}^{t-z^2/4t}\,dt$$

$$\boxed{J'_\nu = \frac{\nu}{z}J_\nu - J_{\nu+1}}$$

From the two relations we have derived you can show that

$$\boxed{J'_\nu = \frac{1}{2}(J_{\nu-1} + J_{\nu+1})} \quad \text{and} \quad \boxed{J'_\nu = J_{\nu-1} - \frac{\nu}{z}J_\nu.}$$

Result 36.3.1 The Bessel function of the first kind, $J_\nu(z)$, is defined,

$$J_\nu(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \oint_C t^{-\nu-1}\,\mathrm{e}^{t-z^2/4t}\,dt.$$

The Bessel function has the expansion,

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu+m+1)}\left(\frac{z}{2}\right)^{\nu+2m}.$$

The Wronskian of $J_\nu(z)$ and $J_{-\nu}(z)$ is

$$W(z) = -\frac{2}{\pi z}\sin(\pi\nu).$$

Thus $J_\nu(z)$ and $J_{-\nu}(z)$ are independent when ν is not an integer. The Bessel functions satisfy the recursion relations,

$$J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{z}J_\nu \qquad J_\nu' = \frac{\nu}{z}J_\nu - J_{\nu+1}$$
$$J_\nu' = \frac{1}{2}(J_{\nu-1} - J_{\nu+1}) \qquad J_\nu' = J_{\nu-1} - \frac{\nu}{z}J_\nu.$$

36.3.5 Bessel Functions of Half-Integral Order

Consider $J_{1/2}(z)$. Start with the series expansion

$$J_{1/2}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(1/2+m+1)} \left(\frac{z}{2}\right)^{1/2+2m}.$$

Use the identity $\Gamma(n+1/2) = \frac{(1)(3)\cdots(2n-1)}{2^n}\sqrt{\pi}$.

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{(-1)^m 2^{m+1}}{m!(1)(3)\cdots(2m+1)\sqrt{\pi}} \left(\frac{z}{2}\right)^{1/2+2m} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m 2^{m+1}}{(2)(4)\cdots(2m)\cdot(1)(3)\cdots(2m+1)\sqrt{\pi}} \left(\frac{1}{2}\right)^{1/2+m} z^{1/2+2m} \\
&= \left(\frac{2}{\pi z}\right)^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} z^{2m+1}
\end{aligned}$$

We recognize the sum as the Taylor series expansion of $\sin z$.

$$= \left(\frac{2}{\pi z}\right)^{1/2} \sin z$$

Using the recurrence relations,

$$J_{\nu+1} = \frac{\nu}{z} J_\nu - J'_\nu \quad \text{and} \quad J_{\nu-1} = \frac{\nu}{z} J_\nu + J'_\nu,$$

we can find $J_{n+1/2}$ for any integral n.

Example 36.3.1 To find $J_{3/2}(z)$,

$$\begin{aligned}
J_{3/2}(z) &= \frac{1/2}{z} J_{1/2}(z) - J'_{1/2}(z) \\
&= \frac{1/2}{z}\left(\frac{2}{\pi}\right)^{1/2} z^{-1/2}\sin z - \left(-\frac{1}{2}\right)\left(\frac{2}{\pi}\right)^{1/2} z^{-3/2}\sin z - \left(\frac{2}{\pi}\right)^{1/2} z^{-1/2}\cos z \\
&= 2^{-1/2}\pi^{-1/2}z^{-3/2}\sin z + 2^{-1/2}\pi^{-1/2}z^{-3/2}\sin z - 2^{-1/2}\pi^{-1/2}\cos z \\
&= \left(\frac{2}{\pi}\right)^{1/2} z^{-3/2}\sin z - \left(\frac{2}{\pi}\right)^{1/2} z^{-1/2}\cos z \\
&= \left(\frac{2}{\pi}\right)^{1/2}\left(z^{-3/2}\sin z - z^{-1/2}\cos z\right).
\end{aligned}$$

You can show that

$$J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\cos z.$$

Note that at a first glance it appears that $J_{3/2} \sim z^{-1/2}$ as $z \to 0$. However, if you expand the sine and cosine you will see that the $z^{-1/2}$ and $z^{1/2}$ terms vanish and thus $J_{3/2}(z) \sim z^{3/2}$ as $z \to 0$ as we showed previously.

Recall that we showed the asymptotic behavior as $x \to +\infty$ of Bessel functions to be linear combinations of

$$x^{-1/2}\sin(x + U_1(x)) \quad \text{and} \quad x^{-1/2}\cos(x + U_2(x))$$

where $U_1, U_2 \to 0$ as $x \to +\infty$.

36.4 Neumann Expansions

Consider expanding an analytic function in a series of Bessel functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_n J_n(z).$$

If $f(z)$ is analytic in the disk $|z| \leq r$ then we can write

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where the path of integration is $|\zeta| = r$ and $|z| < r$. If we were able to expand the function $\frac{1}{\zeta - z}$ in a series of Bessel functions, then we could interchange the order of summation and integration to get a Bessel series expansion of $f(z)$.

The Expansion of $\mathbf{1/(\zeta - z)}$. Assume that $\frac{1}{\zeta - z}$ has the uniformly convergent expansion

$$\frac{1}{\zeta - z} = c_0(\zeta) J_0(z) + 2 \sum_{n=1}^{\infty} c_n(\zeta) J_n(z),$$

where each $c_n(\zeta)$ is analytic. Note that

$$\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial z} \right) \frac{1}{\zeta - z} = \frac{-1}{(\zeta - z)^2} + \frac{1}{(\zeta - z)^2} = 0.$$

Thus we have

$$\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial z} \right) \left[c_0(\zeta) J_0(z) + 2 \sum_{n=1}^{\infty} c_n(\zeta) J_n(z) \right] = 0$$

$$\left[c_0' J_0 + 2 \sum_{n=1}^{\infty} c_n' J_n \right] + \left[c_0 J_0' + 2 \sum_{n=1}^{\infty} c_n J_n' \right] = 0.$$

Using the identity $2 J_n' = J_{n-1} - J_{n+1}$,

$$\left[c_0' J_0 + 2 \sum_{n=1}^{\infty} c_n' J_n \right] + \left[c_0 (-J_1) + \sum_{n=1}^{\infty} c_n (J_{n-1} - J_{n+1}) \right] = 0.$$

Collecting coefficients of J_n,

$$(c_0' + c_1)J_0 + \sum_{n=1}^{\infty}(2c_n' + c_{n+1} - c_{n-1})J_n = 0.$$

Equating the coefficients of J_n, we see that the c_n are given by the relations,

$$c_1 = -c_0', \quad \text{and} \quad c_{n+1} = c_{n-1} - 2c_n'.$$

We can evaluate $c_0(\zeta)$. Setting $z = 0$,

$$\frac{1}{\zeta} = c_0(\zeta)J_0(0) + 2\sum_{n=1}^{\infty} c_n(\zeta)J_n(0)$$
$$\frac{1}{\zeta} = c_0(\zeta).$$

Using the recurrence relations we can calculate the c_n's. The first few are:

$$c_1 = -\frac{-1}{\zeta^2} = \frac{1}{\zeta^2}$$
$$c_2 = \frac{1}{\zeta} - 2\frac{-2}{\zeta^3} = \frac{1}{\zeta} + \frac{4}{\zeta^3}$$
$$c_3 = \frac{1}{\zeta^2} - 2\left(\frac{-1}{\zeta^2} - \frac{12}{\zeta^4}\right) = \frac{3}{\zeta^2} + \frac{24}{\zeta^4}.$$

We see that c_n is a polynomial of degree $n + 1$ in $1/\zeta$. One can show that

$$c_n(\zeta) = \begin{cases} \frac{2^{n-1}n!}{\zeta^{n+1}}\left(1 + \frac{\zeta^2}{2(2n-2)} + \frac{\zeta^4}{2\cdot4\cdot(2n-2)(2n-4)} + \cdots + \frac{\zeta^n}{2\cdot4\cdots n\cdot(2n-2)\cdots(2n-n)}\right) & \text{for even } n \\ \frac{2^{n-1}n!}{\zeta^{n+1}}\left(1 + \frac{\zeta^2}{2(2n-2)} + \frac{\zeta^4}{2\cdot4\cdot(2n-2)(2n-4)} + \cdots + \frac{\zeta^{n-1}}{2\cdot4\cdots(n-1)\cdot(2n-2)\cdots(2n-(n-1))}\right) & \text{for odd } n \end{cases}$$

Uniform Convergence of the Series. We assumed before that the series expansion of $\frac{1}{\zeta - z}$ is uniformly convergent. The behavior of c_n and J_n are

$$c_n(\zeta) = \frac{2^{n-1}n!}{\zeta^{n+1}} + O(\zeta^{-n}), \qquad J_n(z) = \frac{z^n}{2^n n!} + O(z^{n+1}).$$

This gives us

$$c_n(\zeta)J_n(z) = \frac{1}{2\zeta}\left(\frac{z}{\zeta}\right)^n + O\left(\frac{1}{\zeta}\left(\frac{z}{\zeta}\right)^{n+1}\right).$$

If $\left|\frac{z}{\zeta}\right| = \rho < 1$ we can bound the series with the geometric series $\sum \rho^n$. Thus the series is uniformly convergent.

Neumann Expansion of an Analytic Function. Let $f(z)$ be a function that is analytic in the disk $|z| \leq r$. Consider $|z| < r$ and the path of integration along $|\zeta| = r$. Cauchy's integral formula tells us that

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Substituting the expansion for $\frac{1}{\zeta - z}$,

$$\begin{aligned}
&= \frac{1}{2\pi i}\oint f(\zeta)\left(c_o(\zeta)J_0(z) + 2\sum_{n=1}^{\infty} c_n(\zeta)J_n(z)\right)\,d\zeta \\
&= J_0(z)\frac{1}{2\pi i}\oint \frac{f(\zeta)}{\zeta}\,d\zeta + \sum_{n=1}^{\infty}\frac{J_n(z)}{\pi i}\oint c_n(\zeta)f(\zeta)\,d\zeta \\
&= J_0(z)f(0) + \sum_{n=1}^{\infty}\frac{J_n(z)}{\pi i}\oint c_n(\zeta)f(\zeta)\,d\zeta.
\end{aligned}$$

Result 36.4.1 let $f(z)$ be analytic in the disk, $|z| \leq r$. Consider $|z| < r$ and the path of integration along $|\zeta| = r$. $f(z)$ has the Bessel function series expansion

$$f(z) = J_0(z) f(0) + \sum_{n=1}^{\infty} \frac{J_n(z)}{\pi i} \oint c_n(\zeta) f(\zeta) \, d\zeta,$$

where the c_n satisfy

$$\frac{1}{\zeta - z} = c_0(\zeta) J_0(z) + 2 \sum_{n=1}^{\infty} c_n(\zeta) J_n(z).$$

36.5 Bessel Functions of the Second Kind

When ν is an integer, J_ν and $J_{-\nu}$ are not linearly independent. In order to find an second linearly independent solution, we define the Bessel function of the second kind, (also called **Weber's function**),

$$Y_\nu = \begin{cases} \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} & \text{when } \nu \text{ is not an integer} \\ \lim_{\mu \to \nu} \frac{J_\mu(z)\cos(\mu\pi) - J_{-\mu}(z)}{\sin(\mu\pi)} & \text{when } \nu \text{ is an integer.} \end{cases}$$

J_ν and Y_ν are linearly independent for all ν.

In Figure 36.3 Y_0 and Y_1 are plotted in solid and dashed lines, respectively.

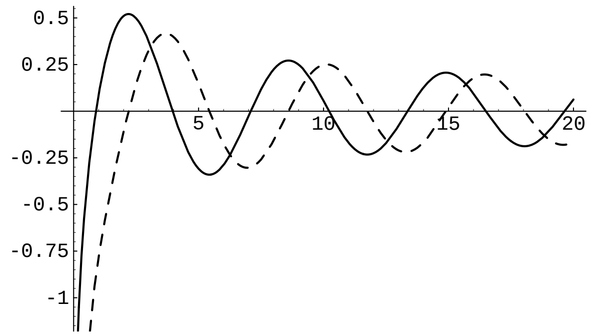

Figure 36.3: Bessel Functions of the Second Kind

Result 36.5.1 The Bessel function of the second kind, $Y_\nu(z)$, is defined,

$$Y_\nu = \begin{cases} \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} & \text{when } \nu \text{ is not an integer} \\ \lim_{\mu\to\nu} \frac{J_\mu(z)\cos(\mu\pi) - J_{-\mu}(z)}{\sin(\mu\pi)} & \text{when } \nu \text{ is an integer.} \end{cases}$$

The Wronskian of $J_\nu(z)$ and $Y_\nu(z)$ is

$$W[J_\nu, Y_\nu] = \frac{2}{\pi z}.$$

Thus $J_\nu(z)$ and $Y_\nu(z)$ are independent for all ν. The Bessel functions of the second kind satisfy the recursion relations,

$$Y_{\nu-1} + Y_{\nu+1} = \frac{2\nu}{z} Y_\nu \qquad Y_\nu' = \frac{\nu}{z} Y_\nu - Y_{\nu+1}$$

$$Y_\nu' = \frac{1}{2}(Y_{\nu-1} - Y_{\nu+1}) \qquad Y_\nu' = Y_{\nu-1} - \frac{\nu}{z} Y_\nu.$$

36.6 Hankel Functions

Another set of solutions to Bessel's equation is the Hankel functions,

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z),$$
$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z)$$

Result 36.6.1 The Hankel functions are defined

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z),$$
$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z)$$

The Wronskian of $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ is

$$W[H_\nu^{(1)}, H_\nu^{(2)}] = -\frac{4i}{\pi z}.$$

The Hankel functions are independent for all ν. The Hankel functions satisfy the same recurrence relations as the other Bessel functions.

36.7 The Modified Bessel Equation

The modified Bessel equation is

$$w'' + \frac{1}{z}w' - \left(1 + \frac{\nu^2}{z^2}\right) w = 0.$$

This equation is identical to the Bessel equation except for a sign change in the last term. If we make the change of variables $\xi = iz$, $u(\xi) = w(z)$ we obtain the equation

$$-u'' - \frac{1}{\xi}u' - \left(1 - \frac{\nu^2}{\xi^2}\right)u = 0$$
$$u'' + \frac{1}{\xi}u' + \left(1 - \frac{\nu^2}{\xi^2}\right)u = 0.$$

This is the Bessel equation. Thus $J_\nu(iz)$ is a solution to the modified Bessel equation. This motivates us to define the modified Bessel function of the first kind

$$I_\nu(z) = i^{-\nu} J_\nu(iz).$$

Since J_ν and $J_{-\nu}$ are linearly independent solutions when ν is not an integer, I_ν and $I_{-\nu}$ are linearly independent solutions to the modified Bessel equation when ν is not an integer.

The Taylor series expansion of $I_\nu(z)$ about $z = 0$ is

$$\begin{aligned}
I_\nu(z) &= i^{-\nu} J_\nu(iz) \\
&= i^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu+m+1)} \left(\frac{iz}{2}\right)^{\nu+2m} \\
&= i^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m i^\nu i^{2m}}{m!\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{\nu+2m} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{\nu+2m}
\end{aligned}$$

Modified Bessel Functions of the Second Kind. In order to have a second linearly independent solution when ν is an integer, we define the modified Bessel function of the second kind

$$K_\nu(z) = \begin{cases} \frac{\pi}{2} \frac{I_{-\nu} - I_\nu}{\sin(\nu\pi)} & \text{when } \nu \text{ is not an integer,} \\ \lim_{\mu \to \nu} \frac{\pi}{2} \frac{I_{-\mu} - I_\mu}{\sin(\mu\pi)} & \text{when } \nu \text{ is an integer.} \end{cases}$$

I_ν and K_ν are linearly independent for all ν. In Figure 36.4 I_0 and K_0 are plotted in solid and dashed lines, respectively.

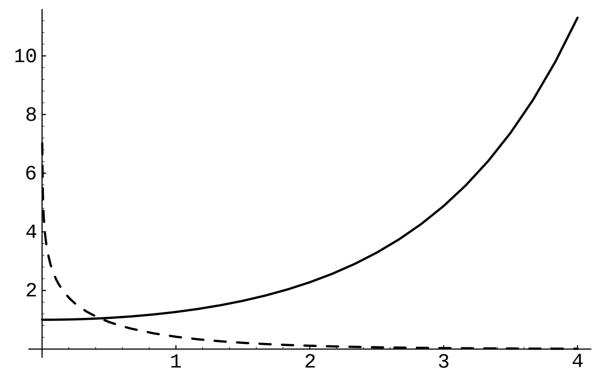

Figure 36.4: Modified Bessel Functions

Result 36.7.1 The modified Bessel functions of the first and second kind, $I_\nu(z)$ and $K_\nu(z)$, are defined,

$$I_\nu(z) = i^{-\nu} J_\nu(iz).$$

$$K_\nu(z) = \begin{cases} \frac{\pi}{2} \frac{I_{-\nu} - I_\nu}{\sin(\nu\pi)} & \text{when } \nu \text{ is not an integer,} \\ \lim_{\mu \to \nu} \frac{\pi}{2} \frac{I_{-\mu} - I_\mu}{\sin(\mu\pi)} & \text{when } \nu \text{ is an integer.} \end{cases}$$

The modified Bessel function of the first kind has the expansion,

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(\nu + m + 1)} \left(\frac{z}{2}\right)^{\nu+2m}$$

The Wronskian of $I_\nu(z)$ and $I_{-\nu}(z)$ is

$$W[I_\nu, I_{-\nu}] = -\frac{2}{\pi z} \sin(\pi\nu).$$

$I_\nu(z)$ and $I_{-\nu}(z)$ are linearly independent when ν is not an integer. The Wronskian of $I_\nu(z)$ and $K_\nu(z)$ is

$$W[I_\nu, K_\nu] = -\frac{1}{z}.$$

$I_\nu(z)$ and $K_\nu(z)$ are independent for all ν. The modified Bessel functions satisfy the recursion relations,

$$A_{\nu-1} - A_{\nu+1} = \frac{2\nu}{z} A_\nu \qquad A'_\nu = A_{\nu+1} + \frac{\nu}{z} A_\nu$$

$$A'_\nu = \frac{1}{2}(A_{\nu-1} + A_{\nu+1}) \qquad A'_\nu = A_{\nu-1} - \frac{\nu}{z} A_\nu.$$

where A stands for either I or K.

36.8 Exercises

Exercise 36.1
Consider Bessel's equation

$$z^2 y''(z) + z y'(z) + \left(z^2 - \nu^2\right) y = 0$$

where $\nu \geq 0$. Find the Frobenius series solution that is asymptotic to t^ν as $t \to 0$. By multiplying this solution by a constant, define the solution

$$J_\nu(z) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu}.$$

This is called the Bessel function of the first kind and order ν. Clearly $J_{-\nu}(z)$ is defined and is linearly independent to $J_\nu(z)$ if ν is not an integer. What happens when ν is an integer?

Exercise 36.2
Consider Bessel's equation for integer n,

$$z^2 y'' + z y' + (z^2 - n^2) y = 0.$$

Using the kernel

$$K(z, t) = \mathrm{e}^{\frac{1}{2} z \left(t - \frac{1}{t}\right)},$$

find two solutions of Bessel's equation. (For $n = 0$ you will find only one solution.) Are the two solutions linearly independent? Define the Bessel function of the first kind and order n,

$$J_n(z) = \frac{1}{i2\pi} \oint_C t^{-n-1} \, \mathrm{e}^{\frac{1}{2} z (t - 1/t)} \, dt,$$

where C is a simple, closed contour about the origin. Verify that

$$\mathrm{e}^{\frac{1}{2} z (t - 1/t)} = \sum_{n=-\infty}^{\infty} J_n(z) t^n.$$

This is the *generating function* for the Bessel functions.

Exercise 36.3
Use the generating function

$$e^{\frac{1}{2}z(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z) t^n$$

to show that J_n satisfies Bessel's equation

$$z^2 y'' + z y' + (z^2 - n^2) y = 0.$$

Exercise 36.4
Using

$$J_{n-1} + J_{n+1} = \frac{2n}{z} J_n \quad \text{and} \quad J_n' = \frac{n}{z} J_n - J_{n+1},$$

show that

$$J_n' = \frac{1}{2}(J_{n-1} - J_{n+1}) \quad \text{and} \quad J_n' = J_{n-1} - \frac{n}{z} J_n.$$

Exercise 36.5
Find the general solution of

$$w'' + \frac{1}{z} w' + \left(1 - \frac{1}{4z^2}\right) w = z.$$

Exercise 36.6
Show that $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent for all ν.

Exercise 36.7
Compute $W[I_\nu, I_{-\nu}]$ and $W[I_\nu, K_\nu]$.

Exercise 36.8
Using the generating function,

$$\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} J_n(z) t^n,$$

verify the following identities:

1.
$$\frac{2n}{z} J_n(z) = J_{n-1}(z) + J_{n+1}(z).$$
This relation is useful for recursively computing the values of the higher order Bessel functions.

2.
$$J_n'(z) = \frac{1}{2}\left(J_{n-1} - J_{n+1}\right).$$
This relation is useful for computing the derivatives of the Bessel functions once you have the values of the Bessel functions of adjacent order.

3.
$$\frac{d}{dz}\left(z^{-n} J_n(z)\right) = -z^{-n} J_{n+1}(z).$$

Exercise 36.9
Use the Wronskian of $J_\nu(z)$ and $J_{-\nu}(z)$,

$$W\left[J_\nu(z), J_{-\nu}(z)\right] = -\frac{2\sin\nu\pi}{\pi z},$$

to derive the identity

$$J_{-\nu+1}(z)J_\nu(z) + J_{-\nu}(z)J_{\nu-1}(z) = \frac{2}{\pi z}\sin\nu\pi.$$

Exercise 36.10
Show that, using the generating function or otherwise,

$$\begin{aligned} J_0(z) + 2J_2(z) + 2J_4(z) + 2J_6(z) + \cdots &= 1 \\ J_0(z) - 2J_2(z) + 2J_4(z) - 2J_6(z) + \cdots &= \cos z \\ 2J_1(z) - 2J_3(z) + 2J_5(z) - \cdots &= \sin z \\ J_0^2(z) + 2J_1^2(z) + 2J_2^2(z) + 2J_3^2(z) + \cdots &= 1 \end{aligned}$$

Exercise 36.11
It is often possible to "solve" certain ordinary differential equations by converting them into the Bessel equation by means of various transformations. For example, show that the solution of

$$y'' + x^{p-2}y = 0,$$

can be written in terms of Bessel functions.

$$y(x) = c_1 x^{1/2} J_{1/p}\left(\frac{2}{p}x^{p/2}\right) + c_2 x^{1/2} Y_{1/p}\left(\frac{2}{p}x^{p/2}\right)$$

Here c_1 and c_2 are arbitrary constants. Thus show that the Airy equation,

$$y'' + xy = 0,$$

can be solved in terms of Bessel functions.

Exercise 36.12
The spherical Bessel functions are defined by

$$j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z),$$
$$y_n(z) = \sqrt{\frac{\pi}{2z}} Y_{n+1/2}(z),$$
$$k_n(z) = \sqrt{\frac{\pi}{2z}} K_{n+1/2}(z),$$
$$i_n(z) = \sqrt{\frac{\pi}{2z}} I_{n+1/2}(z).$$

Show that

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z},$$
$$i_0(z) = \frac{\sinh z}{z},$$
$$k_0(z) = \frac{\pi}{2z} \exp(-z).$$

Exercise 36.13
Show that as $x \to \infty$,

$$K_n(x) \propto \frac{\mathrm{e}^{-x}}{\sqrt{x}} \left(1 + \frac{4n^2 - 1}{8x} + \frac{(4n^2 - 1)(4n^2 - 9)}{128x^2} + \cdots \right).$$

36.9 Hints

Hint 36.2

Hint 36.3

Hint 36.4
Use the generating function

$$e^{\frac{1}{2}z(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z) t^n$$

to show that J_n satisfies Bessel's equation

$$z^2 y'' + z y' + (z^2 - n^2) y = 0.$$

Hint 36.6
Use variation of parameters and the Wronskian that was derived in the text.

Hint 36.7
Compute the Wronskian of $J_\nu(z)$ and $Y_\nu(z)$. Use the relation

$$W\left[J_\nu, J_{-\nu}\right] = -\frac{2}{\pi z} \sin(\pi\nu)$$

Hint 36.8
Derive $W[I_\nu, I_{-\nu}]$ from the value of $W[J_\nu, J_{-\nu}]$. Derive $W[I_\nu, K_\nu]$ from the value of $W[I_\nu, I_{-\nu}]$.

Hint 36.9

Hint 36.10

Hint 36.11

Hint 36.12

Hint 36.13

Hint 36.14

36.10 Solutions

Solution 36.1

Bessel's equation is

$$L[y] \equiv z^2 y'' + z y' + (z^2 - n^2) y = 0.$$

We consider a solution of the form

$$y(z) = \int_C \mathrm{e}^{\frac{1}{2}z(t-1/t)} v(t)\, dt.$$

We substitute the form of the solution into Bessel's equation.

$$\int_C L\left[\mathrm{e}^{\frac{1}{2}z(t-1/t)}\right] v(t)\, dt = 0$$

$$\int_C \left(z^2 \frac{1}{4}\left(t + \frac{1}{t}\right)^2 + z\frac{1}{2}\left(t - \frac{1}{t}\right)^2 + \left(z^2 - n^2\right)\right) \mathrm{e}^{\frac{1}{2}z(t-1/t)} v(t)\, dt = 0 \tag{36.1}$$

By considering

$$\frac{\mathrm{d}}{\mathrm{d}t} t\, \mathrm{e}^{\frac{1}{2}z(t-1/t)} = \left(\frac{1}{2}x\left(t + \frac{1}{t}\right) + 1\right) \mathrm{e}^{\frac{1}{2}z(t-1/t)}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} t^2\, \mathrm{e}^{\frac{1}{2}z(t-1/t)} = \left(\frac{1}{4}x^2\left(t + \frac{1}{t}\right)^2 + x\left(2t + \frac{1}{t}\right) + 2\right) \mathrm{e}^{\frac{1}{2}z(t-1/t)}$$

we see that

$$L\left[\mathrm{e}^{\frac{1}{2}z(t-1/t)}\right] = \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} t^2 - 3\frac{\mathrm{d}}{\mathrm{d}t} t + (1 - n^2)\right) \mathrm{e}^{\frac{1}{2}z(t-1/t)}.$$

Thus Equation 36.1 becomes

$$\int_C \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} t^2\, \mathrm{e}^{\frac{1}{2}z(t-1/t)} - 3\frac{\mathrm{d}}{\mathrm{d}t} t\, \mathrm{e}^{\frac{1}{2}z(t-1/t)} + (1 - n^2)\, \mathrm{e}^{\frac{1}{2}z(t-1/t)}\right) v(t)\, dt = 0$$

We apply integration by parts to move derivatives from the kernel to $v(t)$.

$$\left[t^2\,\mathrm{e}^{\frac{1}{2}z(t-1/t)}v(t)\right]_C - \left[t\,\mathrm{e}^{\frac{1}{2}z(t-1/t)}v'(t)\right]_C + \left[-3t\,\mathrm{e}^{\frac{1}{2}z(t-1/t)}v(t)\right]_C + \int_C \mathrm{e}^{\frac{1}{2}z(t-1/t)}\left(t^2v''(t)+3tv(t)+(1-n^2)v(t)\right)\,dt = 0$$

In order that the integral vanish, $v(t)$ must be a solution of the differential equation

$$t^2v'' + 3tv + (1-n^2)v = 0.$$

This is an Euler equation with the solutions $\{t^{n-1}, t^{-n-1}\}$ for non-zero n and $\{t^{-1}, t^{-1}\log t\}$ for $n = 0$.

Consider the case of non-zero n. Since

$$\mathrm{e}^{\frac{1}{2}z(t-1/t)}\left((t^2-3t)v(t) - tv'(t)\right)$$

is single-valued and analytic for $t \neq 0$ for the functions $v(t) = t^{n-1}$ and $v(t) = t^{-n-1}$, the boundary term will vanish if C is any closed contour that that does not pass through the origin. Note that the integrand in our solution,

$$\mathrm{e}^{\frac{1}{2}z(t-1/t)}v(t),$$

is analytic and single-valued except at the origin and infinity where it has essential singularities. Consider a simple closed contour that does not enclose the origin. The integral along such a path would vanish and give us $y(z) = 0$. This is not an interesting solution. Since

$$\mathrm{e}^{\frac{1}{2}z(t-1/t)}v(t),$$

has non-zero residues for $v(t) = t^{n-1}$ and $v(t) = t^{-n-1}$, choosing any simple, positive, closed contour about the origin will give us a non-trivial solution of Bessel's equation. These solutions are

$$y_1(t) = \int_C t^{n-1}\,\mathrm{e}^{\frac{1}{2}z(t-1/t)}\,dt, \quad y_2(t) = \int_C t^{-n-1}\,\mathrm{e}^{\frac{1}{2}z(t-1/t)}\,dt.$$

Now consider the case $n = 0$. The two solutions above concide and we have the solution

$$y(t) = \int_C t^{-1}\,\mathrm{e}^{\frac{1}{2}z(t-1/t)}\,dt.$$

Choosing $v(t) = t^{-1} \log t$ would make both the boundary terms and the integrand multi-valued. We do not pursue the possibility of a solution of this form.

The solution $y_1(t)$ and $y_2(t)$ are not linearly independent. To demonstrate this we make the change of variables $t \to -1/t$ in the integral representation of $y_1(t)$.

$$\begin{aligned} y_1(t) &= \int_C t^{n-1} \, \mathrm{e}^{\frac{1}{2}z(t-1/t)} \, dt \\ &= \int_C (-1/t)^{n-1} \, \mathrm{e}^{\frac{1}{2}z(-1/t+t)} \frac{-1}{t^2} \, dt \\ &= \int_C (-1)^n t^{-n-1} \, \mathrm{e}^{\frac{1}{2}z(t-1/t)} \, dt \\ &= (-1)^n y_2(t) \end{aligned}$$

Thus we see that a solution of Bessel's equation for integer n is

$$y(t) = \int_C t^{-n-1} \, \mathrm{e}^{\frac{1}{2}z(t-1/t)} \, dt$$

where C is any simple, closed contour about the origin.

Therefore, the Bessel function of the first kind and order n,

$$J_n(z) = \frac{1}{i2\pi} \oint_C t^{-n-1} \, \mathrm{e}^{\frac{1}{2}z(t-1/t)} \, dt$$

is a solution of Bessel's equation for integer n. Note that $J_n(z)$ is the coefficient of t^n in the Laurent series of $\mathrm{e}^{\frac{1}{2}z(t-1/t)}$. This establishes the generating function for the Bessel functions.

$$\mathrm{e}^{\frac{1}{2}z(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z) t^n$$

Solution 36.2
The generating function is

$$\mathrm{e}^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z) t^n.$$

In order to show that J_n satisfies Bessel's equation we seek to show that

$$\sum_{n=-\infty}^{\infty} \left(z^2 J_n''(z) + z J_n(z) + (z^2 - n^2) J_n(z) \right) t^n = 0.$$

To get the appropriate terms in the sum we will differentiate the generating function with respect to z and t. First we differentiate it with respect to z.

$$\frac{1}{2}\left(t - \frac{1}{t}\right) \mathrm{e}^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n'(z) t^n$$

$$\frac{1}{4}\left(t - \frac{1}{t}\right)^2 \mathrm{e}^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n''(z) t^n$$

Now we differentiate with respect to t and multiply by t get the $n^2 J_n$ term.

$$\frac{z}{2}\left(1 + \frac{1}{t^2}\right) \mathrm{e}^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} n J_n(z) t^{n-1}$$

$$\frac{z}{2}\left(t + \frac{1}{t}\right) \mathrm{e}^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} n J_n(z) t^{n}$$

$$\frac{z}{2}\left(1 - \frac{1}{t^2}\right) \mathrm{e}^{\frac{z}{2}(t-1/t)} + \frac{z^2}{4}\left(t + \frac{1}{t}\right)^2 \mathrm{e}^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} n^2 J_n(z) t^{n-1}$$

$$\frac{z}{2}\left(t - \frac{1}{t}\right) \mathrm{e}^{\frac{z}{2}(t-1/t)} + \frac{z^2}{4}\left(t + \frac{1}{t}\right)^2 \mathrm{e}^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} n^2 J_n(z) t^{n}$$

Now we can evaluate the desired sum.

$$\sum_{n=-\infty}^{\infty} \left(z^2 J_n''(z) + zJ_n(z) + (z^2 - n^2)J_n(z)\right) t^n$$
$$= \left(\frac{z^2}{4}\left(t - \frac{1}{t}\right)^2 + \frac{z}{2}\left(t - \frac{1}{t}\right) + z^2 - \frac{z}{2}\left(t - \frac{1}{t}\right) - \frac{z^2}{4}\left(t + \frac{1}{t}\right)^2\right) \mathrm{e}^{\frac{z}{2}(t-1/t)}$$

$$\sum_{n=-\infty}^{\infty} \left(z^2 J_n''(z) + zJ_n(z) + (z^2 - n^2)J_n(z)\right) t^n = 0$$

$$\boxed{z^2 J_n''(z) + zJ_n(z) + (z^2 - n^2)J_n(z) = 0}$$

Thus J_n satisfies Bessel's equation.

Solution 36.3

$$\begin{aligned} J_n' &= \frac{n}{z} J_n - J_{n+1} \\ &= \frac{1}{2}(J_{n-1} + J_{n+1}) - J_{n+1} \\ &= \frac{1}{2}(J_{n-1} - J_{n+1}) \end{aligned}$$

$$\begin{aligned} J_n' &= \frac{n}{z} J_n - J_{n+1} \\ &= \frac{n}{z} J_n - \left(\frac{2n}{z} J_n - J_{n-1}\right) \\ &= J_{n-1} - \frac{n}{z} J_n \end{aligned}$$

Solution 36.4

The linearly independent homogeneous solutions are $J_{1/2}$ and $J_{-1/2}$. The Wronskian is

$$W[J_{1/2}, J_{-1/2}] = -\frac{2}{\pi z}\sin(\pi/2) = -\frac{2}{\pi z}.$$

Using variation of parameters, a particular solution is

$$\begin{aligned} y_p &= -J_{1/2}(z)\int^z \frac{\zeta J_{-1/2}(\zeta)}{-2/\pi\zeta}\,d\zeta + J_{-1/2}(z)\int^z \frac{\zeta J_{1/2}(\zeta)}{-2/\pi\zeta}\,d\zeta \\ &= \frac{\pi}{2}J_{1/2}(z)\int^z \zeta^2 J_{-1/2}(\zeta)\,d\zeta - \frac{\pi}{2}J_{-1/2}(z)\int^z \zeta^2 J_{1/2}(\zeta)\,d\zeta. \end{aligned}$$

Thus the general solution is

$$\boxed{y = c_1 J_{1/2}(z) + c_2 J_{-1/2}(z) + \frac{\pi}{2}J_{1/2}(z)\int^z \zeta^2 J_{-1/2}(\zeta)\,d\zeta - \frac{\pi}{2}J_{-1/2}(z)\int^z \zeta^2 J_{1/2}(\zeta)\,d\zeta.}$$

We could substitute

$$J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\sin z \quad \text{and} \quad J_{-1/2} = \left(\frac{2}{\pi z}\right)^{1/2}\cos z$$

into the solution, but we cannot evaluate the integrals in terms of elementary functions. (You can write the solution in terms of Fresnel integrals.)

Solution 36.5

$$\begin{aligned}
W[J_\nu, Y_\nu] &= \begin{vmatrix} J_\nu & J_\nu \cot(\nu\pi) - J_{-\nu}\csc(\nu\pi) \\ J'_\nu & J'_\nu \cot(\nu\pi) - J'_{-\nu}\csc(\nu\pi) \end{vmatrix} \\
&= \cot(\nu\pi) \begin{vmatrix} J_\nu & J_\nu \\ J'_\nu & J'_\nu \end{vmatrix} - \csc(\nu\pi) \begin{vmatrix} J_\nu & J_{-\nu} \\ J'_\nu & J'_{-\nu} \end{vmatrix} \\
&= -\csc(\nu\pi) \frac{-2}{\pi z} \sin(\pi\nu) \\
&= \frac{2}{\pi z}
\end{aligned}$$

Since the Wronskian does not vanish identically, the functions are independent for all values of ν.

Solution 36.6

$$I_\nu(z) = i^{-\nu} J_\nu(iz)$$

$$\begin{aligned}
W[I_\nu, I_{-\nu}] &= \begin{vmatrix} I_\nu & I_{-\nu} \\ I'_\nu & I'_{-\nu} \end{vmatrix} \\
&= \begin{vmatrix} i^{-\nu} J_\nu(iz) & i^{\nu} J_{-\nu}(iz) \\ i^{-\nu} i J'_\nu(iz) & i^{\nu} i J'_{-\nu}(iz) \end{vmatrix} \\
&= i \begin{vmatrix} J_\nu(iz) & J_{-\nu}(iz) \\ J'_\nu(iz) & J'_{-\nu}(iz) \end{vmatrix} \\
&= i \frac{-2}{\pi i z} \sin(\pi\nu) \\
&= -\frac{2}{\pi z} \sin(\pi\nu)
\end{aligned}$$

$$\begin{aligned}
W\left[I_\nu, K_\nu\right] &= \begin{vmatrix} I_\nu & \frac{\pi}{2}\csc(\pi\nu)(I_{-\nu} - I_\nu) \\ I_\nu' & \frac{\pi}{2}\csc(\pi\nu)(I_{-\nu}' - I_\nu') \end{vmatrix} \\
&= \frac{\pi}{2}\csc(\pi\nu)\left(\begin{vmatrix} I_\nu & I_{-\nu} \\ I_\nu' & I_{-\nu}' \end{vmatrix} - \begin{vmatrix} I_\nu & I_\nu \\ I_\nu' & I_\nu' \end{vmatrix} \right) \\
&= \frac{\pi}{2}\csc(\pi\nu)\frac{-2}{\pi z}\sin(\pi\nu) \\
&= -\frac{1}{z}
\end{aligned}$$

Solution 36.7

1. We diferentiate the generating function with respect to t.

$$\begin{gathered}
\mathrm{e}^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z) t^n \\
\frac{z}{2}\left(1 + \frac{1}{t^2}\right)\mathrm{e}^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z) n t^{n-1} \\
\left(1 + \frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} J_n(z) t^n = \frac{2}{z}\sum_{n=-\infty}^{\infty} J_n(z) n t^{n-1} \\
\sum_{n=-\infty}^{\infty} J_n(z) t^n + \sum_{n=-\infty}^{\infty} J_n(z) t^{n-2} = \frac{2}{z}\sum_{n=-\infty}^{\infty} J_n(z) n t^{n-1} \\
\sum_{n=-\infty}^{\infty} J_{n-1}(z) t^{n-1} + \sum_{n=-\infty}^{\infty} J_{n+1}(z) t^{n-1} = \frac{2}{z}\sum_{n=-\infty}^{\infty} J_n(z) n t^{n-1} \\
J_{n-1}(z) + J_{n+1}(z) = \frac{2}{z} J_n(z) n \\
\boxed{\frac{2n}{z} J_n(z) = J_{n-1}(z) + J_{n+1}(z)}
\end{gathered}$$

2. We diferentiate the generating function with respect to z.

$$\mathrm{e}^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z) t^n$$

$$\frac{1}{2}\left(t - \frac{1}{t}\right) \mathrm{e}^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n'(z) t^n$$

$$\frac{1}{2}\left(t - \frac{1}{t}\right) \sum_{n=-\infty}^{\infty} J_n(z) t^n = \sum_{n=-\infty}^{\infty} J_n'(z) t^n$$

$$\frac{1}{2}\left(\sum_{n=-\infty}^{\infty} J_n(z) t^{n+1} - \sum_{n=-\infty}^{\infty} J_n(z) t^{n-1}\right) = \sum_{n=-\infty}^{\infty} J_n'(z) t^n$$

$$\frac{1}{2}\left(\sum_{n=-\infty}^{\infty} J_{n-1}(z) t^{n} - \sum_{n=-\infty}^{\infty} J_{n+1}(z) t^{n}\right) = \sum_{n=-\infty}^{\infty} J_n'(z) t^n$$

$$\frac{1}{2}\left(J_{n-1}(z) - J_{n+1}(z)\right) = J_n'(z)$$

$$\boxed{J_n'(z) = \frac{1}{2}\left(J_{n-1} - J_{n+1}\right)}$$

3.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-n} J_n(z)\right) &= -n z^{-n-1} J_n(z) + z^{-n} J_n'(z) \\
&= -\frac{1}{2} z^{-n} \frac{2n}{z} J_n(z) + z^{-n} \frac{1}{2}\left(J_{n-1}(z) - J_{n+1}(z)\right) \\
&= -\frac{1}{2} z^{-n}\left(J_{n+1}(z) + J_{n-1}(z)\right) + \frac{1}{2} z^{-n}\left(J_{n-1}(z) - J_{n+1}(z)\right)
\end{aligned}$$

$$\boxed{\frac{d}{dz}\left(z^{-n} J_n(z)\right) = -z^{-n} J_{n+1}(z)}$$

Solution 36.8
For this part we will use the identities

$$J_\nu'(z) = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z), \quad J_\nu'(z) = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z).$$

$$\begin{vmatrix} J_\nu(z) & J_{-\nu}(z) \\ J_\nu'(z) & J_{-\nu}'(z) \end{vmatrix} = -\frac{2\sin(\nu\pi)}{\pi z}$$
$$\begin{vmatrix} J_\nu(z) & J_{-\nu}(z) \\ J_{\nu-1}(z) - \frac{\nu}{z} J_\nu & -\frac{\nu}{z} J_{-\nu}(z) - J_{-\nu+1}(z) \end{vmatrix} = -\frac{2\sin(\nu\pi)}{\pi z}$$
$$\begin{vmatrix} J_\nu(z) & J_{-\nu}(z) \\ J_{\nu-1}(z) & -J_{-\nu+1}(z) \end{vmatrix} - \frac{\nu}{z} \begin{vmatrix} J_\nu(z) & J_{-\nu}(z) \\ J_\nu(z) & J_{-\nu}(z) \end{vmatrix} = -\frac{2\sin(\nu\pi)}{\pi z}$$
$$-J_{\nu+1}(z) J_\nu(z) - J_\nu(z) J_{\nu-1}(z) = -\frac{2\sin(\nu\pi)}{\pi z}$$
$$\boxed{J_{-\nu+1}(z) J_\nu(z) + J_{-\nu}(z) J_{\nu-1}(z) = \frac{2}{\pi z} \sin \nu\pi}$$

Solution 36.9
The generating function for the Bessel functions is

$$\mathrm{e}^{\frac{1}{2} z(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z) t^n. \tag{36.2}$$

1. We substitute $t = 1$ into (36.2).

$$\sum_{n=-\infty}^{\infty} J_n(z) = 1$$
$$J_0(z) + \sum_{n=1}^{\infty} J_n(z) + \sum_{n=1}^{\infty} J_{-n}(z) = 1$$

We use the identity $J_{-n} = (-1)^n J_n$.

$$J_0(z) + \sum_{n=1}^{\infty} (1 + (-1)^n) J_n(z) = 1$$

$$J_0(z) + 2 \sum_{\substack{n=2 \\ \text{even } n}}^{\infty} J_n(z) = 1$$

$$\boxed{J_0(z) + 2 \sum_{n=1}^{\infty} J_{2n}(z) = 1}$$

2. We substitute $t = i$ into (36.2).

$$\sum_{n=-\infty}^{\infty} J_n(z) i^n = \mathrm{e}^{iz}$$

$$J_0(z) + \sum_{n=1}^{\infty} J_n(z) i^n + \sum_{n=1}^{\infty} J_{-n}(z) i^{-n} = \mathrm{e}^{iz}$$

$$J_0(z) + \sum_{n=1}^{\infty} J_n(z) i^n + \sum_{n=1}^{\infty} (-1)^n J_n(z) (-i)^n = \mathrm{e}^{iz}$$

$$J_0(z) + 2 \sum_{n=1}^{\infty} J_n(z) i^n = \mathrm{e}^{iz} \tag{36.3}$$

Substituting $t = -i$ into (36.2) yields

$$J_0(z) + 2 \sum_{n=1}^{\infty} (-1)^n J_n(z) i^n = \mathrm{e}^{-iz} \tag{36.4}$$

Dividing the sum of (36.3) and (36.4) by 2 gives us the desired identity.

$$J_0(z) + \sum_{n=1}^{\infty} (1 + (-1)^n) J_n(z) i^n = \cos z$$

$$J_0(z) + 2 \sum_{\substack{n=2 \\ \text{even } n}}^{\infty} J_n(z) i^n = \cos z$$

$$J_0(z) + 2 \sum_{\substack{n=2 \\ \text{even } n}}^{\infty} (-1)^{n/2} J_n(z) = \cos z$$

$$\boxed{J_0(z) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(z) = \cos z}$$

3. Dividing the difference of (36.3) and (36.4) by $2i$ gives us the other identity.

$$-i \sum_{n=1}^{\infty} (1 - (-1)^n) J_n(z) i^n = \sin z$$

$$2 \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} J_n(z) i^{n-1} = \sin z$$

$$2 \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} (-1)^{(n-1)/2} J_n(z) = \sin z$$

$$\boxed{2 \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(z) = \sin z}$$

4. Substituting $-t$ for t in (36.2) yields

$$e^{-\frac{1}{2}z(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z)(-t)^n. \tag{36.5}$$

We take the product of (36.2) and (36.5) to obtain the final identity.

$$\left(\sum_{n=-\infty}^{\infty} J_n(z)t^n\right)\left(\sum_{m=-\infty}^{\infty} J_m(z)(-t)^m\right) = e^{\frac{1}{2}z(t-1/t)}\, e^{-\frac{1}{2}z(t-1/t)} = 1$$

Note that the coefficients of all powers of t except t^0 in the product of sums must vanish.

$$\sum_{n=-\infty}^{\infty} J_n(z)t^n J_{-n}(z)(-t)^{-n} = 1$$

$$\sum_{n=-\infty}^{\infty} J_n^2(z) = 1$$

$$\boxed{J_0^2(z) + 2\sum_{n=1}^{\infty} J_n^2(z) = 1}$$

Solution 36.10

First we make the change of variables $y(x) = x^{1/2}v(x)$. We compute the derivatives of $y(x)$.

$$y' = x^{1/2}v' + \frac{1}{2}x^{-1/2}v,$$
$$y'' = x^{1/2}v'' + x^{-1/2}v' - \frac{1}{4}x^{-3/2}v.$$

We substitute these into the differential equation for y.

$$y'' + x^{p-2} y = 0$$
$$x^{1/2} v'' + x^{-1/2} v' - \frac{1}{4} x^{-3/2} v + x^{p-3/2} v = 0$$
$$x^2 v'' + x v' + \left(x^p - \frac{1}{4} \right) v = 0$$

Then we make the change of variables $v(x) = u(\xi)$, $\xi = \frac{2}{p} x^{p/2}$. We write the derivatives in terms of ξ.

$$x \frac{\mathrm{d}}{\mathrm{d}x} = x \frac{\mathrm{d}\xi}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}\xi} = x x^{p/2-1} \frac{\mathrm{d}}{\mathrm{d}\xi} = \frac{p}{2} \xi \frac{\mathrm{d}}{\mathrm{d}\xi}$$
$$x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + x \frac{\mathrm{d}}{\mathrm{d}x} = x \frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}}{\mathrm{d}x} = \frac{p}{2} \xi \frac{\mathrm{d}}{\mathrm{d}\xi} \frac{p}{2} \xi \frac{\mathrm{d}}{\mathrm{d}\xi} = \frac{p^2}{4} \xi^2 \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{p^2}{4} \xi \frac{\mathrm{d}}{\mathrm{d}\xi}$$

We write the differential equation for $u(\xi)$.

$$\frac{p^2}{4} \xi^2 u'' + \frac{p^2}{4} \xi u' + \left(\frac{p^2}{4} \xi^2 - \frac{1}{4} \right) u = 0$$
$$u'' + \frac{1}{\xi} u' + \left(1 - \frac{1}{p^2 \xi^2} \right) u = 0$$

This is the Bessel equation of order $1/p$. We can write the general solution for u in terms of Bessel functions of the first kind if $p \neq \pm 1$. Otherwise, we use a Bessel function of the second kind.

$$u(\xi) = c_1 J_{1/p}(\xi) + c_2 J_{-1/p}(\xi) \text{ for } p \neq 0, \pm 1$$
$$u(\xi) = c_1 J_{1/p}(\xi) + c_2 Y_{1/p}(\xi) \text{ for } p \neq 0$$

We write the solution in terms of $y(x)$.

$$\boxed{y(x) = c_1\sqrt{x}J_{1/p}\left(\frac{2}{p}x^{p/2}\right) + c_2\sqrt{x}J_{-1/p}\left(\frac{2}{p}x^{p/2}\right) \quad \text{for } p \neq 0, \pm 1}$$

$$\boxed{y(x) = c_1\sqrt{x}J_{1/p}\left(\frac{2}{p}x^{p/2}\right) + c_2\sqrt{x}Y_{1/p}\left(\frac{2}{p}x^{p/2}\right) \quad \text{for } p \neq 0}$$

The Airy equation $y'' + xy = 0$ is the case $p = 3$. The general solution of the Airy equation is

$$\boxed{y(x) = c_1\sqrt{x}J_{1/3}\left(\frac{2}{3}x^{3/2}\right) + c_2\sqrt{x}J_{-1/3}\left(\frac{2}{3}x^{3/2}\right).}$$

Solution 36.11

Consider $J_{1/2}(z)$. We start with the series expansion.

$$J_{1/2}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(1/2+m+1)}\left(\frac{z}{2}\right)^{1/2+2m}.$$

Use the identity $\Gamma(n+1/2) = \frac{(1)(3)\cdots(2n-1)}{2^n}\sqrt{\pi}$.

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{(-1)^m 2^{m+1}}{m!(1)(3)\cdots(2m+1)\sqrt{\pi}}\left(\frac{z}{2}\right)^{1/2+2m} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m 2^{m+1}}{(2)(4)\cdots(2m)\cdot(1)(3)\cdots(2m+1)\sqrt{\pi}}\left(\frac{1}{2}\right)^{1/2+m} z^{1/2+2m} \\
&= \left(\frac{2}{\pi z}\right)^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} z^{2m+1}
\end{aligned}$$

We recognize the sum as the Taylor series expansion of $\sin z$.

$$= \left(\frac{2}{\pi z}\right)^{1/2} \sin z$$

Using the recurrence relations,

$$J_{\nu+1} = \frac{\nu}{z} J_\nu - J'_\nu \quad \text{and} \quad J_{\nu-1} = \frac{\nu}{z} J_\nu + J'_\nu,$$

we can find $J_{n+1/2}$ for any integral n.

We need $J_{3/2}(z)$ to determine $j_1(z)$. To find $J_{3/2}(z)$,

$$\begin{aligned}
J_{3/2}(z) &= \frac{1/2}{z} J_{1/2}(z) - J'_{1/2}(z) \\
&= \frac{1/2}{z} \left(\frac{2}{\pi}\right)^{1/2} z^{-1/2} \sin z - \left(-\frac{1}{2}\right) \left(\frac{2}{\pi}\right)^{1/2} z^{-3/2} \sin z - \left(\frac{2}{\pi}\right)^{1/2} z^{-1/2} \cos z \\
&= 2^{-1/2} \pi^{-1/2} z^{-3/2} \sin z + 2^{-1/2} \pi^{-1/2} z^{-3/2} \sin z - 2^{-1/2} \pi^{-1/2} \cos z \\
&= \left(\frac{2}{\pi}\right)^{1/2} z^{-3/2} \sin z - \left(\frac{2}{\pi}\right)^{1/2} z^{-1/2} \cos z \\
&= \left(\frac{2}{\pi}\right)^{1/2} \left(z^{-3/2} \sin z - z^{-1/2} \cos z\right).
\end{aligned}$$

The spherical Bessel function $j_1(z)$ is

$$\boxed{j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}.}$$

The modified Bessel function of the first kind is

$$I_\nu(z) = i^{-\nu} J_\nu(iz).$$

We can determine $I_{1/2}(z)$ from $J_{1/2}(z)$.

$$
\begin{aligned}
I_{1/2}(z) &= i^{-1/2}\sqrt{\frac{2}{\pi i z}}\sin(iz) \\
&= -i\sqrt{\frac{2}{\pi z}} i \sinh(z) \\
&= \sqrt{\frac{2}{\pi z}}\sinh(z)
\end{aligned}
$$

The spherical Bessel function $i_0(z)$ is

$$
\boxed{i_0(z) = \frac{\sinh z}{z}.}
$$

The modified Bessel function of the second kind is

$$
K_\nu(z) = \lim_{\mu\to\nu} \frac{\pi}{2}\frac{I_{-\mu} - I_\mu}{\sin(\mu\pi)}
$$

Thus $K_{1/2}(z)$ can be determined in terms of $I_{-1/2}(z)$ and $I_{1/2}(z)$.

$$
K_{1/2}(z) = \frac{\pi}{2}\left(I_{-1/2} - I_{1/2}\right)
$$

We determine $I_{-1/2}$ with the recursion relation

$$
I_{\nu-1}(z) = I_\nu'(z) + \frac{\nu}{z} I_\nu(z).
$$

$$
\begin{aligned}
I_{-1/2}(z) &= I_{1/2}'(z) + \frac{1}{2z} I_{1/2}(z) \\
&= \sqrt{\frac{2}{\pi}} z^{-1/2}\cosh(z) - \frac{1}{2}\sqrt{\frac{2}{\pi}} z^{-3/2}\sinh(z) + \frac{1}{2z}\sqrt{\frac{2}{\pi}} z^{-1/2}\sinh(z) \\
&= \sqrt{\frac{2}{\pi z}}\cosh(z)
\end{aligned}
$$

Now we can determine $K_{1/2}(z)$.

$$K_{1/2}(z) = \frac{\pi}{2}\left(\sqrt{\frac{2}{\pi z}}\cosh(z) - \sqrt{\frac{2}{\pi z}}\sinh(z)\right)$$
$$= \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}$$

The spherical Bessel function $k_0(z)$ is

$$\boxed{k_0(z) = \frac{\pi}{2z}\,\mathrm{e}^{-z}.}$$

Solution 36.12

The Point at Infinity. With the change of variables $z = 1/t$, $w(z) = u(t)$ the modified Bessel equation becomes

$$w'' + \frac{1}{z}w' - \left(1 + \frac{n^2}{z^2}\right)w = 0$$
$$t^4u'' + 2t^3u' + t(-t^2)u' - (1 + n^2t^2)u = 0$$
$$u'' + \frac{1}{t}u' - \left(\frac{1}{t^4} - \frac{n^2}{t^2}\right)u = 0.$$

The point $t = 0$ and hence the point $z = \infty$ is an irregular singular point. We will find the leading order asymptotic behavior of the solutions as $z \to +\infty$.

Controlling Factor. Starting with the modified Bessel equation for real argument

$$y'' + \frac{1}{x}y' - \left(1 + \frac{n^2}{x^2}\right)y = 0,$$

we make the substitution $y = \mathrm{e}^{s(x)}$ to obtain

$$s'' + (s')^2 + \frac{1}{x}s' - 1 - \frac{n^2}{x^2} = 0.$$

We know that $\frac{n^2}{x^2} \ll 1$ as $x \to \infty$; we will assume that $s'' \ll (s')^2$ as $x \to \infty$. This gives us

$$(s')^2 + \frac{1}{x}s' - 1 \sim 0 \qquad \text{as } x \to \infty.$$

To simplify the equation further, we will try the possible two-term balances.

1. $(s')^2 + \frac{1}{x}s' \sim 0 \quad \Rightarrow \quad s' \sim -\frac{1}{x}$ This balance is not consistent as it violates the assumption that 1 is smaller than the other terms.

2. $(s')^2 - 1 \sim 0 \quad \Rightarrow \quad s' \sim \pm 1$ This balance is consistent.

3. $\frac{1}{x}s' - 1 \sim 0 \quad \Rightarrow \quad s' \sim x$ This balance is inconsistent as $(s')^2$ isn't smaller than the other terms.

Thus the only dominant balance is $s' \sim \pm 1$. This balance is consistent with our initial assumption that $s'' \ll (s')^2$. Thus $s \sim \pm x$ and the controlling factor is $\mathrm{e}^{\pm x}$. We are interested in the decaying solution, so we will work with the controlling factor e^{-x}.

Leading Order Behavior. In order to find the leading order behavior, we substitute $s = -x + t(x)$ where $t(x) \ll x$ as $x \to \infty$ into the differential equation for s. We assume that $t' \ll 1$ and $t'' \ll 1/x$.

$$t'' + (-1 + t')^2 + \frac{1}{x}(-1 + t') - 1 - \frac{n^2}{x^2} = 0$$
$$t'' - 2t' + (t')^2 - \frac{1}{x} + \frac{1}{x}t' - \frac{n^2}{x^2} = 0$$

Using our assumptions about the behavior of t' and t'',

$$-2t' - \frac{1}{x} \sim 0$$
$$t' \sim -\frac{1}{2x}$$
$$t \sim -\frac{1}{2}\log x \quad \text{as } x \to \infty.$$

This asymptotic behavior is consistent with our assumptions.

Thus the leading order behavior of the decaying solution is

$$y \sim c\,\mathrm{e}^{-x-\frac{1}{2}\log x+u(x)} = cx^{-1/2}\,\mathrm{e}^{-x+u(x)} \quad \text{as } x \to \infty,$$

where $u(x) \ll \log x$ as $x \to \infty$.

By substituting $t = -\frac{1}{2}\log x + u(x)$ into the differential equation for t, you could show that $u(x) \to \text{const}$ as $x \to \infty$. Thus the full leading order behavior of the decaying solution is

$$y \sim cx^{-1/2}\,\mathrm{e}^{-x} \quad \text{as } x \to \infty$$

where $u(x) \to 0$ as $x \to \infty$.

Asymptotic Series. Now we find the full asymptotic series for $K_n(x)$ as $x \to \infty$. We substitute

$$K_n(x) \propto \frac{\mathrm{e}^{-x}}{\sqrt{x}} w(x)$$

into the modified Bessel equation, where $w(x)$ is a Taylor series about $x = \infty$, i.e.,

$$K_n(x) \propto \frac{\mathrm{e}^{-x}}{\sqrt{x}} \sum_{k=0}^{\infty} a_k x^{-k}.$$

First we differentiate the expression for $K_n(x)$.

$$K_n'(x) \propto \frac{\mathrm{e}^{-x}}{\sqrt{x}} \left(w' - \left(1 + \frac{1}{2x}\right) w \right)$$
$$K_n''(x) \propto \frac{\mathrm{e}^{-x}}{\sqrt{x}} \left(w'' - \left(2 + \frac{1}{x}\right) w' + \left(1 + \frac{1}{x} + \frac{3}{4x^2}\right) w \right)$$

We substitute these expressions into the modified Bessel equation.

$$x^2 y'' + xy' - (x^2 + n^2)y = 0$$
$$x^2 w'' - \left(2x^2 + x\right) w' + \left(x^2 + x + \frac{3}{4}\right) w + xw' - \left(x + \frac{1}{2}\right) w - (x^2 + n^2)w = 0$$
$$x^2 w'' - 2x^2 w' + \left(\frac{1}{4} - n^2\right) w = 0$$

The derivatives of the Taylor series are

$$\begin{aligned} w' &= \sum_{k=1}^{\infty} (-k) a_k x^{-k-1}, \\ &= \sum_{k=0}^{\infty} (-k-1) a_{k+1} x^{-k-2}, \\ w'' &= \sum_{k=1}^{\infty} (-k)(-k-1) a_k x^{-k-2}, \\ &= \sum_{k=0}^{\infty} (-k)(-k-1) a_k x^{-k-2}. \end{aligned}$$

We substitute these expression into the differential equation.

$$x^2 \sum_{k=0}^{\infty} k(k+1) a_k x^{-k-2} + 2x^2 \sum_{k=0}^{\infty} (k+1) a_{k+1} x^{-k-2} + \left(\frac{1}{4} - n^2\right) \sum_{k=0}^{\infty} a_k x^{-k} = 0$$
$$\sum_{k=0}^{\infty} k(k+1) a_k x^{-k} + 2 \sum_{k=0}^{\infty} (k+1) a_{k+1} x^{-k} + \left(\frac{1}{4} - n^2\right) \sum_{k=0}^{\infty} a_k x^{-k} = 0$$

We equation coefficients of x to obtain a recurrence relation for the coefficients.

$$k(k+1)a_k + 2(k+1)a_{k+1} + \left(\frac{1}{4} - n^2\right) a_k = 0$$
$$a_{k+1} = \frac{n^2 - 1/4 - k(k+1)}{2(k+1)} a_k$$
$$a_{k+1} = \frac{n^2 - (k+1/2)^2}{2(k+1)} a_k$$
$$a_{k+1} = \frac{4n^2 - (2k+1)^2}{8(k+1)} a_k$$

We choose $a_0 = 1$ and use the recurrence relation to determine the rest of the coefficients.

$$a_k = (8(n+1))^{-k} \prod_{j=1}^{k} (4n^2 - (2j-1)^2).$$

The asymptotic expansion of the modified Bessel function of the second kind is

$$\boxed{K_n(x) \propto \frac{\mathrm{e}^{-x}}{\sqrt{x}} \sum_{k=0}^{\infty} (8(n+1))^{-k} \left(\prod_{j=1}^{k} (4n^2 - (2j-1)^2) \right) z^{-k}, \quad \text{as } x \to \infty.}$$

Convergence. We determine the domain of convergence of the series with the ratio test. The Taylor series about infinity will converge outside of some circle.

$$\lim_{k\to\infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| < 1$$
$$\lim_{k\to\infty} \left| \frac{a_{n+1} x^{-k-1}}{a_n x^{-k}} \right| < 1$$
$$\lim_{k\to\infty} \left| \frac{4n^2 - (2k+1)^2}{8(k+1)} \right| |x|^{-1} < 1$$
$$\infty < |x|$$

The series does not converge for any x in the finite complex plane. However, if we take only a finite number of terms in the series, it gives a good approximation of $K_n(x)$ for large, positive x. At $x = 10$, the one, two and three term approximations give relative errors of 0.01, 0.0006 and 0.00006, respectively.

Part V

Partial Differential Equations

Chapter 37

Transforming Equations

Let $\{x_i\}$ denote rectangular coordinates. Let $\{\mathbf{a}_i\}$ be unit basis vectors in the orthogonal coordinate system $\{\xi_i\}$. The *distance metric coefficients* h_i can be defined

$$h_i = \sqrt{\left(\frac{\partial x_1}{\partial \xi_i}\right)^2 + \left(\frac{\partial x_2}{\partial \xi_i}\right)^2 + \left(\frac{\partial x_3}{\partial \xi_i}\right)^2}.$$

The gradient, divergence, etc., follow.

$$\nabla u = \frac{\mathbf{a}_1}{h_1}\frac{\partial u}{\partial \xi_1} + \frac{\mathbf{a}_2}{h_2}\frac{\partial u}{\partial \xi_2} + \frac{\mathbf{a}_3}{h_3}\frac{\partial u}{\partial \xi_3}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial \xi_1}(h_2 h_3 v_1) + \frac{\partial}{\partial \xi_2}(h_3 h_1 v_2) + \frac{\partial}{\partial \xi_3}(h_1 h_2 v_3)\right)$$

$$\nabla^2 u = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial \xi_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial u}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial u}{\partial \xi_2}\right) + \frac{\partial}{\partial \xi_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial u}{\partial \xi_3}\right)\right)$$

37.1 Exercises

Exercise 37.1
Find the Laplacian in cylindrical coordinates (r, θ, z).

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z$$

Exercise 37.2
Find the Laplacian in spherical coordinates (r, ϕ, θ).

$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi$$

37.2 Hints

37.3 Solutions

Solution 37.1

$$h_1 = \sqrt{(\cos\theta)^2 + (\sin\theta)^2 + 0} = 1$$
$$h_2 = \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2 + 0} = r$$
$$h_3 = \sqrt{0 + 0 + 1^2} = 1$$

$$\nabla^2 u = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial u}{\partial z}\right)\right)$$
$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

Solution 37.2

$$h_1 = \sqrt{(\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + (\cos\phi)^2} = 1$$
$$h_2 = \sqrt{(r\cos\phi\cos\theta)^2 + (r\cos\phi\sin\theta)^2 + (-r\sin\phi)^2} = r$$
$$h_3 = \sqrt{(-r\sin\phi\sin\theta)^2 + (r\sin\phi\cos\theta)^2 + 0} = r\sin\phi$$

$$\nabla^2 u = \frac{1}{r^2\sin\phi}\left(\frac{\partial}{\partial r}\left(r^2\sin\phi\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\phi}\frac{\partial u}{\partial \theta}\right)\right)$$
$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{r^2\sin\phi}\frac{\partial^2 u}{\partial \theta^2}$$

Chapter 38

Classification of Partial Differential Equations

38.1 Classification of Second Order Quasi-Linear Equations

Consider the general second order quasi-linear partial differential equation in two variables.

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} = F(x, y, u, u_x, u_y) \tag{38.1}$$

We classify the equation by the sign of the discriminant. At a given point x_0, y_0, the equation is classified as one of the following types:

$$\begin{aligned} b^2 - ac > 0 : &\quad \text{hyperbolic} \\ b^2 - ac = 0 : &\quad \text{parabolic} \\ b^2 - ac < 0 : &\quad \text{elliptic} \end{aligned}$$

If an equation has a particular type for all points x, y in a domain then the equation is said to be of that type in the domain. Each of these types has a canonical form that can be obtained through a change of independent variables. The type of an equation indicates much about the nature of its solution.

We seek a change of independent variables, (a different coordinate system), such that Equation 38.1 has a simpler form. We will find that a second order quasi-linear partial differential equation in two variables can be transformed to one of the canonical forms:

$$\begin{aligned} &u_{\xi\eta} = G(\xi, \eta, u, u_\xi, u_\eta), && \text{hyperbolic} \\ &u_{\xi\xi} = G(\xi, \eta, u, u_\xi, u_\eta), && \text{parabolic} \\ &u_{\xi\xi} + u_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta), && \text{elliptic} \end{aligned}$$

Consider the change of independent variables

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y).$$

The partial derivatives of u are

$$\begin{aligned} u_x &= \xi_x u_\xi + \eta_x u_\eta \\ u_y &= \xi_y u_\xi + \eta_y u_\eta \\ u_{xx} &= \xi_x^2 u_{\xi\xi} + 2\xi_x \eta_x u_{\xi\eta} + \eta_x^2 u_{\eta\eta} + \xi_{xx} u_\xi + \eta_{xx} u_\eta \\ u_{xy} &= \xi_x \xi_y u_{\xi\xi} + (\xi_x \eta_y + \xi_y \eta_x) u_{\xi\eta} + \eta_x \eta_y u_{\eta\eta} + \xi_{xy} u_\xi + \eta_{xy} u_\eta \\ u_{yy} &= \xi_y^2 u_{\xi\xi} + 2\xi_y \eta_y u_{\xi\eta} + \eta_y^2 u_{\eta\eta} + \xi_{yy} u_\xi + \eta_{yy} u_\eta. \end{aligned}$$

Substituting these into (**??**) yields an equation in ξ and η.

$$\left(a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2\right) u_{\xi\xi} + 2\left(a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y\right) u_{\xi\eta} \\ + \left(a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2\right) u_{\eta\eta} = H(\xi, \eta, u, u_\xi, u_\eta)$$

$$\alpha(\xi, \eta) u_{\xi\xi} + \beta(\xi, \eta) u_{\xi\eta} + \gamma(\xi, \eta) u_{\eta\eta} = H(\xi, \eta, u, u_\xi, u_\eta) \tag{38.2}$$

38.1.1 Hyperbolic Equations

We start with a hyperbolic equation, ($b^2 - ac > 0$). We seek a change of independent variables that will put Equation 38.1 in the form

$$u_{\xi\eta} = G(\xi, \eta, u, u_\xi, u_\eta) \tag{38.3}$$

We require that the $u_{\xi\xi}$ and $u_{\eta\eta}$ terms vanish. That is $\alpha = \gamma = 0$ in Equation 38.2. This gives us two constraints on ξ and η.

$$\begin{aligned} a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 &= 0, & a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 &= 0 \\ \frac{\xi_x}{\xi_y} &= \frac{-b + \sqrt{b^2 - ac}}{a}, & \frac{\eta_x}{\eta_y} &= \frac{-b - \sqrt{b^2 - ac}}{a}. \end{aligned} \tag{38.4}$$

Here we chose the signs in the quadratic formulas to get different solutions for ξ and η.

Consider $\xi(x, y) = \text{const}$ as an implicit equation for y in terms of x. We differentiate ξ with respect to x.

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = \xi_x + \xi_y \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The derivative of $y(x)$ is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\xi_x}{\xi_y} = \frac{b - \sqrt{b^2 - ac}}{a}.$$

Solving this ordinary differential equation for $y(x)$ determines $\xi(x, y)$. We just write the solution for $y(x)$ in the form $F(x, y(x)) = \text{const}$. We then have $\xi = F(x, y)$. Upon solving for ξ and η we divide Equation 38.2 by $\beta(\xi, \eta)$ to obtain the canonical form.

Note that we could have solved for ξ_y/ξ_x in Equation 38.4.

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{\xi_y}{\xi_x} = \frac{b - \sqrt{b^2 - ac}}{c}$$

This form is useful if a vanishes.

Another canonical form for hyperbolic equations is

$$u_{\sigma\sigma} - u_{\tau\tau} = K(\sigma, \tau, u, u_\sigma, u_\tau). \tag{38.5}$$

We can transform Equation 38.3 to this form with the change of variables

$$\sigma = \xi + \eta, \quad \tau = \xi - \eta.$$

Equation 38.3 becomes

$$u_{\sigma\sigma} - u_{\tau\tau} = G\left(\frac{\sigma+\tau}{2}, \frac{\sigma-\tau}{2}, u, u_\sigma + u_\tau, u_\sigma - u_\tau\right).$$

Example 38.1.1 Consider the wave equation with a source.

$$u_{tt} - c^2 u_{xx} = s(x,t)$$

Since $0 - (1)(-c^2) > 0$, the equation is hyperbolic. We find the new variables.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -c, \quad x = -ct + \text{const}, \quad \xi = x + ct \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= c, \quad x = ct + \text{const}, \quad \eta = x - ct \end{aligned}$$

Then we determine t and x in terms of ξ and η.

$$t = \frac{\xi - \eta}{2c}, \quad x = \frac{\xi + \eta}{2}$$

We calculate the derivatives of ξ and η.

$$\begin{aligned} \xi_t &= c & \xi_x &= 1 \\ \eta_t &= -c & \eta_x &= 1 \end{aligned}$$

Then we calculate the derivatives of u.

$$\begin{aligned} u_{tt} &= c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta} \\ u_{xx} &= u_{\xi\xi} + u_{\eta\eta} \end{aligned}$$

Finally we transform the equation to canonical form.

$$-2c^2 u_{\xi\eta} = s\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right)$$

$$\boxed{u_{\xi\eta} = -\frac{1}{2c^2} s\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right)}$$

If $s(x,t) = 0$, then the equation is $u_{\xi\eta} = 0$ we can integrate with respect to ξ and η to obtain the solution, $u = f(\xi) + g(\eta)$. Here f and g are arbitrary C^2 functions. In terms of t and x, we have

$$\boxed{u(x,t) = f(x+ct) + g(x-ct).}$$

To put the wave equation in the form of Equation 38.5 we make the change of variables

$$\sigma = \xi + \eta = 2x, \quad \tau = \xi - \eta = 2ct.$$

$$u_{tt} - c^2 u_{xx} = s(x,t)$$

$$4c^2 u_{\tau\tau} - 4c^2 u_{\sigma\sigma} = s\left(\frac{\sigma}{2}, \frac{\tau}{2c}\right)$$

$$\boxed{u_{\sigma\sigma} - u_{\tau\tau} = -\frac{1}{4c^2} s\left(\frac{\sigma}{2}, \frac{\tau}{2c}\right)}$$

Example 38.1.2 Consider

$$y^2 u_{xx} - x^2 u_{yy} = 0.$$

For $x \neq 0$ and $y \neq 0$ this equation is hyperbolic. We find the new variables.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sqrt{y^2x^2}}{y^2} = -\frac{x}{y}, \quad y\,dy = -x\,dx, \quad \frac{y^2}{2} = -\frac{x^2}{2} + \text{const}, \quad \xi = y^2 + x^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{y^2x^2}}{y^2} = \frac{x}{y}, \quad y\,dy = x\,dx, \quad \frac{y^2}{2} = \frac{x^2}{2} + \text{const}, \quad \eta = y^2 - x^2$$

We calculate the derivatives of ξ and η.

$$\begin{aligned} \xi_x &= 2x & \xi_y &= 2y \\ \eta_x &= -2x & \eta_y &= 2y \end{aligned}$$

Then we calculate the derivatives of u.

$$\begin{aligned} u_x &= 2x(u_\xi - u_\eta) \\ u_y &= 2y(u_\xi + u_\eta) \\ u_{xx} &= 4x^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) + 2(u_\xi - u_\eta) \\ u_{yy} &= 4y^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) + 2(u_\xi + u_\eta) \end{aligned}$$

Finally we transform the equation to canonical form.

$$y^2 u_{xx} - x^2 u_{yy} = 0$$

$$-8x^2y^2 u_{\xi\eta} - 8x^2y^2 u_{\xi\eta} + 2y^2(u_\xi - u_\eta) + 2x^2(u_\xi + u_\eta) = 0$$

$$16\frac{1}{2}(\xi - \eta)\frac{1}{2}(\xi + \eta)u_{\xi\eta} = 2\xi u_\xi - 2\eta u_\eta$$

$$\boxed{u_{\xi\eta} = \frac{\xi u_\xi - \eta u_\eta}{2(\xi^2 - \eta^2)}}$$

Example 38.1.3 Consider Laplace's equation.

$$u_{xx} + u_{yy} = 0$$

Since $0 - (1)(1) < 0$, the equation is elliptic. We will transform this equation to the canical form of Equation 38.3. We find the new variables.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -i, \quad y = -ix + \text{const}, \quad \xi = x + iy$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = i, \quad y = ix + \text{const}, \quad \eta = x - iy$$

We calculate the derivatives of ξ and η.

$$\xi_x = 1 \quad \xi_y = i$$
$$\eta_x = 1 \quad \eta_y = -i$$

Then we calculate the derivatives of u.

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$
$$u_{yy} = -u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta}$$

Finally we transform the equation to canonical form.

$$4u_{\xi\eta} = 0$$
$$\boxed{u_{\xi\eta} = 0}$$

We integrate with respect to ξ and η to obtain the solution, $u = f(\xi) + g(\eta)$. Here f and g are arbitrary C^2 functions. In terms of x and y, we have

$$\boxed{u(x, y) = f(x + iy) + g(x - iy).}$$

This solution makes a lot of sense, because the real and imaginary parts of an analytic function are harmonic.

38.1.2 Parabolic equations

38.1.3 Elliptic Equations

We start with an elliptic equation, $(b^2 - ac < 0)$. We seek a change of independent variables that will put Equation 38.1 in the form

$$u_{\sigma\sigma} + u_{\tau\tau} = G(\sigma, \tau, u, u_\sigma, u_\tau) \tag{38.6}$$

If we make the change of variables determined by

$$\frac{\xi_x}{\xi_y} = \frac{-b + i\sqrt{ac - b^2}}{a}, \qquad \frac{\eta_x}{\eta_y} = \frac{-b - i\sqrt{ac - b^2}}{a}$$

The equation will have the form

$$u_{\xi\eta} = G(\xi, \eta, u, u_\xi, u_\eta).$$

ξ and η are complex-valued. If we then make the change of variables

$$\sigma = \frac{\xi + \eta}{2}, \quad \tau = \frac{\xi - \eta}{2i}$$

we will obtain the canonical form of Equation 38.6. Note that since ξ and η are complex conjugate, σ and τ are real-valued.

Example 38.1.4 Consider

$$y^2 u_{xx} + x^2 u_{yy} = 0. \tag{38.7}$$

For $x \neq 0$ and $y \neq 0$ this equation is elliptic. We find new variables that will put this equation in the form $u_{\xi\eta} = G(\cdot)$. From Example 38.1.2 we see that they are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -i\frac{\sqrt{y^2x^2}}{y^2} = -i\frac{x}{y}, \quad y\,dy = -ix\,dx, \quad \frac{y^2}{2} = -i\frac{x^2}{2} + \text{const}, \quad \xi = y^2 + ix^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = i\frac{\sqrt{y^2x^2}}{y^2} = i\frac{x}{y}, \quad y\,dy = ix\,dx, \quad \frac{y^2}{2} = i\frac{x^2}{2} + \text{const}, \quad \eta = y^2 - ix^2$$

The variables that will put Equation 38.7 in canonical form are

$$\sigma = \frac{\xi + \eta}{2} = y^2, \quad \tau = \frac{\xi - \eta}{2i} = x^2$$

We calculate the derivatives of σ and τ.

$$\sigma_x = 0 \qquad \sigma_y = 2y$$
$$\tau_x = 2x \qquad \tau_y = 0$$

Then we calculate the derivatives of u.

$$u_x = 2x u_\tau$$
$$u_y = 2y u_\sigma$$
$$u_{xx} = 4x^2 u_{\tau\tau} + 2u_\tau$$
$$u_{yy} = 4y^2 u_{\sigma\sigma} + 2u_\sigma$$

Finally we transform the equation to canonical form.

$$y^2 u_{xx} + x^2 u_{yy} = 0$$
$$\sigma(4\tau u_{\tau\tau} + 2u_\tau) + \tau(4\sigma u_{\sigma\sigma} + 2u_\sigma) = 0$$
$$\boxed{u_{\sigma\sigma} + u_{\tau\tau} = -\frac{1}{2\sigma}u_\sigma - \frac{1}{2\tau}u_\tau}$$

38.2 Equilibrium Solutions

Example 38.2.1 Consider the equilibrium solution for the following problem.

$$u_t = u_{xx}, \qquad u(x,0) = x, \quad u_x(0,t) = u_x(1,t) = 0.$$

Setting $u_t = 0$ we have the ordinary differential equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0.$$

This equation has the solution

$$u = ax + b.$$

Applying the boundary conditions we see that

$$u = b.$$

To determine the constant, we note that the heat energy in the rod is constant in time.

$$\int_0^1 u(x,t)\,dx = \int_0^1 u(x,0)\,dx$$
$$\int_0^1 b\,dx = \int_0^1 x\,dx$$

Thus the equilibrium solution is

$$u(x) = \frac{1}{2}.$$

38.3 Exercises

Exercise 38.1
Classify as hyperbolic, parabolic, or elliptic in a region R each of the equations:

(a) $u_t = (pu_x)_x$

(b) $u_{tt} = c^2 u_{xx} - \gamma u$

(c) $(qu_x)_x + (qu_t)_t = 0$

where $p(x)$, $c(x,t)$, $q(x,t)$, and $\gamma(x)$ are given functions that take on only positive values in a region R of the (x,t) plane.

Exercise 38.2
Transform each of the following equations for $\phi(x,y)$ into canonical form in appropriate regions

(a) $\phi_{xx} - y^2\phi_{yy} + \phi_x - \phi + x^2 = 0$

(b) $\phi_{xx} + x\phi_{yy} = 0$

The equation in part (b) is known as *Tricomi's equation* and is a model for transonic fluid flow in which the flow speed changes from supersonic to subsonic.

38.4 Hints

Hint 38.1

Hint 38.2

38.5 Solutions

Exercise 38.3

1.

$$u_t = (pu_x)_x$$
$$pu_{xx} + 0u_{xt} + 0u_{tt} + p_x u_x - u_t = 0$$

Since $0^2 - p0 = 0$, the equation is parabolic.

2.

$$u_{tt} = c^2 u_{xx} - \gamma u$$
$$u_{tt} + 0u_{tx} - c^2 u_{xx} + \gamma u = 0$$

Since $0^2 - (1)(-c^2) > 0$, the equation is hyperbolic.

3.

$$(qu_x)_x + (qu_t)_t = 0$$
$$qu_{xx} + 0u_{xt} + qu_{tt} + q_x u_x + q_t u_t = 0$$

Since $0^2 - qq < 0$, the equation is elliptic.

Exercise 38.4

1. For $y \neq 0$, the equation is hyperbolic. We find the new independent variables.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{y^2}}{1} = y, \quad y = c\,\mathrm{e}^x, \quad \mathrm{e}^{-x} y = c, \quad \xi = \mathrm{e}^{-x} y$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sqrt{y^2}}{1} = -y, \quad y = c\,\mathrm{e}^{-x}, \quad \mathrm{e}^{x} y = c, \quad \eta = \mathrm{e}^{x} y$$

Next we determine x and y in terms of ξ and η.

$$\xi\eta = y^2, \quad y = \sqrt{\xi\eta}$$

$$\eta = \mathrm{e}^x \sqrt{\xi\eta}, \quad \mathrm{e}^x = \sqrt{\eta/\xi}, \quad x = \frac{1}{2}\log\left(\frac{\eta}{\xi}\right)$$

We calculate the derivatives of ξ and η.

$$\xi_x = -\mathrm{e}^{-x} y = -\xi$$

$$\xi_y = \mathrm{e}^{-x} = \sqrt{\xi/\eta}$$

$$\eta_x = \mathrm{e}^x y = \eta$$

$$\eta_y = \mathrm{e}^x = \sqrt{\eta/\xi}$$

Then we calculate the derivatives of ϕ.

$$\frac{\partial}{\partial x} = -\xi\frac{\partial}{\partial \xi} + \eta\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial y} = \sqrt{\frac{\xi}{\eta}}\frac{\partial}{\partial \xi} + \sqrt{\frac{\eta}{\xi}}\frac{\partial}{\partial \eta}$$

$$\phi_x = -\xi\phi_\xi + \eta\phi_\eta, \quad \phi_y = \sqrt{\frac{\xi}{\eta}}\phi_\xi + \sqrt{\frac{\eta}{\xi}}\phi_\eta$$

$$\phi_{xx} = \xi^2\phi_{\xi\xi} - 2\xi\eta\phi_{\xi\eta} + \eta^2\phi_{\eta\eta} + \xi\phi_\xi + \eta\phi_\eta, \quad \phi_{yy} = \frac{\xi}{\eta}\phi_{\xi\xi} + 2\phi_{\xi\eta} + \frac{\eta}{\xi}\phi_{\eta\eta}$$

Finally we transform the equation to canonical form.

$$\phi_{xx} - y^2\phi_{yy} + \phi_x - \phi + x^2 = 0$$

$$-4\xi\eta\phi_{\xi\eta} + \xi\phi_\xi + \eta\phi_\eta - \xi\phi_\xi + \eta\phi_\eta - \phi + \log\left(\frac{\eta}{\xi}\right) = 0$$

$$\boxed{\phi_{\xi\eta} = \frac{1}{2\xi}\phi_\eta + \phi - \log\left(\frac{\eta}{\xi}\right)}$$

For $y = 0$ we have the ordinary differential equation

$$\phi_{xx} + \phi_x - \phi + x^2 = 0.$$

2. For $x < 0$, the equation is hyperbolic. We find the new independent variables.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{-x}, \quad y = \frac{2}{3}x\sqrt{-x} + c, \quad \xi = \frac{2}{3}x\sqrt{-x} - y$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{-x}, \quad y = -\frac{2}{3}x\sqrt{-x} + c, \quad \eta = \frac{2}{3}x\sqrt{-x} + y$$

Next we determine x and y in terms of ξ and η.

$$x = -\left(\frac{3}{4}(\xi + \eta)\right)^{1/3}, \quad y = \frac{\eta - \xi}{2}$$

We calculate the derivatives of ξ and η.

$$\xi_x = \sqrt{-x} = \left(\frac{3}{4}(\xi + \eta)\right)^{1/6}, \quad \xi_y = -1$$
$$\eta_x = \left(\frac{3}{4}(\xi + \eta)\right)^{1/6}, \quad \eta_y = 1$$

Then we calculate the derivatives of ϕ.

$$\phi_x = \left(\frac{3}{4}(\xi + \eta)\right)^{1/6} (\phi_\xi + \phi_\eta)$$
$$\phi_y = -\phi_\xi + \phi_\eta$$
$$\phi_{xx} = \left(\frac{3}{4}(\xi + \eta)\right)^{1/3} (\phi_{\xi\xi} + \phi_{\eta\eta}) + (6(\xi + \eta))^{1/3}\phi_{\xi\eta} + (6(\xi + \eta))^{-2/3} (\phi_\xi + \phi_\eta)$$
$$\phi_{yy} = \phi_{\xi\xi} - 2\phi_{\xi\eta} + \phi_{\eta\eta}$$

Finally we transform the equation to canonical form.

$$\phi_{xx} + x\phi_{yy} = 0$$
$$(6(\xi+\eta))^{1/3}\phi_{\xi\eta} + (6(\xi+\eta))^{1/3}\phi_{\xi\eta} + (6(\xi+\eta))^{-2/3}(\phi_\xi + \phi_\eta) = 0$$
$$\boxed{\phi_{\xi\eta} = -\frac{\phi_\xi + \phi_\eta}{12(\xi+\eta)}}$$

For $x > 0$, the equation is elliptic. The variables we defined before are complex-valued.

$$\xi = i\frac{2}{3}x^{3/2} - y, \quad \eta = i\frac{2}{3}x^{3/2} + y.$$

We choose the new real-valued variables.

$$\alpha = \xi - \eta, \quad \beta = -i(\xi + \eta)$$

We write the derivatives in terms of α and β.

$$\phi_\xi = \phi_\alpha - i\phi_\beta$$
$$\phi_\eta = -\phi_\alpha - i\phi_\beta$$
$$\phi_{\xi\eta} = -\phi_{\alpha\alpha} - \phi_{\beta\beta}$$

We transform the equation to canonical form.

$$\phi_{\xi\eta} = -\frac{\phi_\xi + \phi_\eta}{12(\xi+\eta)}$$
$$-\phi_{\alpha\alpha} - \phi_{\beta\beta} = -\frac{-2i\phi_\beta}{12i\beta}$$
$$\boxed{\phi_{\alpha\alpha} + \phi_{\beta\beta} = -\frac{\phi_\beta}{6\beta}}$$

Chapter 39

Separation of Variables

39.1 Eigensolutions of Homogeneous Equations

39.2 Homogeneous Equations with Homogeneous Boundary Conditions

The method of separation of variables is a useful technique for finding special solutions of partial differential equations. We can combine these special solutions to solve certain problems. Consider the temperature of a one-dimensional rod of length h [1]. The left end is held at zero temperature, the right end is insulated and the initial temperature distribution is known at time $t = 0$. To find the temperature we solve the problem:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < h, \quad t > 0$$
$$u(0, t) = u_x(h, t) = 0$$
$$u(x, 0) = f(x)$$

[1]Why h? Because l looks like 1 and we use L to denote linear operators

We look for special solutions of the form, $u(x,t) = X(x)T(t)$. Substituting this into the partial differential equation yields

$$X(x)T'(t) = \kappa X''(x)T(t)$$
$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}$$

Since the left side is only dependent on t, the right side in only dependent on x, and the relation is valid for all t and x, both sides of the equation must be constant.

$$\frac{T'}{\kappa T} = \frac{X''}{X} = -\lambda$$

Here $-\lambda$ is an arbitrary constant. (You'll see later that this form is convenient.) $u(x,t) = X(x)T(t)$ will satisfy the partial differential equation if $X(x)$ and $T(t)$ satisfy the ordinary differential equations,

$$T' = -\kappa\lambda T \quad \text{and} \quad X'' = -\lambda X.$$

Now we see how lucky we are that this problem happens to have homogeneous boundary conditions [2]. If the left boundary condition had been $u(0,t) = 1$, this would imply $X(0)T(t) = 1$ which tells us nothing very useful about either X or T. However the boundary condition $u(0,t) = X(0)T(t) = 0$, tells us that either $X(0) = 0$ or $T(t) = 0$. Since the latter case would give us the trivial solution, we must have $X(0) = 0$. Likewise by looking at the right boundary condition we obtain $X'(h) = 0$.

We have a regular Sturm-Liouville problem for $X(x)$.

$$X'' + \lambda X = 0, \quad X(0) = X'(h) = 0$$

The eigenvalues and orthonormal eigenfunctions are

$$\lambda_n = \left(\frac{(2n-1)\pi}{2h}\right)^2, \quad X_n = \sqrt{\frac{2}{h}} \sin\left(\frac{(2n-1)\pi}{2h}x\right), \quad n \in \mathbb{Z}^+.$$

[2] Actually luck has nothing to do with it. I planned it that way.

Now we solve the equation for $T(t)$.

$$T' = -\kappa \lambda_n T$$
$$T = c\,\mathrm{e}^{-\kappa \lambda_n t}$$

The eigen-solutions of the partial differential equation that satisfy the homogeneous boundary conditions are

$$u_n(x,t) = \sqrt{\frac{2}{h}} \sin\left(\sqrt{\lambda_n} x\right) \mathrm{e}^{-\kappa \lambda_n t}.$$

We seek a solution of the problem that is a linear combination of these eigen-solutions.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{h}} \sin\left(\sqrt{\lambda_n} x\right) \mathrm{e}^{-\kappa \lambda_n t}}$$

We apply the initial condition to find the coefficients in the expansion.

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{h}} \sin\left(\sqrt{\lambda_n} x\right) = f(x)$$
$$\boxed{a_n = \sqrt{\frac{2}{h}} \int_0^h \sin\left(\sqrt{\lambda_n} x\right) f(x)\,\mathrm{d}x}$$

39.3 Time-Independent Sources and Boundary Conditions

Consider the temperature in a one-dimensional rod of length h. The ends are held at temperatures α and β, respectively, and the initial temperature is known at time $t = 0$. Additionally, there is a heat source, $s(x)$, that is independent of time. We find the temperature by solving the problem,

$$u_t = \kappa u_{xx} + s(x), \quad u(0,t) = \alpha, \quad u(h,t) = \beta, \quad u(x,0) = f(x). \tag{39.1}$$

Because of the source term, the equation is not separable, so we cannot directly apply separation of variables. Furthermore, we have the added complication of inhomogeneous boundary conditions. Instead of attacking this problem directly, we seek a transformation that will yield a homogeneous equation and homogeneous boundary conditions.

Consider the equilibrium temperature, $\mu(x)$. It satisfies the problem,

$$\mu''(x) = -\frac{s(x)}{\kappa} = 0, \quad \mu(0) = \alpha, \quad \mu(h) = \beta.$$

The Green function for this problem is,

$$G(x;\xi) = \frac{x_<(x_> - h)}{h}.$$

The equilibrium temperature distribution is

$$\mu(x) = \alpha\frac{x-h}{h} + \beta\frac{x}{h} - \frac{1}{\kappa h}\int_0^h x_<(x_> - h)s(\xi)\,\mathrm{d}\xi,$$

$$\boxed{\mu(x) = \alpha + (\beta - \alpha)\frac{x}{h} - \frac{1}{\kappa h}\left((x-h)\int_0^x \xi s(\xi)\,\mathrm{d}\xi + x\int_x^h (\xi - h)s(\xi)\,\mathrm{d}\xi\right).}$$

Now we substitute $u(x,t) = v(x,t) + \mu(x)$ into Equation 39.1.

$$\frac{\partial}{\partial t}(v + \mu(x)) = \kappa\frac{\partial^2}{\partial x^2}(v + \mu(x)) + s(x)$$
$$v_t = \kappa v_{xx} + \kappa\mu''(x) + s(x)$$
$$v_t = \kappa v_{xx} \tag{39.2}$$

Since the equilibrium solution satisfies the inhomogeneous boundary conditions, $v(x,t)$ satisfies homogeneous boundary conditions.

$$v(0,t) = v(h,t) = 0.$$

The initial value of v is

$$v(x, 0) = f(x) - \mu(x).$$

We seek a solution for $v(x, t)$ that is a linear combination of eigen-solutions of the heat equation. We substitute the separation of variables, $v(x, t) = X(x)T(t)$ into Equation 39.2

$$\frac{T'}{\kappa T} = \frac{X''}{X} = -\lambda$$

This gives us two ordinary differential equations.

$$X'' + \lambda X = 0, \qquad X(0) = X(h) = 0$$
$$T' = -\kappa \lambda T.$$

The Sturm-Liouville problem for $X(x)$ has the eigenvalues and orthonormal eigenfunctions,

$$\lambda_n = \left(\frac{n\pi}{h}\right)^2, \quad X_n = \sqrt{\frac{2}{h}} \sin\left(\frac{n\pi x}{h}\right), \quad n \in \mathbf{Z}^+.$$

We solve for $T(t)$.

$$T_n = c\,\mathrm{e}^{-\kappa(n\pi/h)^2 t}.$$

The eigen-solutions of the partial differential equation are

$$v_n(x, t) = \sqrt{\frac{2}{h}} \sin\left(\frac{n\pi x}{h}\right) \mathrm{e}^{-\kappa(n\pi/h)^2 t}.$$

The solution for $v(x, t)$ is a linear combination of these.

$$v(x, t) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{h}} \sin\left(\frac{n\pi x}{h}\right) \mathrm{e}^{-\kappa(n\pi/h)^2 t}$$

We determine the coefficients in the series with the initial condition.

$$v(x,0) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{h}} \sin\left(\frac{n\pi x}{h}\right) = f(x) - \mu(x)$$

$$\boxed{a_n = \sqrt{\frac{2}{h}} \int_0^h \sin\left(\frac{n\pi x}{h}\right) (f(x) - \mu(x))\, \mathrm{d}x}$$

The temperature of the rod is

$$\boxed{u(x,t) = \mu(x) + \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{h}} \sin\left(\frac{n\pi x}{h}\right) \mathrm{e}^{-\kappa(n\pi/h)^2 t}}$$

39.4 Inhomogeneous Equations with Homogeneous Boundary Conditions

Now consider the heat equation with a time dependent source, $s(x,t)$.

$$u_t = \kappa u_{xx} + s(x,t), \quad u(0,t) = u(h,t) = 0, \quad u(x,0) = f(x). \tag{39.3}$$

In general we cannot transform the problem to one with a homogeneous differential equation. Thus we cannot represent the solution in a series of the eigen-solutions of the partial differential equation. Instead, we will do the next best thing and expand the solution in a series of eigenfunctions in $X_n(x)$ where the coefficients depend on time.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x)$$

We will find these eigenfunctions with the separation of variables, $u(x,t) = X(x)T(t)$ applied to the homogeneous equation, $u_t = \kappa u_{xx}$, which yields,

$$X_n(x) = \sqrt{\frac{2}{h}} \sin\left(\frac{n\pi x}{h}\right), \quad n \in \mathbb{Z}^+.$$

We expand the heat source in the eigenfunctions.

$$s(x,t) = \sum_{n=1}^{\infty} s_n(t) \sqrt{\frac{2}{h}} \sin\left(\frac{n\pi x}{h}\right)$$

$$s_n(t) = \sqrt{\frac{2}{h}} \int_0^h \sin\left(\frac{n\pi x}{h}\right) s(x,t) \,\mathrm{d}x,$$

We substitute the series solution into Equation 39.3.

$$\sum_{n=1}^{\infty} u_n'(t) \sqrt{\frac{2}{h}} \sin\left(\frac{n\pi x}{h}\right) = -\kappa \sum_{n=1}^{\infty} u_n(t) \left(\frac{n\pi}{h}\right)^2 \sqrt{\frac{2}{h}} \sin\left(\frac{n\pi x}{h}\right) + \sum_{n=1}^{\infty} s_n(t) \sqrt{\frac{2}{h}} \sin\left(\frac{n\pi x}{h}\right)$$

$$u_n'(t) + \kappa \left(\frac{n\pi}{h}\right)^2 u_n(t) = s_n(t)$$

Now we have a first order, ordinary differential equation for each of the $u_n(t)$. We obtain initial conditions from the initial condition for $u(x,t)$.

$$u(x,0) = \sum_{n=1}^{\infty} u_n(0) \sqrt{\frac{2}{h}} \sin\left(\frac{n\pi x}{h}\right) = f(x)$$

$$u_n(0) = \sqrt{\frac{2}{h}} \int_0^h \sin\left(\frac{n\pi x}{h}\right) f(x) \,\mathrm{d}x \equiv f_n$$

The temperature is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sqrt{\frac{2}{h}} \sin\left(\frac{n\pi x}{h}\right),}$$

$$\boxed{u_n(t) = f_n \,\mathrm{e}^{-\kappa(n\pi/h)^2 t} + \int_0^t \mathrm{e}^{-\kappa(n\pi/h)^2(t-\tau)} s_n(\tau) \,\mathrm{d}\tau.}$$

39.5 Inhomogeneous Boundary Conditions

Consider the temperature of a one-dimensional rod of length h. The left end is held at the temperature $\alpha(t)$, the heat flow at right end is specified, there is a time-dependent source and the initial temperature distribution is known at time $t = 0$. To find the temperature we solve the problem:

$$u_t = \kappa u_{xx} + s(x,t), \quad 0 < x < h, \quad t > 0 \tag{39.4}$$
$$u(0,t) = \alpha(t), \quad u_x(h,t) = \beta(t) \quad u(x,0) = f(x)$$

Transformation to a homogeneous equation. Because of the inhomogeneous boundary conditions, we cannot directly apply the method of separation of variables. However we can transform the problem to an inhomogeneous equation with homogeneous boundary conditions. To do this, we first find a function, $\mu(x,t)$ which satisfies the boundary conditions. We note that

$$\mu(x,t) = \alpha(t) + x\beta(t)$$

does the trick. We make the change of variables

$$u(x,t) = v(x,t) + \mu(x,t)$$

in Equation 39.4.

$$v_t + \mu_t = \kappa\left(v_{xx} + \mu_{xx}\right) + s(x,t)$$
$$v_t = \kappa v_{xx} + s(x,t) - \mu_t$$

The boundary and initial conditions become

$$v(0,t) = 0, \quad v_x(h,t) = 0, \quad v(x,0) = f(x) - \mu(x,0).$$

Thus we have a heat equation with the source $s(x,t) - \mu_t(x,t)$. We could apply separation of variables to find a solution of the form

$$u(x,t) = \mu(x,t) + \sum_{n=1}^{\infty} u_n(t)\sqrt{\frac{2}{h}}\sin\left(\frac{(2n-1)\pi x}{2h}\right).$$

Direct eigenfunction expansion. Alternatively we could seek a direct eigenfunction expansion of $u(x,t)$.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sqrt{\frac{2}{h}} \sin\left(\frac{(2n-1)\pi x}{2h}\right).$$

Note that the eigenfunctions satisfy the homogeneous boundary conditions while $u(x,t)$ does not. If we choose any fixed time $t = t_0$ and form the periodic extension of the function $u(x, t_0)$ to define it for x outside the range $(0, h)$, then this function will have jump discontinuities. This means that our eigenfunction expansion will not converge uniformly. We are not allowed to differentiate the series with respect to x. We can't just plug the series into the partial differential equation to determine the coefficients. Instead, we will multiply Equation 39.4, by an eigenfunction and integrate from $x = 0$ to $x = h$. To avoid differentiating the series with respect to x, we will use integration by parts to move derivatives from $u(x,t)$ to the eigenfunction. (We will denote $\lambda_n = \left(\frac{(2n-1)\pi}{2h}\right)^2$.)

$$\sqrt{\frac{2}{h}} \int_0^h \sin(\sqrt{\lambda_n} x)(u_t - \kappa u_{xx})\,\mathrm{d}x = \sqrt{\frac{2}{h}} \int_0^h \sin(\sqrt{\lambda_n} x) s(x,t)\,\mathrm{d}x$$

$$u_n'(t) - \sqrt{\frac{2}{h}}\kappa \left[u_x \sin(\sqrt{\lambda_n} x)\right]_0^h + \sqrt{\frac{2}{h}}\kappa\sqrt{\lambda_n} \int_0^h u_x \cos(\sqrt{\lambda_n} x)\,\mathrm{d}x = s_n(t)$$

$$u_n'(t) - \sqrt{\frac{2}{h}}\kappa(-1)^n u_x(h,t) + \sqrt{\frac{2}{h}}\kappa\sqrt{\lambda_n}\left[u\cos(\sqrt{\lambda_n} x)\right]_0^h + \sqrt{\frac{2}{h}}\kappa\lambda_n \int_0^h u \sin(\sqrt{\lambda_n} x)\,\mathrm{d}x = s_n(t)$$

$$u_n'(t) - \sqrt{\frac{2}{h}}\kappa(-1)^n \beta(t) - \sqrt{\frac{2}{h}}\kappa\sqrt{\lambda_n} u(0,t) + \kappa\lambda_n u_n(t) = s_n(t)$$

$$u_n'(t) + \kappa\lambda_n u_n(t) = \sqrt{\frac{2}{h}}\kappa\left(\sqrt{\lambda_n}\alpha(t) + (-1)^n \beta(t)\right) + s_n(t)$$

Now we have an ordinary differential equation for each of the $u_n(t)$. We obtain initial conditions for them using the initial condition for $u(x,t)$.

$$u(x,0) = \sum_{n=1}^{\infty} u_n(0) \sqrt{\frac{2}{h}} \sin(\sqrt{\lambda_n} x) = f(x)$$
$$u_n(0) = \sqrt{\frac{2}{h}} \int_0^h \sin(\sqrt{\lambda_n} x) f(x) \, \mathrm{d}x \equiv f_n$$

Thus the temperature is given by

$$\boxed{u(x,t) = \sqrt{\frac{2}{h}} \sum_{n=1}^{\infty} u_n(t) \sin(\sqrt{\lambda_n} x),}$$
$$\boxed{u_n(t) = f_n \, \mathrm{e}^{-\kappa \lambda_n t} + \sqrt{\frac{2}{h}} \kappa \int_0^t \mathrm{e}^{-\kappa \lambda_n (t-\tau)} \left(\sqrt{\lambda_n} \alpha(\tau) + (-1)^n \beta(\tau) \right) \mathrm{d}\tau.}$$

39.6 The Wave Equation

Consider an elastic string with a free end at $x = 0$ and attached to a massless spring at $x = 1$. The partial differential equation that models this problem is

$$u_{tt} = u_{xx}$$
$$u_x(0,t) = 0, \quad u_x(1,t) = -u(1,t), \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

We make the substitution $u(x,t) = \psi(x)\phi(t)$ to obtain

$$\frac{\phi''}{\phi} = \frac{\psi''}{\psi} = -\lambda.$$

First we consider the problem for ψ.

$$\psi'' + \lambda\psi = 0, \qquad \psi'(0) = \psi(1) + \psi'(1) = 0.$$

To find the eigenvalues we consider the following three cases:

$\boldsymbol{\lambda < 0}$. The general solution is

$$\psi = a\cosh(\sqrt{-\lambda}x) + b\sinh(\sqrt{-\lambda}x).$$

$$\begin{aligned} \psi'(0) = 0 \quad &\Rightarrow \quad b = 0. \\ \psi(1) + \psi'(1) = 0 \quad &\Rightarrow \quad a\cosh(\sqrt{-\lambda}) + a\sqrt{-\lambda}\sinh(\sqrt{-\lambda}) = 0 \\ &\Rightarrow \quad a = 0. \end{aligned}$$

Since there is only the trivial solution, there are no negative eigenvalues.

$\boldsymbol{\lambda = 0}$. The general solution is

$$\psi = ax + b.$$

$$\begin{aligned} \psi'(0) = 0 \quad &\Rightarrow \quad a = 0. \\ \psi(1) + \psi'(1) = 0 \quad &\Rightarrow \quad b + 0 = 0. \end{aligned}$$

Thus $\lambda = 0$ is not an eigenvalue.

$\boldsymbol{\lambda > 0}$. The general solution is

$$\psi = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x).$$

$$\begin{aligned}
\psi'(0) &\Rightarrow \quad b = 0. \\
\psi(1) + \psi'(1) = 0 &\Rightarrow \quad a\cos(\sqrt{\lambda}) - a\sqrt{\lambda}\sin(\sqrt{\lambda}) = 0 \\
&\Rightarrow \quad \cos(\sqrt{\lambda}) = \sqrt{\lambda}\sin(\sqrt{\lambda}) \\
&\Rightarrow \quad \sqrt{\lambda} = \cot(\sqrt{\lambda})
\end{aligned}$$

By looking at Figure 39.1, (the plot shows the functions $f(x) = x$, $f(x) = \cot x$ and has lines at $x = n\pi$), we see that there are an infinite number of positive eigenvalues and that

$$\lambda_n \to (n\pi)^2 \text{ as } n \to \infty.$$

The eigenfunctions are

$$\psi_n = \cos(\sqrt{\lambda_n}x).$$

The solution for ϕ is

$$\phi_n = a_n\cos(\sqrt{\lambda_n}t) + b_n\sin(\sqrt{\lambda_n}t).$$

Thus the solution to the differential equation is

$$u(x,t) = \sum_{n=1}^{\infty}\cos(\sqrt{\lambda_n}x)[a_n\cos(\sqrt{\lambda_n}t) + b_n\sin(\sqrt{\lambda_n}t)].$$

Let

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} f_n\cos(\sqrt{\lambda_n}x) \\
g(x) &= \sum_{n=1}^{\infty} g_n\cos(\sqrt{\lambda_n}x).
\end{aligned}$$

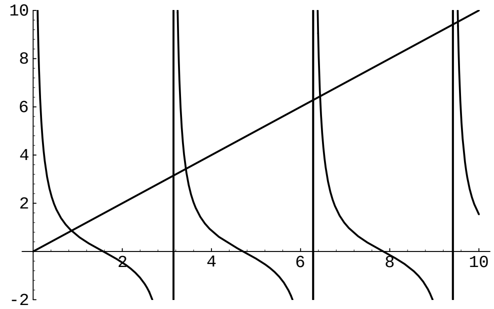

Figure 39.1: Plot of x and $\cot x$.

From the initial value we have

$$\sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n x}) a_n = \sum_{n=1}^{\infty} f_n \cos(\sqrt{\lambda_n} x)$$
$$a_n = f_n.$$

The initial velocity condition gives us

$$\sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n} x) \sqrt{\lambda_n} b_n = \sum_{n=1}^{\infty} g_n \cos(\sqrt{\lambda_n} x)$$
$$b_n = \frac{g_n}{\sqrt{\lambda_n}}.$$

Thus the solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n} x) \left[f_n \cos(\sqrt{\lambda_n} t) + \frac{g_n}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n} t) \right].}$$

39.7 General Method

Here is an outline detailing the method of separation of variables for a linear partial differential equation for $u(x, y, z, \ldots)$.

1. Substitute $u(x, y, z, \ldots) = X(x)Y(y)Z(z)\cdots$ into the partial differential equation. Separate the equation into ordinary differential equations.

2. Translate the boundary conditions for u into boundary conditions for X, Y, Z, ... The continuity of u may give additional boundary conditions and boundedness conditions.

3. Solve the differential equation(s) that determine the eigenvalues. Make sure to consider all cases. The eigenfunctions will be determined up to a multiplicative constant.

4. Solve the rest of the differential equations subject to the homogeneous boundary conditions. The eigenvalues will be a parameter in the solution. The solutions will be determined up to a multiplicative constant.

5. The eigen-solutions are the product of the solutions of the ordinary differential equations. $\phi_n = X_n Y_n Z_n \cdots$. The solution of the partial differential equation is a linear combination of the eigen-solutions.

$$u(x, y, z, \ldots) = \sum a_n \phi_n$$

6. Solve for the coefficients, a_n using the inhomogeneous boundary conditions.

39.8 Exercises

Exercise 39.1
Obtain Poisson's formula to solve the Dirichlet problem for the circular region $0 \leq r < R$, $0 \leq \theta < 2\pi$. That is, determine a solution $\phi(r, \theta)$ to Laplace's equation

$$\nabla^2 \phi = 0$$

in polar coordinates given $\phi(R, \theta)$. Show that

$$\phi(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \phi(R, \alpha) \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \alpha)} \, \mathrm{d}\alpha$$

Exercise 39.2
Consider the temperature of a ring of unit radius. Solve the problem

$$u_t = \kappa u_{\theta\theta}, \quad u(\theta, 0) = f(\theta)$$

with separation of variables.

Exercise 39.3
Solve the Laplace's equation by separation of variables.

$$\Delta u \equiv u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$
$$u(x, 0) = f(x), \quad u(x, 1) = 0, \quad u(0, y) = 0, \quad u(1, y) = 0$$

Here $f(x)$ is an arbitrary function which is known.

Exercise 39.4
Solve the following problem by separation of variables:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad 0 < r < 1,$$

$$u(1, \theta) = f(\theta).$$

Thus, you must find the function $u = u(r, \theta)$ which satisfies the partial differential equation inside the unit circle and which takes on the values of $f(\theta)$ on the circumference.

Exercise 39.5

Find the normal modes of oscillation of a drum head of unit radius. The drum head obeys the wave equation with zero displacement on the boundary.

$$\Delta v \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}, \qquad v(1, \theta, t) = 0$$

Exercise 39.6

Solve the equation

$$\phi_t = a^2 \phi_{xx}, \quad 0 < x < l, \quad t > 0$$

with boundary conditions $\phi(0, t) = \phi(l, t) = 0$, and initial conditions

$$\phi(x, 0) = \begin{cases} x, & 0 \leq x \leq l/2, \\ l - x, & l/2 < x \leq l. \end{cases}$$

Comment on the differentiability (that is the number of finite derivatives with respect to x) at time $t = 0$ and at time $t = \epsilon$, where $\epsilon > 0$ and $\epsilon \ll 1$.

Exercise 39.7

Consider a one-dimensional rod of length L with initial temperature distribution $f(x)$. The temperatures at the left and right ends of the rod are held at T_0 and T_1, respectively. To find the temperature of the rod for $t > 0$, solve

$$u_t = \kappa u_{xx}, \quad 0 < x < L, \quad t > 0$$
$$u(0, t) = T_0, \quad u(L, t) = T_1, \quad u(x, 0) = f(x),$$

with separation of variables.

Exercise 39.8
For $0 < x < l$ solve the problem

$$\phi_t = a^2 \phi_{xx} + w(x,t) \tag{39.5}$$
$$\phi(0,t) = 0, \quad \phi_x(l,t) = 0, \quad \phi(x,0) = f(x)$$

by means of a series expansion involving the eigenfunctions of

$$\frac{\mathrm{d}^2\beta(x)}{\mathrm{d}x^2} + \lambda\beta(x) = 0,$$
$$\beta(0) = \beta'(l) = 0.$$

Here $w(x,t)$ and $f(x)$ are prescribed functions.

Exercise 39.9
Solve the heat equation of Exercise 39.8 with the same initial conditions but with the boundary conditions

$$\phi(0,t) = 0, \quad c\phi(l,t) + \phi_x(l,t) = 0.$$

Here $c > 0$ is a constant. Although it is not possible to solve for the eigenvalues λ in closed form, show that the eigenvalues assume a simple form for large values of λ.

Exercise 39.10
Use a series expansion technique to solve the problem

$$\phi_t = a^2 \phi_{xx} + 1, \qquad t > 0, \quad 0 < x < l$$

with boundary and initial conditions given by

$$\phi(x,0) = 0, \quad \phi(0,t) = t, \quad \phi_x(l,t) = -c\phi(l,t)$$

where $c > 0$ is a constant.

Exercise 39.11
Let $\phi(x,t)$ satisfy the equation

$$\phi_t = a^2 \phi_{xx}$$

for $0 < x < l$, $t > 0$ with initial conditions $\phi(x,0) = 0$ for $0 < x < l$, with boundary conditions $\phi(0,t) = 0$ for $t > 0$, and $\phi(l,t) + \phi_x(l,t) = 1$ for $t > 0$. Obtain two series solutions for this problem, one which is useful for large t and the other useful for small t.

Exercise 39.12
A rod occupies the portion $1 < x < 2$ of the x-axis. The thermal conductivity depends on x in such a manner that the temperature $\phi(x,t)$ satisfies the equation

$$\phi_t = A^2 (x^2 \phi_x)_x \tag{39.6}$$

where A is a constant. For $\phi(1,t) = \phi(2,t) = 0$ for $t > 0$, with $\phi(x,0) = f(x)$ for $1 < x < 2$, show that the appropriate series expansion involves the eigenfunctions

$$\beta_n(x) = \frac{1}{\sqrt{x}} \sin\left(\frac{\pi n \log x}{\log 2}\right).$$

Work out the series expansion for the given boundary and initial conditions.

Exercise 39.13
Consider a string of length L with a fixed left end a free right end. Initially the string is at rest with displacement $f(x)$. Find the motion of the string by solving,

$$\begin{gathered} u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u(0,t) = 0, \quad u_x(L,t) = 0, \\ u(x,0) = f(x), \quad u_t(x,0) = 0, \end{gathered}$$

with separation of variables.

Exercise 39.14
Consider the equilibrium temperature distribution in a two-dimensional block of width a and height b. There is a heat source given by the function $f(x, y)$. The vertical sides of the block are held at zero temperature; the horizontal sides are insulated. To find this equilibrium temperature distribution, solve the potential equation,

$$u_{xx} + u_{yy} = f(x, y), \quad 0 < x < a, \quad 0 < y < b,$$
$$u(0, y) = u(a, y) = 0, \quad u_y(x, 0) = u_y(x, b) = 0,$$

with separation of variables.

Exercise 39.15
Consider the vibrations of a stiff beam of length L. More precisely, consider the transverse vibrations of an unloaded beam, whose weight can be neglected compared to its stiffness. The beam is simply supported at $x = 0, L$. (That is, it is resting on fulcrums there. $u(0, t) = 0$ means that the beam is resting on the fulcrum; $u_{xx}(0, t) = 0$ indicates that there is no bending force at that point.) The beam has initial displacement $f(x)$ and velocity $g(x)$. To determine the motion of the beam, solve

$$u_{tt} + a^2 u_{xxxx} = 0, \quad 0 < x < L, \quad t > 0,$$
$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x),$$
$$u(0, t) = u_{xx}(0, t) = 0, \quad u(L, t) = u_{xx}(L, t) = 0,$$

with separation of variables.

Exercise 39.16
The temperature along a magnet winding of length L carrying a current I satisfies, (for some $\alpha > 0$):

$$u_t = \kappa u_{xx} + I^2 \alpha u.$$

The ends of the winding are kept at zero, i.e.,

$$u(0, t) = u(L, t) = 0;$$

and the initial temperature distribution is

$$u(x, 0) = g(x).$$

Find $u(x, t)$ and determine the critical current I_{CR} which is defined as the least current at which the winding begins to heat up exponentially. Suppose that $\alpha < 0$, so that the winding has a negative coefficient of resistance with respect to temperature. What can you say about the critical current in this case?

Exercise 39.17
The "e-folding" time of a decaying function of time is the time interval, Δ_e, in which the magnitude of the function is reduced by at least $\frac{1}{e}$. Thus if $u(x, t) = \mathrm{e}^{-\alpha t} f(x) + \mathrm{e}^{-\beta t} g(x)$ with $\alpha > \beta > 0$ then $\Delta_e = \frac{1}{\beta}$. A body with heat conductivity κ has its exterior surface maintained at temperature zero. Initially the interior of the body is at the uniform temperature $T > 0$. Find the e-folding time of the body if it is:

a) An infinite slab of thickness a.

b) An infinite cylinder of radius a.

c) A sphere of radius a.

Note that in (a) the temperature varies only in the z direction and in time; in (b) and (c) the temperature varies only in the radial direction and in time.

d) What are the e-folding times if the surfaces are perfectly insulated, (i.e., $\frac{\partial u}{\partial n} = 0$, where n is the exterior normal at the surface)?

Exercise 39.18
Solve the heat equation with a time-dependent diffusivity in the rectangle $0 < x < a$, $0 < y < b$. The top and bottom sides are held at temperature zero; the lateral sides are insulated. We have the initial-boundary value

problem:

$$
\begin{gathered}
u_t = \kappa(t)\left(u_{xx} + u_{yy}\right), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0, \\
u(x, 0, t) = u(x, b, t) = 0, \\
u_x(0, y, t) = u_x(a, y, t) = 0, \\
u(x, y, 0) = f(x, y).
\end{gathered}
$$

The diffusivity, $\kappa(t)$, is a known, positive function.

Exercise 39.19
A semi-circular rod of infinite extent is maintained at temperature $T = 0$ on the flat side and at $T = 1$ on the curved surface:

$$x^2 + y^2 = 1, \quad y > 0.$$

Find the steady state temperature in a cross section of the rod using separation of variables.

Exercise 39.20
Use separation of variables to find the steady state temperature $u(x, y)$ in a slab: $x \geq 0$, $0 \leq y \leq 1$, which has zero temperature on the faces $y = 0$ and $y = 1$ and has a given distribution: $u(y, 0) = f(y)$ on the edge $x = 0$, $0 \leq y \leq 1$.

Exercise 39.21
Find $u(r, \theta)$ which satisfies:

$$\Delta u = 0, \quad 0 < \theta < \alpha, a < r < b,$$

subject to the boundary conditions:

$$u(r, 0) = u(r, \alpha) = 0, \quad u(a, \theta) = 0, \quad u(b, \theta) = f(\theta).$$

Exercise 39.22

a) A piano string of length L is struck, at time $t = 0$, by a flat hammer of width $2d$ centered at a point ξ, having velocity v. Find the ensuing motion, $u(x,t)$, of the string for which the wave speed is c.

b) Suppose the hammer is curved, rather than flat as above, so that the initial velocity distribution is

$$u_t(x,0) = \begin{cases} v\cos\left(\frac{\pi(x-\xi)}{2d}\right), & |x-\xi| < d \\ 0 & |x-\xi| > d. \end{cases}$$

Find the ensuing motion.

c) Compare the kinetic energies of each harmonic in the two solutions. Where should the string be struck in order to maximize the energy in the n^{th} harmonic in each case?

Exercise 39.23

If the striking hammer is not perfectly rigid, then its effect must be included as a time dependent forcing term of the form:

$$s(x,t) = \begin{cases} v\cos\left(\frac{\pi(x-\xi)}{2d}\right)\sin\left(\frac{\pi t}{\delta}\right), & \text{for } |x-\xi| < d, \quad 0 < t < \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Find the motion of the string for $t > \delta$. Discuss the effects of the width of the hammer and duration of the blow with regard to the energy in overtones.

Exercise 39.24

Find the propagating modes in a square waveguide of side L for harmonic signals of frequency ω when the propagation speed of the medium is c. That is, we seek those solutions of

$$u_{tt} - c^2\Delta u = 0,$$

where $u = u(x,y,z,t)$ has the form $u(x,y,z,t) = v(x,y,z)\,\mathrm{e}^{i\omega t}$, which satisfy the conditions:

$$u(x,y,z,t) = 0 \quad \text{for} \quad x = 0, L, \quad y = 0, L, \quad z > 0,$$
$$\lim_{z\to\infty} |u| \neq \infty \text{ and } \neq 0.$$

Indicate in terms of inequalities involving $k = \omega/c$ and appropriate eigenvalues, $\lambda_{n,m}$ say, for which n and m the solutions $u_{n,m}$ satisfy the conditions.

Exercise 39.25
Find the modes of oscillation and their frequencies for a rectangular drum head of width a and height b. The modes of oscillation are eigensolutions of

$$u_{tt} = c^2 \Delta u, \quad 0 < x < a,\ 0 < y < b,$$
$$u(0,y) = u(a,y) = u(x,0) = u(x,b) = 0.$$

Exercise 39.26
Using separation of variables solve the heat equation

$$\phi_t = a^2 \left(\phi_{xx} + \phi_{yy}\right)$$

in the rectangle $0 < x < l_x$, $0 < y < l_y$ with initial conditions

$$\phi(x,y,0) = 1,$$

and boundary conditions

$$\phi(0,y,t) = \phi(l_x,y,t) = 0, \quad \phi_y(x,0,t) = \phi_y(x,l_y,t) = 0.$$

Exercise 39.27

Using polar coordinates and separation of variables solve the heat equation

$$\phi_t = a^2 \nabla^2 \phi$$

in the circle $0 < r < R_0$ with initial conditions

$$\phi(r, \theta, 0) = V$$

where V is a constant, and boundary conditions

$$\phi(R_0, \theta, t) = 0.$$

(a) Show that for $t > 0$,

$$\phi(r, \theta, t) = 2V \sum_{n=1}^{\infty} \exp\left(-\frac{a^2 j_{0,n}^2}{R_0^2} t\right) \frac{J_0\left(j_{0,n} r/R_0\right)}{j_{0,n} J_1(j_{0,n})},$$

where $j_{0,n}$ are the roots of $J_0(x)$:

$$J_0(j_{0,n}) = 0, \quad n = 1, 2, \ldots$$

Hint: The following identities may be of some help:

$$\int_0^{R_0} r J_0\left(j_{0,n} r/R_0\right) J_0\left(j_{0,m} r/R_0\right) dr = 0, \qquad m \neq n,$$

$$\int_0^{R_0} r J_0^2\left(j_{0,n} r/R_0\right) dr = \frac{R_0^2}{2} J_1^2(j_{0,n}),$$

$$\int_0^{r} r J_0(\beta r) dr = \frac{r}{\beta} J_1(\beta r) \qquad \text{for any } \beta.$$

(b) For any fixed r, $0 < r < R_0$, use the asymptotic approximation for the J_n Bessel functions for large argument (this can be found in the notes for second quarter, AMa95b or in any standard math tables) to determine the rate of decay of the terms of the series solution for ϕ at time $t = 0$.

Exercise 39.28

Consider the solution of the diffusion equation in spherical coordinates given by

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta, \end{aligned}$$

where r is the radius, θ is the polar angle, and ϕ is the azimuthal angle. We wish to solve the equation on the **surface** of the sphere given by $r = R$, $0 < \theta < \pi$, and $0 < \phi < 2\pi$. The diffusion equation for the solution $\Psi(\theta, \phi, t)$ in these coordinates on the surface of the sphere becomes

$$\frac{\partial \Psi}{\partial t} = \frac{a^2}{R^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right). \tag{39.7}$$

where a is a positive constant.

(a) Using separation of variables show that a solution Ψ can be found in the form

$$\Psi(\theta, \phi, t) = T(t)\Theta(\theta)\Phi(\phi),$$

where T,Θ,Φ obey ordinary differential equations in t,θ, and ϕ respectively. Derive the ordinary differential equations for T and Θ, and show that the differential equation obeyed by Φ is given by

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} - c\Phi = 0,$$

where c is a constant.

(b) Assuming that $\Psi(\theta, \phi, t)$ is determined over the full range of the azimuthal angle, $0 < \phi < 2\pi$, determine the allowable values of the separation constant c and the corresponding allowable functions Φ. Using these values of c and letting $x = \cos\theta$ rewrite in terms of the variable x the differential equation satisfied by Θ. What are appropriate boundary conditions for Θ? The resulting equation is known as the generalized or associated Legendre equation.

(c) Assume next that the initial conditions for Ψ are chosen such that

$$\Psi(\theta, \phi, t = 0) = f(\theta),$$

where $f(\theta)$ is a specified function which is regular at the north and south poles (that is $\theta = 0$ and $\theta = \pi$). Note that the initial condition is independent of the azimuthal angle ϕ. Show that in this case the method of separation of variables gives a series solution for Ψ of the form

$$\Psi(\theta, t) = \sum_{l=0}^{\infty} A_l \exp(-\lambda_l^2 t) P_l(\cos\theta),$$

where $P_l(x)$ is the l'th Legendre polynomial, and determine the constants λ_l as a function of the index l.

(d) Solve for $\Psi(\theta, t)$, $t > 0$ given that $f(\theta) = 2\cos^2\theta - 1$.

Useful facts:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1 - x^2)\frac{\mathrm{d}P_l(x)}{\mathrm{d}x}\right] + l(l+1)P_l(x) = 0$$

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2} \end{aligned}$$

$$\int_{-1}^{1} dx P_l(x) P_m(x) = \begin{cases} 0 & \text{if } l \neq m \\ \dfrac{2}{2l+1} & \text{if } l = m \end{cases}$$

Exercise 39.29
Let $\phi(x, y)$ satisfy Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0$$

in the rectangle $0 < x < 1$, $0 < y < 2$, with $\phi(x, 2) = x(1 - x)$, and with $\phi = 0$ on the other three sides. Use a series solution to determine ϕ inside the rectangle. How many terms are required to give $\phi(\frac{1}{2}, 1)$ with about 1% (also 0.1%) accuracy; how about $\phi_x(\frac{1}{2}, 1)$?

Exercise 39.30
Let $\psi(r, \theta, \phi)$ satisfy Laplace's equation in spherical coordinates in each of the two regions $r < a$, $r > a$, with $\psi \to 0$ as $r \to \infty$. Let

$$\lim_{r \to a^+} \psi(r, \theta, \phi) - \lim_{r \to a^-} \psi(r, \theta, \phi) = 0,$$
$$\lim_{r \to a^+} \psi_r(r, \theta, \phi) - \lim_{r \to a^-} \psi_r(r, \theta, \phi) = P_n^m(\cos\theta) \sin(m\phi),$$

where m and $n \geq m$ are integers. Find ψ in $r < a$ and $r > a$. In electrostatics, this problem corresponds to that of determining the potential of a spherical harmonic type charge distribution over the surface of the sphere. In this way one can determine the potential due to an arbitrary surface charge distribution since any charge distribution can be expressed as a series of spherical harmonics.

Exercise 39.31
Obtain a formula analogous to the Poisson formula to solve the Neumann problem for the circular region $0 \leq r < R$, $0 \leq \theta < 2\pi$. That is, determine a solution $\phi(r, \theta)$ to Laplace's equation

$$\nabla^2 \phi = 0$$

in polar coordinates given $\phi_r(R, \theta)$. Show that

$$\phi(r, \theta) = -\frac{R}{2\pi} \int_0^{2\pi} \phi_r(R, \alpha) \ln\left[1 - \frac{2r}{R}\cos(\theta - \alpha) + \frac{r^2}{R^2}\right] \mathrm{d}\alpha$$

within an arbitrary additive constant.

Exercise 39.32
Investigate solutions of

$$\phi_t = a^2 \phi_{xx}$$

obtained by setting the separation constant $C = (\alpha + i\beta)^2$ in the equations obtained by assuming $\phi = X(x)T(t)$:

$$\frac{T'}{T} = C, \quad \frac{X''}{X} = \frac{C}{a^2}.$$

39.9 Hints

Hint 39.1

Hint 39.2

Impose the boundary conditions

$$u(0, t) = u(2\pi, t), \quad u_\theta(0, t) = u_\theta(2\pi, t).$$

Hint 39.3

Apply the separation of variables $u(x, y) = X(x)Y(y)$. Solve an eigenvalue problem for $X(x)$.

Hint 39.4

Hint 39.5

Hint 39.6

Hint 39.7

There are two ways to solve the problem. For the first method, expand the solution in a series of the form

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

Because of the inhomogeneous boundary conditions, the convergence of the series will not be uniform. You can differentiate the series with respect to t, but not with respect to x. Multiply the partial differential equation by the eigenfunction $\sin(n\pi x/L)$ and integrate from $x = 0$ to $x = L$. Use integration by parts to move derivatives in x from u to the eigenfunctions. This process will yield a first order, ordinary differential equation for each of the a_n's.

For the second method: Make the change of variables $v(x,t) = u(x,t) - \mu(x)$, where $\mu(x)$ is the equilibrium temperature distribution to obtain a problem with homogeneous boundary conditions.

Hint 39.8

Hint 39.9

Hint 39.10

Hint 39.11

Hint 39.12

Hint 39.13

Use separation of variables to find eigen-solutions of the partial differential equation that satisfy the homogeneous boundary conditions. There will be two eigen-solutions for each eigenvalue. Expand $u(x,t)$ in a series of the eigen-solutions. Use the two initial conditions to determine the constants.

Hint 39.14
Expand the solution in a series of eigenfunctions in x. Determine these eigenfunctions by using separation of variables on the homogeneous partial differential equation. You will find that the answer has the form,

$$u(x, y) = \sum_{n=1}^{\infty} u_n(y) \sin\left(\frac{n\pi x}{a}\right).$$

Substitute this series into the partial differential equation to determine ordinary differential equations for each of the u_n's. The boundary conditions on $u(x, y)$ will give you boundary conditions for the u_n's. Solve these ordinary differential equations with Green functions.

Hint 39.15
Solve this problem by expanding the solution in a series of eigen-solutions that satisfy the partial differential equation and the homogeneous boundary conditions. Use the initial conditions to determine the coefficients in the expansion.

Hint 39.16
Use separation of variables to find eigen-solutions that satisfy the partial differential equation and the homogeneous boundary conditions. The solution is a linear combination of the eigen-solutions. The whole solution will be exponentially decaying if each of the eigen-solutions is exponentially decaying.

Hint 39.17
For parts (a), (b) and (c) use separation of variables. For part (b) the eigen-solutions will involve Bessel functions. For part (c) the eigen-solutions will involve spherical Bessel functions. Part (d) is trivial.

Hint 39.18
The solution is a linear combination of eigen-solutions of the partial differential equation that satisfy the homogeneous boundary conditions. Determine the coefficients in the expansion with the initial condition.

Hint 39.19
The problem is

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 0 < r < 1, \quad 0 < \theta < \pi$$
$$u(r, 0) = u(r, \pi) = 0, \quad u(0, \theta) = 0, \quad u(1, \theta) = 1$$

The solution is a linear combination of eigen-solutions that satisfy the partial differential equation and the three homogeneous boundary conditions.

Hint 39.20

Hint 39.21

Hint 39.22

Hint 39.23

Hint 39.24

Hint 39.25

Hint 39.26

Hint 39.27

Hint 39.28

Hint 39.29

Hint 39.30

Hint 39.31

Hint 39.32

39.10 Solutions

Solution 39.1

We expand the solution in a Fourier series.

$$\phi = \frac{1}{2}a_0(r) + \sum_{n=1}^{\infty} a_n(r)\cos(n\theta) + \sum_{n=1}^{\infty} b_n(r)\sin(n\theta)$$

We substitute the series into the Laplace's equation to determine ordinary differential equations for the coefficients.

$$\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$$a_0'' + \frac{1}{r}a_0' = 0, \quad a_n'' + \frac{1}{r}a_n' - n^2 a_n = 0, \quad b_n'' + \frac{1}{r}b_n' - n^2 b_n = 0$$

The solutions that are bounded at $r = 0$ are, (to within multiplicative constants),

$$a_0(r) = 1, \quad a_n(r) = r^n, \quad b_n(r) = r^n.$$

Thus $\phi(r, \theta)$ has the form

$$\phi(r, \theta) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} d_n r^n \sin(n\theta)$$

We apply the boundary condition at $r = R$.

$$\phi(R, \theta) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n R^n \cos(n\theta) + \sum_{n=1}^{\infty} d_n R^n \sin(n\theta)$$

The coefficients are

$$c_0 = \frac{1}{\pi}\int_0^{2\pi} \phi(R, \alpha)\,\mathrm{d}\alpha, \quad c_n = \frac{1}{\pi R^n}\int_0^{2\pi} \phi(R, \alpha)\cos(n\alpha)\,\mathrm{d}\alpha, \quad d_n = \frac{1}{\pi R^n}\int_0^{2\pi} \phi(R, \alpha)\sin(n\alpha)\,\mathrm{d}\alpha.$$

We substitute the coefficients into our series solution.

$$\phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \phi(R,\alpha)\,\mathrm{d}\alpha + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \int_0^{2\pi} \phi(R,\alpha)\cos(n(\theta-\alpha))\,\mathrm{d}\alpha$$

$$\phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \phi(R,\alpha)\,\mathrm{d}\alpha + \frac{1}{\pi}\int_0^{2\pi} \phi(R,\alpha)\Re\left(\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \mathrm{e}^{in(\theta-\alpha)}\right)\mathrm{d}\alpha$$

$$\phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \phi(R,\alpha)\,\mathrm{d}\alpha + \frac{1}{\pi}\int_0^{2\pi} \phi(R,\alpha)\Re\left(\frac{\frac{r}{R}\mathrm{e}^{i(\theta-\alpha)}}{1-\frac{r}{R}\mathrm{e}^{i(\theta-\alpha)}}\right)\mathrm{d}\alpha$$

$$\phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \phi(R,\alpha)\,\mathrm{d}\alpha + \frac{1}{\pi}\int_0^{2\pi} \phi(R,\alpha)\Re\left(\frac{\frac{r}{R}\mathrm{e}^{i(\theta-\alpha)} - \left(\frac{r}{R}\right)^2}{1-2\frac{r}{R}\cos(\theta-\alpha)+\left(\frac{r}{R}\right)^2}\right)\mathrm{d}\alpha$$

$$\phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \phi(R,\alpha)\,\mathrm{d}\alpha + \frac{1}{\pi}\int_0^{2\pi} \phi(R,\alpha)\frac{Rr\cos(\theta-\alpha)-r^2}{R^2+r^2-2Rr\cos(\theta-\alpha)}\,\mathrm{d}\alpha$$

$$\boxed{\phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \phi(R,\alpha)\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\alpha)}\,\mathrm{d}\alpha}$$

Solution 39.2

In order that the solution is continuously differentiable, (which it must be in order to satisfy the differential equation), we impose the boundary conditions

$$u(0,t) = u(2\pi,t), \quad u_\theta(0,t) = u_\theta(2\pi,t).$$

We apply the separation of variables $u(\theta,t) = \Theta(\theta)T(t)$.

$$u_t = \kappa u_{\theta\theta}$$
$$\Theta T' = \kappa \Theta'' T$$
$$\frac{T'}{\kappa T} = \frac{\Theta''}{\Theta} = -\lambda$$

We have the self-adjoint eigenvalue problem

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)$$

which has the eigenvalues and orthonormal eigenfunctions

$$\lambda_n = n^2, \quad \Theta_n = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{in\theta}, \quad n \in \mathbb{Z}.$$

Now we solve the problems for $T_n(t)$ to obtain eigen-solutions of the heat equation.

$$T_n' = -n^2\kappa T_n$$
$$T_n = \mathrm{e}^{-n^2\kappa t}$$

The solution is a linear combination of the eigen-solutions.

$$\boxed{u(\theta, t) = \sum_{n=-\infty}^{\infty} u_n \frac{1}{\sqrt{2\pi}} \mathrm{e}^{in\theta} \mathrm{e}^{-n^2\kappa t}}$$

We use the initial conditions to determine the coefficients.

$$u(\theta, 0) = \sum_{n=-\infty}^{\infty} u_n \frac{1}{\sqrt{2\pi}} \mathrm{e}^{in\theta} = f(\theta)$$
$$\boxed{u_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathrm{e}^{-in\theta} f(\theta) \,\mathrm{d}\theta}$$

Solution 39.3

Substituting $u(x, y) = X(x)Y(y)$ into the partial differential equation yields

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

With the homogeneous boundary conditions, we have the two problems

$$X'' + \lambda X = 0, \qquad X(0) = X(1) = 0,$$

$$Y'' - \lambda Y = 0, \qquad Y(1) = 0.$$

The eigenvalues and orthonormal eigenfunctions for $X(x)$ are

$$\lambda_n = (n\pi)^2, \quad X_n = \sqrt{2}\sin(n\pi x).$$

The general solution for Y is

$$Y_n = a\cosh(n\pi y) + b\sinh(n\pi y).$$

The solution for that satisfies the right homogeneous boundary condition, (up to a multiplicative constant), is

$$Y_n = \sinh(n\pi(1 - y))$$

$u(x, y)$ is a linear combination of the eigen-solutions.

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} u_n \sqrt{2}\sin(n\pi x)\sinh(n\pi(1 - y))}$$

We use the inhomogeneous boundary condition to determine coefficients.

$$u(x, 0) = \sum_{n=1}^{\infty} u_n \sqrt{2}\sin(n\pi x)\sinh(n\pi) = f(x)$$

$$\boxed{u_n = \sqrt{2}\int_0^1 \sin(n\pi\xi) f(\xi)\, \mathrm{d}\xi}$$

Solution 39.4
Substituting $u(r, \theta) = R(r)\Theta(\theta)$ yields

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$$r^2R'' + rR' - \lambda R = 0, \qquad \Theta'' + \lambda\Theta = 0$$

We assume that u is a strong solution of the partial differential equation and is thus twice continuously differentiable, ($u \in C^2$). In particular, this implies that R and Θ are bounded and that Θ is continuous and has a continuous first derivative along $\theta = 0$. This gives us the problems

$$\Theta'' + \lambda\Theta = 0, \qquad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi),$$

$$r^2R'' + rR' - \lambda R = 0, \qquad R \text{ is bounded}$$

We consider negative, zero and positive values of λ in solving the equation for Θ.

$\lambda < 0$. The general solution for Θ is

$$\Theta = a\cosh(\sqrt{-\lambda}\theta) + b\sinh(\sqrt{-\lambda}\theta).$$

$$\Theta(0) = \Theta(2\pi) \quad \Rightarrow \quad a = 0, \qquad \Theta'(0) = \Theta'(2\pi) \quad \Rightarrow \quad b = 0.$$

There are no negative eigenvalues.

$\lambda = 0$. The general solution for Θ is

$$\Theta = a + b\theta.$$

$$\Theta(0) = \Theta(2\pi) \quad \Rightarrow \quad b = 0, \qquad \Theta'(0) = \Theta'(2\pi) \quad \Rightarrow \quad a \text{ is arbitrary.}$$

We have the eigenvalue and eigenfunction

$$\lambda_0 = 0, \qquad \Theta_0 = \frac{1}{2}.$$

$\lambda > 0$. The general solution is

$$\Theta = a\cos(\sqrt{\lambda}\theta) + b\sin(\sqrt{\lambda}\theta).$$

Applying the boundary conditions to find the eigenvalues and eigenfunctions,

$$\begin{aligned}
\Theta(0) = \Theta(2\pi) \quad &\Rightarrow \quad a = a\cos(\sqrt{\lambda}2\pi) + b\sin(\sqrt{\lambda}2\pi) \\
&\Rightarrow \quad \cos(\sqrt{\lambda}2\pi) = 1, \quad \sin(\sqrt{\lambda}2\pi) = 0 \\
&\Rightarrow \quad \sqrt{\lambda} = n, \quad \text{for } n = 1, 2, 3, \ldots \\
&\Rightarrow \quad \lambda = n^2, \quad \text{for } n = 1, 2, 3, \ldots
\end{aligned}$$

The boundary condition, $\Theta'(0) = \Theta'(2\pi)$ is satisfied for these values of λ. This gives us the eigenvalues and eigenfunctions

$$\lambda_n = n^2, \qquad \Theta_n^{(1)} = \cos(n\theta), \quad \Theta_n^{(2)} = \sin(n\theta), \qquad \text{for} n = 1, 2, 3, \ldots$$

Now to find the bounded solutions of the equation for R. Substituting $R = r^\alpha$ yields

$$\alpha(\alpha - 1) + \alpha - \lambda = 0$$

$$\alpha = \pm\sqrt{\lambda}.$$

There are two cases to consider.

$\lambda_0 = 0$.

$$R = a + b \log r$$

Boundedness demands that $b = 0$. Thus we have the solution

$$R = 1$$

$\lambda_n = n^2 > 0$

$$R = ar^n + br^{-n}$$

Boundedness demands that $b = 0$. Thus we have the solution

$$R = r^n.$$

The general solution for u is

$$\boxed{u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] r^n.}$$

The inhomogeneous boundary condition will determine the coefficients.

$$u(1, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] = f(\theta)$$

The coefficients are the Fourier coefficients of $f(\theta)$.

$$\boxed{\begin{aligned} a_n &= \tfrac{1}{\pi} \textstyle\int_0^{2\pi} f(\theta) \cos(n\theta) \, \mathrm{d}\theta \\ b_n &= \tfrac{1}{\pi} \textstyle\int_0^{2\pi} f(\theta) \sin(n\theta) \, \mathrm{d}\theta \end{aligned}}$$

Solution 39.5
A normal mode of frequency ω satisfies

$$v(r,\theta,t) = u(r,\theta)\,\mathrm{e}^{i\omega t}.$$

Substituting this into the partial differential equation and the boundary condition yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{\omega^2}{c^2}u, \qquad u(1,\theta) = 0,$$
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + k^2 u = 0, \qquad u(1,\theta) = 0,$$

where $k = \frac{\omega}{c}$. Applying separation of variables to the partial differential equation for u, with $u = R(r)\Theta(\theta)$,

$$r^2 R''\Theta + rR'\Theta + R\Theta'' + k^2 r^2 R\Theta = 0,$$
$$r^2\frac{R''}{R} + r\frac{R'}{R} + k^2 r^2 = -\frac{\Theta''}{\Theta} = \lambda^2.$$

Now we have the two ordinary differential equations,

$$R'' + \frac{1}{r}R' + \left(k^2 - \frac{\lambda^2}{r^2}\right)R = 0, \quad R(0) \text{ is bounded}, \quad R(1) = 0,$$
$$\Theta'' + \lambda^2\Theta = 0, \quad \Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi).$$

The eigenvalues and eigenfunctions for Θ are

$$\lambda_n = n, \quad n = 0, 1, 2, \ldots,$$

$$\Theta_0 = \frac{1}{2}, \quad \Theta_n^{(1)} = \cos(n\theta), \quad \Theta_n^{(2)} = \sin(n\theta), \quad n = 1, 2, 3, \ldots$$

The differential equation for R is then

$$R'' + \frac{1}{r}R' + \left(k^2 - \frac{n^2}{r^2}\right)R = 0, \quad R(0) \text{ is bounded}, \quad R(1) = 0.$$

The general solution of the differential equation is a linear combination of Bessel functions of order n.

$$R(r) = c_1 J_n(kr) + c_2 Y_n(kr)$$

Since $Y_n(kr)$ is unbounded at $r = 0$, the solution has the form

$$R(r) = c J_n(kr).$$

Applying the second boundary condition yields

$$J_n(k) = 0.$$

Thus the eigenvalues and eigenfunctions for R are

$$k_{nm} = j_{nm}, \quad R_{nm} = J_n(j_{nm} r),$$

where j_{nm} is the m^{th} positive root of J_n. Combining the above results, the normal modes of oscillation are

$$v_{0m} = \frac{1}{2} J_0(j_{0m} r)\, \mathrm{e}^{i\omega t}, \qquad m = 1, 2, 3, \ldots,$$
$$v_{nm} = \cos(n\theta + \alpha) J_{nm}(j_{nm} r)\, \mathrm{e}^{i\omega t}, \qquad n, m = 1, 2, 3, \ldots.$$

u_{22} and u_{33} are plotted in Figure 39.2.

Solution 39.6

We will expand the solution in a complete, orthogonal set of functions $\{X_n(x)\}$, where the coefficients are functions of t.

$$\phi = \sum_n T_n(t) X_n(x)$$

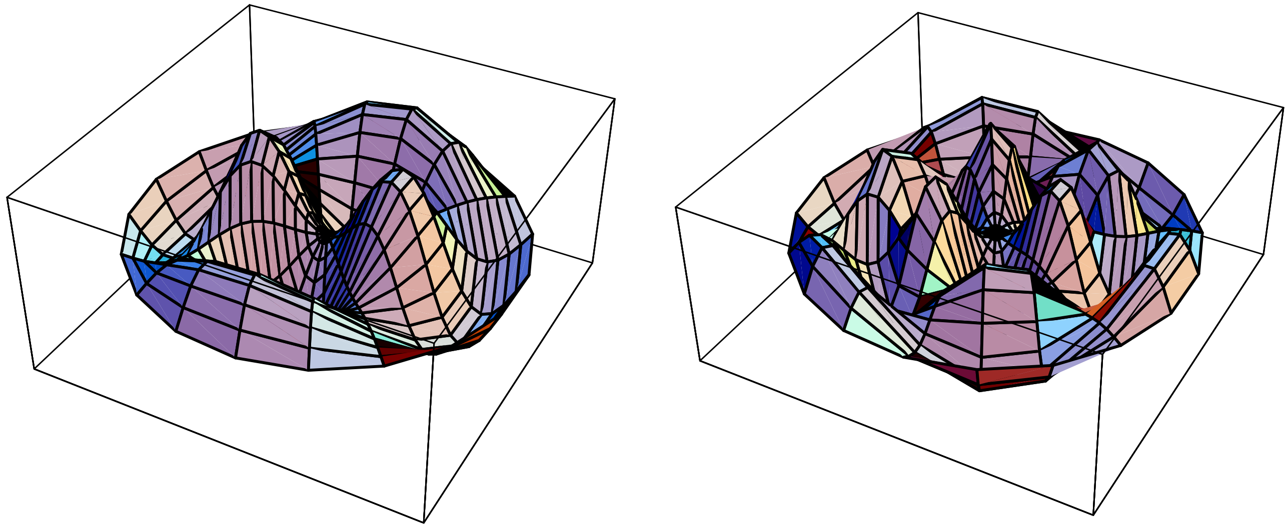

Figure 39.2: The Normal Modes u_{22} and u_{33}

We will use separation of variables to determine a convenient set $\{X_n\}$. We substitite $\phi = T(t)X(x)$ into the diffusion equation.

$$\phi_t = a^2 \phi_{xx}$$
$$XT' = a^2 X'' T$$
$$\frac{T'}{a^2 T} = \frac{X''}{X} = -\lambda$$
$$T' = -a^2 \lambda T, \quad X'' + \lambda X = 0$$

Note that in order to satisfy $\phi(0, t) = \phi(l, t) = 0$, the X_n must satisfy the same homogeneous boundary conditions,

$X_n(0) = X_n(l) = 0$. This gives us a Sturm-Liouville problem for $X(x)$.

$$X'' + \lambda X = 0, \quad X(0) = X(l) = 0$$
$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n = \sin\left(\frac{n\pi x}{l}\right), \quad n \in \mathbb{Z}^+$$

Thus we seek a solution of the form

$$\phi = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{l}\right). \tag{39.8}$$

This solution automatically satisfies the boundary conditions. We will assume that we can differentiate it. We will substitite this form into the diffusion equation and the initial condition to determine the coefficients in the series, $T_n(t)$. First we substitute Equation 39.8 into the partial differential equation for ϕ to determine ordinary differential equations for the T_n.

$$\phi_t = a^2 \phi_{xx}$$
$$\sum_{n=1}^{\infty} T_n'(t) \sin\left(\frac{n\pi x}{l}\right) = -a^2 \sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right)^2 T_n(t) \sin\left(\frac{n\pi x}{l}\right)$$
$$T_n' = -\left(\frac{an\pi}{l}\right)^2 T_n$$

Now we substitute Equation 39.8 into the initial condition for ϕ to determine initial conditions for the T_n.

$$\sum_{n=1}^{\infty} T_n(0) \sin\left(\frac{n\pi x}{l}\right) = \phi(x,0)$$

$$T_n(0) = \frac{\int_0^l \sin\left(\frac{n\pi x}{l}\right) \phi(x,0)\,\mathrm{d}x}{\int_0^l \sin^2\left(\frac{n\pi x}{l}\right)\,\mathrm{d}x}$$

$$T_n(0) = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right) \phi(x,0)\,\mathrm{d}x$$

$$T_n(0) = \frac{2}{l}\int_0^{l/2} \sin\left(\frac{n\pi x}{l}\right) x\,\mathrm{d}x + \frac{2}{l}\int_0^{l/2} \sin\left(\frac{n\pi x}{l}\right)(l-x)\,\mathrm{d}x$$

$$T_n(0) = \frac{4l}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$T_{2n-1}(0) = (-1)^n \frac{4l}{(2n-1)^2\pi^2}, \quad T_{2n}(0) = 0, \quad n \in \mathbb{Z}^+$$

We solve the ordinary differential equations for T_n subject to the initial conditions.

$$T_{2n-1}(t) = (-1)^n \frac{4l}{(2n-1)^2\pi^2} \exp\left(-\left(\frac{a(2n-1)\pi}{l}\right)^2 t\right), \quad T_{2n}(t) = 0, \quad n \in \mathbb{Z}^+$$

This determines the series representation of the solution.

$$\boxed{\phi = \frac{4}{l}\sum_{n=1}^{\infty}(-1)^n \left(\frac{l}{(2n-1)\pi}\right)^2 \exp\left(-\left(\frac{a(2n-1)\pi}{l}\right)^2 t\right) \sin\left(\frac{(2n-1)\pi x}{l}\right)}$$

From the initial condition, we know that the the solution at $t = 0$ is C^0. That is, it is continuous, but not differentiable. The series representation of the solution at $t = 0$ is

$$\phi = \frac{4}{l}\sum_{n=1}^{\infty}(-1)^n \left(\frac{l}{(2n-1)\pi}\right)^2 \sin\left(\frac{(2n-1)\pi x}{l}\right).$$

That the coefficients decay as $1/n^2$ corroborates that $\phi(x,0)$ is C^0.

The derivatives of ϕ with respect to x are

$$\frac{\partial^{2m-1}}{\partial x^{2m-1}}\phi = \frac{4(-1)^{m+1}}{l}\sum_{n=1}^{\infty}(-1)^n\left(\frac{(2n-1)\pi}{l}\right)^{2m-3}\exp\left(-\left(\frac{a(2n-1)\pi}{l}\right)^2 t\right)\cos\left(\frac{(2n-1)\pi x}{l}\right)$$
$$\frac{\partial^{2m}}{\partial x^{2m}}\phi = \frac{4(-1)^{m}}{l}\sum_{n=1}^{\infty}(-1)^n\left(\frac{(2n-1)\pi}{l}\right)^{2m-2}\exp\left(-\left(\frac{a(2n-1)\pi}{l}\right)^2 t\right)\sin\left(\frac{(2n-1)\pi x}{l}\right)$$

For any fixed $t > 0$, the coefficients in the series for $\frac{\partial^n}{\partial x}\phi$ decay exponentially. These series are uniformly convergent in x. Thus for any fixed $t > 0$, ϕ is C^∞ in x.

Solution 39.7

$$u_t = \kappa u_{xx}, \quad 0 < x < L, \quad t > 0$$
$$u(0,t) = T_0, \quad u(L,t) = T_1, \quad u(x,0) = f(x),$$

Method 1. We solve this problem with an eigenfunction expansion in x. To find an appropriate set of eigenfunctions, we apply the separation of variables, $u(x,t) = X(x)T(t)$ to the partial differential equation with the homogeneous boundary conditions, $u(0,t) = u(L,t) = 0$.

$$(XT)_t = (XT)_{xx}$$
$$XT' = X''T$$
$$\frac{T'}{T} = \frac{X''}{X} = -\lambda^2$$

We have the eigenvalue problem,

$$X'' + \lambda^2 X = 0, \quad X(0) = X(L) = 0,$$

which has the solutions,

$$\lambda_n = \frac{n\pi x}{L}, \quad X_n = \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}.$$

We expand the solution of the partial differential equation in terms of these eigenfunctions.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right)}$$

Because of the inhomogeneous boundary conditions, the convergence of the series will not be uniform. We can differentiate the series with respect to t, but not with respect to x. We multiply the partial differential equation by an eigenfunction and integrate from $x = 0$ to $x = L$. We use integration by parts to move derivatives from u to the eigenfunction.

$$u_t - \kappa u_{xx} = 0$$

$$\int_0^L (u_t - \kappa u_{xx}) \sin\left(\frac{m\pi x}{L}\right) \, \mathrm{d}x = 0$$

$$\int_0^L \left(\sum_{n=1}^{\infty} a_n'(t) \sin\left(\frac{n\pi x}{L}\right)\right) \sin\left(\frac{m\pi x}{L}\right) \, \mathrm{d}x - \kappa \left[u_x \sin\left(\frac{m\pi x}{L}\right)\right]_0^L + \kappa \frac{m\pi}{L} \int_0^L u_x \cos\left(\frac{m\pi x}{L}\right) \, \mathrm{d}x = 0$$

$$\frac{L}{2} a_m'(t) + \kappa \frac{m\pi}{L} \left[u \cos\left(\frac{m\pi x}{L}\right)\right]_0^L + \kappa \left(\frac{m\pi}{L}\right)^2 \int_0^L u \sin\left(\frac{m\pi x}{L}\right) \, \mathrm{d}x = 0$$

$$\frac{L}{2} a_m'(t) + \kappa \frac{m\pi}{L} \left((-1)^m u(L,t) - u(0,t)\right) + \kappa \left(\frac{m\pi}{L}\right)^2 \int_0^L \left(\sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right)\right) \sin\left(\frac{m\pi x}{L}\right) \, \mathrm{d}x = 0$$

$$\frac{L}{2} a_m'(t) + \kappa \frac{m\pi}{L} \left((-1)^m T_1 - T_0\right) + \kappa \frac{L}{2} \left(\frac{m\pi}{L}\right)^2 a_m(t) = 0$$

$$a_m'(t) + \kappa \left(\frac{m\pi}{L}\right)^2 a_m(t) = \kappa \frac{2m\pi}{L^2} \left(T_0 - (-1)^m T_1\right)$$

Now we have a first order differential equation for each of the a_n's. We obtain initial conditions for each of the a_n's from the initial condition for $u(x,t)$.

$$u(x,0) = f(x)$$
$$\sum_{n=1}^{\infty} a_n(0) \sin\left(\frac{n\pi x}{L}\right) = f(x)$$
$$a_n(0) = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \equiv f_n$$

By solving the first order differential equation for $a_n(t)$, we obtain

$$\boxed{a_n(t) = \frac{2(T_0 - (-1)^n T_1)}{n\pi} + \mathrm{e}^{-\kappa(n\pi/L)^2 t}\left(f_n - \frac{2(T_0 - (-1)^n T_1)}{n\pi}\right).}$$

Note that the series does not converge uniformly due to the $1/n$ term.

Method 2. For our second method we transform the problem to one with homogeneous boundary conditions so that we can use the partial differential equation to determine the time dependence of the eigen-solutions. We make the change of variables $v(x,t) = u(x,t) - \mu(x)$ where $\mu(x)$ is some function that satisfies the inhomogeneous boundary conditions. If possible, we want $\mu(x)$ to satisfy the partial differential equation as well. For this problem we can choose $\mu(x)$ to be the equilibrium solution which satisfies

$$\mu''(x) = 0, \quad \mu(0)T_0, \quad \mu(L) = T_1.$$

This has the solution

$$\mu(x) = T_0 + \frac{T_1 - T_0}{L} x.$$

With the change of variables,

$$v(x,t) = u(x,t) - \left(T_0 + \frac{T_1 - T_0}{L} x\right),$$

we obtain the problem

$$v_t = \kappa v_{xx}, \quad 0 < x < L, \quad t > 0$$
$$v(0,t) = 0, \quad v(L,t) = 0, \quad v(x,0) = f(x) - \left(T_0 + \frac{T_1 - T_0}{L} x \right).$$

Now we substitute the separation of variables $v(x,t) = X(x)T(t)$ into the partial differential equation.

$$(XT)_t = \kappa (XT)_{xx}$$
$$\frac{T'}{\kappa T} = \frac{X''}{X} = -\lambda^2$$

Utilizing the boundary conditions at $x = 0, L$ we obtain the two ordinary differential equations,

$$T' = -\kappa \lambda^2 T,$$
$$X'' = -\lambda^2 X, \quad X(0) = X(L) = 0.$$

The problem for X is a regular Sturm-Liouville problem and has the solutions

$$\lambda_n = \frac{n\pi}{L}, \quad X_n = \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}.$$

The ordinary differential equation for T becomes,

$$T_n' = -\kappa \left(\frac{n\pi}{L}\right)^2 T_n,$$

which, (up to a multiplicative constant), has the solution,

$$T_n = \mathrm{e}^{-\kappa (npi/L)^2 t}.$$

Thus the eigenvalues and eigen-solutions of the partial differential equation are,

$$\lambda_n = \frac{n\pi}{L}, \quad v_n = \sin\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\kappa (n\pi/L)^2 t}, \quad n \in \mathbb{N}.$$

Let $v(x,t)$ have the series expansion,

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\kappa(n\pi/L)^2 t}.$$

We determine the coefficients in the expansion from the initial condition,

$$v(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x) - \left(T_0 + \frac{T_1 - T_0}{L} x\right).$$

The coefficients in the expansion are the Fourier sine coefficients of $f(x) - \left(T_0 + \frac{T_1-T_0}{L}x\right)$.

$$a_n = \frac{2}{L} \int_0^L \left(f(x) - \left(T_0 + \frac{T_1 - T_0}{L} x\right)\right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

$$a_n = f_n - \frac{2(T_0 - (-1)^n T_1)}{n\pi}$$

With the coefficients defined above, the solution for $u(x,t)$ is

$$\boxed{u(x,t) = T_0 + \frac{T_1 - T_0}{L} x + \sum_{n=1}^{\infty} \left(f_n - \frac{2(T_0 - (-1)^n T_1)}{n\pi}\right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\kappa(n\pi/L)^2 t}.}$$

Since the coefficients in the sum decay exponentially for $t > 0$, we see that the series is uniformly convergent for positive t. It is clear that the two solutions we have obtained are equivalent.

Solution 39.8

First we solve the eigenvalue problem for $\beta(x)$, which is the problem we would obtain if we applied separation of variables to the partial differential equation, $\phi_t = \phi_{xx}$. We have the eigenvalues and orthonormal eigenfunctions

$$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2, \quad \beta_n(x) = \sqrt{\frac{2}{l}} \sin\left(\frac{(2n-1)\pi x}{2l}\right), \quad n \in \mathbb{Z}^+.$$

We expand the solution and inhomogeneity in Equation 39.5 in a series of the eigenvalues.

$$\phi(x,t) = \sum_{n=1}^{\infty} T_n(t)\beta_n(x)$$
$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\beta_n(x), \quad w_n(t) = \int_0^l \beta_n(x) w(x,t)\,\mathrm{d}x$$

Since ϕ satisfies the same homgeneous boundary conditions as β, we substitute the series into Equation 39.5 to determine differential equations for the $T_n(t)$.

$$\sum_{n=1}^{\infty} T_n'(t)\beta_n(x) = a^2 \sum_{n=1}^{\infty} T_n(t)(-\lambda_n)\beta_n(x) + \sum_{n=1}^{\infty} w_n(t)\beta_n(x)$$
$$T_n'(t) = -a^2\left(\frac{(2n-1)\pi}{2l}\right)^2 T_n(t) + w_n(t)$$

Now we substitute the series for ϕ into its initial condition to determine initial conditions for the T_n.

$$\phi(x,0) = \sum_{n=1}^{\infty} T_n(0)\beta_n(x) = f(x)$$
$$T_n(0) = \int_0^l \beta_n(x) f(x)\,\mathrm{d}x$$

We solve for $T_n(t)$ to determine the solution, $\phi(x,t)$.

$$T_n(t) = \exp\left(-\left(\frac{(2n-1)a\pi}{2l}\right)^2 t\right)\left(T_n(0) + \int_0^t w_n(\tau)\exp\left(\left(\frac{(2n-1)a\pi}{2l}\right)^2 \tau\right)\mathrm{d}\tau\right)$$

Solution 39.9
Separation of variables leads to the eigenvalue problem

$$\beta'' + \lambda\beta = 0, \quad \beta(0) = 0, \quad \beta(l) + c\beta'(l) = 0.$$

First we consider the case $\lambda = 0$. A set of solutions of the differential equation is $\{1, x\}$. The solution that satisfies the left boundary condition is $\beta(x) = x$. The right boundary condition imposes the constraint $l + c = 0$. Since c is positive, this has no solutions. $\lambda = 0$ is not an eigenvalue.

Now we consider $\lambda \neq 0$. A set of solutions of the differential equation is $\{\cos(\sqrt{\lambda}x), \sin(\sqrt{\lambda}x)\}$. The solution that satisfies the left boundary condition is $\beta = \sin(\sqrt{\lambda}x)$. The right boundary condition imposes the constraint

$$c \sin\left(\sqrt{\lambda}l\right) + \sqrt{\lambda}\cos\left(\sqrt{\lambda}l\right) = 0$$

$$\tan\left(\sqrt{\lambda}l\right) = -\frac{\sqrt{\lambda}}{c}$$

For large λ, the we can determine approximate solutions.

$$\sqrt{\lambda_n}l \approx \frac{(2n-1)\pi}{2}, n \in \mathbb{Z}^+$$

$$\lambda_n \approx \left(\frac{(2n-1)\pi}{2l}\right)^2, n \in \mathbb{Z}^+$$

The eigenfunctions are

$$\beta_n(x) = \frac{\sin\left(\sqrt{\lambda_n}x\right)}{\sqrt{\int_0^l \sin^2\left(\sqrt{\lambda_n}x\right)\,\mathrm{d}x}}, n \in \mathbb{Z}^+.$$

We expand $\phi(x,t)$ and $w(x,t)$ in series of the eigenfunctions.

$$\phi(x,t) = \sum_{n=1}^{\infty} T_n(t)\beta_n(x)$$

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\beta_n(x), \quad w_n(t) = \int_0^l \beta_n(x) w(x,t)\,\mathrm{d}x$$

Since ϕ satisfies the same homgeneous boundary conditions as β, we substitute the series into Equation 39.5 to determine differential equations for the $T_n(t)$.

$$\sum_{n=1}^{\infty} T_n'(t)\beta_n(x) = a^2 \sum_{n=1}^{\infty} T_n(t)(-\lambda_n)\beta_n(x) + \sum_{n=1}^{\infty} w_n(t)\beta_n(x)$$
$$T_n'(t) = -a^2\lambda_n T_n(t) + w_n(t)$$

Now we substitute the series for ϕ into its initial condition to determine initial conditions for the T_n.

$$\phi(x,0) = \sum_{n=1}^{\infty} T_n(0)\beta_n(x) = f(x)$$
$$T_n(0) = \int_0^l \beta_n(x) f(x)\,\mathrm{d}x$$

We solve for $T_n(t)$ to determine the solution, $\phi(x,t)$.

$$T_n(t) = \exp\left(-a^2\lambda_n t\right)\left(T_n(0) + \int_0^t w_n(\tau)\exp\left(a^2\lambda_n\tau\right)\,\mathrm{d}\tau\right)$$

Solution 39.10

First we seek a function $u(x,t)$ that satisfies the boundary conditions $u(0,t) = t$, $u_x(l,t) = -cu(l,t)$. We try a function of the form $u = (ax+b)t$. The left boundary condition imposes the constraint $b = 1$. We then apply the right boundary condition no determine u.

$$at = -c(al+1)t$$
$$a = -\frac{c}{1+cl}$$
$$u(x,t) = \left(1 - \frac{cx}{1+cl}\right)t$$

Now we define ψ to be the difference of ϕ and u.

$$\psi(x,t) = \phi(x,t) - u(x,t)$$

ψ satisfies an inhomogeneous diffusion equation with homogeneous boundary conditions.

$$(\psi + u)_t = a^2(\psi + u)_{xx} + 1$$
$$\psi_t = a^2\psi_{xx} + 1 + a^2 u_{xx} - u_t$$
$$\psi_t = a^2\psi_{xx} + \frac{cx}{1+cl}$$

The initial and boundary conditions for ψ are

$$\psi(x,0) = 0, \quad \psi(0,t) = 0, \quad \psi_x(l,t) = -c\psi(l,t).$$

We solved this system in problem 2. Just take

$$w(x,t) = \frac{cx}{1+cl}, \quad f(x) = 0.$$

The solution is

$$\psi(x,t) = \sum_{n=1}^{\infty} T_n(t)\beta_n(x),$$
$$T_n(t) = \int_0^t w_n \exp\left(-a^2\lambda_n(t-\tau)\right) \, \mathrm{d}\tau,$$
$$w_n(t) = \int_0^l \beta_n(x)\frac{cx}{1+cl} \, \mathrm{d}x.$$

This determines the solution for ϕ.

Solution 39.11

First we solve this problem with a series expansion. We transform the problem to one with homogeneous boundary conditions. Note that

$$u(x) = \frac{x}{l+1}$$

satisfies the boundary conditions. (It is the equilibrium solution.) We make the change of variables $\psi = \phi - u$. The problem for ψ is

$$\psi_t = a^2 \psi_{xx},$$
$$\psi(0,t) = \psi(l,t) + \psi_x(l,t) = 0, \quad \psi(x,0) = \frac{x}{l+1}.$$

This is a particular case of what we solved in Exercise 39.9. We apply the result of that problem. The solution for $\phi(x,t)$ is

$$\phi(x,t) = \frac{x}{l+1} + \sum_{n=1}^{\infty} T_n(t) \beta_n(x)$$
$$\beta_n(x) = \frac{\sin\left(\sqrt{\lambda_n} x\right)}{\sqrt{\int_0^l \sin^2\left(\sqrt{\lambda_n} x\right) \, \mathrm{d}x}}, \quad n \in \mathbb{Z}^+$$
$$\tan\left(\sqrt{\lambda} l\right) = -\sqrt{\lambda}$$
$$T_n(t) = T_n(0) \exp\left(-a^2 \lambda_n t\right)$$
$$T_n(0) = \int_0^l \beta_n(x) \frac{x}{l+1} \, \mathrm{d}x$$

This expansion is useful for large t because the coefficients decay exponentially with increasing t.

Now we solve this problem with the Laplace transform.

$$\phi_t = a^2\phi_{xx}, \quad \phi(0,t) = 0, \quad \phi(l,t) + \phi_x(l,t) = 1, \quad \phi(x,0) = 0$$
$$s\hat{\phi} = a^2\hat{\phi}_{xx}, \quad \hat{\phi}(0,s) = 0, \quad \hat{\phi}(l,s) + \hat{\phi}_x(l,s) = \frac{1}{s}$$
$$\hat{\phi}_{xx} - \frac{s}{a^2}\hat{\phi} = 0, \quad \hat{\phi}(0,s) = 0, \quad \hat{\phi}(l,s) + \hat{\phi}_x(l,s) = \frac{1}{s}$$

The solution that satisfies the left boundary condition is

$$\hat{\phi} = c \sinh\left(\frac{\sqrt{s}x}{a}\right).$$

We apply the right boundary condition to determine the constant.

$$\hat{\phi} = \frac{\sinh\left(\frac{\sqrt{s}x}{a}\right)}{s\left(\sinh\left(\frac{\sqrt{s}l}{a}\right) + \frac{\sqrt{s}}{a}\cosh\left(\frac{\sqrt{s}l}{a}\right)\right)}$$

We expand this in a series of simpler functions of s.

$$\hat{\phi} = \frac{2\sinh\left(\frac{\sqrt{s}x}{a}\right)}{s\left(\exp\left(\frac{\sqrt{s}l}{a}\right) - \exp\left(-\frac{\sqrt{s}l}{a}\right) + \frac{\sqrt{s}}{a}\left(\exp\left(\frac{\sqrt{s}l}{a}\right) + \exp\left(-\frac{\sqrt{s}l}{a}\right)\right)\right)}$$

$$\hat{\phi} = \frac{2\sinh\left(\frac{\sqrt{s}x}{a}\right)}{s\exp\left(\frac{\sqrt{s}l}{a}\right)} \frac{1}{1+\frac{\sqrt{s}}{a} - \left(1-\frac{\sqrt{s}}{a}\right)\exp\left(-\frac{2\sqrt{s}l}{a}\right)}$$

$$\hat{\phi} = \frac{\exp\left(\frac{\sqrt{s}x}{a}\right) - \exp\left(-\frac{\sqrt{s}x}{a}\right)}{s\left(1+\frac{\sqrt{s}}{a}\right)\exp\left(\frac{\sqrt{s}l}{a}\right)} \frac{1}{1 - \left(\frac{1-\sqrt{s}/a}{1+\sqrt{s}/a}\right)\exp\left(-\frac{2\sqrt{s}l}{a}\right)}$$

$$\hat{\phi} = \frac{\exp\left(\frac{\sqrt{s}(x-l)}{a}\right) - \exp\left(\frac{\sqrt{s}(-x-l)}{a}\right)}{s\left(1+\frac{\sqrt{s}}{a}\right)} \sum_{n=0}^{\infty}\left(\frac{1-\sqrt{s}/a}{1+\sqrt{s}/a}\right)^n \exp\left(-\frac{2\sqrt{s}ln}{a}\right)$$

$$\hat{\phi} = \frac{1}{s}\left(\sum_{n=0}^{\infty}\frac{(1-\sqrt{s}/a)^n}{(1+\sqrt{s}/a)^{n+1}}\exp\left(-\frac{\sqrt{s}((2n+1)l-x)}{a}\right)\right.$$
$$\left. - \sum_{n=0}^{\infty}\frac{(1-\sqrt{s}/a)^n}{(1+\sqrt{s}/a)^{n+1}}\exp\left(-\frac{\sqrt{s}((2n+1)l+x)}{a}\right)\right)$$

By expanding

$$\frac{(1-\sqrt{s}/a)^n}{(1+\sqrt{s}/a)^{n+1}}$$

in binomial series all the terms would be of the form

$$s^{-m/2-3/2}\exp\left(-\frac{\sqrt{s}((2n\pm 1)l \mp x)}{a}\right).$$

Taking the first term in each series yields

$$\hat{\phi} \sim \frac{a}{s^{3/2}} \left(\exp\left(-\frac{\sqrt{s}(l-x)}{a} \right) - \exp\left(-\frac{\sqrt{s}(l+x)}{a} \right) \right), \quad \text{as } s \to \infty.$$

We take the inverse Laplace transform to obtain an appoximation of the solution for $t \ll 1$.

$$\phi(x,t) \sim 2a^2\sqrt{\pi t} \left(\frac{\exp\left(-\frac{(l-x)^2}{4a^2 t}\right)}{l-x} - \frac{\exp\left(-\frac{(l+x)^2}{4a^2 t}\right)}{l+x} \right) - \pi \left(\operatorname{erfc}\left(\frac{l-x}{2a\sqrt{t}} \right) - \operatorname{erfc}\left(\frac{l+x}{2a\sqrt{t}} \right) \right), \quad \text{for } t \ll 1$$

Solution 39.12

We apply the separation of variables $\phi(x,t) = X(x)T(t)$.

$$\begin{aligned} \phi_t &= A^2 \left(x^2 \phi_x \right)_x \\ XT' &= TA^2 \left(x^2 X' \right)' \\ \frac{T'}{A^2 T} &= \frac{(x^2 X')'}{X} = -\lambda \end{aligned}$$

This gives us a regular Sturm-Liouville problem.

$$\left(x^2 X' \right)' + \lambda X = 0, \quad X(1) = X(2) = 0$$

This is an Euler equation. We make the substitution $X = x^\alpha$ to find the solutions.

$$x^2 X'' + 2xX' + \lambda X = 0 \tag{39.9}$$

$$\begin{aligned} \alpha(\alpha - 1) + 2\alpha + \lambda &= 0 \\ \alpha &= \frac{-1 \pm \sqrt{1 - 4\lambda}}{2} \\ \alpha &= -\frac{1}{2} \pm i\sqrt{\lambda - 1/4} \end{aligned}$$

First we consider the case of a double root when $\lambda = 1/4$. The solutions of Equation 39.9 are $\{x^{-1/2}, x^{-1/2} \ln x\}$. The solution that satisfies the left boundary condition is $X = x^{-1/2} \ln x$. Since this does not satisfy the right boundary condition, $\lambda = 1/4$ is not an eigenvalue.

Now we consider $\lambda \neq 1/4$. The solutions of Equation 39.9 are

$$\left\{ \frac{1}{\sqrt{x}} \cos\left(\sqrt{\lambda - 1/4} \ln x\right), \frac{1}{\sqrt{x}} \sin\left(\sqrt{\lambda - 1/4} \ln x\right) \right\}.$$

The solution that satisfies the left boundary condition is

$$\frac{1}{\sqrt{x}} \sin\left(\sqrt{\lambda - 1/4} \ln x\right).$$

The right boundary condition imposes the constraint

$$\sqrt{\lambda - 1/4} \ln 2 = n\pi, \quad n \in \mathbb{Z}^+.$$

This gives us the eigenvalues and eigenfunctions.

$$\lambda_n = \frac{1}{4} + \left(\frac{n\pi}{\ln 2}\right)^2, \quad X_n(x) = \frac{1}{\sqrt{x}} \sin\left(\frac{n\pi \ln x}{\ln 2}\right), \quad n \in \mathbb{Z}^+.$$

We normalize the eigenfunctions.

$$\int_1^2 \frac{1}{x} \sin^2\left(\frac{n\pi \ln x}{\ln 2}\right) \mathrm{d}x = \ln 2 \int_0^1 \sin^2(n\pi\xi) \, \mathrm{d}\xi = \frac{\ln 2}{2}$$

$$X_n(x) = \sqrt{\frac{2}{\ln 2}} \frac{1}{\sqrt{x}} \sin\left(\frac{n\pi \ln x}{\ln 2}\right), \quad n \in \mathbb{Z}^+.$$

From separation of variables, we have differential equations for the T_n.

$$T_n' = -A^2 \left(\frac{1}{4} + \left(\frac{n\pi}{\ln 2}\right)^2\right) T_n$$

$$T_n(t) = \exp\left(-A^2 \left(\frac{1}{4} + \left(\frac{n\pi}{\ln 2}\right)^2\right) t\right)$$

We expand ϕ in a series of the eigensolutions.

$$\phi(x,t) = \sum_{n=1}^{\infty} \phi_n X_n(x) T_n(t)$$

We substitute the expansion for ϕ into the initial condition to determine the coefficients.

$$\phi(x,0) = \sum_{n=1}^{\infty} \phi_n X_n(x) = f(x)$$
$$\phi_n = \int_1^2 X_n(x) f(x)\,\mathrm{d}x$$

Solution 39.13

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$$
$$u(0,t) = 0, \quad u_x(L,t) = 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = 0,$$

We substitute the separation of variables $u(x,t) = X(x)T(t)$ into the partial differential equation.

$$(XT)_{tt} = c^2 (XT)_{xx}$$
$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda^2$$

With the boundary conditions at $x = 0, L$, we have the ordinary differential equations,

$$T'' = -c^2 \lambda^2 T,$$
$$X'' = -\lambda^2 X, \quad X(0) = X'(L) = 0.$$

The problem for X is a regular Sturm-Liouville eigenvalue problem. From the Rayleigh quotient,

$$\lambda^2 = \frac{-\left[\phi\phi'\right]_0^L + \int_0^L (\phi')^2 \, \mathrm{d}x}{\int_0^L \phi^2 \, \mathrm{d}x} = \frac{\int_0^L (\phi')^2 \, \mathrm{d}x}{\int_0^L \phi^2 \, \mathrm{d}x}$$

we see that there are only positive eigenvalues. For $\lambda^2 > 0$ the general solution of the ordinary differential equation is

$$X = a_1 \cos(\lambda x) + a_2 \sin(\lambda x).$$

The solution that satisfies the left boundary condition is

$$X = a \sin(\lambda x).$$

For non-trivial solutions, the right boundary condition imposes the constraint,

$$\cos(\lambda L) = 0,$$

$$\lambda = \frac{\pi}{L}\left(n - \frac{1}{2}\right), \quad n \in \mathbb{N}.$$

The eigenvalues and eigenfunctions are

$$\lambda_n = \frac{(2n-1)\pi}{2L}, \quad X_n = \sin\left(\frac{(2n-1)\pi x}{2L}\right), \quad n \in \mathbb{N}.$$

The differential equation for T becomes

$$T'' = -c^2 \left(\frac{(2n-1)\pi}{2L}\right)^2 T,$$

which has the two linearly independent solutions,

$$T_n^{(1)} = \cos\left(\frac{(2n-1)c\pi t}{2L}\right), \quad T_n^{(2)} = \sin\left(\frac{(2n-1)c\pi t}{2L}\right).$$

The eigenvalues and eigen-solutions of the partial differential equation are,

$$\lambda_n = \frac{(2n-1)\pi}{2L}, \quad n \in \mathbb{N},$$

$$u_n^{(1)} = \sin\left(\frac{(2n-1)\pi x}{2L}\right)\cos\left(\frac{(2n-1)c\pi t}{2L}\right), \quad u_n^{(2)} = \sin\left(\frac{(2n-1)\pi x}{2L}\right)\sin\left(\frac{(2n-1)c\pi t}{2L}\right).$$

We expand $u(x,t)$ in a series of the eigen-solutions.

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi x}{2L}\right)\left(a_n \cos\left(\frac{(2n-1)c\pi t}{2L}\right) + b_n \sin\left(\frac{(2n-1)c\pi t}{2L}\right)\right).$$

We impose the initial condition $u_t(x,0) = 0$,

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \frac{(2n-1)c\pi}{2L} \sin\left(\frac{(2n-1)\pi x}{2L}\right) = 0,$$

$$b_n = 0.$$

The initial condition $u(x,0) = f(x)$ allows us to determine the remaining coefficients,

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{(2n-1)\pi x}{2L}\right) = f(x),$$

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{(2n-1)\pi x}{2L}\right) \,\mathrm{d}x.}$$

The series solution for $u(x,t)$ is,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{(2n-1)\pi x}{2L}\right)\cos\left(\frac{(2n-1)c\pi t}{2L}\right).}$$

Solution 39.14

$$u_{xx} + u_{yy} = f(x,y), \quad 0 < x < a, \quad 0 < y < b,$$
$$u(0,y) = u(a,y) = 0, \quad u_y(x,0) = u_y(x,b) = 0,$$

We will solve this problem with an eigenfunction expansion in x. To determine a suitable set of eigenfunctions, we substitute the separation of variables $u(x,y) = X(x)Y(y)$ into the homogeneous partial differential equation.

$$u_{xx} + u_{yy} = 0$$
$$(XY)_{xx} + (XY)_{yy} = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

With the boundary conditions at $x = 0, a$, we have the regular Sturm-Liouville problem,

$$X'' = -\lambda^2 X, \quad X(0) = X(a) = 0,$$

which has the solutions,

$$\lambda_n = \frac{n\pi}{a}, \quad X_n = \sin\left(\frac{n\pi x}{a}\right), \quad n \in \mathbb{N}.$$

We expand $u(x,y)$ in a series of the eigenfunctions,

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} u_n(y) \sin\left(\frac{n\pi x}{a}\right).}$$

Substituting this series into the partial differential equation and boundary conditions at $y = 0, b$, we obtain,

$$\sum_{n=1}^{\infty} \left(-\left(\frac{n\pi}{a}\right)^2 u_n(y) \sin\left(\frac{n\pi x}{a}\right) + u_n''(y) \sin\left(\frac{n\pi x}{a}\right) \right) = f(x),$$
$$\sum_{n=1}^{\infty} u_n'(0) \sin\left(\frac{n\pi x}{a}\right) = \sum_{n=1}^{\infty} u_n'(b) \sin\left(\frac{n\pi x}{a}\right) = 0.$$

Expanding $f(x,y)$ in the Fourier sine series,

$$f(x,y) = \sum_{n=1}^{\infty} f_n(y) \sin\left(\frac{n\pi x}{a}\right),$$

$$\boxed{f_n(y) = \frac{2}{a}\int_0^a f(x,y)\sin\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x,}$$

we obtain the ordinary differential equations,

$$u_n''(y) - \left(\frac{n\pi}{a}\right)^2 u_n(y) = f_n(y), \quad u_n'(0) = u_n'(b) = 0, \quad n \in \mathbb{N}.$$

We will solve these ordinary differential equations with Green functions. Consider the Green function problem,

$$g_n''(y;\eta) - \left(\frac{n\pi}{a}\right)^2 g_n(y;\eta) = \delta(y-\eta), \quad g_n'(0;\eta) = g_n'(b;\eta) = 0.$$

The homogeneous solutions

$$\cosh\left(\frac{n\pi y}{a}\right) \quad \text{and} \quad \cosh\left(\frac{n\pi(y-b)}{a}\right)$$

satisfy the left and right boundary conditions, respectively. The Wronskian of these two solutions is

$$\begin{aligned}
W(y) &= \begin{vmatrix} \cosh(n\pi y/a) & \cosh(n\pi(y-b)/a) \\ \frac{n\pi}{a}\sinh(n\pi y/a) & \frac{n\pi}{a}\sinh(n\pi(y-b)/a) \end{vmatrix} \\
&= \frac{n\pi}{a}\left(\cosh\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi(y-b)}{a}\right) - \sinh\left(\frac{n\pi y}{a}\right)\cosh\left(\frac{n\pi(y-b)}{a}\right)\right) \\
&= -\frac{n\pi}{a}\sinh\left(\frac{n\pi b}{a}\right).
\end{aligned}$$

Thus the Green function is

$$\boxed{g_n(y;\eta) = -\frac{a\cosh(n\pi y_</a)\cosh(n\pi(y_> - b)/a)}{n\pi\sinh(n\pi b/a)}.}$$

The solutions for the coefficients in the expansion are

$$\boxed{u_n(y) = \int_0^b g_n(y;\eta) f_n(\eta)\, d\eta.}$$

Solution 39.15

$$u_{tt} + a^2 u_{xxxx} = 0, \quad 0 < x < L, t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x),$$
$$u(0,t) = u_{xx}(0,t) = 0, \quad u(L,t) = u_{xx}(L,t) = 0,$$

We will solve this problem by expanding the solution in a series of eigen-solutions that satisfy the partial differential equation and the homogeneous boundary conditions. We will use the initial conditions to determine the coefficients in the expansion. We substitute the separation of variables, $u(x,t) = X(x)T(t)$ into the partial differential equation.

$$(XT)_{tt} + a^2(XT)_{xxxx} = 0$$
$$\frac{T''}{a^2T} = -\frac{X''''}{X} = -\lambda^4$$

Here we make the assumption that $0 \leq \arg(\lambda) < \pi/2$, i.e., λ lies in the first quadrant of the complex plane. Note that λ^4 covers the entire complex plane. We have the ordinary differential equation,

$$T'' = -a^2\lambda^4 T,$$

and with the boundary conditions at $x = 0, L$, the eigenvalue problem,

$$X'''' = \lambda^4 X, \quad X(0) = X''(0) = X(L) = X''(L) = 0.$$

For $\lambda = 0$, the general solution of the differential equation is

$$X = c_1 + c_2 x + c_3 x^2 + c_4 x^3.$$

Only the trivial solution satisfies the boundary conditions. $\lambda = 0$ is not an eigenvalue. For $\lambda \neq 0$, a set of linearly independent solutions is

$$\{\mathrm{e}^{\lambda x}, \mathrm{e}^{i\lambda x}, \mathrm{e}^{-\lambda x}, \mathrm{e}^{-i\lambda x}\}.$$

Another linearly independent set, (which will be more useful for this problem), is

$$\{\cos(\lambda x), \sin(\lambda x), \cosh(\lambda x), \sinh(\lambda x)\}.$$

Both $\sin(\lambda x)$ and $\sinh(\lambda x)$ satisfy the left boundary conditions. Consider the linear combination $c_1 \cos(\lambda x) + c_2 \cosh(\lambda x)$. The left boundary conditions impose the two constraints $c_1 + c_2 = 0$, $c_1 - c_2 = 0$. Only the trivial linear combination of $\cos(\lambda x)$ and $\cosh(\lambda x)$ can satisfy the left boundary condition. Thus the solution has the form,

$$X = c_1 \sin(\lambda x) + c_2 \sinh(\lambda x).$$

The right boundary conditions impose the constraints,

$$\begin{cases} c_1 \sin(\lambda L) + c_2 \sinh(\lambda L) = 0, \\ -c_1 \lambda^2 \sin(\lambda L) + c_2 \lambda^2 \sinh(\lambda L) = 0 \end{cases}$$

$$\begin{cases} c_1 \sin(\lambda L) + c_2 \sinh(\lambda L) = 0, \\ -c_1 \sin(\lambda L) + c_2 \sinh(\lambda L) = 0 \end{cases}$$

This set of equations has a nontrivial solution if and only if the determinant is zero,

$$\begin{vmatrix} \sin(\lambda L) & \sinh(\lambda L) \\ -\sin(\lambda L) & \sinh(\lambda L) \end{vmatrix} = 2\sin(\lambda L)\sinh(\lambda L) = 0.$$

Since $\sinh(z)$ is nonzero in $0 \leq \arg(z) < \pi/2$, $z \neq 0$, and $\sin(z)$ has the zeros $z = n\pi$, $n \in \mathbb{N}$ in this domain, the eigenvalues and eigenfunctions are,

$$\lambda_n = \frac{n\pi}{L}, \quad X_n = \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}.$$

The differential equation for T becomes,

$$T'' = -a^2 \left(\frac{n\pi}{L}\right)^4 T,$$

which has the solutions,

$$\left\{ \cos\left(a \left(\frac{n\pi}{L}\right)^2 t\right), \sin\left(a \left(\frac{n\pi}{L}\right)^2 t\right) \right\}.$$

The eigen-solutions of the partial differential equation are,

$$u_n^{(1)} = \sin\left(\frac{n\pi x}{L}\right) \cos\left(a \left(\frac{n\pi}{L}\right)^2 t\right), \quad u_n^{(2)} = \sin\left(\frac{n\pi x}{L}\right) \sin\left(a \left(\frac{n\pi}{L}\right)^2 t\right), \quad n \in \mathbb{N}.$$

We expand the solution of the partial differential equation in a series of the eigen-solutions.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left(c_n \cos\left(a \left(\frac{n\pi}{L}\right)^2 t\right) + d_n \sin\left(a \left(\frac{n\pi}{L}\right)^2 t\right)\right)}$$

The initial condition for $u(x,t)$ and $u_t(x,t)$ allow us to determine the coefficients in the expansion.

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} d_n a \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

c_n and d_n are coefficients in Fourier sine series.

$$\boxed{c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x}$$

$$\boxed{d_n = \frac{2L}{a\pi^2 n^2} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x}$$

Solution 39.16

$$u_t = \kappa u_{xx} + I^2 \alpha u, \quad 0 < x < L, \quad t > 0,$$
$$u(0,t) = u(L,t) = 0, \quad u(x,0) = g(x).$$

We will solve this problem with an expansion in eigen-solutions of the partial differential equation. We substitute the separation of variables $u(x,t) = X(x)T(t)$ into the partial differential equation.

$$(XT)_t = \kappa (XT)_{xx} + I^2 \alpha XT$$
$$\frac{T'}{\kappa T} - \frac{I^2 \alpha}{\kappa} = \frac{X''}{X} = -\lambda^2$$

Now we have an ordinary differential equation for T and a Sturm-Liouville eigenvalue problem for X. (Note that we have followed the rule of thumb that the problem will be easier if we move all the parameters out of the eigenvalue problem.)

$$T' = -\left(\kappa \lambda^2 - I^2 \alpha\right) T$$
$$X'' = -\lambda^2 X, \quad X(0) = X(L) = 0$$

The eigenvalues and eigenfunctions for X are

$$\lambda_n = \frac{n\pi}{L}, \quad X_n = \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}.$$

The differential equation for T becomes,

$$T_n' = -\left(\kappa\left(\frac{n\pi}{L}\right)^2 - I^2\alpha\right)T_n,$$

which has the solution,

$$T_n = c\exp\left(-\left(\kappa\left(\frac{n\pi}{L}\right)^2 - I^2\alpha\right)t\right).$$

From this solution, we see that the critical current is

$$\boxed{I_{CR} = \sqrt{\frac{\kappa}{\alpha}\frac{\pi}{L}}.}$$

If I is greater that this, then the eigen-solution for $n = 1$ will be exponentially growing. This would make the whole solution exponentially growing. For $I < I_{CR}$, each of the T_n is exponentially decaying. The eigen-solutions of the partial differential equation are,

$$u_n = \exp\left(-\left(\kappa\left(\frac{n\pi}{L}\right)^2 - I^2\alpha\right)t\right)\sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}.$$

We expand $u(x,t)$ in its eigen-solutions, u_n.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(-\left(\kappa\left(\frac{n\pi}{L}\right)^2 - I^2\alpha\right)t\right)\sin\left(\frac{n\pi x}{L}\right)}$$

We determine the coefficients a_n from the initial condition.

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

$$\boxed{a_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.}$$

If $\alpha < 0$, then the solution is exponentially decaying regardless of current. Thus there is no critical current.

Solution 39.17

a) The problem is

$$u_t(x,y,z,t) = \kappa \Delta u(x,y,z,t), \quad -\infty < x < \infty, \quad -\infty < y < \infty, \quad 0 < z < a, \quad t > 0,$$
$$u(x,y,z,0) = T, \quad u(x,y,0,t) = u(x,y,a,t) = 0.$$

Because of symmetry, the partial differential equation in four variables is reduced to a problem in two variables,

$$u_t(z,t) = \kappa u_{zz}(z,t), \quad 0 < z < a, \quad t > 0,$$
$$u(z,0) = T, \quad u(0,t) = u(a,t) = 0.$$

We will solve this problem with an expansion in eigen-solutions of the partial differential equation that satisfy the homogeneous boundary conditions. We substitute the separation of variables $u(z,t) = Z(z)T(t)$ into the partial differential equation.

$$ZT' = \kappa Z'' T$$
$$\frac{T'}{\kappa T} = \frac{Z''}{Z} = -\lambda^2$$

With the boundary conditions at $z = 0, a$ we have the Sturm-Liouville eigenvalue problem,

$$Z'' = -\lambda^2 Z, \quad Z(0) = Z(a) = 0,$$

which has the solutions,

$$\lambda_n = \frac{n\pi}{a}, \quad Z_n = \sin\left(\frac{n\pi z}{a}\right), \quad n \in \mathbb{N}.$$

The problem for T becomes,

$$T_n' = -\kappa \left(\frac{n\pi}{a}\right)^2 T_n,$$

with the solution,

$$T_n = \exp\left(-\kappa \left(\frac{n\pi}{a}\right)^2 t\right).$$

The eigen-solutions are

$$u_n(z,t) = \sin\left(\frac{n\pi z}{a}\right) \exp\left(-\kappa \left(\frac{n\pi}{a}\right)^2 t\right).$$

The solution for u is a linear combination of the eigen-solutions. The slowest decaying eigen-solution is

$$u_1(z,t) = \sin\left(\frac{\pi z}{a}\right) \exp\left(-\kappa \left(\frac{\pi}{a}\right)^2 t\right).$$

Thus the e-folding time is

$$\boxed{\Delta_e = \frac{a^2}{\kappa \pi^2}.}$$

b) The problem is

$$u_t(r,\theta,z,t) = \kappa \Delta u(r,\theta,z,t), \quad 0 < r < a, \quad 0 < \theta < 2\pi, \quad -\infty < z < \infty, \quad t > 0,$$
$$u(r,\theta,z,0) = T, \quad u(0,\theta,z,t) \text{ is bounded}, \quad u(a,\theta,z,t) = 0.$$

The Laplacian in cylindrical coordinates is

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz}.$$

Because of symmetry, the solution does not depend on θ or z.

$$u_t(r,t) = \kappa \left(u_{rr}(r,t) + \frac{1}{r} u_r(r,t)\right), \quad 0 < r < a, \quad t > 0,$$
$$u(r,0) = T, \quad u(0,t) \text{ is bounded}, \quad u(a,t) = 0.$$

We will solve this problem with an expansion in eigen-solutions of the partial differential equation that satisfy the homogeneous boundary conditions at $r = 0$ and $r = a$. We substitute the separation of variables $u(r, t) = R(r)T(t)$ into the partial differential equation.

$$RT' = \kappa \left(R''T + \frac{1}{r} R'T \right)$$
$$\frac{T'}{\kappa T} = \frac{R''}{R} + \frac{R'}{rR} = -\lambda^2$$

We have the eigenvalue problem,

$$R'' + \frac{1}{r} R' + \lambda^2 R = 0, \quad R(0) \text{ is bounded}, R(a) = 0.$$

Recall that the Bessel equation,

$$y'' + \frac{1}{x} y' + \left(\lambda^2 - \frac{\nu^2}{x^2} \right) y = 0,$$

has the general solution $y = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$. We discard the Bessel function of the second kind, Y_ν, as it is unbounded at the origin. The solution for $R(r)$ is

$$R(r) = J_0(\lambda r).$$

Applying the boundary condition at $r = a$, we see that the eigenvalues and eigenfunctions are

$$\lambda_n = \frac{\beta_n}{a}, \quad R_n = J_0 \left(\frac{\beta_n r}{a} \right), \quad n \in \mathbb{N},$$

where $\{\beta_n\}$ are the positive roots of the Bessel function J_0.

The differential equation for T becomes,

$$T_n' = -\kappa \left(\frac{\beta_n}{a} \right)^2 T_n,$$

which has the solutions,

$$T_n = \exp\left(-\kappa\left(\frac{\beta_n}{a}\right)^2 t\right).$$

The eigen-solutions of the partial differential equation for $u(r,t)$ are,

$$u_n(r,t) = J_0\left(\frac{\beta_n r}{a}\right)\exp\left(-\kappa\left(\frac{\beta_n}{a}\right)^2 t\right).$$

The solution $u(r,t)$ is a linear combination of the eigen-solutions, u_n. The slowest decaying eigenfunction is,

$$u_1(r,t) = J_0\left(\frac{\beta_1 r}{a}\right)\exp\left(-\kappa\left(\frac{\beta_1}{a}\right)^2 t\right).$$

Thus the e-folding time is

$$\boxed{\Delta_e = \frac{a^2}{\kappa\beta_1^2}.}$$

c) The problem is

$$u_t(r,\theta,\phi,t) = \kappa\Delta u(r,\theta,\phi,t), \quad 0 < r < a, \quad 0 < \theta < 2\pi, \quad 0 < \phi < \pi, \quad t > 0,$$
$$u(r,\theta,\phi,0) = T, \quad u(0,\theta,\phi,t) \text{ is bounded}, \quad u(a,\theta,\phi,t) = 0.$$

The Laplacian in spherical coordinates is,

$$\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\cos\theta}{r^2\sin\theta}u_\theta + \frac{1}{r^2\sin^2\theta}u_{\phi\phi}.$$

Because of symmetry, the solution does not depend on θ or ϕ.

$$u_t(r,t) = \kappa\left(u_{rr}(r,t) + \frac{2}{r}u_r(r,t)\right), \quad 0 < r < a, \quad t > 0,$$
$$u(r,0) = T, \quad u(0,t) \text{ is bounded}, \quad u(a,t) = 0$$

We will solve this problem with an expansion in eigen-solutions of the partial differential equation that satisfy the homogeneous boundary conditions at $r = 0$ and $r = a$. We substitute the separation of variables $u(r,t) = R(r)T(t)$ into the partial differential equation.

$$RT' = \kappa\left(R''T + \frac{2}{r}R'T\right)$$
$$\frac{T'}{\kappa T} = \frac{R''}{R} + \frac{2}{r}\frac{R'}{R} = -\lambda^2$$

We have the eigenvalue problem,

$$R'' + \frac{2}{r}R' + \lambda^2 R = 0, \quad R(0) \text{ is bounded}, \quad R(a) = 0.$$

Recall that the equation,

$$y'' + \frac{2}{x}y' + \left(\lambda^2 - \frac{\nu(\nu+1)}{x^2}\right)y = 0,$$

has the general solution $y = c_1 j_\nu(\lambda x) + c_2 y_\nu(\lambda x)$, where j_ν and y_ν are the spherical Bessel functions of the first and second kind. We discard y_ν as it is unbounded at the origin. (The spherical Bessel functions are related to the Bessel functions by

$$j_\nu(x) = \sqrt{\frac{\pi}{2x}} J_{\nu+1/2}(x).)$$

The solution for $R(r)$ is

$$R_n = j_0(\lambda r).$$

Applying the boundary condition at $r = a$, we see that the eigenvalues and eigenfunctions are

$$\lambda_n = \frac{\gamma_n}{a}, \quad R_n = j_0\left(\frac{\gamma_n r}{a}\right), \quad n \in \mathbb{N}.$$

The problem for T becomes

$$T_n' = -\kappa\left(\frac{\gamma_n}{a}\right)^2 T_n,$$

which has the solutions,

$$T_n = \exp\left(-\kappa\left(\frac{\gamma_n}{a}\right)^2 t\right).$$

The eigen-solutions of the partial differential equation are,

$$u_n(r,t) = j_0\left(\frac{\gamma_n r}{a}\right) \exp\left(-\kappa\left(\frac{\gamma_n}{a}\right)^2 t\right).$$

The slowest decaying eigen-solution is,

$$u_1(r,t) = j_0\left(\frac{\gamma_1 r}{a}\right) \exp\left(-\kappa\left(\frac{\gamma_1}{a}\right)^2 t\right).$$

Thus the e-folding time is

$$\boxed{\Delta_e = \frac{a^2}{\kappa\gamma_1^2}.}$$

d) If the edges are perfectly insulated, then no heat escapes through the boundary. The temperature is constant for all time. There is no e-folding time.

Solution 39.18

We will solve this problem with an eigenfunction expansion. Since the partial differential equation is homogeneous, we will find eigenfunctions in both x and y. We substitute the separation of variables $u(x, y, t) = X(x)Y(y)T(t)$ into the partial differential equation.

$$XYT' = \kappa(t)\left(X''YT + XY''T\right)$$
$$\frac{T'}{\kappa(t)T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2$$
$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda^2 = -\mu^2$$

First we have a Sturm-Liouville eigenvalue problem for X,

$$X'' = \mu^2 X, \quad X'(0) = X'(a) = 0,$$

which has the solutions,

$$\mu_m = \frac{m\pi}{a}, \quad X_m = \cos\left(\frac{m\pi x}{a}\right), \quad m = 0, 1, 2, \ldots .$$

Now we have a Sturm-Liouville eigenvalue problem for Y,

$$Y'' = -\left(\lambda^2 - \left(\frac{m\pi}{a}\right)^2\right) Y, \quad Y(0) = Y(b) = 0,$$

which has the solutions,

$$\lambda_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}, \quad Y_n = \sin\left(\frac{n\pi y}{b}\right), \quad m = 0, 1, 2, \ldots, \quad n = 1, 2, 3, \ldots .$$

A few of the eigenfunctions, $\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$, are shown in Figure 39.3.

The differential equation for T becomes,

$$T'_{mn} = -\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)\kappa(t)T_{mn},$$

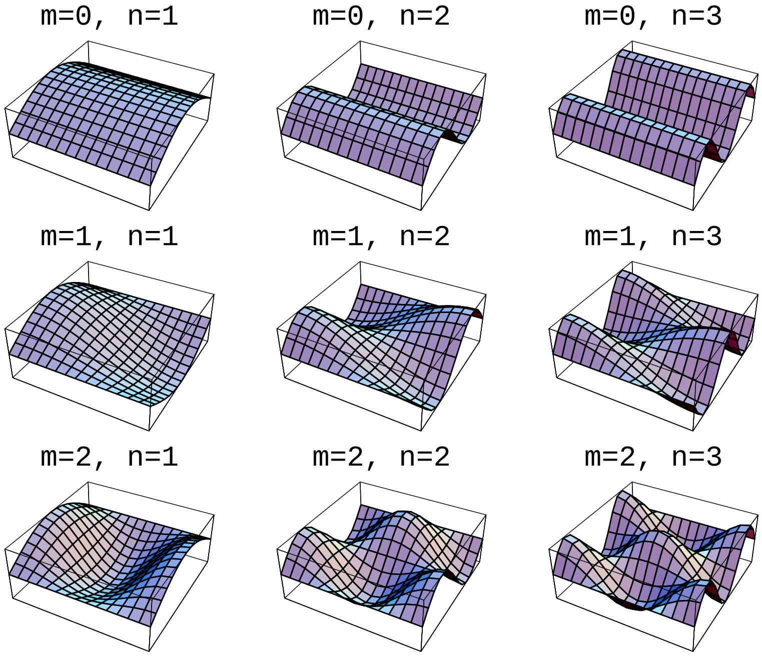

Figure 39.3: The eigenfunctions $\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$

which has the solutions,

$$T_{mn} = \exp\left(-\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)\int_0^t \kappa(\tau)\,\mathrm{d}\tau\right).$$

The eigen-solutions of the partial differential equation are,

$$u_{mn} = \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\exp\left(-\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)\int_0^t \kappa(\tau)\,\mathrm{d}\tau\right).$$

The solution of the partial differential equation is,

$$\boxed{u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \exp\left(-\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right) \int_0^t \kappa(\tau)\,\mathrm{d}\tau\right).}$$

We determine the coefficients from the initial condition.

$$u(x,y,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = f(x,y)$$

$$\boxed{c_{0n} = \frac{2}{ab}\int_0^a \int_0^b f(x,y) \sin\left(\frac{n\pi}{b}\right)\,\mathrm{d}y\,\mathrm{d}x}$$

$$\boxed{c_{mn} = \frac{4}{ab}\int_0^a \int_0^b f(x,y) \cos\left(\frac{m\pi}{a}\right) \sin\left(\frac{n\pi}{b}\right)\,\mathrm{d}y\,\mathrm{d}x}$$

Solution 39.19

The steady state temperature satisfies Laplace's equation, $\Delta u = 0$. The Laplacian in cylindrical coordinates is,

$$\Delta u(r,\theta,z) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}.$$

Because of the homogeneity in the z direction, we reduce the partial differential equation to,

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < 1, \quad 0 < \theta < \pi.$$

The boundary conditions are,

$$u(r,0) = u(r,\pi) = 0, \quad u(0,\theta) = 0, \quad u(1,\theta) = 1.$$

We will solve this problem with an eigenfunction expansion. We substitute the separation of variables $u(r,\theta) = R(r)T(\theta)$ into the partial differential equation.

$$R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' = 0$$
$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{T''}{T} = \lambda^2$$

We have the regular Sturm-Liouville eigenvalue problem,

$$T'' = -\lambda^2 T, \quad T(0) = T(\pi) = 0,$$

which has the solutions,

$$\lambda_n = n, \quad T_n = \sin(n\theta), \quad n \in \mathbb{N}.$$

The problem for R becomes,

$$r^2 R'' + rR' - n^2 R = 0, \quad R(0) = 0.$$

This is an Euler equation. We substitute $R = r^\alpha$ into the differential equation to obtain,

$$\alpha(\alpha - 1) + \alpha - n^2 = 0,$$
$$\alpha = \pm n.$$

The general solution of the differential equation for R is

$$R_n = c_1 r^n + c_2 r^{-n}.$$

The solution that vanishes at $r = 0$ is

$$R_n = cr^n.$$

The eigen-solutions of the differential equation are,

$$u_n = r^n \sin(n\theta).$$

The solution of the partial differential equation is

$$u(r, \theta) = \sum_{n=1}^{\infty} a_n r^n \sin(n\theta).$$

We determine the coefficients from the boundary condition at $r = 1$.

$$u(1, \theta) = \sum_{n=1}^{\infty} a_n \sin(n\theta) = 1$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin(n\theta) \, \mathrm{d}\theta = \frac{2}{\pi n} \left(1 - (-1)^n\right)$$

The solution of the partial differential equation is

$$\boxed{u(r, \theta) = \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} r^n \sin(n\theta).}$$

Solution 39.20
The problem is

$$u_{xx} + u_{yy} = 0, \quad 0 < x, \quad 0 < y < 1,$$
$$u(x, 0) = u(x, 1) = 0, \quad u(0, y) = f(y).$$

We substitute the separation of variables $u(x, y) = X(x)Y(y)$ into the partial differential equation.

$$X''Y + XY'' = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2$$

We have the regular Sturm-Liouville problem,

$$Y'' = -\lambda^2 Y, \quad Y(0) = Y(1) = 0,$$

which has the solutions,

$$\lambda_n = n\pi, \quad Y_n = \sin(n\pi y), \quad n \in \mathbb{N}.$$

The problem for X becomes,

$$X_n'' = (n\pi)^2 X,$$

which has the general solution,

$$X_n = c_1 \mathrm{e}^{n\pi x} + c_2 \mathrm{e}^{-n\pi x}.$$

The solution that is bounded as $x \to \infty$ is,

$$X_n = c\,\mathrm{e}^{-n\pi x}.$$

The eigen-solutions of the partial differential equation are,

$$u_n = \mathrm{e}^{-n\pi x} \sin(n\pi y), \quad n \in \mathbb{N}.$$

The solution of the partial differential equation is,

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-n\pi x} \sin(n\pi y).}$$

We find the coefficients from the boundary condition at $x = 0$.

$$u(0, y) = \sum_{n=1}^{\infty} a_n \sin(n\pi y) = f(y)$$

$$\boxed{a_n = 2 \int_0^1 f(y) \sin(n\pi y)\,\mathrm{d}y}$$

Solution 39.21

The Laplacian in circular coordinates is

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

Since we have homogeneous boundary conditions at $\theta = 0$ and $\theta = \alpha$, we will solve this problem with an eigenfunction expansion. We substitute the separation of variables $u(r, \theta) = R(r)T(\theta)$ into the partial differential equation.

$$R''T + \frac{1}{r} R'T + \frac{1}{r^2} RT'' = 0$$
$$r^2 \frac{R''}{R} + r\frac{R'}{R} = -\frac{T''}{T} = \lambda^2.$$

We have the regular Sturm-Liouville eigenvalue problem,

$$T'' = -\lambda^2 T, \quad T(0) = T(\alpha) = 0,$$

which has the solutions,

$$\lambda_n = \frac{n\pi}{\alpha}, \quad T_n = \sin\left(\frac{n\pi x}{\alpha}\right), \quad n \in \mathbb{N}.$$

The differential equation for R becomes,

$$r^2 R'' + rR' - \left(\frac{n\pi}{\alpha}\right)^2 R = 0, \quad R(a) = 0.$$

This is an Euler equation. We make the substitution, $R = r^\beta$.

$$\beta(\beta - 1) + \beta - \left(\frac{n\pi}{\alpha}\right)^2 = 0$$
$$\beta = \pm\frac{n\pi}{\alpha}$$

The general solution of the equation for R is

$$R = c_1 r^{n\pi/\alpha} + c_2 r^{-n\pi/\alpha}.$$

The solution, (up to a multiplicative constant), that vanishes at $r = a$ is

$$R = r^{n\pi/\alpha} - a^{2n\pi/\alpha} r^{-n\pi/\alpha}.$$

Thus the series expansion of our solution is,

$$\boxed{u(r, \theta) = \sum_{n=1}^{\infty} c_n \left(r^{n\pi/\alpha} - a^{2n\pi/\alpha} r^{-n\pi/\alpha} \right) \sin\left(\frac{n\pi\theta}{\alpha} \right).}$$

We determine the coefficients from the boundary condition at $r = b$.

$$u(b, \theta) = \sum_{n=1}^{\infty} c_n \left(b^{n\pi/\alpha} - a^{2n\pi/\alpha} b^{-n\pi/\alpha} \right) \sin\left(\frac{n\pi\theta}{\alpha} \right) = f(\theta)$$

$$\boxed{c_n = \frac{2}{\alpha \left(b^{n\pi/\alpha} - a^{2n\pi/\alpha} b^{-n\pi/\alpha} \right)} \int_0^{\alpha} f(\theta) \sin\left(\frac{n\pi\theta}{\alpha} \right) \mathrm{d}\theta}$$

Solution 39.22

a) The mathematical statement of the problem is

$$\begin{gathered} u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u(0, t) = u(L, t) = 0, \\ u(x, 0) = 0, \quad u_t(x, 0) = \begin{cases} v & \text{for } |x - \xi| < d \\ 0 & \text{for } |x - \xi| > d. \end{cases} \end{gathered}$$

Because we are interest in the harmonics of the motion, we will solve this problem with an eigenfunction expansion in x. We substitute the separation of variables $u(x,t) = X(x)T(t)$ into the wave equation.

$$XT'' = c^2 X'' T$$
$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda^2$$

The eigenvalue problem for X is,

$$X'' = -\lambda^2 X, \quad X(0) = X(L) = 0,$$

which has the solutions,

$$\lambda_n = \frac{n\pi}{L}, \quad X_n = \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}.$$

The ordinary differential equation for the T_n are,

$$T_n'' = -\left(\frac{n\pi c}{L}\right)^2 T_n,$$

which have the linearly independent solutions,

$$\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi ct}{L}\right).$$

The solution for $u(x,t)$ is a linear combination of the eigen-solutions.

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right)$$

Since the string initially has zero displacement, each of the a_n are zero.

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right)$$

Now we use the initial velocity to determine the coefficients in the expansion. Because the position is a continuous function of x, and there is a jump discontinuity in the velocity as a function of x, the coefficients in the expansion will decay as $1/n^2$.

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} v & \text{for } |x-\xi| < d \\ 0 & \text{for } |x-\xi| > d. \end{cases}$$

$$\frac{n\pi c}{L} b_n = \frac{2}{L}\int_0^L u_t(x,0) \sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

$$\begin{aligned} b_n &= \frac{2}{n\pi c}\int_{\xi-d}^{\xi+d} v \sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \\ &= \frac{4Lv}{n^2\pi^2 c}\sin\left(\frac{n\pi d}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right) \end{aligned}$$

The solution for $u(x,t)$ is,

$$\boxed{u(x,t) = \frac{4Lv}{\pi^2 c}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi d}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).}$$

b) The form of the solution is again,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

We determine the coefficients in the expansion from the initial velocity.

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} v\cos\left(\frac{\pi(x-\xi)}{2d}\right) & \text{for } |x-\xi| < d \\ 0 & \text{for } |x-\xi| > d. \end{cases}$$

$$\frac{n\pi c}{L} b_n = \frac{2}{L}\int_0^L u_t(x,0) \sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

$$b_n = \frac{2}{n\pi c} \int_{\xi-d}^{\xi+d} v \cos\left(\frac{\pi(x-\xi)}{2d}\right) \sin\left(\frac{n\pi x}{L}\right) \,\mathrm{d}x$$

$$b_n = \begin{cases} \frac{8dL^2 v}{n\pi^2 c(L^2-4d^2n^2)} \cos\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi\xi}{L}\right) & \text{for } d \neq \frac{L}{2n}, \\ \frac{v}{n^2\pi^2 c} \left(2n\pi d + L\sin\left(\frac{2n\pi d}{L}\right)\right) \sin\left(\frac{n\pi\xi}{L}\right) & \text{for } d = \frac{L}{2n} \end{cases}$$

The solution for $u(x,t)$ is,

$$\boxed{u(x,t) = \frac{8dL^2 v}{\pi^2 c} \sum_{n=1}^{\infty} \frac{1}{n(L^2-4d^2n^2)} \cos\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi\xi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) \quad \text{for } d \neq \frac{L}{2n},}$$

$$\boxed{u(x,t) = \frac{v}{\pi^2 c} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(2n\pi d + L\sin\left(\frac{2n\pi d}{L}\right)\right) \sin\left(\frac{n\pi\xi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) \quad \text{for } d = \frac{L}{2n}.}$$

c) The kinetic energy of the string is

$$E = \frac{1}{2} \int_0^L \rho \left(u_t(x,t)\right)^2 \,\mathrm{d}x,$$

where ρ is the density of the string per unit length.

Flat Hammer. The n^{th} harmonic is

$$u_n = \frac{4Lv}{n^2\pi^2 c} \sin\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi\xi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right).$$

The kinetic energy of the n^{th} harmonic is

$$E_n = \frac{\rho}{2} \int_0^L \left(\frac{\partial u_n}{\partial t}\right)^2 \,\mathrm{d}x = \frac{4Lv^2}{n^2\pi^2} \sin^2\left(\frac{n\pi d}{L}\right) \sin^2\left(\frac{n\pi\xi}{L}\right) \cos^2\left(\frac{n\pi ct}{L}\right).$$

This will be maximized if

$$\sin^2\left(\frac{n\pi\xi}{L}\right) = 1,$$

$$\frac{n\pi\xi}{L} = \frac{\pi(2m-1)}{2}, \quad m = 1, \dots, n,$$

$$\boxed{\xi = \frac{(2m-1)L}{2n}, \quad m = 1, \dots, n}$$

We note that the kinetic energies of the n^{th} harmonic decay as $1/n^2$.

Curved Hammer. We assume that $d \neq \frac{L}{2n}$. The n^{th} harmonic is

$$u_n = \frac{8dL^2v}{n\pi^2c(L^2 - 4d^2n^2)} \cos\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi\xi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right).$$

The kinetic energy of the n^{th} harmonic is

$$E_n = \frac{\rho}{2}\int_0^L \left(\frac{\partial u_n}{\partial t}\right)^2 \mathrm{d}x = \frac{16d^2L^3v^2}{\pi^2(L^2 - 4d^2n^2)^2} \cos^2\left(\frac{n\pi d}{L}\right) \sin^2\left(\frac{n\pi\xi}{L}\right) \cos^2\left(\frac{n\pi ct}{L}\right).$$

This will be maximized if

$$\sin^2\left(\frac{n\pi\xi}{L}\right) = 1,$$

$$\boxed{\xi = \frac{(2m-1)L}{2n}, \quad m = 1, \dots, n}$$

We note that the kinetic energies of the n^{th} harmonic decay as $1/n^4$.

Solution 39.23

In mathematical notation, the problem is

$$u_{tt} - c^2 u_{xx} = s(x,t), \quad 0 < x < L, \quad t > 0,$$
$$u(0,t) = u(L,t) = 0,$$
$$u(x,0) = u_t(x,0) = 0.$$

Since this is an inhomogeneous partial differential equation, we will expand the solution in a series of eigenfunctions in x for which the coefficients are functions of t. The solution for u has the form,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

Substituting this expression into the inhomogeneous partial differential equation will give us ordinary differential equations for each of the u_n.

$$\sum_{n=1}^{\infty} \left(u_n'' + c^2 \left(\frac{n\pi}{L}\right)^2 u_n \right) \sin\left(\frac{n\pi x}{L}\right) = s(x,t).$$

We expand the right side in a series of the eigenfunctions.

$$s(x,t) = \sum_{n=1}^{\infty} s_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

For $0 < t < \delta$ we have

$$\begin{aligned}
s_n(t) &= \frac{2}{L} \int_0^L s(x,t) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x \\
&= \frac{2}{L} \int_0^L v \cos\left(\frac{\pi(x-\xi)}{2d}\right) \sin\left(\frac{\pi t}{\delta}\right) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x \\
&= \frac{8dLv}{\pi(L^2 - 4d^2n^2)} \cos\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi \xi}{L}\right) \sin\left(\frac{\pi t}{\delta}\right).
\end{aligned}$$

For $t > \delta$, $s_n(t) = 0$. Substituting this into the partial differential equation yields,

$$u_n'' + \left(\frac{n\pi c}{L}\right)^2 u_n = \begin{cases} \frac{8dLv}{\pi(L^2-4d^2n^2)} \cos\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi\xi}{L}\right) \sin\left(\frac{\pi t}{\delta}\right), & \text{for } t < \delta, \\ 0 & \text{for } t > \delta. \end{cases}$$

Since the initial position and velocity of the string is zero, we have

$$u_n(0) = u_n'(0) = 0.$$

First we solve the differential equation on the range $0 < t < \delta$. The homogeneous solutions are

$$\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi ct}{L}\right).$$

Since the right side of the ordinary differential equation is a constant times $\sin(\pi t/\delta)$, which is an eigenfunction of the differential operator, we can guess the form of a particular solution, $p_n(t)$.

$$p_n(t) = d \sin\left(\frac{\pi t}{\delta}\right)$$

We substitute this into the ordinary differential equation to determine the multiplicative constant d.

$$p_n(t) = -\frac{8d\delta^2L^3v}{\pi^3(L^2 - c^2\delta^2n^2)(L^2 - 4d^2n^2)} \cos\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi\xi}{L}\right) \sin\left(\frac{\pi t}{\delta}\right)$$

The general solution for $u_n(t)$ is

$$u_n(t) = a\cos\left(\frac{n\pi ct}{L}\right) + b\sin\left(\frac{n\pi ct}{L}\right) - \frac{8d\delta^2L^3v}{\pi^3(L^2 - c^2\delta^2n^2)(L^2 - 4d^2n^2)} \cos\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi\xi}{L}\right) \sin\left(\frac{\pi t}{\delta}\right).$$

We use the initial conditions to determine the constants a and b. The solution for $0 < t < \delta$ is

$$u_n(t) = \frac{8d\delta^2L^3v}{\pi^3(L^2 - c^2\delta^2n^2)(L^2 - 4d^2n^2)} \cos\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi\xi}{L}\right) \left(\frac{L}{\delta cn} \sin\left(\frac{n\pi ct}{L}\right) - \sin\left(\frac{\pi t}{\delta}\right)\right).$$

The solution for $t > \delta$, the solution is a linear combination of the homogeneous solutions. This linear combination is determined by the position and velocity at $t = \delta$. We use the above solution to determine these quantities.

$$u_n(\delta) = \frac{8d\delta^2 L^4 v}{\pi^3 \delta c n (L^2 - c^2\delta^2 n^2)(L^2 - 4d^2 n^2)} \cos\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi\xi}{L}\right) \sin\left(\frac{n\pi c\delta}{L}\right)$$

$$u_n'(\delta) = \frac{8d\delta^2 L^3 v}{\pi^2 \delta (L^2 - c^2\delta^2 n^2)(L^2 - 4d^2 n^2)} \cos\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi\xi}{L}\right) \left(1 + \cos\left(\frac{n\pi c\delta}{L}\right)\right)$$

The fundamental set of solutions at $t = \delta$ is

$$\left\{\cos\left(\frac{n\pi c(t-\delta)}{L}\right), \frac{L}{n\pi c} \sin\left(\frac{n\pi c(t-\delta)}{L}\right)\right\}$$

From the initial conditions at $t = \delta$, we see that the solution for $t > \delta$ is

$$u_n(t) = \frac{8d\delta^2 L^3 v}{\pi^3 (L^2 - c^2\delta^2 n^2)(L^2 - 4d^2 n^2)} \cos\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi\xi}{L}\right)$$
$$\left(\frac{L}{\delta c n} \sin\left(\frac{n\pi c\delta}{L}\right) \cos\left(\frac{n\pi c(t-\delta)}{L}\right) + \frac{\pi}{\delta}\left(1 + \cos\left(\frac{n\pi c\delta}{L}\right)\right) \sin\left(\frac{n\pi c(t-\delta)}{L}\right)\right).$$

Width of the Hammer. The n^{th} harmonic has the width dependent factor,

$$\frac{d}{L^2 - 4d^2 n^2} \cos\left(\frac{n\pi d}{L}\right).$$

Differentiating this expression and trying to find zeros to determine extrema would give us an equation with both algebraic and transcendental terms. Thus we don't attempt to find the maxima exactly. We know that $d < L$. The cosine factor is large when

$$\frac{n\pi d}{L} \approx m\pi, \quad m = 1, 2, \ldots, n-1,$$
$$d \approx \frac{mL}{n}, \quad m = 1, 2, \ldots, n-1.$$

Substituting $d = mL/n$ into the width dependent factor gives us

$$\frac{d}{L^2(1-4m^2)}(-1)^m.$$

Thus we see that the amplitude of the n^{th} harmonic and hence its kinetic energy will be maximized for

$$\boxed{d \approx \frac{L}{n}}$$

The cosine term in the width dependent factor vanishes when

$$d = \frac{(2m-1)L}{2n}, \quad m = 1, 2, \dots, n.$$

The kinetic energy of the n^{th} harmonic is minimized for these widths.

For the lower harmonics, $n \ll \frac{L}{2d}$, the kinetic energy is proportional to d^2; for the higher harmonics, $n \gg \frac{L}{2d}$, the kinetic energy is proportional to $1/d^2$.

Duration of the Blow. The n^{th} harmonic has the duration dependent factor,

$$\frac{\delta^2}{L^2 - n^2c^2\delta^2}\left(\frac{L}{nc\delta}\sin\left(\frac{n\pi c\delta}{L}\right)\cos\left(\frac{n\pi c(t-\delta)}{L}\right) + \frac{\pi}{\delta}\left(1+\cos\left(\frac{n\pi c\delta}{L}\right)\right)\sin\left(\frac{n\pi c(t-\delta)}{L}\right)\right).$$

If we assume that δ is small, then

$$\frac{L}{nc\delta}\sin\left(\frac{n\pi c\delta}{L}\right) \approx \pi.$$

and

$$\frac{\pi}{\delta}\left(1+\cos\left(\frac{n\pi c\delta}{L}\right)\right) \approx \frac{2\pi}{\delta}.$$

Thus the duration dependent factor is about,

$$\frac{\delta}{L^2 - n^2c^2\delta^2} \sin\left(\frac{n\pi c(t-\delta)}{L}\right).$$

Thus for the lower harmonics, (those satisfying $n \ll \frac{L}{c\delta}$), the amplitude is proportional to δ, which means that the kinetic energy is proportional to δ^2. For the higher harmonics, (those with $n \gg \frac{L}{c\delta}$), the amplitude is proportional to $1/\delta$, which means that the kinetic energy is proportional to $1/\delta^2$.

Solution 39.24

Substituting $u(x, y, z, t) = v(x, y, z)\,\mathrm{e}^{\imath\omega t}$ into the wave equation will give us a Helmholtz equation.

$$-\omega^2 v\,\mathrm{e}^{\imath\omega t} - c^2(v_{xx} + v_{yy} + v_{zz})\,\mathrm{e}^{\imath\omega t} = 0$$
$$v_{xx} + v_{yy} + v_{zz} + k^2 v = 0.$$

We find the propagating modes with separation of variables. We substitute $v = X(x)Y(y)Z(z)$ into the Helmholtz equation.

$$X''YZ + XY''Z + XYZ'' + k^2XYZ = 0$$
$$-\frac{X''}{X} = \frac{Y''}{Y} + \frac{Z''}{Z} + k^2 = \nu^2$$

The eigenvalue problem in x is

$$X'' = -\nu^2 X, \quad X(0) = X(L) = 0,$$

which has the solutions,

$$\nu_n = \frac{n\pi}{L}, \quad X_n = \sin\left(\frac{n\pi x}{L}\right).$$

We continue with the separation of variables.

$$-\frac{Y''}{Y} = \frac{Z''}{Z} + k^2 - \left(\frac{n\pi}{L}\right)^2 = \mu^2$$

The eigenvalue problem in y is

$$Y'' = -\mu^2 Y, \quad Y(0) = Y(L) = 0,$$

which has the solutions,

$$\mu_n = \frac{m\pi}{L}, \quad Y_m = \sin\left(\frac{m\pi y}{L}\right).$$

Now we have an ordinary differential equation for Z,

$$Z'' + \left(k^2 - \left(\frac{\pi}{L}\right)^2 (n^2 + m^2)\right) Z = 0.$$

We define the eigenvalues,

$$\boxed{\lambda_{n,m}^2 = k^2 - \left(\frac{\pi}{L}\right)^2 (n^2 + m^2).}$$

If $k^2 - \left(\frac{\pi}{L}\right)^2 (n^2 + m^2) < 0$, then the solutions for Z are,

$$\exp\left(\pm\sqrt{\left(\left(\frac{\pi}{L}\right)^2 (n^2 + m^2) - k^2\right)} z\right).$$

We discard this case, as the solutions are not bounded as $z \to \infty$.

If $k^2 - \left(\frac{\pi}{L}\right)^2 (n^2 + m^2) = 0$, then the solutions for Z are,

$$\{1, z\}$$

The solution $Z = 1$ satisfies the boundedness and nonzero condition at infinity. This corresponds to a standing wave.

If $k^2 - \left(\frac{\pi}{L}\right)^2 (n^2 + m^2) > 0$, then the solutions for Z are,

$$\mathrm{e}^{\pm i\lambda_{n,m} z}.$$

These satisfy the boundedness and nonzero conditions at infinity. For values of n, m satisfying $k^2 - \left(\frac{\pi}{L}\right)^2 (n^2 + m^2) \geq 0$, there are the propagating modes,

$$\boxed{u_{n,m} = \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) \mathrm{e}^{i(\omega t \pm \lambda_{n,m} z)}.}$$

Solution 39.25

$$u_{tt} = c^2 \Delta u, \quad 0 < x < a, \ 0 < y < b, \tag{39.10}$$
$$u(0, y) = u(a, y) = u(x, 0) = u(x, b) = 0.$$

We substitute the separation of variables $u(x, y, t) = X(x)Y(y)T(t)$ into Equation 39.10.

$$\frac{T''}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y} = -\nu$$
$$\frac{X''}{X} = -\frac{Y''}{Y} - \nu = -\mu$$

This gives us differential equations for $X(x)$, $Y(y)$ and $T(t)$.

$$X'' = -\mu X, \quad X(0) = X(a) = 0$$
$$Y'' = -(\nu - \mu)Y, \quad Y(0) = Y(b) = 0$$
$$T'' = -c^2 \nu T$$

First we solve the problem for X.

$$\mu_m = \left(\frac{m\pi}{a}\right)^2, \quad X_m = \sin\left(\frac{m\pi x}{a}\right)$$

Then we solve the problem for Y.

$$\nu_{m,n} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2, \quad Y_{m,n} = \sin\left(\frac{n\pi y}{b}\right)$$

Finally we determine T.

$$T_{m,n} = \begin{matrix}\cos\\ \sin\end{matrix}\left(c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\, t\right)$$

The modes of oscillation are

$$u_{m,n} = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\begin{matrix}\cos\\ \sin\end{matrix}\left(c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\, t\right).$$

The frequencies are

$$\omega_{m,n} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}.$$

Figure 39.4 shows a few of the modes of oscillation in surface and density plots.

Solution 39.26

We substitute the separation of variables $\phi = X(x)Y(y)T(t)$ into the differential equation.

$$\phi_t = a^2\left(\phi_{xx} + \phi_{yy}\right) \tag{39.11}$$

$$XYT' = a^2\left(X''YT + XY''T\right)$$

$$\frac{T'}{a^2T} = \frac{X''}{X} + \frac{Y''}{Y} = -\nu$$

$$\frac{T'}{a^2T} = -\nu, \quad \frac{X''}{X} = -\nu - \frac{Y''}{Y} = -\mu$$

First we solve the eigenvalue problem for X.

$$X'' + \mu X = 0, \quad X(0) = X(l_x) = 0$$

$$\mu_m = \left(\frac{m\pi}{l_x}\right)^2, \quad X_m(x) = \sin\left(\frac{m\pi x}{l_x}\right), \quad m \in \mathbb{Z}^+$$

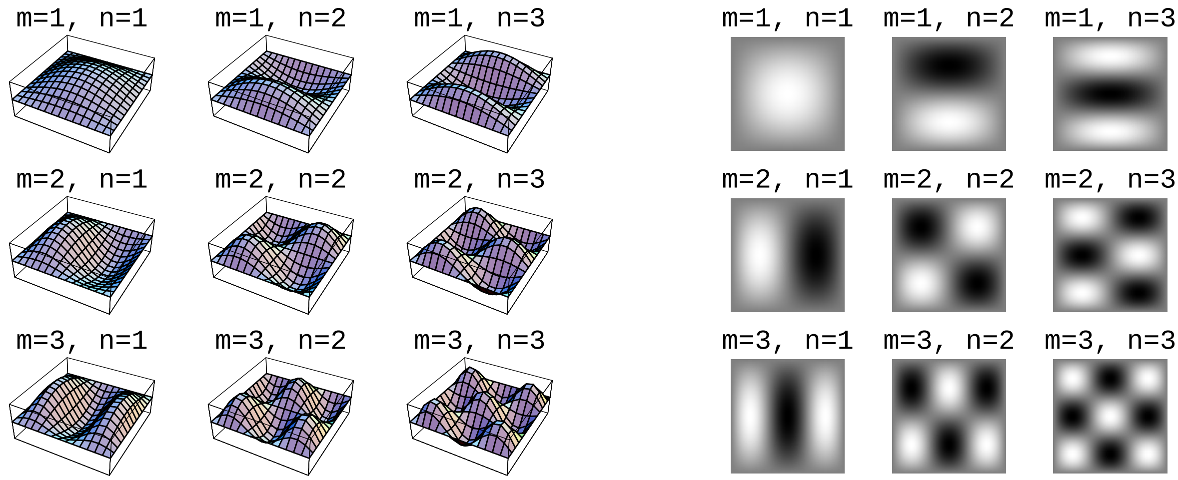

Figure 39.4: The modes of oscillation of a rectangular drum head.

Then we solve the eigenvalue problem for Y.

$$Y'' + (\nu - \mu_m)Y = 0, \quad Y'(0) = Y'(l_y) = 0$$

$$\nu_{mn} = \mu_m + \left(\frac{n\pi}{l_y}\right)^2, \quad Y_{mn}(y) = \cos\left(\frac{n\pi y}{l_y}\right), \quad n \in \mathbb{Z}^{0+}$$

Next we solve the differential equation for T, (up to a multiplicative constant).

$$T' = -a^2\nu_{mn}T$$

$$T(t) = \exp\left(-a^2\nu_{mn}t\right)$$

The eigensolutions of Equation 39.11 are

$$\sin(\mu_m x)\cos(\nu_{mn}y)\exp\left(-a^2\nu_{mn}t\right), \quad m \in \mathbb{Z}^+,\ n \in \mathbb{Z}^{0+}.$$

We choose the eigensolutions ϕ_{mn} to be orthonormal on the xy domain at $t = 0$.

$$\phi_{m0}(x, y, t) = \sqrt{\frac{2}{l_x l_y}} \sin(\mu_m x) \exp\left(-a^2 \nu_{mn} t\right), \quad m \in \mathbb{Z}^+$$
$$\phi_{mn}(x, y, t) = \frac{2}{\sqrt{l_x l_y}} \sin(\mu_m x) \cos(\nu_{mn} y) \exp\left(-a^2 \nu_{mn} t\right), \quad m \in \mathbb{Z}^+,\ n \in \mathbb{Z}^+$$

The solution of Equation 39.11 is a linear combination of the eigensolutions.

$$\phi(x, y, t) = \sum_{\substack{m=1 \\ n=0}}^{\infty} c_{mn} \phi_{mn}(x, y, t)$$

We determine the coefficients from the initial condition.

$$\phi(x, y, 0) = 1$$

$$\sum_{\substack{m=1\\n=0}}^{\infty} c_{mn}\phi_{mn}(x, y, 0) = 1$$

$$c_{mn} = \int_0^{l_x}\int_0^{l_y} \phi_{mn}(x, y, 0)\,\mathrm{d}y\,\mathrm{d}x$$

$$c_{m0} = \sqrt{\frac{2}{l_x l_y}}\int_0^{l_x}\int_0^{l_y} \sin(\mu_m x)\,\mathrm{d}y\,\mathrm{d}x$$

$$c_{m0} = \sqrt{2 l_x l_y}\,\frac{1-(-1)^m}{m\pi}, \quad m \in \mathbb{Z}^+$$

$$c_{mn} = \frac{2}{\sqrt{l_x l_y}}\int_0^{l_x}\int_0^{l_y} \sin(\mu_m x)\cos(\nu_{mn} y)\,\mathrm{d}y\,\mathrm{d}x$$

$$c_{mn} = 0, \quad m \in \mathbb{Z}^+,\ n \in \mathbb{Z}^+$$

$$\phi(x, y, t) = \sum_{m=1}^{\infty} c_{m0}\phi_{m0}(x, y, t)$$

$$\boxed{\phi(x, y, t) = \sum_{\substack{m=1\\ \text{odd } m}}^{\infty} \frac{2\sqrt{2 l_x l_y}}{m\pi}\sin(\mu_m x)\exp\left(-a^2\nu_{mn} t\right)}$$

Addendum. Note that an equivalent problem to the one specified is

$$\phi_t = a^2\left(\phi_{xx} + \phi_{yy}\right), \quad 0 < x < l_x,\ -\infty < y < \infty,$$
$$\phi(x, y, 0) = 1, \quad \phi(0, y, t) = \phi(l_y, y, t) = 0.$$

Here we have done an even periodic continuation of the problem in the y variable. Thus the boundary conditions

$$\phi_y(x, 0, t) = \phi_y(x, l_y, t) = 0$$

are automatically satisfied. Note that this problem does not depend on y. Thus we only had to solve

$$\phi_t = a^2 \phi_{xx}, \quad 0 < x < l_x$$
$$\phi(x, 0) = 1, \quad \phi(0, t) = \phi(l_y, t) = 0.$$

Solution 39.27

1. Since the initial and boundary conditions do not depend on θ, neither does ϕ. We apply the separation of variables $\phi = u(r)T(t)$.

$$\phi_t = a^2 \Delta \phi \tag{39.12}$$
$$\phi_t = a^2 \frac{1}{r} (r\phi_r)_r \tag{39.13}$$
$$\frac{T'}{a^2 T} = \frac{1}{r}(ru')' = -\lambda \tag{39.14}$$

We solve the eigenvalue problem for $u(r)$.

$$(ru')' + \lambda u = 0, \quad u(0) \text{ bounded}, \quad u(R) = 0$$

First we write the general solution.

$$u(r) = c_1 J_0 \left(\sqrt{\lambda} r \right) + c_2 Y_0 \left(\sqrt{\lambda} r \right)$$

The Bessel function of the second kind, Y_0, is not bounded at $r = 0$, so $c_2 = 0$. We use the boundary condition at $r = R$ to determine the eigenvalues.

$$\lambda_n = \left(\frac{j_{0,n}}{R} \right)^2, \quad u_n(r) = c J_0 \left(\frac{j_{0,n} r}{R} \right)$$

We choose the constant c so that the eigenfunctions are orthonormal with respect to the weighting function r.

$$
\begin{aligned}
u_n(r) &= \frac{J_0\left(\frac{j_{0,n} r}{R}\right)}{\sqrt{\int_0^R r J_0^2\left(\frac{j_{0,n} r}{R}\right)}} \\
&= \frac{\sqrt{2}}{R J_1(j_{0,n})} J_0\left(\frac{j_{0,n} r}{R}\right)
\end{aligned}
$$

Now we solve the differential equation for T.

$$
\begin{aligned}
T' &= -a^2 \lambda_n T \\
T_n &= \exp\left(-\left(\frac{a j_{0,n}}{R^2}\right)^2 t\right)
\end{aligned}
$$

The eigensolutions of Equation 39.12 are

$$
\phi_n(r,t) = \frac{\sqrt{2}}{R J_1(j_{0,n})} J_0\left(\frac{j_{0,n} r}{R}\right) \exp\left(-\left(\frac{a j_{0,n}}{R^2}\right)^2 t\right)
$$

The solution is a linear combination of the eigensolutions.

$$
\phi = \sum_{n=1}^{\infty} c_n \frac{\sqrt{2}}{R J_1(j_{0,n})} J_0\left(\frac{j_{0,n} r}{R}\right) \exp\left(-\left(\frac{a j_{0,n}}{R^2}\right)^2 t\right)
$$

We determine the coefficients from the initial condition.

$$\phi(r,\theta,0) = V$$

$$\sum_{n=1}^{\infty} c_n \frac{\sqrt{2}}{R J_1(j_{0,n})} J_0\left(\frac{j_{0,n} r}{R}\right) = V$$

$$c_n = \int_0^R V r \frac{\sqrt{2}}{R J_1(j_{0,n})} J_0\left(\frac{j_{0,n} r}{R}\right) \mathrm{d}r$$

$$c_n = V \frac{\sqrt{2}}{R J_1(j_{0,n})} \frac{R}{j_{0,n}/R} J_1\left(j_{0,n}\right)$$

$$c_n = \frac{\sqrt{2}\, V R}{j_{0,n}}$$

$$\boxed{\phi(r,\theta,t) = 2V \sum_{n=1}^{\infty} \frac{J_0\left(\frac{j_{0,n} r}{R}\right)}{j_{0,n} J_1(j_{0,n})} \exp\left(-\left(\frac{a j_{0,n}}{R^2}\right)^2 t\right)}$$

2.

$$J_\nu(r) \sim \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi\nu}{2} - \frac{\pi}{4}\right), \quad r \to +\infty$$

$$j_{\nu,n} \sim \left(n + \frac{\nu}{2} - \frac{1}{4}\right)\pi$$

For large n, the terms in the series solution at $t = 0$ are

$$\frac{J_0\left(\frac{j_{0,n} r}{R}\right)}{j_{0,n} J_1(j_{0,n})} \sim \frac{\sqrt{\frac{2R}{\pi j_{0,n} r}} \cos\left(\frac{j_{0,n} r}{R} - \frac{\pi}{4}\right)}{j_{0,n} \sqrt{\frac{2}{\pi j_{0,n}}} \cos\left(j_{0,n} - \frac{3\pi}{4}\right)}$$

$$\sim \frac{R}{r(n-1/4)\pi} \frac{\cos\left(\frac{(n-1/4)\pi r}{R} - \frac{\pi}{4}\right)}{\cos\left((n-1)\pi\right)}.$$

The coefficients decay as $1/n$.

Solution 39.28

1. We substitute the separation of variables $\Psi = T(t)\Theta(\theta)\Phi(\phi)$ into Equation 39.7

$$T'\Theta\Phi = \frac{a^2}{R^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, T\Theta'\Phi) + \frac{1}{\sin^2\theta}T\Theta\Phi''\right)$$
$$\frac{R^2 T'}{a^2 T} = \left(\frac{1}{\sin\theta\,\Theta}(\sin\theta\,\Theta')' + \frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi}\right) = -\mu$$
$$\frac{\sin\theta}{\Theta}(\sin\theta\,\Theta')' + \mu\sin^2\theta = -\frac{\Phi''}{\Phi} = \nu$$

We have differential equations for each of T, Θ and Φ.

$$T' = -\mu\frac{a^2}{R^2}T, \quad \frac{1}{\sin\theta}(\sin\theta\,\Theta')' + \left(\mu - \frac{\nu}{\sin^2\theta}\right)\Theta = 0, \quad \Phi'' + \nu\Phi = 0$$

2. In order that the solution be continuously differentiable, we need the periodic boundary conditions

$$\Phi(0) = \Phi(2\pi), \quad \Phi'(0) = \Phi'(2\pi).$$

The eigenvalues and eigenfunctions for Φ are

$$\nu_n = n^2, \quad \Phi_n = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{in\phi}, \quad n \in \mathbb{Z}.$$

Now we deal with the equation for Θ.

$$x = \cos\theta, \quad \Theta(\theta) = P(x), \quad \sin^2\theta = 1 - x^2, \quad \frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}$$
$$\frac{1}{\sin\theta}(\sin^2\theta\frac{1}{\sin\theta}\,\Theta')' + \left(\mu - \frac{\nu}{\sin^2\theta}\right)\Theta = 0$$
$$\left(\left(1 - x^2\right)P'\right)' + \left(\mu - \frac{n^2}{1 - x^2}\right)P = 0$$

$P(x)$ should be bounded at the endpoints, $x = -1$ and $x = 1$.

3. If the solution does not depend on θ, then the only one of the Φ_n that will appear in the solution is $\Phi_0 = 1/\sqrt{2\pi}$. The equations for T and P become

$$\left(\left(1 - x^2\right) P'\right)' + \mu P = 0, \quad P(\pm 1) \text{ bounded},$$
$$T' = -\mu \frac{a^2}{R^2} T.$$

The solutions for P are the Legendre polynomials.

$$\mu_l = l(l+1), \quad P_l(\cos\theta), \quad l \in \mathbb{Z}^{0+}$$

We solve the differential equation for T.

$$T' = -l(l+1)\frac{a^2}{R^2} T$$
$$T_l = \exp\left(-\frac{a^2 l(l+1)}{R^2} t\right)$$

The eigensolutions of the partial differential equation are

$$\Psi_l = P_l(\cos\theta) \exp\left(-\frac{a^2 l(l+1)}{R^2} t\right).$$

The solution is a linear combination of the eigensolutions.

$$\Psi = \sum_{l=0}^{\infty} A_l P_l(\cos\theta) \exp\left(-\frac{a^2 l(l+1)}{R^2} t\right)$$

4. We determine the coefficients in the expansion from the initial condition.

$$\Psi(\theta, 0) = 2\cos^2\theta - 1$$
$$\sum_{l=0}^{\infty} A_l P_l(\cos\theta) = 2\cos^2\theta - 1$$
$$A_0 + A_1 \cos\theta + A_2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots = 2\cos^2\theta - 1$$
$$A_0 = -\frac{1}{3}, \quad A_1 = 0, \quad A_2 = \frac{4}{3}, \quad A_3 = A_4 = \cdots = 0$$
$$\Psi(\theta, t) = -\frac{1}{3}P_0(\cos\theta) + \frac{4}{3}P_2(\cos\theta)\exp\left(-\frac{6a^2}{R^2}t\right)$$
$$\boxed{\Psi(\theta, t) = -\frac{1}{3} + \left(2\cos^2\theta - \frac{2}{3}\right)\exp\left(-\frac{6a^2}{R^2}t\right)}$$

Solution 39.29

Since we have homogeneous boundary conditions at $x = 0$ and $x = 1$, we will expand the solution in a series of eigenfunctions in x. We determine a suitable set of eigenfunctions with the separation of variables, $\phi = X(x)Y(y)$.

$$\phi_{xx} + \phi_{yy} = 0 \tag{39.15}$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

We have differential equations for X and Y.

$$X'' + \lambda X = 0, \quad X(0) = X(1) = 0$$
$$Y'' - \lambda Y = 0, \quad Y(0) = 0$$

The eigenvalues and orthonormal eigenfunctions for X are

$$\lambda_n = (n\pi)^2, \quad X_n(x) = \sqrt{2}\sin(n\pi x), \quad n \in \mathbb{Z}^+.$$

The solutions for Y are, (up to a multiplicative constant),

$$Y_n(y) = \sinh(n\pi y).$$

The solution of Equation 39.15 is a linear combination of the eigensolutions.

$$\phi(x, y) = \sum_{n=1}^{\infty} a_n \sqrt{2} \sin(n\pi x) \sinh(n\pi y)$$

We determine the coefficients from the boundary condition at $y = 2$.

$$x(1-x) = \sum_{n=1}^{\infty} a_n \sqrt{2} \sin(n\pi x) \sinh(n\pi 2)$$

$$a_n \sinh(2n\pi) = \sqrt{2} \int_0^1 x(1-x) \sin(n\pi x) \, \mathrm{d}x$$

$$a_n = \frac{2\sqrt{2}(1-(-1)^n)}{n^3 \pi^3 \sinh(2n\pi)}$$

$$\boxed{\phi(x, y) = \frac{8}{\pi^3} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n^3} \sin(n\pi x) \frac{\sinh(n\pi y)}{\sinh(2n\pi)}}$$

The solution at $x = 1/2$, $y = 1$ is

$$\phi(1/2, 1) = -\frac{8}{\pi^3} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n^3} \frac{\sinh(n\pi)}{\sinh(2n\pi)}.$$

Let R_k be the relative error at that point incurred by taking k terms.

$$R_k = \left| \frac{-\frac{8}{\pi^3} \sum_{\substack{n=k+2 \\ \text{odd } n}}^{\infty} \frac{1}{n^3} \frac{\sinh(n\pi)}{\sinh(2n\pi)}}{-\frac{8}{\pi^3} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n^3} \frac{\sinh(n\pi)}{\sinh(2n\pi)}} \right|$$

$$R_k = \frac{\sum_{\substack{n=k+2 \\ \text{odd } n}}^{\infty} \frac{1}{n^3} \frac{\sinh(n\pi)}{\sinh(2n\pi)}}{\sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n^3} \frac{\sinh(n\pi)}{\sinh(2n\pi)}}$$

Since $R_1 \approx 0.0000693169$ we see that one term is sufficient for 1% or 0.1% accuracy.

Now consider $\phi_x(1/2, 1)$.

$$\phi_x(x, y) = \frac{8}{\pi^2} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n^2} \cos(n\pi x) \frac{\sinh(n\pi y)}{\sinh(2n\pi)}$$

$$\phi_x(1/2, 1) = 0$$

Since all the terms in the series are zero, accuracy is not an issue.

Solution 39.30

The solution has the form

$$\psi = \begin{cases} \alpha r^{-n-1} P_n^m(\cos\theta)\sin(m\phi), & r > a \\ \beta r^n P_n^m(\cos\theta)\sin(m\phi), & r < a. \end{cases}$$

The boundary condition on ψ at $r = a$ gives us the constraint

$$\alpha a^{-n-1} - \beta a^n = 0$$

$$\beta = \alpha a^{-2n-1}.$$

Then we apply the boundary condition on ψ_r at $r = a$.

$$-(n+1)\alpha a^{-n-2} - n\alpha a^{-2n-1} a^{n-1} = 1$$

$$\alpha = -\frac{a^{n+2}}{2n+1}$$

$$\boxed{\psi = \begin{cases} -\frac{a^{n+2}}{2n+1} r^{-n-1} P_n^m(\cos\theta)\sin(m\phi), & r > a \\ -\frac{a^{-n+1}}{2n+1} r^n P_n^m(\cos\theta)\sin(m\phi), & r < a \end{cases}}$$

Solution 39.31
We expand the solution in a Fourier series.

$$\phi = \frac{1}{2}a_0(r) + \sum_{n=1}^{\infty} a_n(r)\cos(n\theta) + \sum_{n=1}^{\infty} b_n(r)\sin(n\theta)$$

We substitute the series into the Laplace's equation to determine ordinary differential equations for the coefficients.

$$\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

$$a_0'' + \frac{1}{r}a_0' = 0, \quad a_n'' + \frac{1}{r}a_n' - n^2 a_n = 0, \quad b_n'' + \frac{1}{r}b_n' - n^2 b_n = 0$$

The solutions that are bounded at $r = 0$ are, (to within multiplicative constants),

$$a_0(r) = 1, \quad a_n(r) = r^n, \quad b_n(r) = r^n.$$

Thus $\phi(r, \theta)$ has the form

$$\phi(r,\theta) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} d_n r^n \sin(n\theta)$$

We apply the boundary condition at $r = R$.

$$\phi_r(R, \theta) = \sum_{n=1}^{\infty} n c_n R^{n-1} \cos(n\theta) + \sum_{n=1}^{\infty} n d_n R^{n-1} \sin(n\theta)$$

In order that $\phi_r(R, \theta)$ have a Fourier series of this form, it is necessary that

$$\int_0^{2\pi} \phi_r(R, \theta) \, \mathrm{d}\theta = 0.$$

In that case c_0 is arbitrary in our solution. The coefficients are

$$c_n = \frac{1}{\pi n R^{n-1}} \int_0^{2\pi} \phi_r(R, \alpha) \cos(n\alpha) \, \mathrm{d}\alpha, \quad d_n = \frac{1}{\pi n R^{n-1}} \int_0^{2\pi} \phi_r(R, \alpha) \sin(n\alpha) \, \mathrm{d}\alpha.$$

We substitute the coefficients into our series solution to determine it up to the additive constant.

$$\phi(r,\theta) = \frac{R}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{R}\right)^n \int_0^{2\pi} \phi_r(R,\alpha)\cos(n(\theta-\alpha))\,\mathrm{d}\alpha$$

$$\phi(r,\theta) = \frac{R}{\pi}\int_0^{2\pi} \phi_r(R,\alpha)\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{R}\right)^n \cos(n(\theta-\alpha))\,\mathrm{d}\alpha$$

$$\phi(r,\theta) = \frac{R}{\pi}\int_0^{2\pi} \phi_r(R,\alpha)\sum_{n=1}^{\infty}\int_0^r \frac{\rho^{n-1}}{R^n}\,\mathrm{d}\rho\Re\left(\mathrm{e}^{in(\theta-\alpha)}\right)\,\mathrm{d}\alpha$$

$$\phi(r,\theta) = \frac{R}{\pi}\int_0^{2\pi} \phi_r(R,\alpha)\Re\left(\int_0^r \frac{1}{\rho}\sum_{n=1}^{\infty}\frac{\rho^n}{R^n}\,\mathrm{e}^{in(\theta-\alpha)}\,\mathrm{d}\rho\right)\,\mathrm{d}\alpha$$

$$\phi(r,\theta) = \frac{R}{\pi}\int_0^{2\pi} \phi_r(R,\alpha)\Re\left(\int_0^r \frac{1}{\rho}\frac{\frac{\rho}{R}\,\mathrm{e}^{i(\theta-\alpha)}}{1-\frac{\rho}{R}\,\mathrm{e}^{i(\theta-\alpha)}}\,\mathrm{d}\rho\right)\,\mathrm{d}\alpha$$

$$\phi(r,\theta) = -\frac{R}{\pi}\int_0^{2\pi} \phi_r(R,\alpha)\Re\left(\log\left(1-\frac{r}{R}\,\mathrm{e}^{i(\theta-\alpha)}\right)\right)\,\mathrm{d}\alpha$$

$$\phi(r,\theta) = -\frac{R}{\pi}\int_0^{2\pi} \phi_r(R,\alpha)\log\left|1-\frac{r}{R}\,\mathrm{e}^{i(\theta-\alpha)}\right|\,\mathrm{d}\alpha$$

$$\boxed{\phi(r,\theta) = -\frac{R}{2\pi}\int_0^{2\pi} \phi_r(R,\alpha)\log\left(1-2\frac{r}{R}\cos(\theta-\alpha)+\frac{r^2}{R^2}\right)\,\mathrm{d}\alpha}$$

Solution 39.32

We will assume that both α and β are nonzero. The cases of real and pure imaginary have already been covered. We solve the ordinary differential equations, (up to a multiplicative constant), to find special solutions of the diffusion equation.

$$\frac{T'}{T} = (\alpha + i\beta)^2, \quad \frac{X''}{X} = \frac{(\alpha + i\beta)^2}{a^2}$$
$$T = \exp\left((\alpha + i\beta)^2 t\right), \quad X = \exp\left(\pm\frac{\alpha + i\beta}{a} x\right)$$
$$T = \exp\left(\left(\alpha^2 - \beta^2\right) t + i2\alpha\beta t\right), \quad X = \exp\left(\pm\frac{\alpha}{a} x \pm i\frac{\beta}{a} x\right)$$
$$\phi = \exp\left(\left(\alpha^2 - \beta^2\right) t \pm \frac{\alpha}{a} x + i\left(2\alpha\beta t \pm \frac{\beta}{a} x\right)\right)$$

We take the sum and difference of these solutions to obtain

$$\boxed{\phi = \exp\left(\left(\alpha^2 - \beta^2\right) t \pm \frac{\alpha}{a} x\right) \begin{matrix}\cos\\ \sin\end{matrix}\left(2\alpha\beta t \pm \frac{\beta}{a} x\right)}$$

Chapter 40

Finite Transforms

Example 40.0.1 Consider the problem

$$\Delta u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \delta(x-\xi)\delta(y-\eta)\,\mathrm{e}^{-i\omega t} \quad \text{on } -\infty < x < \infty,\ 0 < y < b,$$

with

$$u_y(x,0,t) = u_y(x,b,t) = 0.$$

Substituting $u(x,y,t) = v(x,y)\,\mathrm{e}^{-i\omega t}$ into the partial differential equation yields the problem

$$\Delta v + k^2 v = \delta(x-\xi)\delta(y-\eta) \quad \text{on } -\infty < x < \infty,\ 0 < y < b,$$

with

$$v_y(x,0) = v_y(x,b) = 0.$$

We assume that the solution has the form

$$v(x,y) = \frac{1}{2}c_0(x) + \sum_{n=1}^{\infty} c_n(x)\cos\left(\frac{n\pi y}{b}\right), \tag{40.1}$$

and apply a finite cosine transform in the y direction. Integrating from 0 to b yields

$$\int_0^b v_{xx} + v_{yy} + k^2 v \, dy = \int_0^b \delta(x-\xi)\delta(y-\eta) \, dy,$$

$$\left[v_y\right]_0^b + \int_0^b v_{xx} + k^2 v \, dy = \delta(x-\xi),$$

$$\int_0^b v_{xx} + k^2 v \, dy = \delta(x-\xi).$$

Substituting in Equation 40.1 and using the orthogonality of the cosines gives us

$$\boxed{c_0''(x) + k^2 c_0(x) = \frac{2}{b}\delta(x-\xi).}$$

Multiplying by $\cos(n\pi y/b)$ and integrating form 0 to b yields

$$\int_0^b \left(v_{xx} + v_{yy} + k^2 v\right) \cos\left(\frac{n\pi y}{b}\right) \, dy = \int_0^b \delta(x-\xi)\delta(y-\eta) \cos\left(\frac{n\pi y}{b}\right) \, dy.$$

The v_{yy} term becomes

$$\begin{aligned}\int_0^b v_{yy} \cos\left(\frac{n\pi y}{b}\right) \, dy &= \left[v_y \cos\left(\frac{n\pi y}{b}\right)\right]_0^b - \int_0^b -\frac{n\pi}{b} v_y \sin\left(\frac{n\pi y}{b}\right) \, dy \\ &= \left[\frac{n\pi}{b} v \sin\left(\frac{n\pi y}{b}\right)\right]_0^b - \int_0^b \left(\frac{n\pi}{b}\right)^2 v \cos\left(\frac{n\pi y}{b}\right) \, dy.\end{aligned}$$

The right-hand-side becomes

$$\int_0^b \delta(x-\xi)\delta(y-\eta) \cos\left(\frac{n\pi y}{b}\right) \, dy = \delta(x-\xi) \cos\left(\frac{n\pi \eta}{b}\right).$$

Thus the partial differential equation becomes

$$\int_0^b \left(v_{xx} - \left(\frac{n\pi}{b}\right)^2 v + k^2 v \right) \cos\left(\frac{n\pi y}{b}\right) dy = \delta(x-\xi)\cos\left(\frac{n\pi\eta}{b}\right).$$

Substituting in Equation 40.1 and using the orthogonality of the cosines gives us

$$\boxed{c_n''(x) + \left[k^2 - \left(\frac{n\pi}{b}\right)^2\right] c_n(x) = \frac{2}{b}\delta(x-\xi)\cos\left(\frac{n\pi\eta}{b}\right).}$$

Now we need to solve for the coefficients in the expansion of $v(x,y)$. The homogeneous solutions for $c_0(x)$ are $e^{\pm ikx}$. The solution for $u(x,y,t)$ must satisfy the radiation condition. The waves at $x = -\infty$ travel to the left and the waves at $x = +\infty$ travel to the right. The two solutions of that will satisfy these conditions are, respectively,

$$y_1 = e^{-ikx}, \qquad y_2 = e^{ikx}.$$

The Wronskian of these two solutions is $2ik$. Thus the solution for $c_0(x)$ is

$$\boxed{c_0(x) = \frac{e^{-ikx_<} e^{ikx_>}}{ibk}}$$

We need to consider three cases for the equation for c_n.

$\mathbf{k > n\pi/b}$ Let $\alpha = \sqrt{k^2 - (n\pi/b)^2}$. The homogeneous solutions that satisfy the radiation condition are

$$y_1 = e^{-i\alpha x}, \qquad y_2 = e^{i\alpha x}.$$

The Wronskian of the two solutions is $2i\alpha$. Thus the solution is

$$\boxed{c_n(x) = \frac{e^{-i\alpha x_<} e^{i\alpha x_>}}{ib\alpha}\cos\left(\frac{n\pi\eta}{b}\right).}$$

In the case that $\cos\left(\frac{n\pi\eta}{b}\right) = 0$ this reduces to the trivial solution.

$\mathbf{k = n\pi/b}$ The homogeneous solutions that are bounded at infinity are

$$y_1 = 1, \qquad y_2 = 1.$$

If the right-hand-side is nonzero there is no way to combine these solutions to satisfy both the continuity and the derivative jump conditions. Thus if $\cos\left(\frac{n\pi\eta}{b}\right) \neq 0$ there is no bounded solution. If $\cos\left(\frac{n\pi\eta}{b}\right) = 0$ then the solution is not unique.

$$c_n(x) = \text{const.}$$

$\mathbf{k < n\pi/b}$ Let $\beta = \sqrt{(n\pi/b)^2 - k^2}$. The homogeneous solutions that are bounded at infinity are

$$y_1 = \mathrm{e}^{\beta x}, \qquad y_2 = \mathrm{e}^{-\beta x}.$$

The Wronskian of these solutions is -2β. Thus the solution is

$$\boxed{c_n(x) = -\frac{\mathrm{e}^{\beta x_<}\,\mathrm{e}^{-\beta x_>}}{b\beta} \cos\left(\frac{n\pi\eta}{b}\right)}$$

In the case that $\cos\left(\frac{n\pi\eta}{b}\right) = 0$ this reduces to the trivial solution.

40.1 Exercises

Exercise 40.1
A slab is perfectly insulated at the surface $x = 0$ and has a specified time varying temperature $f(t)$ at the surface $x = L$. Initially the temperature is zero. Find the temperature $u(x,t)$ if the heat conductivity in the slab is $\kappa = 1$.

Exercise 40.2
Solve

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad y > 0,$$
$$u(x,0) = f(x), \quad u(0,y) = g(y), \quad u(L,y) = h(y),$$

with an eigenfunction expansion.

40.2 Hints

Hint 40.1

Hint 40.2

40.3 Solutions

Solution 40.1
The problem is

$$u_t = u_{xx}, \quad 0 < x < L, t > 0,$$
$$u_x(0,t) = 0, \quad u(L,t) = f(t), \quad u(x,0) = 0.$$

We will solve this problem with an eigenfunction expansion. We find these eigenfunction by replacing the inhomogeneous boundary condition with the homogeneous one, $u(L,t) = 0$. We substitute the separation of variables $v(x,t) = X(x)T(t)$ into the homogeneous partial differential equation.

$$XT' = X''T$$
$$\frac{T'}{T} = \frac{X''}{X} = -\lambda^2.$$

This gives us the regular Sturm-Liouville eigenvalue problem,

$$X'' = -\lambda^2 X, \quad X'(0) = X(L) = 0,$$

which has the solutions,

$$\lambda_n = \frac{\pi(2n-1)}{2L}, \quad X_n = \cos\left(\frac{\pi(2n-1)x}{2L}\right), \quad n \in \mathbb{N}.$$

Our solution for $u(x,t)$ will be an eigenfunction expansion in these eigenfunctions. Since the inhomogeneous boundary condition is a function of t, the coefficients will be functions of t.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t) \cos(\lambda_n x)}$$

Since $u(x,t)$ does not satisfy the homogeneous boundary conditions of the eigenfunctions, the series is not uniformly convergent and we are not allowed to differentiate it with respect to x. We substitute the expansion into the partial differential equation, multiply by the eigenfunction and integrate from $x = 0$ to $x = L$. We use integration by parts to move derivatives from u to the eigenfunctions.

$$
\begin{gathered}
u_t = u_{xx} \\
\int_0^L u_t \cos(\lambda_m x)\, dx = \int_0^L u_{xx} \cos(\lambda_m x)\, dx \\
\int_0^L \left(\sum_{n=1}^{\infty} a_n'(t) \cos(\lambda_n x) \right) \cos(\lambda_m x)\, dx = [u_x \cos(\lambda_m x)]_0^L + \int_0^L u_x \lambda_m \sin(\lambda_m x)\, dx \\
\frac{L}{2} a_m'(t) = [u \lambda_m \sin(\lambda_m x)]_0^L - \int_0^L u \lambda_m^2 \cos(\lambda_m x)\, dx \\
\frac{L}{2} a_m'(t) = \lambda_m u(L,t) \sin(\lambda_m L) - \lambda_m^2 \int_0^L \left(\sum_{n=1}^{\infty} a_n(t) \cos(\lambda_n x) \right) \cos(\lambda_m x)\, dx \\
\frac{L}{2} a_m'(t) = \lambda_m (-1)^n f(t) - \lambda_m^2 \frac{L}{2} a_m(t) \\
a_m'(t) + \lambda_m^2 a_m(t) = (-1)^n \lambda_m f(t)
\end{gathered}
$$

From the initial condition $u(x, 0) = 0$ we see that $a_m(0) = 0$. Thus we have a first order differential equation and an initial condition for each of the $a_m(t)$.

$$
a_m'(t) + \lambda_m^2 a_m(t) = (-1)^n \lambda_m f(t), \quad a_m(0) = 0
$$

This equation has the solution,

$$
\boxed{a_m(t) = (-1)^n \lambda_m \int_0^t \mathrm{e}^{-\lambda_m^2 (t-\tau)} f(\tau)\, d\tau.}
$$

Solution 40.2

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad y > 0,$$
$$u(x, 0) = f(x), \quad u(0, y) = g(y), \quad u(L, y) = h(y),$$

We seek a solution of the form,

$$u(x, y) = \sum_{n=1}^{\infty} u_n(y) \sin\left(\frac{n\pi x}{L}\right).$$

Since we have inhomogeneous boundary conditions at $x = 0, L$, we cannot differentiate the series representation with respect to x. We multiply Laplace's equation by the eigenfunction and integrate from $x = 0$ to $x = L$.

$$\int_0^L (u_{xx} + u_{yy}) \sin\left(\frac{m\pi x}{L}\right) \, dx = 0$$

We use integration by parts to move derivatives from u to the eigenfunctions.

$$\left[u_x \sin\left(\frac{m\pi x}{L}\right)\right]_0^L - \frac{m\pi}{L} \int_0^L u_x \cos\left(\frac{m\pi x}{L}\right) \, dx + \frac{L}{2} u_m''(y) = 0$$
$$\left[-\frac{m\pi}{L} u \cos\left(\frac{m\pi x}{L}\right)\right]_0^L - \left(\frac{m\pi}{L}\right)^2 \int_0^L u \sin\left(\frac{m\pi x}{L}\right) \, dx + \frac{L}{2} u_m''(y) = 0$$
$$-\frac{m\pi}{L} h(y)(-1)^m + \frac{m\pi}{L} g(y) - \frac{L}{2} \left(\frac{m\pi}{L}\right)^2 u_m(y) + \frac{L}{2} u_m''(y) = 0$$
$$u_m''(y) - \left(\frac{m\pi}{L}\right)^2 u_m(y) = 2m\pi \left((-1)^m h(y) - g(y)\right)$$

Now we have an ordinary differential equation for the $u_n(y)$. In order that the solution is bounded, we require that each $u_n(y)$ is bounded as $y \to \infty$. We use the boundary condition $u(x, 0) = f(x)$ to determine boundary conditions for the $u_m(y)$ at $y = 0$.

$$u(x,0) = \sum_{n=1}^{\infty} u_n(0) \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$u_n(0) = f_n \equiv \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Thus we have the problems,

$$u_n''(y) - \left(\frac{n\pi}{L}\right)^2 u_n(y) = 2n\pi \left((-1)^n h(y) - g(y)\right), \quad u_n(0) = f_n, \quad u_n(+\infty) \text{ bounded},$$

for the coefficients in the expansion. We will solve these with Green functions. Consider the associated Green function problem

$$G_n''(y;\eta) - \left(\frac{n\pi}{L}\right)^2 G_n(y;\eta) = \delta(y-\eta), \quad G_n(0;\eta) = 0, \quad G_n(+\infty;\eta) \text{ bounded}.$$

The homogeneous solutions that satisfy the boundary conditions are

$$\sinh\left(\frac{n\pi y}{L}\right) \quad \text{and} \quad \mathrm{e}^{-n\pi y/L},$$

respectively. The Wronskian of these solutions is

$$\begin{vmatrix} \sinh\left(\frac{n\pi y}{L}\right) & \mathrm{e}^{-n\pi y/L} \\ \frac{n\pi}{L} \sinh\left(\frac{n\pi y}{L}\right) & -\frac{n\pi}{L} \mathrm{e}^{-n\pi y/L} \end{vmatrix} = -\frac{n\pi}{L} \mathrm{e}^{-2n\pi y/L}.$$

Thus the Green function is

$$G_n(y;\eta) = -\frac{L \sinh\left(\frac{n\pi y_<}{L}\right) \mathrm{e}^{-n\pi y_>/L}}{n\pi \, \mathrm{e}^{-2n\pi\eta/L}}.$$

Using the Green function we determine the $u_n(y)$ and thus the solution of Laplace's equation.

$$\boxed{u_n(y) = f_n \,\mathrm{e}^{-n\pi y/L} + 2n\pi \int_0^\infty G_n(y;\eta)\left((-1)^n h(\eta) - g(\eta)\right) d\eta}$$

$$\boxed{u(x,y) = \sum_{n=1}^\infty u_n(y) \sin\left(\frac{n\pi x}{L}\right).}$$

Chapter 41

Waves

41.1 Exercises

Exercise 41.1
Sketch the solution to the wave equation:

$$u(x,t) = \frac{1}{2}\left(u(x+ct,0) + u(x-ct,0)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(\tau,0)\, d\tau,$$

for various values of t corresponding to the initial conditions:

1. $u(x,0) = 0, \qquad u_t(x,0) = \sin \omega x \qquad$ where ω is a constant,
2. $u(x,0) = 0, \qquad u_t(x,0) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ -1 & \text{for } -1 < x < 0 \\ 0 & \text{for } |x| > 1. \end{cases}$

Exercise 41.2

1. Consider the solution of the wave equation for $u(x,t)$:

$$u_{tt} = c^2 u_{xx}$$

on the infinite interval $-\infty < x < \infty$ with initial displacement of the form

$$u(x,0) = \begin{cases} h(x) & \text{for } x > 0, \\ -h(-x) & \text{for } x < 0, \end{cases}$$

and with initial velocity

$$u_t(x,0) = 0.$$

Show that the solution of the wave equation satisfying these initial conditions also solves the following semi-infinite problem: Find $u(x,t)$ satisfying the wave equation $u_{tt} = c^2 u_{xx}$ in $0 < x < \infty$, $t > 0$, with initial conditions $u(x,0) = h(x)$, $u_t(x,0) = 0$, and with the fixed end condition $u(0,t) = 0$. Here $h(x)$ is any given function with $h(0) = 0$.

2. Use a similar idea to explain how you could use the general solution of the wave equation to solve the finite interval problem ($0 < x < l$) in which $u(0, t) = u(l, t) = 0$ for all t, with $u(x, 0) = h(x)$ and $u_t(x, 0) = 0$. Take $h(0) = h(l) = 0$.

Exercise 41.3

The deflection $u(x, T) = \phi(x)$ and velocity $u_t(x, T) = \psi(x)$ for an infinite string (governed by $u_{tt} = c^2 u_{xx}$) are measured at time T, and we are asked to determine what the initial displacement and velocity profiles $u(x, 0)$ and $u_t(x, 0)$ must have been. An alert AMa95c student suggests that this problem is equivalent to that of determining the solution of the wave equation at time T when initial conditions $u(x, 0) = \phi(x)$, $u_t(x, 0) = -\psi(x)$ are prescribed. Is she correct? If not, can you rescue her idea?

Exercise 41.4

In obtaining the general solution of the wave equation the interval was chosen to be infinite in order to simplify the evaluation of the functions $\alpha(\xi)$ and $\beta(\xi)$ in the general solution

$$u(x, t) = \alpha(x + ct) + \beta(x - ct).$$

But this general solution is in fact valid for any interval be it infinite or finite. We need only choose appropriate functions $\alpha(\xi)$, $\beta(\xi)$ to satisfy the appropriate initial and boundary conditions. This is not always convenient but there are other situations besides the solution for $u(x, t)$ in an infinite domain in which the general solution is of use. Consider the "whip-cracking" problem (this is not meant to be a metaphor for AMa95c):

$$u_{tt} = c^2 u_{xx},$$

(with c a constant) in the domain $x > 0, t > 0$ with initial conditions

$$u(x, 0) = u_t(x, 0) = 0 \qquad x > 0,$$

and boundary conditions

$$u(0, t) = \gamma(t)$$

prescribed for all $t > 0$. Here $\gamma(0) = 0$. Find α and β so as to determine u for $x > 0$, $t > 0$.

Hint: (From physical considerations conclude that you can take $\alpha(\xi) = 0$. Your solution will corroborate this.) Use the initial conditions to determine $\alpha(\xi)$ and $\beta(\xi)$ for $\xi > 0$. Then use the initial condition to determine $\beta(\xi)$ for $\xi < 0$.

Exercise 41.5
Let $u(x, t)$ satisfy the equation

$$u_{tt} = c^2 u_{xx};$$

(with c a constant) in some region of the (x, t) plane.

1. Show that the quantity $(u_t - cu_x)$ is constant along each straight line defined by $x - ct =$ constant, and that $(u_t + cu_x)$ is constant along each straight line of the form $x + ct =$ constant. These straight lines are called *characteristics*; we will refer to typical members of the two families as C_+ and C_- characteristics, respectively. Thus the line $x - ct =$ constant is a C_+ characteristic.

2. Let $u(x, 0)$ and $u_t(x, 0)$ be prescribed for all values of x in $-\infty < x < \infty$, and let (x_0, t_0) be some point in the (x, t) plane, with $t_0 > 0$. Draw the C_+ and C_- characteristics through (x_0, t_0) and let them intersect the x-axis at the points A,B. Use the properties of these curves derived in part (a) to determine $u_t(x_0, t_0)$ in terms of initial data at points A and B. Using a similar technique to obtain $u_t(x_0, \tau)$ with $0 < \tau < t$, determine $u(x_0, t_0)$ by integration with respect to τ, and compare this with the solution derived in class:

$$u(x, t) = \frac{1}{2}\left(u(x + ct, 0) + u(x - ct, 0)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(\tau, 0)\, d\tau.$$

   Observe that this "method of characteristics" again shows that $u(x_0, t_0)$ depends only on that part of the initial data between points A and B.

Exercise 41.6
The temperature $u(x,t)$ at a depth x below the Earth's surface at time t satisfies

$$u_t = \kappa u_{xx}.$$

The surface $x = 0$ is heated by the sun according to the periodic rule:

$$u(0,t) = T\cos(\omega t).$$

Seek a solution of the form

$$u(x,t) = \Re\left(A\,\mathrm{e}^{i\omega t - \alpha x}\right).$$

a) Find $u(x,t)$ satisfying $u \to 0$ as $x \to +\infty$, (i.e. deep into the Earth).

b) Find the temperature variation at a fixed depth, h, below the surface.

c) Find the phase lag $\delta(x)$ such that when the maximum temperature occurs at t_0 on the surface, the maximum at depth x occurs at $t_0 + \delta(x)$.

d) Show that the seasonal, (i.e. yearly), temperature changes and daily temperature changes penetrate to depths in the ratio:

$$\frac{x_{\text{year}}}{x_{\text{day}}} = \sqrt{365},$$

where x_{year} and x_{day} are the depths of same temperature variation caused by the different periods of the source.

Exercise 41.7
An infinite cylinder of radius a produces an external acoustic pressure field u satisfying:

$$u_{tt} = c^2 \delta u,$$

by a pure harmonic oscillation of its surface at $r = a$. That is, it moves so that

$$u(a, \theta, t) = f(\theta) \, \mathrm{e}^{i\omega t}$$

where $f(\theta)$ is a known function. Note that the waves must be outgoing at infinity, (radiation condition at infinity). Find the solution, $u(r, \theta, t)$. We seek a periodic solution of the form,

$$u(r, \theta, t) = v(r, \theta) \, \mathrm{e}^{i\omega t}.$$

Exercise 41.8

Plane waves are incident on a "soft" cylinder of radius a whose axis is parallel to the plane of the waves. Find the field scattered by the cylinder. In particular, examine the leading term of the solution when a is much smaller than the wavelength of the incident waves. If $v(x, y, t)$ is the scattered field it must satisfy:

$$\begin{array}{ll}
\text{Wave Equation:} & v_{tt} = c^2 \Delta v, \quad x^2 + y^2 > a^2; \\
\text{Soft Cylinder:} & v(x, y, t) = -\,\mathrm{e}^{i(ka\cos\theta - \omega t}, \text{ on } r = a, \quad 0 \leq \theta < 2\pi; \\
\text{Scattered:} & v \text{ is outgoing as } r \to \infty.
\end{array}$$

Here $k = \omega / c$. Use polar coordinates in the (x, y) plane.

Exercise 41.9

Consider the flow of electricity in a transmission line. The current, $I(x, t)$, and the voltage, $V(x, t)$, obey the telegrapher's system of equations:

$$\begin{aligned}
-I_x &= C V_t + G V, \\
-V_x &= L I_t + R I,
\end{aligned}$$

where C is the capacitance, G is the conductance, L is the inductance and R is the resistance.

a) Show that both I and V satisfy a damped wave equation.

b) Find the relationship between the physical constants, C, G, L and R such that there exist damped traveling wave solutions of the form:

$$V(x,t) = \mathrm{e}^{-\gamma t}(f(x - at) + g(x + at)).$$

What is the wave speed?

41.2 Hints

Hint 41.1

Hint 41.2

Hint 41.3

Hint 41.4
From physical considerations conclude that you can take $\alpha(\xi) = 0$. Your solution will corroborate this. Use the initial conditions to determine $\alpha(\xi)$ and $\beta(\xi)$ for $\xi > 0$. Then use the initial condition to determine $\beta(\xi)$ for $\xi < 0$.

Hint 41.5

Hint 41.6

a) Substitute $u(x, t) = \Re(A\,\mathrm{e}^{i\omega t - \alpha x})$ into the partial differential equation and solve for α. Assume that α has positive real part so that the solution vanishes as $x \to +\infty$.

Hint 41.7
Seek a periodic solution of the form,

$$u(r, \theta, t) = v(r, \theta)\, \mathrm{e}^{i\omega t}.$$

Solve the Helmholtz equation for v with a Fourier series expansion,

$$v(r, \theta) = \sum_{n=-\infty}^{\infty} v_n(r)\, \mathrm{e}^{in\theta}.$$

You will find that the v_n satisfy Bessel's equation. Choose the v_n so that u satisfies the boundary condition at $r = a$ and the radiation condition at infinity.

The Bessel functions have the asymptotic behavior,

$$\begin{aligned}
J_n(\rho) &\sim \sqrt{\frac{2}{\pi\rho}} \cos(\rho - n\pi/2 - \pi/4), && \text{as } \rho \to \infty, \\
Y_n(\rho) &\sim \sqrt{\frac{2}{\pi\rho}} \sin(\rho - n\pi/2 - \pi/4), && \text{as } \rho \to \infty, \\
H_n^{(1)}(\rho) &\sim \sqrt{\frac{2}{\pi\rho}}\, \mathrm{e}^{i(\rho - n\pi/2 - \pi/4)}, && \text{as } \rho \to \infty, \\
H_n^{(2)}(\rho) &\sim \sqrt{\frac{2}{\pi\rho}}\, \mathrm{e}^{-i(\rho - n\pi/2 - \pi/4)}, && \text{as } \rho \to \infty.
\end{aligned}$$

Hint 41.8

Hint 41.9

41.3 Solutions

Solution 41.1

1.

$$u(x,t) = \frac{1}{2}(u(x+ct,0) + u(x-ct,0)) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(\tau,0)\,d\tau$$
$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(\omega\tau)\,d\tau$$
$$u(x,t) = \frac{\sin(\omega x)\sin(\omega ct)}{\omega c}$$

Figure 41.1 shows the solution for $c = 1$ and $\omega = 1/10$.

2. We can write the initial velocity in terms of the Heaviside function.

$$u_t(x,0) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ -1 & \text{for } -1 < x < 0 \\ 0 & \text{for } |x| > 1. \end{cases}$$
$$u_t(x,0) = -H(x+1) + 2H(x) - H(x-1)$$

We integrate the Heaviside function.

$$\int_a^b H(x-c)\,dx = \begin{cases} 0 & \text{for } b < c \\ b-a & \text{for } a > c \\ b-c & \text{otherwise} \end{cases}$$

If $a < b$, we can express this as

$$\int_a^b H(x-c)\,dx = \min(b-a, \max(b-c, 0)).$$

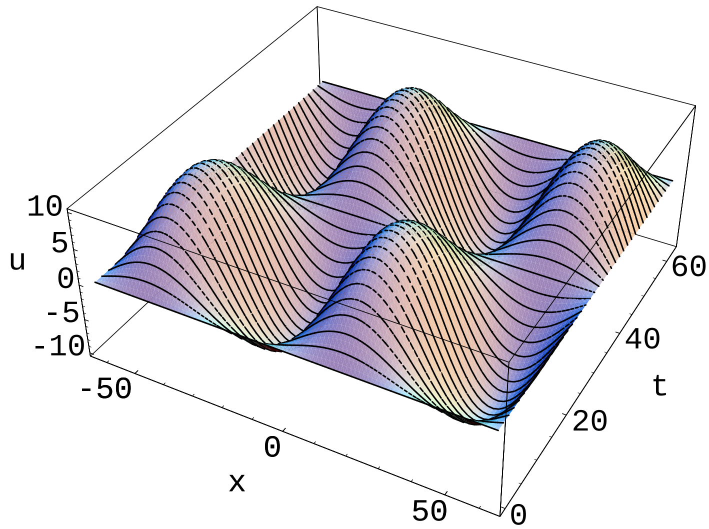

Figure 41.1: Solution of the wave equation.

Now we find an expression for the solution.

$$u(x,t) = \frac{1}{2}\left(u(x+ct,0) + u(x-ct,0)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(\tau,0)\,d\tau$$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \left(-H(\tau+1) + 2H(\tau) - H(\tau-1)\right)\,d\tau$$

$$u(x,t) = -\min(2ct, \max(x+ct+1,0)) + 2\min(2ct, \max(x+ct,0)) - \min(2ct, \max(x+ct-1,0))$$

Figure 41.2 shows the solution for $c = 1$.

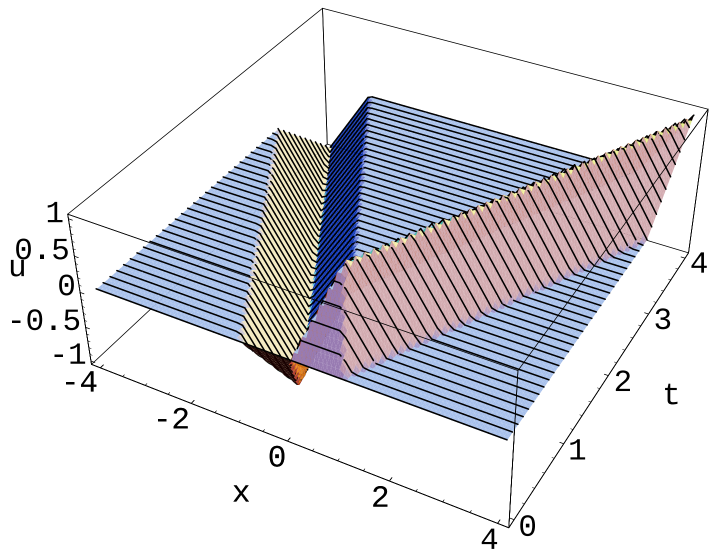

Figure 41.2: Solution of the wave equation.

Solution 41.2

1. The solution on the interval $(-\infty \dots \infty)$ is

$$u(x,t) = \frac{1}{2}(h(x+ct) + h(x-ct)).$$

Now we solve the problem on $(0 \dots \infty)$. We define the odd extension of $h(x)$.

$$\hat{h}(x) = \begin{cases} h(x) & \text{for } x > 0, \\ -h(-x) & \text{for } x < 0, \end{cases} = \operatorname{sign}(x) h(|x|)$$

Note that

$$\hat{h}'(0^-) = \frac{\mathrm{d}}{\mathrm{d}x}(-h(-x))\big|_{x \to 0^+} = h'(0^+) = \hat{h}'(0^+).$$

Thus $\hat{h}(x)$ is piecewise C^2. Clearly

$$u(x,t) = \frac{1}{2}(\hat{h}(x+ct) + \hat{h}(x-ct))$$

satisfies the differential equation on $(0 \ldots \infty)$. We verify that it satisfies the initial condition and boundary condition.

$$u(x,0) = \frac{1}{2}(\hat{h}(x) + \hat{h}(x)) = h(x)$$
$$u(0,t) = \frac{1}{2}(\hat{h}(ct) + \hat{h}(-ct)) = \frac{1}{2}(h(ct) - h(ct)) = 0$$

2. First we define the odd extension of $h(x)$ on the interval $(-l \ldots l)$.

$$\hat{h}(x) = \operatorname{sign}(x) h(|x|), \quad x \in (-l \ldots l)$$

Then we form the odd periodic extension of $h(x)$ defined on $(-\infty \ldots \infty)$.

$$\hat{h}(x) = \operatorname{sign}\left(x - 2l\left\lfloor \frac{x+l}{2l} \right\rfloor\right) h\left(\left|x - 2l\left\lfloor \frac{x+l}{2l} \right\rfloor\right|\right), \quad x \in (-\infty \ldots \infty)$$

We note that $\hat{h}(x)$ is piecewise C^2. Also note that $\hat{h}(x)$ is odd about the points $x = nl$, $n \in \mathbb{Z}$. That is, $\hat{h}(nl - x) = -\hat{h}(nl + x)$. Clearly

$$u(x,t) = \frac{1}{2}(\hat{h}(x+ct) + \hat{h}(x-ct))$$

satisfies the differential equation on $(0 \ldots l)$. We verify that it satisfies the initial condition and boundary conditions.

$$u(x,0) = \frac{1}{2}(\hat{h}(x) + \hat{h}(x))$$
$$u(x,0) = \hat{h}(x)$$
$$u(x,0) = \operatorname{sign}\left(x - 2l\left\lfloor\frac{x+l}{2l}\right\rfloor\right) h\left(\left|x - 2l\left\lfloor\frac{x+l}{2l}\right\rfloor\right|\right)$$
$$u(x,0) = h(x)$$
$$u(0,t) = \frac{1}{2}(\hat{h}(ct) + \hat{h}(-ct)) = \frac{1}{2}(\hat{h}(ct) - \hat{h}(ct)) = 0$$
$$u(l,t) = \frac{1}{2}(\hat{h}(l+ct) + \hat{h}(l-ct)) = \frac{1}{2}(\hat{h}(l+ct) - \hat{h}(l+ct)) = 0$$

Solution 41.3
Change of Variables. Let $u(x,t)$ be the solution of the problem with deflection $u(x,T) = \phi(x)$ and velocity $u_t(x,T) = \psi(x)$. Define

$$v(x,\tau) = u(x, T-\tau).$$

We note that $u(x,0) = v(x,T)$. $v(\tau)$ satisfies the wave equation.

$$v_{\tau\tau} = c^2 v_{xx}$$

The initial conditions for v are

$$v(x,0) = u(x,T) = \phi(x), \quad v_\tau(x,0) = -u_t(x,T) = -\psi(x).$$

Thus we see that the student was correct.

Direct Solution. D'Alembert's solution is valid for all x and t. We formally substitute $t-T$ for t in this solution to solve the problem with deflection $u(x,T) = \phi(x)$ and velocity $u_t(x,T) = \psi(x)$.

$$u(x,t) = \frac{1}{2}\left(\phi(x + c(t-T)) + \phi(x - c(t-T))\right) + \frac{1}{2c}\int_{x-c(t-T)}^{x+c(t-T)} \psi(\tau)\, d\tau$$

This satisfies the wave equation, because the equation is shift-invariant. It also satisfies the initial conditions.

$$u(x,T) = \frac{1}{2}(\phi(x) + \phi(x)) + \frac{1}{2c}\int_x^x \psi(\tau)\,d\tau = \phi(x)$$

$$u_t(x,t) = \frac{1}{2}(c\phi'(x + c(t-T)) - c\phi'(x - c(t-T))) + \frac{1}{2}(\psi(x + c(t-T)) + \psi(x - c(t-T)))$$

$$u_t(x,T) = \frac{1}{2}(c\phi'(x) - c\phi'(x)) + \frac{1}{2}(\psi(x) + \psi(x)) = \psi(x)$$

Solution 41.4

Since the solution is a wave moving to the right, we conclude that we could take $\alpha(\xi) = 0$. Our solution will corroborate this.

The form of the solution is

$$u(x,t) = \alpha(x + ct) + \beta(x - ct).$$

We substitute the solution into the initial conditions.

$$u(x,0) = \alpha(\xi) + \beta(\xi) = 0, \quad \xi > 0$$
$$u_t(x,0) = c\alpha'(\xi) - c\beta'(\xi) = 0, \quad \xi > 0$$

We integrate the second equation to obtain the system

$$\alpha(\xi) + \beta(\xi) = 0, \quad \xi > 0,$$
$$\alpha(\xi) - \beta(\xi) = 2k, \quad \xi > 0,$$

which has the solution

$$\alpha(\xi) = k, \quad \beta(\xi) = -k, \quad \xi > 0.$$

Now we substitute the solution into the initial condition.

$$u(0,t) = \alpha(ct) + \beta(-ct) = \gamma(t), \quad t > 0$$
$$\alpha(\xi) + \beta(-\xi) = \gamma(\xi/c), \quad \xi > 0$$
$$\beta(\xi) = \gamma(-\xi/c) - k, \quad \xi < 0$$

This determines $u(x,t)$ for $x > 0$ as it depends on $\alpha(\xi)$ only for $\xi > 0$. The constant k is arbitrary. Changing k does not change $u(x,t)$. For simplicity, we take $k = 0$.

$$u(x,t) = \beta(x - ct)$$

$$u(x,t) = \begin{cases} 0 & \text{for } x - ct < 0 \\ \gamma(t - x/c) & \text{for } x - ct > 0 \end{cases}$$

$$\boxed{u(x,t) = \gamma(t - x/c)H(ct - x)}$$

Solution 41.5

1. We write the value of u along the line $x - ct = k$ as a function of t: $u(k + ct, t)$. We differentiate $u_t - cu_x$ with respect to t to see how the quantity varies.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(u_t(k+ct,t) - cu_x(k+ct,t)\right) &= cu_{xt} + u_{tt} - c^2u_{xx} - cu_{xt} \\ &= u_{tt} - c^2u_{xx} \\ &= 0\end{aligned}$$

Thus $u_t - cu_x$ is constant along the line $x - ct = k$. Now we examine $u_t + cu_x$ along the line $x + ct = k$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(u_t(k-ct,t) + cu_x(k-ct,t)\right) &= -cu_{xt} + u_{tt} - c^2u_{xx} + cu_{xt} \\ &= u_{tt} - c^2u_{xx} \\ &= 0\end{aligned}$$

$u_t + cu_x$ is constant along the line $x + ct = k$.

2. From part (a) we know

$$\begin{aligned}u_t(x_0,t_0) - cu_x(x_0,t_0) &= u_t(x_0 - ct_0, 0) - cu_x(x_0 - ct_0, 0) \\ u_t(x_0,t_0) + cu_x(x_0,t_0) &= u_t(x_0 + ct_0, 0) + cu_x(x_0 + ct_0, 0).\end{aligned}$$

We add these equations to find $u_t(x_0, t_0)$.

$$u_t(x_0, t_0) = \frac{1}{2} \left(u_t(x_0 - ct_0, 0) - cu_x(x_0 - ct_0, 0) u_t(x_0 + ct_0, 0) + cu_x(x_0 + ct_0, 0) \right)$$

Since t_0 was arbitrary, we have

$$u_t(x_0, \tau) = \frac{1}{2} \left(u_t(x_0 - c\tau, 0) - cu_x(x_0 - c\tau, 0) u_t(x_0 + c\tau, 0) + cu_x(x_0 + c\tau, 0) \right)$$

for $0 < \tau < t_0$. We integrate with respect to τ to determine $u(x_0, t_0)$.

$$\begin{aligned}
u(x_0, t_0) &= u(x_0, 0) + \int_0^{t_0} \frac{1}{2} \left(u_t(x_0 - c\tau, 0) - cu_x(x_0 - c\tau, 0) u_t(x_0 + c\tau, 0) + cu_x(x_0 + c\tau, 0) \right) \, d\tau \\
&= u(x_0, 0) + \frac{1}{2} \int_0^{t_0} \left(-cu_x(x_0 - c\tau, 0) + cu_x(x_0 + c\tau, 0) \right) \, d\tau \\
&\quad + \frac{1}{2} \int_0^{t_0} \left(u_t(x_0 - c\tau, 0) + u_t(x_0 + c\tau, 0) \right) \, d\tau \\
&= u(x_0, 0) + \frac{1}{2} \left(u(x_0 - ct_0, 0) - u(x_0, 0) + u(x_0 + ct_0, 0) - u(x_0, 0) \right) \\
&\quad + \frac{1}{2c} \int_{x_0}^{x_0 - ct_0} -u_t(\tau, 0) \, d\tau + \frac{1}{2c} \int_{x_0}^{x_0 + ct_0} u_t(\tau, 0) \, d\tau \\
&= \frac{1}{2} \left(u(x_0 - ct_0, 0) + u(x_0 + ct_0, 0) \right) + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} u_t(\tau, 0) \, d\tau
\end{aligned}$$

We have D'Alembert's solution.

$$\boxed{u(x, t) = \frac{1}{2} \left(u(x - ct, 0) + u(x + ct, 0) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} u_t(\tau, 0) \, d\tau}$$

Solution 41.6

a) We substitute $u(x,t) = A\,\mathrm{e}^{\imath\omega t - \alpha x}$ into the partial differential equation and take the real part as the solution. We assume that α has positive real part so the solution vanishes as $x \to +\infty$.

$$\imath\omega A\,\mathrm{e}^{\imath\omega t-\alpha x} = \kappa\alpha^2 A\,\mathrm{e}^{\imath\omega t-\alpha x}$$
$$\imath\omega = \kappa\alpha^2$$
$$\alpha = (1+\imath)\sqrt{\frac{\omega}{2\kappa}}$$

A solution of the partial differential equation is,

$$u(x,t) = \Re\left(A\exp\left(\imath\omega t - (1+\imath)\sqrt{\frac{\omega}{2\kappa}}x\right)\right),$$

$$u(x,t) = A\exp\left(-\sqrt{\frac{\omega}{2\kappa}}x\right)\cos\left(\omega t - \sqrt{\frac{\omega}{2\kappa}}x\right).$$

Applying the initial condition, $u(0,t) = T\cos(\omega t)$, we obtain,

$$\boxed{u(x,t) = T\exp\left(-\sqrt{\frac{\omega}{2\kappa}}x\right)\cos\left(\omega t - \sqrt{\frac{\omega}{2\kappa}}x\right).}$$

b) At a fixed depth $x = h$, the temperature is

$$u(h,t) = T\exp\left(-\sqrt{\frac{\omega}{2\kappa}}h\right)\cos\left(\omega t - \sqrt{\frac{\omega}{2\kappa}}h\right).$$

Thus the temperature variation is

$$\boxed{-T\exp\left(-\sqrt{\frac{\omega}{2\kappa}}h\right) \leq u(h,t) \leq T\exp\left(-\sqrt{\frac{\omega}{2\kappa}}h\right).}$$

c) The solution is an exponentially decaying, traveling wave that propagates into the Earth with speed $\omega/\sqrt{\omega/(2\kappa)} = \sqrt{2\kappa\omega}$. More generally, the wave

$$\mathrm{e}^{-bt}\cos(\omega t - ax)$$

travels in the positive direction with speed ω/a. Figure 41.3 shows such a wave for a sequence of times.

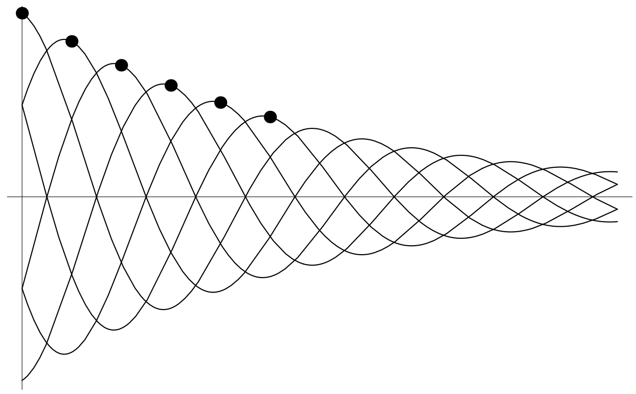

Figure 41.3: An Exponentially Decaying, Traveling Wave

The phase lag, $\delta(x)$ is the time that it takes for the wave to reach a depth of x. It satisfies,

$$\omega\delta(x) - \sqrt{\frac{\omega}{2\kappa}}x = 0,$$

$$\boxed{\delta(x) = \frac{x}{\sqrt{2\kappa\omega}}.}$$

d) Let ω_{year} be the frequency for annual temperature variation, then $\omega_{\text{day}} = 365\omega_{\text{year}}$. If x_{year} is the depth that a particular yearly temperature variation reaches and x_{day} is the depth that this same variation in daily temperature reaches, then

$$\exp\left(-\sqrt{\frac{\omega_{\text{year}}}{2\kappa}}x_{\text{year}}\right) = \exp\left(-\sqrt{\frac{\omega_{\text{day}}}{2\kappa}}x_{\text{day}}\right),$$

$$\sqrt{\frac{\omega_{\text{year}}}{2\kappa}}x_{\text{year}} = \sqrt{\frac{\omega_{\text{day}}}{2\kappa}}x_{\text{day}},$$

$$\boxed{\frac{x_{\text{year}}}{x_{\text{day}}} = \sqrt{365}.}$$

Solution 41.7
We seek a periodic solution of the form,

$$u(r,\theta,t) = v(r,\theta)\,\mathrm{e}^{\imath\omega t}.$$

Substituting this into the wave equation will give us a Helmholtz equation for v.

$$-\omega^2 v = c^2 \Delta v$$
$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} + \frac{\omega^2}{c^2}v = 0$$

We have the boundary condition $v(a,\theta) = f(\theta)$ and the radiation condition at infinity. We expand v in a Fourier series in θ in which the coefficients are functions of r. You can check that $\mathrm{e}^{\imath n\theta}$ are the eigenfunctions obtained with separation of variables.

$$v(r,\theta) = \sum_{n=-\infty}^{\infty} v_n(r)\,\mathrm{e}^{\imath n\theta}$$

We substitute this expression into the Helmholtz equation to obtain ordinary differential equations for the coefficients v_n.

$$\sum_{n=-\infty}^{\infty} \left(v_n'' + \frac{1}{r} v_n' + \left(\frac{\omega^2}{c^2} - \frac{n^2}{r^2} \right) v_n \right) \mathrm{e}^{in\theta} = 0$$

The differential equations for the v_n are

$$v_n'' + \frac{1}{r} v_n' + \left(\frac{\omega^2}{c^2} - \frac{n^2}{r^2} \right) v_n = 0.$$

which has as linearly independent solutions the Bessel and Neumann functions,

$$J_n\left(\frac{\omega r}{c}\right), \quad Y_n\left(\frac{\omega r}{c}\right),$$

or the Hankel functions,

$$H_n^{(1)}\left(\frac{\omega r}{c}\right), \quad H_n^{(2)}\left(\frac{\omega r}{c}\right).$$

The functions have the asymptotic behavior,

$$\begin{aligned}
J_n(\rho) &\sim \sqrt{\frac{2}{\pi\rho}} \cos(\rho - n\pi/2 - \pi/4), \quad &&\text{as } \rho \to \infty, \\
Y_n(\rho) &\sim \sqrt{\frac{2}{\pi\rho}} \sin(\rho - n\pi/2 - \pi/4), \quad &&\text{as } \rho \to \infty, \\
H_n^{(1)}(\rho) &\sim \sqrt{\frac{2}{\pi\rho}} \mathrm{e}^{i(\rho - n\pi/2 - \pi/4)}, \quad &&\text{as } \rho \to \infty, \\
H_n^{(2)}(\rho) &\sim \sqrt{\frac{2}{\pi\rho}} \mathrm{e}^{-i(\rho - n\pi/2 - \pi/4)}, \quad &&\text{as } \rho \to \infty.
\end{aligned}$$

$u(r, \theta, t)$ will be an outgoing wave at infinity if it is the sum of terms of the form $e^{i(\omega t - \text{const} r)}$. Thus the v_n must have the form

$$v_n(r) = b_n H_n^{(2)}\left(\frac{\omega r}{c}\right)$$

for some constants, b_n. The solution for $v(r, \theta)$ is

$$v(r, \theta) = \sum_{n=-\infty}^{\infty} b_n H_n^{(2)}\left(\frac{\omega r}{c}\right) e^{in\theta}.$$

We determine the constants b_n from the boundary condition at $r = a$.

$$v(a, \theta) = \sum_{n=-\infty}^{\infty} b_n H_n^{(2)}\left(\frac{\omega a}{c}\right) e^{in\theta} = f(\theta)$$

$$\boxed{b_n = \frac{1}{2\pi H_n^{(2)}(\omega a/c)} \int_0^{2\pi} f(\theta)\, e^{-in\theta}\, d\theta}$$

$$\boxed{u(r, \theta, t) = e^{i\omega t} \sum_{n=-\infty}^{\infty} b_n H_n^{(2)}\left(\frac{\omega r}{c}\right) e^{in\theta}}$$

Solution 41.8

We substitute the form $v(x, y, t) = u(r, \theta)\, e^{-i\omega t}$ into the wave equation to obtain a Helmholtz equation.

$$c^2 \Delta u + \omega^2 u = 0$$
$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + k^2 u = 0$$

We solve the Helmholtz equation with separation of variables. We expand u in a Fourier series.

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} u_n(r)\, e^{in\theta}$$

We substitute the sum into the Helmholtz equation to determine ordinary differential equations for the coefficients.

$$u_n'' + \frac{1}{r} u_n' + \left(k^2 - \frac{n^2}{r^2} \right) u_n = 0$$

This is Bessel's equation, which has as solutions the Bessel and Neumann functions, $\{J_n(kr), Y_n(kr)\}$ or the Hankel functions, $\{H_n^{(1)}(kr), H_n^{(2)}(kr)\}$.

Recall that the solutions of the Bessel equation have the asymptotic behavior,

$$\begin{aligned} J_n(\rho) &\sim \sqrt{\frac{2}{\pi\rho}} \cos(\rho - n\pi/2 - \pi/4), \quad &\text{as } \rho \to \infty, \\ Y_n(\rho) &\sim \sqrt{\frac{2}{\pi\rho}} \sin(\rho - n\pi/2 - \pi/4), \quad &\text{as } \rho \to \infty, \\ H_n^{(1)}(\rho) &\sim \sqrt{\frac{2}{\pi\rho}} \,\mathrm{e}^{i(\rho - n\pi/2 - \pi/4)}, \quad &\text{as } \rho \to \infty, \\ H_n^{(2)}(\rho) &\sim \sqrt{\frac{2}{\pi\rho}} \,\mathrm{e}^{-i(\rho - n\pi/2 - \pi/4)}, \quad &\text{as } \rho \to \infty. \end{aligned}$$

From this we see that only the Hankel function of the first kink will give us outgoing waves as $\rho \to \infty$. Our solution for u becomes,

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} b_n H_n^{(1)}(kr) \,\mathrm{e}^{in\theta}.$$

We determine the coefficients in the expansion from the boundary condition at $r = a$.

$$u(a, \theta) = \sum_{n=-\infty}^{\infty} b_n H_n^{(1)}(ka) \,\mathrm{e}^{in\theta} = -\,\mathrm{e}^{ika\cos\theta}$$

$$b_n = -\frac{1}{2\pi H_n^{(1)}(ka)} \int_0^{2\pi} \mathrm{e}^{ika\cos\theta} \,\mathrm{e}^{-in\theta} \, d\theta$$

We evaluate the integral with the identities,

$$J_n(x) = \frac{1}{2\pi i^n} \int_0^{2\pi} \mathrm{e}^{ix\cos\theta}\, \mathrm{e}^{in\theta}\, d\theta,$$
$$J_{-n}(x) = (-1)^n J_n(x).$$

Thus we obtain,

$$\boxed{u(r,\theta) = -\sum_{n=-\infty}^{\infty} \frac{(-i)^n J_n(ka)}{H_n^{(1)}(ka)} H_n^{(1)}(kr)\, \mathrm{e}^{in\theta}.}$$

When $a \ll 1/k$, i.e. $ka \ll 1$, the Bessel function has the behavior,

$$J_n(ka) \sim \frac{(ka/2)^n}{n!}.$$

In this case, the $n \neq 0$ terms in the sum are much smaller than the $n = 0$ term. The approximate solution is,

$$u(r,\theta) \sim -\frac{H_0^{(1)}(kr)}{H_0^{(1)}(ka)},$$
$$\boxed{v(r,\theta,t) \sim -\frac{H_0^{(1)}(kr)}{H_0^{(1)}(ka)}\, \mathrm{e}^{-i\omega t}.}$$

Solution 41.9

a)

$$\begin{cases} -I_x = CV_t + GV, \\ -V_x = LI_t + RI \end{cases}$$

First we derive a single partial differential equation for I. We differentiate the two partial differential equations with respect to x and t, respectively and then eliminate the V_{xt} terms.

$$\begin{cases} -I_{xx} = CV_{tx} + GV_x, \\ -V_{xt} = LI_{tt} + RI_t \end{cases}$$

$$-I_{xx} + LCI_{tt} + RCI_t = GV_x$$

We use the initial set of equations to write V_x in terms of I.

$$-I_{xx} + LCI_{tt} + RCI_t + G(LI_t + RI) = 0$$

$$\boxed{I_{tt} + \frac{RC + GL}{LC} I_t + \frac{GR}{LC} I - \frac{1}{LC} I_{xx} = 0}$$

Now we derive a single partial differential equation for V. We differentiate the two partial differential equations with respect to t and x, respectively and then eliminate the I_{xt} terms.

$$\begin{cases} -I_{xt} = CV_{tt} + GV_t, \\ -V_{xx} = LI_{tx} + RI_x \end{cases}$$

$$-V_{xx} = RI_x - LCV_{tt} - LGV_t$$

We use the initial set of equations to write I_x in terms of V.

$$LCV_{tt} + LGV_t - V_{xx} + R(CV_t + GV) = 0$$

$$\boxed{V_{tt} + \frac{RC + LG}{LC} V_t + \frac{RG}{LC} V - \frac{1}{LC} V_{xx} = 0.}$$

Thus we see that I and V both satisfy the same damped wave equation.

b) We substitute $V(x,t) = \mathrm{e}^{-\gamma t}(f(x - at) + g(x + at))$ into the damped wave equation for V.

$$\left(\gamma^2 - \frac{RC + LG}{LC}\gamma + \frac{RG}{LC}\right) \mathrm{e}^{-\gamma t}(f + g) + \left(-2\gamma + \frac{RC + LG}{LC}\right) a\,\mathrm{e}^{-\gamma t}(-f' + g') \\ + a^2\,\mathrm{e}^{-\gamma t}(f'' + g'') - \frac{1}{LC}\mathrm{e}^{-\gamma t}(f'' + g'') = 0$$

Since f and g are arbitrary functions, the coefficients of $\mathrm{e}^{-\gamma t}(f+g)$, $\mathrm{e}^{-\gamma t}(-f'+g')$ and $\mathrm{e}^{-\gamma t}(f''+g'')$ must vanish. This gives us three constraints.

$$a^2 - \frac{1}{LC} = 0, \quad -2\gamma + \frac{RC + LG}{LC} = 0, \quad \gamma^2 - \frac{RC + LG}{LC}\gamma + \frac{RG}{LC} = 0$$

The first equation determines the wave speed to be $a = 1/\sqrt{LC}$. We substitute the value of γ from the second equation into the third equation.

$$\gamma = \frac{RC + LG}{2LC}, \quad -\gamma^2 + \frac{RG}{LC} = 0$$

In order for damped waves to propagate, the physical constants must satisfy,

$$\frac{RG}{LC} - \left(\frac{RC + LG}{2LC}\right)^2 = 0,$$
$$4RGLC - (RC + LG)^2 = 0,$$
$$(RC - LG)^2 = 0,$$
$$\boxed{RC = LG.}$$

Chapter 42

The Diffusion Equation

42.1 Exercises

Exercise 42.1
Derive the heat equation for a general 3 dimensional body, with non-uniform density $\rho(\mathbf{x})$, specific heat $c(\mathbf{x})$, and conductivity $k(\mathbf{x})$. Show that

$$\frac{\partial u(\mathbf{x}, t)}{\partial t} = \frac{1}{c\rho} \nabla \cdot (k \nabla u(\mathbf{x}, t))$$

where u is the temperature, and you may assume there are no internal sources or sinks.

Exercise 42.2
Verify Duhamel's Principal: If $u(x, t, \tau)$ is the solution of the initial value problem:

$$u_t = \kappa u_{xx}, \quad u(x, 0, \tau) = f(x, \tau),$$

then the solution of

$$w_t = \kappa w_{xx} + f(x, t), \quad w(x, 0) = 0$$

is

$$w(x, t) = \int_0^t u(x, t - \tau, \tau)\, d\tau.$$

Exercise 42.3
Modify the derivation of the diffusion equation

$$\phi_t = a^2 \phi_{xx}, \quad a^2 = \frac{k}{c\rho}, \tag{42.1}$$

so that it is valid for diffusion in a non-homogeneous medium for which c and k are functions of x and ϕ and so that it is valid for a geometry in which A is a function of x. Show that Equation (42.1) above is in this case replaced by

$$c\rho A\phi_t = (kA\phi_x)_x .$$

Recall that c is the specific heat, k is the thermal conductivity, ρ is the density, ϕ is the temperature and A is the cross-sectional area.

42.2 Hints

Hint 42.1

Hint 42.2
Check that the expression for $w(x,t)$ satisfies the partial differential equation and initial condition. Recall that

$$\frac{\partial}{\partial x}\int_a^x h(x,\xi)\,d\xi = \int_a^x h_x(x,\xi)\,d\xi + h(x,x).$$

Hint 42.3

42.3 Solutions

Exercise 42.4

Consider a Region of material, R. Let u be the temperature and $\boldsymbol{\phi}$ be the heat flux. The amount of heat energy in the region is

$$\int_R c\rho u \,\mathrm{d}\mathbf{x}.$$

We equate the rate of change of heat energy in the region with the heat flux across the boundary of the region.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R c\rho u \,\mathrm{d}\mathbf{x} = -\int_{\partial R} \boldsymbol{\phi}\cdot\mathbf{n}\,\mathrm{d}s$$

We apply the divergence theorem to change the surface integral to a volume integral.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R c\rho u \,\mathrm{d}\mathbf{x} = -\int_R \nabla\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}$$

$$\int_R \left(c\rho\frac{\partial u}{\partial t} + \nabla\cdot\boldsymbol{\phi}\right)\mathrm{d}\mathbf{x} = 0$$

Since the region is arbitrary, the integral must vanish identically.

$$c\rho\frac{\partial u}{\partial t} = -\nabla\cdot\boldsymbol{\phi}$$

We apply Fourier's law of heat conduction, $\boldsymbol{\phi} = -k\nabla u$, to obtain the heat equation.

$$\frac{\partial u}{\partial t} = \frac{1}{c\rho}\nabla\cdot(k\nabla u)$$

Solution 42.1
We verify Duhamel's principal by showing that the integral expression for $w(x,t)$ satisfies the partial differential equation and the initial condition. Clearly the initial condition is satisfied.

$$w(x,0) = \int_0^0 u(x, 0-\tau, \tau)\, d\tau = 0$$

Now we substitute the expression for $w(x,t)$ into the partial differential equation.

$$\begin{gathered}
\frac{\partial}{\partial t} \int_0^t u(x, t-\tau, \tau)\, d\tau = \kappa \frac{\partial^2}{\partial x^2} \int_0^t u(x, t-\tau, \tau)\, d\tau + f(x,t) \\
u(x, t-t, t) + \int_0^t u_t(x, t-\tau, \tau)\, d\tau = \kappa \int_0^t u_{xx}(x, t-\tau, \tau)\, d\tau + f(x,t) \\
f(x,t) + \int_0^t u_t(x, t-\tau, \tau)\, d\tau = \kappa \int_0^t u_{xx}(x, t-\tau, \tau)\, d\tau + f(x,t) \\
\int_0^t \left(u_t(x, t-\tau, \tau)\, d\tau - \kappa u_{xx}(x, t-\tau, \tau) \right)\, d\tau
\end{gathered}$$

Since $u_t(x, t-\tau, \tau)\, d\tau - \kappa u_{xx}(x, t-\tau, \tau) = 0$, this equation is an identity.

Solution 42.2
We equate the rate of change of thermal energy in the segment $(\alpha \ldots \beta)$ with the heat entering the segment

through the endpoints.

$$\int_\alpha^\beta \phi_t c\rho A\,dx = k(\beta,\phi(\beta))A(\beta)\phi_x(\beta,t) - k(\alpha,\phi(\alpha))A(\alpha)\phi_x(\alpha,t)$$

$$\int_\alpha^\beta \phi_t c\rho A\,dx = [kA\phi_x]_\alpha^\beta$$

$$\int_\alpha^\beta \phi_t c\rho A\,dx = \int_\alpha^\beta (kA\phi_x)_x\;dx$$

$$\int_\alpha^\beta c\rho A\phi_t - (kA\phi_x)_x\;dx = 0$$

Since the domain is arbitrary, we conclude that

$$\boxed{c\rho A\phi_t = (kA\phi_x)_x\,.}$$

Chapter 43

Similarity Methods

Introduction. Consider the partial differential equation (not necessarily linear)

$$F\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, u, t, x\right) = 0.$$

Say the solution is

$$u(x,t) = \frac{x}{t} \sin\left(\frac{t^{1/2}}{x^{1/2}}\right).$$

Making the change of variables $\xi = x/t$, $f(\xi) = u(x,t)$, we could rewrite this equation as

$$f(\xi) = \xi \sin\left(\xi^{-1/2}\right).$$

We see now that if we had guessed that the solution of this partial differential equation was only dependent on powers of x/t we could have changed variables to ξ and f and instead solved the ordinary differential equation

$$G\left(\frac{\mathrm{d}f}{\mathrm{d}\xi}, f, \xi\right) = 0.$$

By using similarity methods one can reduce the number of independent variables in some PDE's.

Example 43.0.1 Consider the partial differential equation

$$x\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} - u = 0.$$

One way to find a similarity variable is to introduce a transformation to the temporary variables u', t', x', and the parameter λ.

$$\begin{aligned} u &= u'\lambda \\ t &= t'\lambda^m \\ x &= x'\lambda^n \end{aligned}$$

where n and m are unknown. Rewriting the partial differential equation in terms of the temporary variables,

$$\begin{aligned} x'\lambda^n \frac{\partial u'}{\partial t'}\lambda^{1-m} + t'\lambda^m \frac{\partial u'}{\partial x'}\lambda^{1-n} - u'\lambda = 0 \\ x'\frac{\partial u'}{\partial t'}\lambda^{-m+n} + t'\frac{\partial u'}{\partial x'}\lambda^{m-n} - u' = 0 \end{aligned}$$

There is a similarity variable if λ can be eliminated from the equation. Equating the coefficients of the powers of λ in each term,

$$-m + n = m - n = 0.$$

This has the solution $m = n$. The similarity variable, ξ, will be unchanged under the transformation to the temporary variables. One choice is

$$\xi = \frac{t}{x} = \frac{t'\lambda^n}{x'\lambda^m} = \frac{t'}{x'}.$$

Writing the two partial derivative in terms of ξ,

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\mathrm{d}}{\mathrm{d}\xi} = \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}\xi} \\ \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\mathrm{d}}{\mathrm{d}\xi} = -\frac{t}{x^2}\frac{\mathrm{d}}{\mathrm{d}\xi} \end{aligned}$$

The partial differential equation becomes

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} - \xi^2 \frac{\mathrm{d}u}{\mathrm{d}\xi} - u = 0$$
$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = \frac{u}{1-\xi^2}$$

Thus we have reduced the partial differential equation to an ordinary differential equation that is much easier to solve.

$$u(\xi) = \exp\left(\int^{\xi} \frac{d\xi}{1-\xi^2}\right)$$
$$u(\xi) = \exp\left(\int^{\xi} \frac{1/2}{1-\xi} + \frac{1/2}{1+\xi}\, d\xi\right)$$
$$u(\xi) = \exp\left(-\frac{1}{2}\log(1-\xi) + \frac{1}{2}\log(1+\xi)\right)$$
$$u(\xi) = (1-\xi)^{-1/2}(1+\xi)^{1/2}$$
$$u(x,t) = \left(\frac{1+t/x}{1-t/x}\right)^{1/2}$$

Thus we have found a similarity solution to the partial differential equation. Note that the existence of a similarity solution does not mean that all solutions of the differential equation are similarity solutions.

Another Method. Another method is to substitute $\xi = x^\alpha t$ and determine if there is an α that makes ξ a similarity variable. The partial derivatives become

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\mathrm{d}}{\mathrm{d}\xi} = x^\alpha \frac{\mathrm{d}}{\mathrm{d}\xi}$$
$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\mathrm{d}}{\mathrm{d}\xi} = \alpha x^{\alpha-1} t \frac{\mathrm{d}}{\mathrm{d}\xi}$$

The partial differential equation becomes

$$x^{\alpha+1}\frac{\mathrm{d}u}{\mathrm{d}\xi} + \alpha x^{\alpha-1}t^2\frac{\mathrm{d}u}{\mathrm{d}\xi} - u = 0.$$

If there is a value of α such that we can write this equation in terms of ξ, then $\xi = x^\alpha t$ is a similarity variable. If $\alpha = -1$ then the coefficient of the first term is trivially in terms of ξ. The coefficient of the second term then becomes $-x^{-2}t^2$. Thus we see $\xi = x^{-1}t$ is a similarity variable.

Example 43.0.2 To see another application of similarity variables, any partial differential equation of the form

$$F\left(tx, u, \frac{u_t}{x}, \frac{u_x}{t}\right) = 0$$

is equivalent to the ODE

$$F\left(\xi, u, \frac{\mathrm{d}u}{\mathrm{d}\xi}, \frac{\mathrm{d}u}{\mathrm{d}\xi}\right) = 0$$

where $\xi = tx$. Performing the change of variables,

$$\begin{aligned}
\frac{1}{x}\frac{\partial u}{\partial t} &= \frac{1}{x}\frac{\partial \xi}{\partial t}\frac{\mathrm{d}u}{\mathrm{d}\xi} = \frac{1}{x}x\frac{\mathrm{d}u}{\mathrm{d}\xi} = \frac{\mathrm{d}u}{\mathrm{d}\xi} \\
\frac{1}{t}\frac{\partial u}{\partial x} &= \frac{1}{t}\frac{\partial \xi}{\partial x}\frac{\mathrm{d}u}{\mathrm{d}\xi} = \frac{1}{t}t\frac{\mathrm{d}u}{\mathrm{d}\xi} = \frac{\mathrm{d}u}{\mathrm{d}\xi}.
\end{aligned}$$

For example the partial differential equation

$$u\frac{\partial u}{\partial t} + \frac{x}{t}\frac{\partial u}{\partial x} + tx^2 u = 0$$

which can be rewritten

$$u\frac{1}{x}\frac{\partial u}{\partial t} + \frac{1}{t}\frac{\partial u}{\partial x} + txu = 0,$$

is equivalent to

$$u\frac{\mathrm{d}u}{\mathrm{d}\xi} + \frac{\mathrm{d}u}{\mathrm{d}\xi} + \xi u = 0$$

where $\xi = tx$.

43.1 Exercises

Exercise 43.1
With $\xi = x^\alpha t$, find α such that for some function f, $\phi = f(\xi)$ is a solution of

$$\phi_t = a^2 \phi_{xx}.$$

Find $f(\xi)$ as well.

43.2 Hints

Hint 43.1

43.3 Solutions

Solution 43.1

We write the derivatives of ϕ in terms of f.

$$\begin{gathered}
\phi_t = \frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} f = x^\alpha f' = t^{-1} \xi f' \\
\phi_x = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} f = \alpha x^{\alpha-1} t f' \\
\phi_{xx} = f' \frac{\partial}{\partial x} \left(\alpha x^{\alpha-1} t \right) + \alpha x^{\alpha-1} t \alpha x^{\alpha-1} t \frac{\partial}{\partial \xi} f' \\
\phi_{xx} = \alpha^2 x^{2\alpha-2} t^2 f'' + \alpha(\alpha-1) x^{\alpha-2} t f' \\
\phi_{xx} = x^{-2} \left(\alpha^2 \xi^2 f'' + \alpha(\alpha-1) \xi f' \right)
\end{gathered}$$

We substitute these expressions into the diffusion equation.

$$\xi f' = x^{-2} t \left(\alpha^2 \xi^2 f'' + \alpha(\alpha-1) \xi f' \right)$$

In order for this equation to depend only on the variable ξ, we must have $\alpha = -2$. For this choice we obtain an ordinary differential equation for $f(\xi)$.

$$f' = 4\xi^2 f'' + 6\xi f'$$
$$\frac{f''}{f'} = \frac{1}{4\xi^2} - \frac{3}{2\xi}$$
$$\log(f') = -\frac{1}{4\xi} - \frac{3}{2}\log\xi + c$$
$$f' = c_1 \xi^{-3/2}\,\mathrm{e}^{-1/(4\xi)}$$
$$f(\xi) = c_1 \int^{\xi} t^{-3/2}\,\mathrm{e}^{-1/(4t)}\,dt + c_2$$
$$f(\xi) = c_1 \int^{1/(2\sqrt{\xi})} \mathrm{e}^{-t^2}\,dt + c_2$$
$$\boxed{f(\xi) = c_1 \operatorname{erf}\left(\frac{1}{2\sqrt{\xi}}\right) + c_2}$$

Chapter 44

Method of Characteristics

44.1 The Method of Characteristics and the Wave Equation

Consider the one dimensional wave equation

$$u_{tt} = c^2 u_{xx}.$$

With the change of variables, $v = u_x$, $w = u_t$, we have the system of equations,

$$\begin{aligned} v_t - w_x &= 0, \\ w_t - c^2 v_x &= 0. \end{aligned}$$

We can write this as the matrix equation,

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x = 0.$$

The eigenvalues and eigenvectors of the matrix are

$$\lambda_1 = -c, \quad \lambda_2 = c, \qquad \phi_1 = \begin{pmatrix} 1 \\ c \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 1 \\ -c \end{pmatrix}.$$

The matrix is diagonalized by the similarity transformation,

$$\begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix}.$$

We make the change of variables

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix} \begin{pmatrix} \nu \\ \omega \end{pmatrix}.$$

The partial differential equation becomes

$$\begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix} \begin{pmatrix} \nu \\ \omega \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix} \begin{pmatrix} \nu \\ \omega \end{pmatrix}_x = 0.$$

Now we left multiply by the inverse of the matrix of eigenvectors to obtain

$$\begin{pmatrix} \nu \\ \omega \end{pmatrix}_t + \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} \nu \\ \omega \end{pmatrix}_x = 0.$$

This is two un-coupled partial differential equations of first order with solutions

$$\nu(x,t) = p(x+ct), \quad \omega(x,t) = q(x-ct),$$

where $p, q \in C^2$ are arbitrary functions. Changing variables back to v and w,

$$v(x,t) = p(x+ct) + q(x-ct), \qquad w(x,t) = cp(x+ct) - cq(x-ct).$$

Since $v = u_x$, $w = u_t$, we have

$$\boxed{u = f(x+ct) + g(x-ct).}$$

where $f, g \in C^2$ are arbitrary functions. This is the general solution of the one-dimensional wave equation. Note that for any given problem, f and g are only determined to whithin an additive constant. For any constant k, adding k to f and subtracting it from g does not change the solution.

$$u = (f(x+ct) + k) + (g(x-ct) - k).$$

44.2 The Method of Characteristics for an Infinite Domain

Consider the problem

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty,\ t > 0$$
$$u(x, 0) = p(x), \qquad u_t(x, 0) = q(x).$$

We know that the solution has the form

$$u(x, t) = f(x + ct) + g(x - ct). \tag{44.1}$$

The initial conditions give us the two equations

$$f(x) + g(x) = p(x), \quad cf'(x) - cg'(x) = q(x).$$

We integrate the second equation.

$$f(x) - g(x) = \frac{1}{c}Q(x) + 2k$$

Here $Q(x) = \int q(x)\,\mathrm{d}x$ and k is an arbitrary constant. We solve the system of equations for f and g.

$$f(x) = \frac{1}{2}p(x) + \frac{1}{2c}Q(x) + k, \quad g(x) = \frac{1}{2}p(x) - \frac{1}{2c}Q(x) - k$$

Note that the value of k does not affect the solution, $u(x, t)$. For simplicity we take $k = 0$. We substitute f and g into Equation 44.1 to determine the solution.

$$u(x, t) = \frac{1}{2}\left(p(x + ct) + p(x - ct)\right) + \frac{1}{2c}\left(Q(x + ct) - Q(x - ct)\right)$$
$$u(x, t) = \frac{1}{2}\left(p(x + ct) + p(x - ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} q(\xi)\,\mathrm{d}\xi$$
$$\boxed{u(x, t) = \frac{1}{2}\left(u(x + ct, 0) + u(x - ct, 0)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(\xi, 0)\,\mathrm{d}\xi}$$

44.3 The Method of Characteristics for a Semi-Infinite Domain

Consider the problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x < \infty, \ t > 0$$

$$u(x,0) = f(x), \qquad \frac{\partial u(x,0)}{\partial t} = 0, \qquad u(0,t) = h(t).$$

We assume that $f(0) = h(0) = u_0$. Following the previous example we see that

$$G(\xi) = \frac{1}{2} f(\xi) + k, \quad \text{for } \xi > 0$$

$$F(\xi) = \frac{1}{2} f(\xi) - k, \quad \text{for } \xi > 0$$

The boundary condition yields

$$F(-ct) + G(ct) = h(t), \quad \text{for } t > 0$$

$$F(\xi) + G(-\xi) = h(-\xi/c), \quad \text{for } \xi < 0$$

$$F(\xi) = h(-\xi/c) - \frac{1}{2} f(-\xi) - k, \quad \text{for } \xi < 0.$$

Since $u(0,0) = F(0) + G(0) = f(0) = h(0) = u_0$,

$$F(\xi) = h(-\xi/c) - \frac{1}{2} f(-\xi) + \frac{1}{2} u_0, \quad \text{for } \xi < 0.$$

Now F and G are

$$F(\xi) = \begin{cases} \frac{1}{2} f(\xi) + \frac{1}{2} u_0, & \text{for } \xi > 0 \\ h(-\xi/c) - \frac{1}{2} f(-\xi) + \frac{1}{2} u_0, & \text{for } \xi < 0 \end{cases}$$

$$G(\xi) = \frac{1}{2} f(\xi) - \frac{1}{2} u_0, \quad \text{for } \xi > 0.$$

Thus the solution is

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}\left[f(x-ct) + f(x+ct)\right], & \text{for } x - ct > 0 \\ h(t - x/c) - \frac{1}{2}f(ct-x) + \frac{1}{2}u_0, & \text{for } x - ct < 0. \end{cases}}$$

44.4 Envelopes of Curves

Consider the tangent lines to the parabola $y = x^2$. The slope of the tangent at the point (x, x^2) is $2x$. The set of tangents form a one parameter family of lines,

$$f(x,t) = t^2 + (x-t)2t = 2tx - t^2.$$

The parabola and some of its tangents are plotted in Figure 44.1.

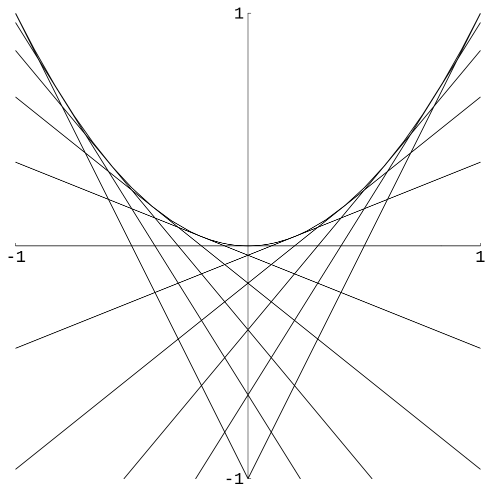

Figure 44.1: A parabola and its tangents.

The parabola is the *envelope* of the family of tangent lines. Each point on the parabola is tangent to one of the lines. Given a curve, we can generate a family of lines that envelope the curve. We can also do the opposite, given a family of lines, we can determine the curve that they envelope. More generally, given a family of curves, we can determine the curve that they envelope. Let the one parameter family of curves be given by the equation $F(x, y, t) = 0$. For the example of the tangents to the parabola this equation would be $y - 2tx + t^2 = 0$.

Let $y(x)$ be the envelope of $F(x, y, t) = 0$. Then the points on $y(x)$ must lie on the family of curves. Thus $y(x)$ must satisfy the equation $F(x, y, t) = 0$. The points that lie on the envelope have the property,

$$\frac{\partial}{\partial t} F(x, y, t) = 0.$$

We can solve this equation for t in terms of x and y, $t = t(x, y)$. The equation for the envelope is then

$$F(x, y, t(x, y)) = 0.$$

Consider the example of the tangents to the parabola. The equation of the one-parameter family of curves is

$$F(x, y, t) \equiv y - 2tx + t^2 = 0.$$

The condition $F_t(x, y, t) = 0$ gives us the constraint,

$$-2x + 2t = 0.$$

Solving this for t gives us $t(x, y) = x$. The equation for the envelope is then,

$$y - 2xx + x^2 = 0,$$

$$y = x^2.$$

Example 44.4.1 Consider the one parameter family of curves,

$$(x - t)^2 + (y - t)^2 - 1 = 0.$$

These are circles of unit radius and center (t, t). To determine the envelope of the family, we first use the constraint $F_t(x, y, t)$ to solve for $t(x, y)$.

$$F_t(x, y, t) = -2(x - t) - 2(y - t) = 0$$

$$t(x, y) = \frac{x + y}{2}$$

Now we substitute this into the equation $F(x, y, t) = 0$ to determine the envelope.

$$F\left(x, y, \frac{x + y}{2}\right) = \left(x - \frac{x + y}{2}\right)^2 + \left(y - \frac{x + y}{2}\right)^2 - 1 = 0$$

$$\left(\frac{x - y}{2}\right)^2 + \left(\frac{y - x}{2}\right)^2 - 1 = 0$$

$$(x - y)^2 = 2$$

$$\boxed{y = x \pm \sqrt{2}}$$

The one parameter family of curves and its envelope is shown in Figure 44.2.

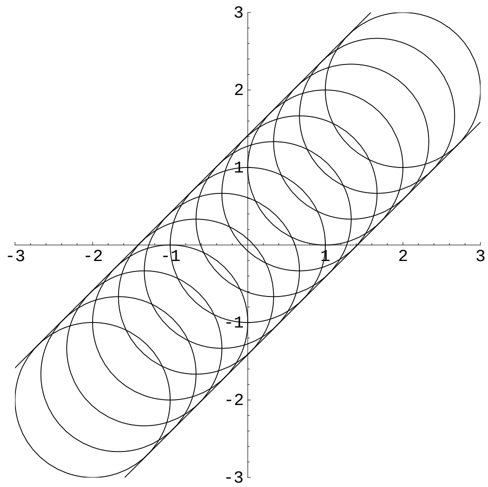

Figure 44.2: The envelope of $(x-t)^2 + (y-t)^2 - 1 = 0$.

44.5 Exercises

Exercise 44.1
Consider a semi-infinite string, $x > 0$. For all time the end of the string is displaced according to $u(0,t) = f(t)$. Find the motion of the string, $u(x,t)$ with the method of characteristics and then with a Fourier transform in time. The wave speed is c.

Exercise 44.2
Solve using characteristics:

$$uu_x + u_y = 1, \quad u\big|_{x=y} = \frac{x}{2}.$$

Exercise 44.3
Solve using characteristics:

$$(y+u)u_x + yu_y = x - y, \quad u\Big|_{y=1} = 1 + x.$$

44.6 Hints

Hint 44.1

Hint 44.2

Hint 44.3

44.7 Solutions

Solution 44.1

Method of characteristics. The problem is

$$u_{tt} - c^2 u_{xx} = 0, \quad x > 0, \quad -\infty < t < \infty,$$
$$u(0,t) = f(t).$$

By the method of characteristics, we know that the solution has the form,

$$u(x,t) = F(x - ct).$$

That is, it is a wave moving to the right with speed c. Substituting this into the boundary condition yields,

$$F(-ct) = f(t)$$
$$F(\xi) = f\left(-\frac{\xi}{c}\right)$$

Now we can write the solution.

$$\boxed{u(x,t) = f(t - x/c)}$$

Fourier transform. We take the Fourier transform in time of the wave equation and the boundary condition.

$$u_{tt} = c^2 u_{xx}, \quad u(0,t) = f(t)$$
$$-\omega^2 \hat{u} = c^2 \hat{u}_{xx}, \quad \hat{u}(0,\omega) = \hat{f}(\omega)$$
$$\hat{u}_{xx} + \frac{\omega^2}{c^2}\hat{u} = 0, \quad \hat{u}(0,\omega) = \hat{f}(\omega)$$

The general solution of this ordinary differential equation is

$$\hat{u}(x,\omega) = a(\omega)\,\mathrm{e}^{i\omega x/c} + b(\omega)\,\mathrm{e}^{-i\omega x/c}.$$

The radiation condition, ($u(x,t)$ must be a wave traveling in the positive direction), and the boundary condition at $x = 0$ will determine the constants a and b. u is the inverse Fourier transform of $\hat{u}$.

$$u(x,t) = \int_{-\infty}^{\infty} \left(a(\omega)\, \mathrm{e}^{i\omega x/c} + b(\omega)\, \mathrm{e}^{-i\omega x/c} \right) \mathrm{e}^{i\omega t}\, \mathrm{d}\omega$$
$$u(x,t) = \int_{-\infty}^{\infty} \left(a(\omega)\, \mathrm{e}^{i\omega(t+x/c)} + b(\omega)\, \mathrm{e}^{i\omega(t-x/c)} \right) \mathrm{d}\omega$$

The first and second terms in the integrand are left and right traveling waves, respectively. In order that u is a right traveling wave, it must be a superposition of right traveling waves. We conclude that $a(\omega) = 0$. Applying the boundary condition at $x = 0$, we solve for $\hat{u}$.

$$\hat{u}(x,\omega) = \hat{f}(\omega)\, \mathrm{e}^{-i\omega x/c}$$

Finally we take the inverse Fourier transform.

$$u(x,t) = \int_{-\infty}^{\infty} \hat{f}(\omega)\, \mathrm{e}^{i\omega(t-x/c)}\, \mathrm{d}\omega$$
$$\boxed{u(x,t) = f(t - x/c)}$$

Solution 44.2

$$u u_x + u_y = 1, \quad u\Big|_{x=y} = \frac{x}{2} \tag{44.2}$$

We form $\frac{\mathrm{d}u}{\mathrm{d}y}$.

$$\frac{\mathrm{d}u}{\mathrm{d}y} = u_x \frac{\mathrm{d}x}{\mathrm{d}y} + u_y$$

We compare this with Equation 44.2 to obtain differential equations for x and u.

$$\frac{\mathrm{d}x}{\mathrm{d}y} = u, \quad \frac{\mathrm{d}u}{\mathrm{d}y} = 1. \tag{44.3}$$

The initial data is

$$x(y = \alpha) = \alpha, \quad u(y = \alpha) = \frac{\alpha}{2}. \tag{44.4}$$

We solve the differenial equation for u (44.3) subject to the initial condition (44.4).

$$u(x(y), y) = y - \frac{\alpha}{2}$$

The differential equation for x becomes

$$\frac{\mathrm{d}x}{\mathrm{d}y} = y - \frac{\alpha}{2}.$$

We solve this subject to the initial condition (44.4).

$$x(y) = \frac{1}{2}(y^2 + \alpha(2 - y))$$

This defines the characteristic starting at the point (α, α). We solve for α.

$$\alpha = \frac{y^2 - 2x}{y - 2}$$

We substitute this value for α into the solution for u.

$$u(x, y) = \frac{y(y - 4) + 2x}{2(y - 2)}$$

This solution is defined for $y \neq 2$. This is because at $(x, y) = (2, 2)$, the characteristic is parallel to the line $x = y$. Figure 44.3 has a plot of the solution that shows the singularity at $y = 2$.

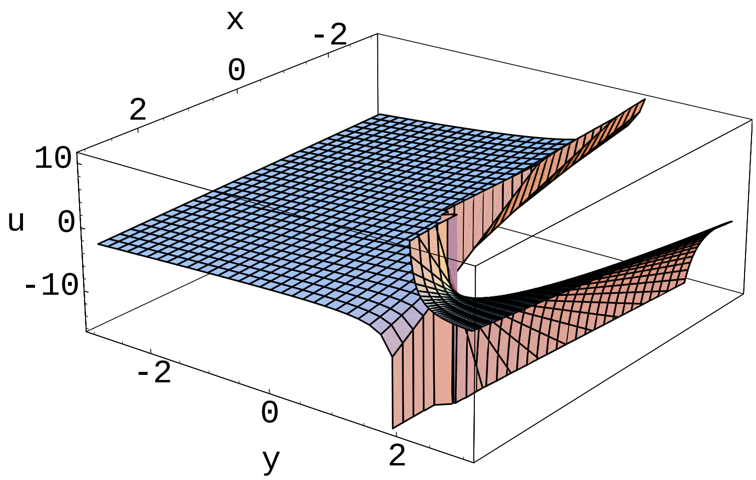

Figure 44.3: The solution $u(x, y)$.

Solution 44.3

$$(y + u)u_x + yu_y = x - y, \quad u\big|_{y=1} = 1 + x \tag{44.5}$$

We differentiate u with respect to s.

$$\frac{\mathrm{d}u}{\mathrm{d}s} = u_x \frac{\mathrm{d}x}{\mathrm{d}s} + u_y \frac{\mathrm{d}y}{\mathrm{d}s}$$

We compare this with Equation 44.5 to obtain differential equations for x, y and u.

$$\frac{\mathrm{d}x}{\mathrm{d}s} = y + u, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = y, \quad \frac{\mathrm{d}u}{\mathrm{d}s} = x - y$$

We parametrize the initial data in terms of s.

$$x(s = 0) = \alpha, \quad y(s = 0) = 1, \quad u(s = 0) = 1 + \alpha$$

We solve the equation for y subject to the inital condition.

$$y(s) = \mathrm{e}^s$$

This gives us a coupled set of differential equations for x and u.

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \mathrm{e}^s + u, \quad \frac{\mathrm{d}u}{\mathrm{d}s} = x - \mathrm{e}^s$$

The solutions subject to the initial conditions are

$$x(s) = (\alpha + 1)\,\mathrm{e}^s - \mathrm{e}^{-s}, \quad u(s) = \alpha\,\mathrm{e}^s + \mathrm{e}^{-s}.$$

We substitute $y(s) = \mathrm{e}^s$ into these solutions.

$$x(s) = (\alpha + 1)y - \frac{1}{y}, \quad u(s) = \alpha y + \frac{1}{y}$$

We solve the first equation for α and substitute it into the second equation to obtain the solution.

$$u(x, y) = \frac{2 + xy - y^2}{y}$$

This solution is valid for $y > 0$. The characteristic passing through $(\alpha, 1)$ is

$$x(s) = (\alpha + 1)\,\mathrm{e}^s - \mathrm{e}^{-s}, \quad y(s) = \mathrm{e}^s.$$

Hence we see that the characteristics satisfy $y(s) \geq 0$ for all real s. Figure 44.4 shows some characteristics in the (x, y) plane with starting points from $(-5, 1)$ to $(5, 1)$ and a plot of the solution.

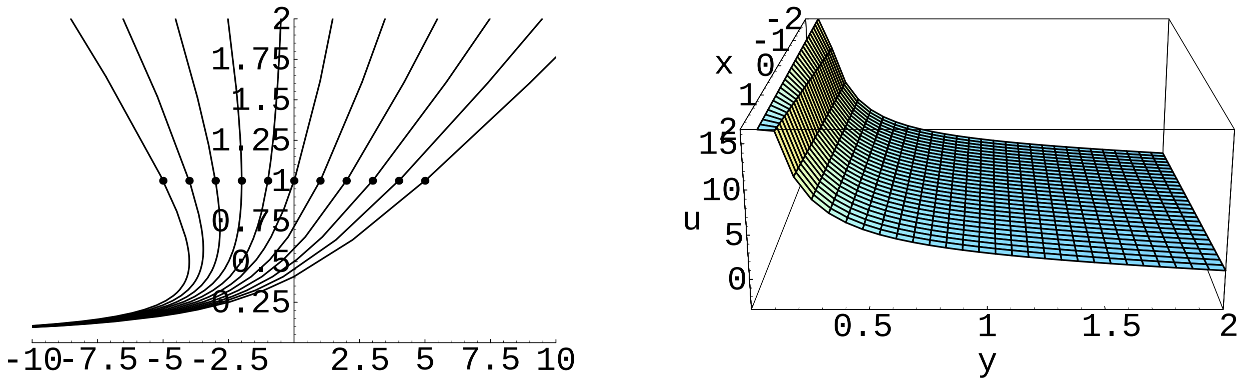

Figure 44.4: Some characteristics and the solution $u(x, y)$.

Chapter 45

Transform Methods

45.1 Fourier Transform for Partial Differential Equations

Solve Laplace's equation in the upper half plane

$$\begin{aligned} &\nabla^2 u = 0 && -\infty < x < \infty,\ y > 0 \\ &u(x, 0) = f(x) && -\infty < x < \infty \end{aligned}$$

Taking the Fourier transform in the x variable of the equation and the boundary condition,

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = 0, \qquad \mathcal{F}[u(x, 0)] = \mathcal{F}[f(x)]$$

$$-\omega^2 U(\omega, y) + \frac{\partial^2}{\partial y^2} U(\omega, y) = 0, \qquad U(\omega, 0) = F(\omega).$$

The general solution to the equation is

$$U(\omega, y) = a\,\mathrm{e}^{\omega y} + b\,\mathrm{e}^{-\omega y}.$$

Remember that in solving the differential equation here we consider ω to be a parameter. Requiring that the solution be bounded for $y \in [0, \infty)$ yields

$$U(\omega, y) = a\,\mathrm{e}^{-|\omega| y}.$$

Applying the boundary condition,

$$U(\omega, y) = F(\omega)\,\mathrm{e}^{-|\omega| y}.$$

The inverse Fourier transform of $\mathrm{e}^{-|\omega| y}$ is

$$\mathcal{F}^{-1}\left[\mathrm{e}^{-|\omega| y}\right] = \frac{2y}{x^2 + y^2}.$$

Thus

$$U(\omega, y) = F(\omega)\ \mathcal{F}\left[\frac{2y}{x^2 + y^2}\right]$$
$$\mathcal{F}\left[u(x, y)\right] = \mathcal{F}\left[f(x)\right]\ \mathcal{F}\left[\frac{2y}{x^2 + y^2}\right].$$

Recall that the convolution theorem is

$$\mathcal{F}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x - \xi) g(\xi)\, d\xi\right] = F(\omega) G(\omega).$$

Applying the convolution theorem to the equation for U,

$$u(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(x - \xi) 2y}{\xi^2 + y^2}\, d\xi$$

$$\boxed{u(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(x - \xi)}{\xi^2 + y^2}\, d\xi.}$$

45.2 The Fourier Sine Transform

Consider the problem

$$u_t = \kappa u_{xx}, \quad x > 0, \quad t > 0$$
$$u(0,t) = 0, \quad u(x,0) = f(x)$$

Since we are given the position at $x = 0$ we apply the Fourier sine transform.

$$\hat{u}_t = \kappa \left(-\omega^2 \hat{u} + \frac{2}{\pi} \omega u(0,t) \right)$$
$$\hat{u}_t = -\kappa \omega^2 \hat{u}$$
$$\hat{u}(\omega, t) = c(\omega) \, \mathrm{e}^{-\kappa \omega^2 t}$$

The initial condition is

$$\hat{u}(\omega, 0) = \hat{f}(\omega).$$

We solve the first order differential equation to determine $\hat{u}$.

$$\hat{u}(\omega, t) = \hat{f}(\omega) \, \mathrm{e}^{-\kappa \omega^2 t}$$
$$\hat{u}(\omega, t) = \hat{f}(\omega) \mathcal{F}_c \left[\frac{1}{\sqrt{4\pi\kappa t}} \, \mathrm{e}^{-x^2/(4\kappa t)} \right]$$

We take the inverse sine transform with the convolution theorem.

$$\boxed{u(x,t) = \frac{1}{4\pi^{3/2}\sqrt{\kappa t}} \int_0^\infty f(\xi) \left(\mathrm{e}^{-|x-\xi|^2/(4\kappa t)} - \mathrm{e}^{-(x+\xi)^2/(4\kappa t)} \right) d\xi}$$

45.3 Fourier Transform

Consider the problem

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + u = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = f(x).$$

Taking the Fourier Transform of the partial differential equation and the initial condition yields

$$\frac{\partial U}{\partial t} - i\omega U + U = 0,$$

$$U(\omega, 0) = F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-i\omega x}\, dx.$$

Now we have a first order differential equation for $U(\omega, t)$ with the solution

$$U(\omega, t) = F(\omega)\, \mathrm{e}^{(-1+i\omega)t}.$$

Now we apply the inverse Fourier transform.

$$u(x, t) = \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{(-1+i\omega)t}\, \mathrm{e}^{i\omega x}\, d\omega$$

$$u(x, t) = \mathrm{e}^{-t} \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{i\omega(x+t)}\, d\omega$$

$$\boxed{u(x, t) = \mathrm{e}^{-t} f(x + t)}$$

45.4 Exercises

Exercise 45.1
Find an integral representation of the solution $u(x, y)$, of

$$u_{xx} + u_{yy} = 0 \text{ in } -\infty < x < \infty,\ 0 < y < \infty,$$

subject to the boundary conditions:

$$u(x, 0) = f(x),\ -\infty < x < \infty;$$
$$u(x, y) \to 0 \text{ as } x^2 + y^2 \to \infty.$$

Exercise 45.2
Solve the Cauchy problem for the one-dimensional heat equation in the domain $-\infty < x < \infty$, $t > 0$,

$$u_t = \kappa u_{xx}, \quad u(x, 0) = f(x),$$

with the Fourier transform.

Exercise 45.3
Let $\phi(x, t)$ satisfy the equation

$$\phi_t = a^2 \phi_{xx}, \tag{45.1}$$

for $-\infty < x < \infty$, $t > 0$ with initial conditions $\phi(x, 0) = f(x)$ in $-\infty < x < \infty$. Boundary conditions cannot be given here because both endpoints are infinite. In this case all we can ask is that the solution be regular as $x \to \pm\infty$. Show that the Laplace transform of $\phi(x, t)$ is given by

$$\Phi(x, s) = \frac{1}{2a\sqrt{s}} \int_{-\infty}^{\infty} f(\xi) \exp\left(-\frac{\sqrt{s}}{a} |x - \xi| \right) d\xi,$$

and hence deduce that

$$\phi(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) \exp\left(-\frac{(x - \xi)^2}{4a^2 t} \right) d\xi.$$

Exercise 45.4

1. In Exercise 45.3 above, let $f(-x) = -f(x)$ for all x and verify that $\phi(x,t)$ so obtained is the solution, for $x > 0$, of the following problem: find $\phi(x,t)$ satisfying

$$\phi_t = a^2 \phi_{xx}$$

in $0 < x < \infty$, $t > 0$, with boundary condition $\phi(0,t) = 0$ and initial condition $\phi(x,0) = f(x)$. This technique, in which the solution for a semi-infinite interval is obtained from that for an infinite interval, is an example of what is called the *method of images*.

2. How would you modify the result of part (a) if the boundary condition $\phi(0,t) = 0$ was replaced by $\phi_x(0,t) = 0$?

Exercise 45.5
Solve the Cauchy problem for the one-dimensional wave equation in the domain $-\infty < x < \infty$, $t > 0$,

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x),$$

with the Fourier transform.

Exercise 45.6
Solve the Cauchy problem for the one-dimensional wave equation in the domain $-\infty < x < \infty$, $t > 0$,

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x),$$

with the Laplace transform.

Exercise 45.7
Consider the problem of determining $\phi(x,t)$ in the region $0 < x < \infty$, $0 < t < \infty$, such that

$$\phi_t = a^2 \phi_{xx}, \tag{45.2}$$

with initial and boundary conditions

$$\phi(x,0) = 0 \qquad \text{for all } x > 0,$$
$$\phi(0,t) = f(t) \quad \text{for all } t > 0,$$

where $f(t)$ is a given function.

1. Obtain the formula for the Laplace transform of $\phi(x,t)$, $\Phi(x,s)$ and use the convolution theorem for Laplace transforms to show that

$$\phi(x,t) = \frac{x}{2a\sqrt{\pi}} \int_0^t f(t-\tau) \frac{1}{\tau^{3/2}} \exp\left(-\frac{x^2}{4a^2\tau}\right) d\tau.$$

2. Discuss the special case obtained by setting $f(t) = 1$ and also that in which $f(t) = 1$ for $0 < t < T$, with $f(t) = 0$ for $t > T$. Here T is some positive constant.

Exercise 45.8
Solve the radiating half space problem:

$$u_t = \kappa u_{xx}, \quad x > 0, \quad t > 0,$$
$$u_x(0,t) - \alpha u(0,t) = 0, \quad u(x,0) = f(x).$$

To do this, define

$$v(x,t) = u_x(x,t) - \alpha u(x,t)$$

and find the half space problem that v satisfies. Solve this problem and then show that

$$u(x,t) = -\int_x^\infty \mathrm{e}^{-\alpha(\xi - x)} v(\xi,t)\, d\xi.$$

Exercise 45.9
Show that

$$\int_0^\infty \omega\, \mathrm{e}^{-c\omega^2} \sin(\omega x)\, d\omega = \frac{x\sqrt{\pi}}{4c^{3/2}}\, \mathrm{e}^{-x^2/(4c)}.$$

Use the sine transform to solve:

$$u_t = u_{xx}, \quad x > 0, \quad t > 0,$$
$$u(0,t) = g(t), \quad u(x,0) = 0.$$

Exercise 45.10
Use the Fourier sine transform to find the steady state temperature $u(x,y)$ in a slab: $x \geq 0$, $0 \leq y \leq 1$, which has zero temperature on the faces $y = 0$ and $y = 1$ and has a given distribution: $u(y,0) = f(y)$ on the edge $x = 0$, $0 \leq y \leq 1$.

Exercise 45.11
Find a harmonic function $u(x,y)$ in the upper half plane which takes on the value $g(x)$ on the x-axis. Assume that u and u_x vanish as $|x| \to \infty$. Use the Fourier transform with respect to x. Express the solution as a single integral by using the convolution formula.

Exercise 45.12
Find the bounded solution of

$$u_t = \kappa u_{xx} - a^2 u, \quad 0 < x < \infty, t > 0,$$
$$-u_x(0,t) = f(t), \quad u(x,0) = 0.$$

Exercise 45.13
The left end of a taut string of length L is displaced according to $u(0,t) = f(t)$. The right end is fixed, $u(L,t) = 0$. Initially the string is at rest with no displacement. If c is the wave speed for the string, find it's motion for all $t > 0$.

Exercise 45.14
Let $\nabla^2\phi = 0$ in the (x, y)-plane region defined by $0 < y < l$, $-\infty < x < \infty$, with $\phi(x, 0) = \delta(x - \xi)$, $\phi(x, l) = 0$, and $\phi \to 0$ as $|x| \to \infty$. Solve for ϕ using Fourier transforms. You may leave your answer in the form of an integral but in fact it is possible to use techniques of contour integration to show that

$$\phi(x, y|\xi) = \frac{1}{2l}\left[\frac{\sin(\pi y/l)}{\cosh[\pi(x - \xi)/l] - \cos(\pi y/l)}\right].$$

Note that as $l \to \infty$ we recover the result derived in class:

$$\phi \to \frac{1}{\pi}\frac{y}{(x - \xi)^2 + y^2},$$

which clearly approaches $\delta(x - \xi)$ as $y \to 0$.

45.5 Hints

Hint 45.1
The desired solution form is: $u(x,y) = \int_{-\infty}^{\infty} K(x-\xi, y) f(\xi)\, d\xi$. You must find the correct K. Take the Fourier transform with respect to x and solve for $\hat{u}(\omega, y)$ recalling that $\hat{u}_{xx} = -\omega^2 \hat{u}$. By $\hat{u}_{xx}$ we denote the Fourier transform with respect to x of $u_{xx}(x,y)$.

Hint 45.2
Use the Fourier convolution theorem and the table of Fourier transforms in the appendix.

Hint 45.3

Hint 45.4

Hint 45.5
Use the Fourier convolution theorem. The transform pairs,

$$\mathcal{F}[\pi(\delta(x+\tau) + \delta(x-\tau))] = \cos(\omega\tau),$$
$$\mathcal{F}[\pi(H(x+\tau) - H(x-\tau))] = \frac{\sin(\omega\tau)}{\omega},$$

will be useful.

Hint 45.6

Hint 45.7

Hint 45.8
$v(x,t)$ satisfies the same partial differential equation. You can solve the problem for $v(x,t)$ with the Fourier sine transform. Use the convolution theorem to invert the transform.

To show that

$$u(x,t) = -\int_x^\infty \mathrm{e}^{-\alpha(\xi - x)} v(\xi, t)\, d\xi,$$

find the solution of

$$u_x - \alpha u = v$$

that is bounded as $x \to \infty$.

Hint 45.9
Note that

$$\int_0^\infty \omega\, \mathrm{e}^{-c\omega^2} \sin(\omega x)\, d\omega = -\frac{\partial}{\partial x} \int_0^\infty \mathrm{e}^{-c\omega^2} \cos(\omega x)\, d\omega.$$

Write the integral as a Fourier transform.

Take the Fourier sine transform of the heat equation to obtain a first order, ordinary differential equation for $\hat{u}(\omega, t)$. Solve the differential equation and do the inversion with the convolution theorem.

Hint 45.10

Hint 45.11

Hint 45.12

Hint 45.13

Hint 45.14

45.6 Solutions

Solution 45.1

1. We take the Fourier transform of the integral equation, noting that the left side is the convolution of $u(x)$ and $\frac{1}{x^2+a^2}$.

$$2\pi \hat{u}(\omega) \mathcal{F}\left[\frac{1}{x^2+a^2}\right] = \mathcal{F}\left[\frac{1}{x^2+b^2}\right]$$

We find the Fourier transform of $f(x) = \frac{1}{x^2+c^2}$. Note that since $f(x)$ is an even, real-valued function, $\hat{f}(\omega)$ is an even, real-valued function.

$$\mathcal{F}\left[\frac{1}{x^2+c^2}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+c^2} \,\mathrm{e}^{-\imath\omega x}\, dx$$

For $x > 0$ we close the path of integration in the upper half plane and apply Jordan's Lemma to evaluate the integral in terms of the residues.

$$\begin{aligned}
&= \frac{1}{2\pi}\imath 2\pi \operatorname{Res}\left(\frac{\mathrm{e}^{-\imath\omega x}}{(x-\imath c)(x+\imath c)}, x = \imath c\right) \\
&= \imath \frac{\mathrm{e}^{-\imath\omega\imath c}}{2\imath c} \\
&= \frac{1}{2c}\,\mathrm{e}^{-c\omega}
\end{aligned}$$

Since $\hat{f}(\omega)$ is an even function, we have

$$\mathcal{F}\left[\frac{1}{x^2+c^2}\right] = \frac{1}{2c}\,\mathrm{e}^{-c|\omega|}.$$

Our equation for $\hat{u}(\omega)$ becomes,

$$2\pi\hat{u}(\omega)\frac{1}{2a}\,\mathrm{e}^{-a|\omega|} = \frac{1}{2b}\,\mathrm{e}^{-b|\omega|}$$
$$\hat{u}(\omega) = \frac{a}{2\pi b}\,\mathrm{e}^{-(b-a)|omega|}.$$

We take the inverse Fourier transform using the transform pair we derived above.

$$u(x) = \frac{a}{2\pi b}\frac{2(b-a)}{x^2+(b-a)^2}$$
$$\boxed{u(x) = \frac{a(b-a)}{\pi b(x^2+(b-a)^2)}}$$

2. We take the Fourier transform of the partial differential equation and the boundary condtion.

$$u_{xx} + u_{yy} = 0, \quad u(x,0) = f(x)$$
$$-\omega^2\hat{u}(\omega,y) + \hat{u}_{yy}(\omega,y) = 0, \quad \hat{u}(\omega,0) = \hat{f}(\omega)$$

This is an ordinary differential equation for $\hat{u}$ in which ω is a parameter. The general solution is

$$\hat{u} = c_1\,\mathrm{e}^{\omega y} + c_2\,\mathrm{e}^{-\omega y}.$$

We apply the boundary conditions that $\hat{u}(\omega,0) = \hat{f}(\omega)$ and $\hat{u} \to 0$ and $y \to \infty$.

$$\hat{u}(\omega,y) = \hat{f}(\omega)\,\mathrm{e}^{-\omega y}$$

We take the inverse transform using the convolution theorem.

$$\boxed{u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-(x-\xi)y} f(\xi)\,d\xi}$$

Solution 45.2

$$u_t = \kappa u_{xx}, \quad u(x,0) = f(x),$$

We take the Fourier transform of the heat equation and the initial condition.

$$\hat{u}_t = -\kappa\omega^2\hat{u}, \quad \hat{u}(\omega,0) = \hat{f}(\omega)$$

This is a first order ordinary differential equation which has the solution,

$$\hat{u}(\omega,t) = \hat{f}(\omega)\,\mathrm{e}^{-\kappa\omega^2 t}.$$

Using a table of Fourier transforms we can write this in a form that is conducive to applying the convolution theorem.

$$\hat{u}(\omega,t) = \hat{f}(\omega)\mathcal{F}\left[\sqrt{\frac{\pi}{\kappa t}}\,\mathrm{e}^{-x^2/(4\kappa t)}\right]$$

$$\boxed{u(x,t) = \sqrt{\frac{\pi}{\kappa t}}\int_{-\infty}^{\infty}\mathrm{e}^{-(x-\xi)^2/(4\kappa t)}f(\xi)\,d\xi}$$

Solution 45.3
We take the Laplace transform of Equation 45.1.

$$s\hat{\phi} - \phi(x,0) = a^2\hat{\phi}_{xx}$$

$$\hat{\phi}_{xx} - \frac{s}{a^2}\hat{\phi} = -\frac{f(x)}{a^2} \qquad (45.3)$$

The Green function problem for Equation 45.3 is

$$G'' - \frac{s}{a^2}G = \delta(x-\xi), \quad G(\pm\infty;\xi) \text{ is bounded.}$$

The homogeneous solutions that satisfy the left and right boundary conditions are, respectively,

$$\exp\left(\frac{\sqrt{s}a}{x}\right), \quad \exp\left(-\frac{\sqrt{s}a}{x}\right).$$

We compute the Wronskian of these solutions.

$$W = \begin{vmatrix} \exp\left(\frac{\sqrt{s}}{a}x\right) & \exp\left(-\frac{\sqrt{s}}{a}x\right) \\ \frac{\sqrt{s}}{a}\exp\left(\frac{\sqrt{s}a}{x}\right) & -\frac{\sqrt{s}}{a}\exp\left(-\frac{\sqrt{s}a}{x}\right) \end{vmatrix} = -\frac{2\sqrt{s}}{a}$$

The Green function is

$$G(x;\xi) = \frac{\exp\left(\frac{\sqrt{s}}{a}x_<\right)\exp\left(-\frac{\sqrt{s}}{a}x_>\right)}{-\frac{2\sqrt{s}}{a}}$$

$$G(x;\xi) = -\frac{a}{2\sqrt{s}}\exp\left(-\frac{\sqrt{s}}{a}|x-\xi|\right).$$

Now we solve Equation 45.3 using the Green function.

$$\hat{\phi}(x,s) = \int_{-\infty}^{\infty} -\frac{f(\xi)}{a^2}G(x;\xi)\,d\xi$$

$$\hat{\phi}(x,s) = \frac{1}{2a\sqrt{s}}\int_{-\infty}^{\infty} f(\xi)\exp\left(-\frac{\sqrt{s}}{a}|x-\xi|\right)d\xi$$

Finally we take the inverse Laplace transform to obtain the solution of Equation 45.1.

$$\boxed{\phi(x,t) = \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty} f(\xi)\exp\left(-\frac{(x-\xi)^2}{4a^2t}\right)d\xi}$$

Solution 45.4

1. Clearly the solution satisfies the differential equation. We must verify that it satisfies the boundary condition, $\phi(0,t) = 0$.

$$\phi(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right) d\xi$$

$$\phi(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{0} f(\xi) \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right) d\xi + \frac{1}{2a\sqrt{\pi t}} \int_{0}^{\infty} f(\xi) \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right) d\xi$$

$$\phi(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{0}^{\infty} f(-\xi) \exp\left(-\frac{(x+\xi)^2}{4a^2 t}\right) d\xi + \frac{1}{2a\sqrt{\pi t}} \int_{0}^{\infty} f(\xi) \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right) d\xi$$

$$\phi(x,t) = -\frac{1}{2a\sqrt{\pi t}} \int_{0}^{\infty} f(\xi) \exp\left(-\frac{(x+\xi)^2}{4a^2 t}\right) d\xi + \frac{1}{2a\sqrt{\pi t}} \int_{0}^{\infty} f(\xi) \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right) d\xi$$

$$\phi(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{0}^{\infty} f(\xi) \left(\exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right) \exp\left(-\frac{(x+\xi)^2}{4a^2 t}\right)\right) d\xi$$

$$\phi(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{0}^{\infty} f(\xi) \exp\left(-\frac{x^2+\xi^2}{4a^2 t}\right) \left(\exp\left(\frac{x\xi}{2a^2 t}\right) - \exp\left(-\frac{x\xi}{2a^2 t}\right)\right) d\xi$$

$$\boxed{\phi(x,t) = \frac{1}{a\sqrt{\pi t}} \int_{0}^{\infty} f(\xi) \exp\left(-\frac{x^2+\xi^2}{4a^2 t}\right) \sinh\left(\frac{x\xi}{2a^2 t}\right) d\xi}$$

Since the integrand is zero for $x = 0$, the solution satisfies the boundary condition there.

2. For the boundary condition $\phi_x(0,t) = 0$ we would choose $f(x)$ to be even. $f(-x) = f(x)$. The solution is

$$\boxed{\phi(x,t) = \frac{1}{a\sqrt{\pi t}} \int_{0}^{\infty} f(\xi) \exp\left(-\frac{x^2+\xi^2}{4a^2 t}\right) \cosh\left(\frac{x\xi}{2a^2 t}\right) d\xi}$$

The derivative with respect to x is

$$\phi_x(x,t) = \frac{1}{2a^3\sqrt{\pi} t^{3/2}} \int_{0}^{\infty} f(\xi) \exp\left(-\frac{x^2+\xi^2}{4a^2 t}\right) \left(\xi \sinh\left(\frac{x\xi}{2a^2 t}\right) - x \cosh\left(\frac{x\xi}{2a^2 t}\right)\right) d\xi.$$

Since the integrand is zero for $x = 0$, the solution satisfies the boundary condition there.

Solution 45.5
With the change of variables

$$\tau = ct, \quad \frac{\partial}{\partial \tau} = \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}, \quad v(x,\tau) = u(x,t),$$

the problem becomes

$$v_{\tau\tau} = v_{xx}, \quad v(x,0) = f(x), \quad v_\tau(x,0) = \frac{1}{c}g(x).$$

(This change of variables isn't necessary, it just gives us fewer constants to carry around.) We take the Fourier transform in x of the equation and the initial conditions, (we consider τ to be a parameter),

$$\hat{v}_{\tau\tau}(\omega,\tau) = -\omega^2 \hat{v}(\omega,\tau), \quad \hat{v}(\omega,\tau) = \hat{f}(\omega), \quad \hat{v}_\tau(\omega,\tau) = \frac{1}{c}\hat{g}(\omega).$$

Now we have an ordinary differential equation for $\hat{v}(\omega,\tau)$, (now we consider ω to be a parameter). The general solution of this constant coefficient differential equation is,

$$\hat{v}(\omega,\tau) = a(\omega)\cos(\omega\tau) + b(\omega)\sin(\omega\tau),$$

where a and b are constants that depend on the parameter ω. Applying the initial conditions, we see that

$$\hat{v}(\omega,\tau) = \hat{f}(\omega)\cos(\omega\tau) + \frac{1}{c\omega}\hat{g}(\omega)\sin(\omega\tau),$$

With the Fourier transform pairs

$$\mathcal{F}[\pi(\delta(x+\tau) + \delta(x-\tau))] = \cos(\omega\tau),$$
$$\mathcal{F}[\pi(H(x+\tau) - H(x-\tau))] = \frac{\sin(\omega\tau)}{\omega},$$

we can write $\hat{v}(\omega, \tau)$ in a form that is conducive to applying the Fourier convolution theorem.

$$\hat{v}(\omega, \tau) = \mathcal{F}[f(x)]\mathcal{F}[\pi(\delta(x+\tau) + \delta(x-\tau))] + \frac{1}{c}\mathcal{F}[g(x)]\mathcal{F}[\pi(H(x+\tau) - H(x-\tau))]$$

$$v(x, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\pi(\delta(x-\xi+\tau) + \delta(x-\xi-\tau))\, d\xi \\ + \frac{1}{c}\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\pi(H(x-\xi+\tau) - H(x-\xi-\tau))\, d\xi$$

$$v(x, \tau) = \frac{1}{2}(f(x+\tau) + f(x-\tau)) + \frac{1}{2c}\int_{x-\tau}^{x+\tau} g(\xi)\, d\xi$$

Finally we make the change of variables $t = \tau/c$, $u(x, t) = v(x, \tau)$ to obtain D'Alembert's solution of the wave equation,

$$\boxed{u(x, t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi.}$$

Solution 45.6
With the change of variables

$$\tau = ct, \quad \frac{\partial}{\partial \tau} = \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}, \quad v(x, \tau) = u(x, t),$$

the problem becomes

$$v_{\tau\tau} = v_{xx}, \quad v(x, 0) = f(x), \quad v_\tau(x, 0) = \frac{1}{c}g(x).$$

We take the Laplace transform in τ of the equation, (we consider x to be a parameter),

$$s^2 V(x,s) - s v(x,0) - v_\tau(x,0) = V_{xx}(x,s),$$

$$V_{xx}(x,s) - s^2 V(x,s) = -s f(x) - \frac{1}{c} g(x),$$

Now we have an ordinary differential equation for $V(x,s)$, (now we consider s to be a parameter). We impose the boundary conditions that the solution is bounded at $x = \pm\infty$. Consider the Green's function problem

$$g_{xx}(x;\xi) - s^2 g(x;\xi) = \delta(x-\xi), \quad g(\pm\infty;\xi) \text{ bounded.}$$

e^{sx} is a homogeneous solution that is bounded at $x = -\infty$. e^{-sx} is a homogeneous solution that is bounded at $x = +\infty$. The Wronskian of these solutions is

$$W(x) = \begin{vmatrix} \mathrm{e}^{sx} & \mathrm{e}^{-sx} \\ s\,\mathrm{e}^{sx} & -s\,\mathrm{e}^{-sx} \end{vmatrix} = -2s.$$

Thus the Green's function is

$$g(x;\xi) = \begin{cases} -\frac{1}{2s}\,\mathrm{e}^{sx}\,\mathrm{e}^{-s\xi} & \text{for } x < \xi, \\ -\frac{1}{2s}\,\mathrm{e}^{s\xi}\,\mathrm{e}^{-sx} & \text{for } x > \xi, \end{cases} = -\frac{1}{2s}\,\mathrm{e}^{-s|x-\xi|}.$$

The solution for $V(x,s)$ is

$$V(x,s) = -\frac{1}{2s} \int_{-\infty}^{\infty} \mathrm{e}^{-s|x-\xi|}\left(-s f(\xi) - \frac{1}{c} g(\xi)\right) d\xi,$$

$$V(x,s) = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{e}^{-s|x-\xi|} f(\xi)\, d\xi + \frac{1}{2cs} \int_{-\infty}^{\infty} \mathrm{e}^{-s|x-\xi|} g(\xi))\, d\xi,$$

$$V(x,s) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-s|\xi|} f(x-\xi)\,d\xi + \frac{1}{2c}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-s|\xi|}}{s} g(x-\xi))\,d\xi.$$

Now we take the inverse Laplace transform and interchange the order of integration.

$$v(x,\tau) = \frac{1}{2}\mathcal{L}^{-1}\left[\int_{-\infty}^{\infty} \mathrm{e}^{-s|\xi|} f(x-\xi)\,d\xi\right] + \frac{1}{2c}\mathcal{L}^{-1}\left[\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-s|\xi|}}{s} g(x-\xi))\,d\xi\right]$$

$$v(x,\tau) = \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{L}^{-1}\left[\mathrm{e}^{-s|\xi|}\right] f(x-\xi)\,d\xi + \frac{1}{2c}\int_{-\infty}^{\infty} \mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-s|\xi|}}{s}\right] g(x-\xi))\,d\xi$$

$$v(x,\tau) = \frac{1}{2}\int_{-\infty}^{\infty} \delta(\tau-|\xi|) f(x-\xi)\,d\xi + \frac{1}{2c}\int_{-\infty}^{\infty} H(\tau-|\xi|) g(x-\xi))\,d\xi$$

$$v(x,\tau) = \frac{1}{2}(f(x-\tau)+f(x+\tau)) + \frac{1}{2c}\int_{-\tau}^{\tau} g(x-\xi)\,d\xi$$

$$v(x,\tau) = \frac{1}{2}(f(x-\tau)+f(x+\tau)) + \frac{1}{2c}\int_{-x-\tau}^{-x+\tau} g(-\xi)\,d\xi$$

$$v(x,\tau) = \frac{1}{2}(f(x-\tau)+f(x+\tau)) + \frac{1}{2c}\int_{x-\tau}^{x+\tau} g(\xi)\,d\xi$$

Now we write make the change of variables $t = \tau/c$, $u(x,t) = v(x,\tau)$ to obtain D'Alembert's solution of the wave equation,

$$\boxed{u(x,t) = \frac{1}{2}(f(x-ct)+f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi.}$$

Solution 45.7

1. We take the Laplace transform of Equation 45.2.

$$s\hat{\phi} - \phi(x, 0) = a^2 \hat{\phi}_{xx}$$
$$\hat{\phi}_{xx} - \frac{s}{a^2}\hat{\phi} = 0 \tag{45.4}$$

We take the Laplace transform of the initial condition, $\phi(0, t) = f(t)$, and use that $\hat{\phi}(x, s)$ vanishes as $x \to \infty$ to obtain boundary conditions for $\hat{\phi}(x, s)$.

$$\hat{\phi}(0, s) = \hat{f}(s), \quad \hat{\phi}(\infty, s) = 0$$

The solutions of Equation 45.4 are

$$\exp\left(\pm\frac{\sqrt{s}}{a}x\right).$$

The solution that satisfies the boundary conditions is

$$\hat{\phi}(x, s) = \hat{f}(s) \exp\left(-\frac{\sqrt{s}}{a}x\right).$$

We write this as the product of two Laplace transforms.

$$\hat{\phi}(x, s) = \hat{f}(s)\mathcal{L}\left[\frac{x}{2a\sqrt{\pi}t^{3/2}} \exp\left(-\frac{x^2}{4a^2 t}\right)\right]$$

We invert using the convolution theorem.

$$\boxed{\phi(x, t) = \frac{x}{2a\sqrt{\pi}} \int_0^t f(t - \tau)\frac{1}{\tau^{3/2}} \exp\left(-\frac{x^2}{4a^2\tau}\right) d\tau.}$$

2. Consider the case $f(t) = 1$.

$$\phi(x,t) = \frac{x}{2a\sqrt{\pi}} \int_0^t \frac{1}{\tau^{3/2}} \exp\left(-\frac{x^2}{4a^2\tau}\right) d\tau$$

$$\xi = \frac{x}{2a\sqrt{\tau}}, \quad d\xi = -\frac{x}{4a\tau^{3/2}}$$

$$\phi(x,t) = -\frac{2}{\sqrt{\pi}} \int_\infty^{x/(2a\sqrt{t})} \mathrm{e}^{-\xi^2}\, d\xi$$

$$\boxed{\phi(x,t) = \operatorname{erfc}\left(\frac{x}{2a\sqrt{t}}\right)}$$

Now consider the case in which $f(t) = 1$ for $0 < t < T$, with $f(t) = 0$ for $t > T$. For $t < T$, ϕ is the same as before.

$$\phi(x,t) = \operatorname{erfc}\left(\frac{x}{2a\sqrt{t}}\right), \quad \text{for } 0 < t < T$$

Consider $t > T$.

$$\phi(x,t) = \frac{x}{2a\sqrt{\pi}} \int_{t-T}^t \frac{1}{\tau^{3/2}} \exp\left(-\frac{x^2}{4a^2\tau}\right) d\tau$$

$$\phi(x,t) = -\frac{2}{\sqrt{\pi}} \int_{x/(2a\sqrt{t-T})}^{x/(2a\sqrt{t})} \mathrm{e}^{-\xi^2}\, d\xi$$

$$\boxed{\phi(x,t) = \operatorname{erf}\left(\frac{x}{2a\sqrt{t-T}}\right) - \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right)}$$

Solution 45.8

$$u_t = \kappa u_{xx}, \quad x > 0, \quad t > 0,$$
$$u_x(0,t) - \alpha u(0,t) = 0, \quad u(x,0) = f(x).$$

First we find the partial differential equation that v satisfies. We start with the partial differential equation for u,

$$u_t = \kappa u_{xx}.$$

Differentiating this equation with respect to x yields,

$$u_{tx} = \kappa u_{xxx}.$$

Subtracting α times the former equation from the latter yields,

$$u_{tx} - \alpha u_t = \kappa u_{xxx} - \alpha\kappa u_{xx},$$
$$\frac{\partial}{\partial t}(u_x - \alpha u) = \kappa \frac{\partial^2}{\partial x^2}(u_x - \alpha u),$$
$$v_t = \kappa v_{xx}.$$

Thus v satisfies the same partial differential equation as u. This is because the equation for u is linear and homogeneous and v is a linear combination of u and its derivatives. The problem for v is,

$$v_t = \kappa v_{xx}, \quad x > 0, \quad t > 0,$$
$$v(0,t) = 0, \quad v(x,0) = f'(x) - \alpha f(x).$$

With this new boundary condition, we can solve the problem with the Fourier sine transform. We take the sine transform of the partial differential equation and the initial condition.

$$\hat{v}_t(\omega,t) = \kappa\left(-\omega^2\hat{v}(\omega,t) + \frac{1}{\pi}\omega v(0,t)\right),$$
$$\hat{v}(\omega,0) = \mathcal{F}_s\left[f'(x) - \alpha f(x)\right]$$

$$\hat{v}_t(\omega,t) = -\kappa\omega^2\hat{v}(\omega,t)$$
$$\hat{v}(\omega,0) = \mathcal{F}_s\left[f'(x) - \alpha f(x)\right]$$

Now we have a first order, ordinary differential equation for $\hat{v}$. The general solution is,

$$\hat{v}(\omega, t) = c\,\mathrm{e}^{-\kappa\omega^2 t}.$$

The solution subject to the initial condition is,

$$\hat{v}(\omega, t) = \mathcal{F}_s\left[f'(x) - \alpha f(x)\right] \mathrm{e}^{-\kappa\omega^2 t}.$$

Now we take the inverse sine transform to find v. We utilize the Fourier cosine transform pair,

$$\mathcal{F}_c^{-1}\left[\mathrm{e}^{-\kappa\omega^2 t}\right] = \sqrt{\frac{\pi}{\kappa t}}\,\mathrm{e}^{-x^2/(4\kappa t)},$$

to write $\hat{v}$ in a form that is suitable for the convolution theorem.

$$\hat{v}(\omega, t) = \mathcal{F}_s\left[f'(x) - \alpha f(x)\right] \mathcal{F}_c\left[\sqrt{\frac{\pi}{\kappa t}}\,\mathrm{e}^{-x^2/(4\kappa t)}\right]$$

Recall that the Fourier sine convolution theorem is,

$$\mathcal{F}_s\left[\frac{1}{2\pi}\int_0^\infty f(\xi)\left(g(|x-\xi|) - g(x+\xi)\right) d\xi\right] = \mathcal{F}_s[f(x)]\mathcal{F}_c[g(x)].$$

Thus $v(x, t)$ is

$$\boxed{v(x, t) = \frac{1}{2\sqrt{\pi\kappa t}}\int_0^\infty \left(f'(\xi) - \alpha f(\xi)\right)\left(\mathrm{e}^{-|x-\xi|^2/(4\kappa t)} - \mathrm{e}^{-(x+\xi)^2/(4\kappa t)}\right) d\xi.}$$

With v determined, we have a first order, ordinary differential equation for u,

$$u_x - \alpha u = v.$$

We solve this equation by multiplying by the integrating factor and integrating.

$$\frac{\partial}{\partial x}\left(\mathrm{e}^{-\alpha x} u\right) = \mathrm{e}^{-\alpha x} v$$

$$\mathrm{e}^{-\alpha x} u = \int^x \mathrm{e}^{-\alpha \xi} v(x,t)\, d\xi + c(t)$$

$$u = \int^x \mathrm{e}^{-\alpha(\xi - x)} v(x,t)\, d\xi + \mathrm{e}^{\alpha x} c(t)$$

The solution that vanishes as $x \to \infty$ is

$$\boxed{u(x,t) = -\int_x^\infty \mathrm{e}^{-\alpha(\xi - x)} v(\xi,t)\, d\xi.}$$

Solution 45.9

$$\begin{aligned}
\int_0^\infty \omega\, \mathrm{e}^{-c\omega^2} \sin(\omega x)\, d\omega &= -\frac{\partial}{\partial x} \int_0^\infty e^{-c\omega^2} \cos(\omega x)\, d\omega \\
&= -\frac{1}{2}\frac{\partial}{\partial x} \int_{-\infty}^\infty e^{-c\omega^2} \cos(\omega x)\, d\omega \\
&= -\frac{1}{2}\frac{\partial}{\partial x} \int_{-\infty}^\infty e^{-c\omega^2 + i\omega x}\, d\omega \\
&= -\frac{1}{2}\frac{\partial}{\partial x} \int_{-\infty}^\infty e^{-c(\omega + ix/(2c))^2}\, \mathrm{e}^{-x^2/(4c)}\, d\omega \\
&= -\frac{1}{2}\frac{\partial}{\partial x} \mathrm{e}^{-x^2/(4c)} \int_{-\infty}^\infty e^{-c\omega^2}\, d\omega \\
&= -\frac{1}{2}\sqrt{\frac{\pi}{c}} \frac{\partial}{\partial x} \mathrm{e}^{-x^2/(4c)} \\
&= \frac{x\sqrt{\pi}}{4c^{3/2}}\, \mathrm{e}^{-x^2/(4c)}
\end{aligned}$$

$$u_t = u_{xx}, \quad x > 0, \quad t > 0,$$
$$u(0,t) = g(t), \quad u(x,0) = 0.$$

We take the Fourier sine transform of the partial differential equation and the initial condition.

$$\hat{u}_t(\omega,t) = -\omega^2 \hat{u}(\omega,t) + \frac{\omega}{\pi} g(t), \qquad \hat{u}(\omega,0) = 0$$

Now we have a first order, ordinary differential equation for $\hat{u}(\omega,t)$.

$$\frac{\partial}{\partial t}\left(\mathrm{e}^{\omega^2 t}\hat{u}_t(\omega,t)\right) = \frac{\omega}{\pi} g(t)\,\mathrm{e}^{\omega^2 t}$$
$$\hat{u}(\omega,t) = \frac{\omega}{\pi}\,\mathrm{e}^{-\omega^2 t}\int_0^t g(\tau)\,\mathrm{e}^{\omega^2 \tau}\,d\tau + c(\omega)\,\mathrm{e}^{-\omega^2 t}$$

The initial condition is satisfied for $c(\omega) = 0$.

$$\hat{u}(\omega,t) = \frac{\omega}{\pi}\int_0^t g(\tau)\,\mathrm{e}^{-\omega^2(t-\tau)}\,d\tau$$

We take the inverse sine transform to find u.

$$u(x,t) = \mathcal{F}_s^{-1}\left[\frac{\omega}{\pi}\int_0^t g(\tau)\,\mathrm{e}^{-\omega^2(t-\tau)}\,d\tau\right]$$

$$u(x,t) = \int_0^t g(\tau)\mathcal{F}_s^{-1}\left[\frac{\omega}{\pi}\,\mathrm{e}^{-\omega^2(t-\tau)}\right]\,d\tau$$

$$u(x,t) = \int_0^t g(\tau)\frac{x}{2\sqrt{\pi}(t-\tau)^{3/2}}\,\mathrm{e}^{-x^2/(4(t-\tau))}\,d\tau$$

$$\boxed{u(x,t) = \frac{x}{2\sqrt{\pi}}\int_0^t g(\tau)\frac{\mathrm{e}^{-x^2/(4(t-\tau))}}{(t-\tau)^{3/2}}\,d\tau}$$

Solution 45.10
The problem is

$$u_{xx} + u_{yy} = 0, \quad 0 < x, 0 < y < 1,$$
$$u(x,0) = u(x,1) = 0, \quad u(0,y) = f(y).$$

We take the Fourier sine transform of the partial differential equation and the boundary conditions.

$$-\omega^2 \hat{u}(\omega, y) + \frac{k}{\pi} u(0,y) + \hat{u}_{yy}(\omega, y) = 0$$
$$\hat{u}_{yy}(\omega, y) - \omega^2 \hat{u}(\omega, y) = -\frac{k}{\pi} f(y), \quad \hat{u}(\omega, 0) = \hat{u}(\omega, 1) = 0$$

This is an inhomogeneous, ordinary differential equation that we can solve with Green functions. The homogeneous solutions are

$$\{\cosh(\omega y), \sinh(\omega y)\}.$$

The homogeneous solutions that satisfy the left and right boundary conditions are

$$y_1 = \sinh(\omega y), \quad y_2 = \sinh(\omega(y-1)).$$

The Wronskian of these two solutions is,

$$\begin{aligned} W(x) &= \begin{vmatrix} \sinh(\omega y) & \sinh(\omega(y-1)) \\ \omega \cosh(\omega y) & \omega \cosh(\omega(y-1)) \end{vmatrix} \\ &= \omega \left(\sinh(\omega y) \cosh(\omega(y-1)) - \cosh(\omega y) \sinh(\omega(y-1)) \right) \\ &= \omega \sinh(\omega). \end{aligned}$$

The Green function is

$$G(y|\eta) = \frac{\sinh(\omega y_<) \sinh(\omega(y_> - 1))}{\omega \sinh(\omega)}.$$

The solution of the ordinary differential equation for $\hat{u}(\omega, y)$ is

$$\begin{aligned}\hat{u}(\omega, y) &= -\frac{\omega}{\pi} \int_0^1 f(\eta) G(y|\eta)\, d\eta \\ &= -\frac{1}{\pi} \int_0^y f(\eta) \frac{\sinh(\omega\eta)\sinh(\omega(y-1))}{\sinh(\omega)}\, d\eta - \frac{1}{\pi} \int_y^1 f(\eta) \frac{\sinh(\omega y)\sinh(\omega(\eta-1))}{\sinh(\omega)}\, d\eta.\end{aligned}$$

With some uninteresting grunge, you can show that,

$$2 \int_0^\infty \frac{\sinh(\omega\eta)\sinh(\omega(y-1))}{\sinh(\omega)} \sin(\omega x)\, d\omega = -2 \frac{\sin(\pi\eta)\sin(\pi y)}{(\cosh(\pi x) - \cos(\pi(y-\eta)))(\cosh(\pi x) - \cos(\pi(y+\eta)))}.$$

Taking the inverse Fourier sine transform of $\hat{u}(\omega, y)$ and interchanging the order of integration yields,

$$\begin{aligned}u(x, y) = \frac{2}{\pi} \int_0^y f(\eta) &\frac{\sin(\pi\eta)\sin(\pi y)}{(\cosh(\pi x) - \cos(\pi(y-\eta)))(\cosh(\pi x) - \cos(\pi(y+\eta)))}\, d\eta \\ &+ \frac{2}{\pi} \int_y^1 f(\eta) \frac{\sin(\pi y)\sin(\pi\eta)}{(\cosh(\pi x) - \cos(\pi(\eta-y)))(\cosh(\pi x) - \cos(\pi(\eta+y)))}\, d\eta.\end{aligned}$$

$$\boxed{u(x, y) = \frac{2}{\pi} \int_0^1 f(\eta) \frac{\sin(\pi\eta)\sin(\pi y)}{(\cosh(\pi x) - \cos(\pi(y-\eta)))(\cosh(\pi x) - \cos(\pi(y+\eta)))}\, d\eta}$$

Solution 45.11

The problem for $u(x, y)$ is,

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty,\, y > 0,$$
$$u(x, 0) = g(x).$$

We take the Fourier transform of the partial differential equation and the boundary condition.

$$-\omega^2 \hat{u}(\omega, y) + \hat{u}_{yy}(\omega, y) = 0, \quad \hat{u}(\omega, 0) = \hat{g}(\omega).$$

This is an ordinary differential equation for $\hat{u}(\omega, y)$. So far we only have one boundary condition. In order that u is bounded we impose the second boundary condition $\hat{u}(\omega, y)$ is bounded as $y \to \infty$. The general solution of the differential equation is

$$\hat{u}(\omega, y) = \begin{cases} c_1(\omega)\,\mathrm{e}^{\omega y} + c_2(\omega)\,\mathrm{e}^{-\omega y}, & \text{for } \omega \neq 0, \\ c_1(\omega) + c_2(\omega) y, & \text{for } \omega = 0. \end{cases}$$

Note that $\mathrm{e}^{\omega y}$ is the bounded solution for $\omega < 0$, 1 is the bounded solution for $\omega = 0$ and $\mathrm{e}^{-\omega y}$ is the bounded solution for $\omega > 0$. Thus the bounded solution is

$$\hat{u}(\omega, y) = c(\omega)\,\mathrm{e}^{-|\omega| y}.$$

The boundary condition at $y = 0$ determines the constant of integration.

$$\hat{u}(\omega, y) = \hat{g}(\omega)\,\mathrm{e}^{-|\omega| y}$$

Now we take the inverse Fourier transform to obtain the solution for $u(x, y)$. To do this we use the Fourier transform pair,

$$\mathcal{F}\left[\frac{2c}{x^2 + c^2}\right] = \mathrm{e}^{-c|\omega|},$$

and the convolution theorem,

$$\mathcal{F}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi) g(x - \xi)\, d\xi\right] = \hat{f}(\omega)\hat{g}(\omega).$$

$$\boxed{u(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\frac{2y}{(x - \xi)^2 + y^2}\, d\xi.}$$

Solution 45.12

Since the derivative of u is specified at $x = 0$, we take the cosine transform of the partial differential equation and the initial condition.

$$\hat{u}_t(\omega, t) = \kappa\left(-\omega^2\hat{u}(\omega, t) - \frac{1}{\pi}u_x(0, t)\right) - a^2\hat{u}(\omega, t), \quad \hat{u}(\omega, 0) = 0$$

$$\hat{u}_t + \left(\kappa\omega^2 + a^2\right)\hat{u} = \frac{\kappa}{\pi}f(t), \quad \hat{u}(\omega, 0) = 0$$

This first order, ordinary differential equation for $\hat{u}(\omega, t)$ has the solution,

$$\hat{u}(\omega, t) = \frac{\kappa}{\pi}\int_0^t \mathrm{e}^{-(\kappa\omega^2+a^2)(t-\tau)} f(\tau)\, d\tau.$$

We take the inverse Fourier cosine transform to find the solution $u(x, t)$.

$$u(x, t) = \frac{\kappa}{\pi}\mathcal{F}_c^{-1}\left[\int_0^t \mathrm{e}^{-(\kappa\omega^2+a^2)(t-\tau)} f(\tau)\, d\tau\right]$$

$$u(x, t) = \frac{\kappa}{\pi}\int_0^t \mathcal{F}_c^{-1}\left[\mathrm{e}^{-\kappa\omega^2(t-\tau)}\right] \mathrm{e}^{-a^2(t-\tau)} f(\tau)\, d\tau$$

$$u(x, t) = \frac{\kappa}{\pi}\int_0^t \sqrt{\frac{\pi}{\kappa(t-\tau)}}\, \mathrm{e}^{-x^2/(4\kappa(t-\tau))}\, \mathrm{e}^{-a^2(t-\tau)} f(\tau)\, d\tau$$

$$\boxed{u(x, t) = \sqrt{\frac{\kappa}{\pi}}\int_0^t \frac{\mathrm{e}^{-x^2/(4\kappa(t-\tau))-a^2(t-\tau)}}{\sqrt{t-\tau}} f(\tau)\, d\tau}$$

Solution 45.13

Mathematically stated we have

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$$
$$u(x, 0) = u_t(x, 0) = 0,$$
$$u(0, t) = f(t), \quad u(L, t) = 0.$$

We take the Laplace transform of the partial differential equation and the boundary conditions.

$$s^2\hat{u}(x,s) - su(x,0) - u_t(x,0) = c^2\hat{u}_{xx}(x,s)$$
$$\hat{u}_{xx} = \frac{s^2}{c^2}\hat{u}, \quad \hat{u}(0,s) = \hat{f}(s), \quad \hat{u}(L,s) = 0$$

Now we have an ordinary differential equation. A set of solutions is

$$\left\{\cosh\left(\frac{sx}{c}\right), \sinh\left(\frac{sx}{c}\right)\right\}.$$

The solution that satisfies the right boundary condition is

$$\hat{u} = a\sinh\left(\frac{s(L-x)}{c}\right).$$

The left boundary condition determines the multiplicative constant.

$$\hat{u}(x,s) = \hat{f}(s)\frac{\sinh(s(L-x)/c)}{\sinh(sL/c)}$$

If we can find the inverse Laplace transform of

$$\hat{u}(x,s) = \frac{\sinh(s(L-x)/c)}{\sinh(sL/c)}$$

then we can use the convolution theorem to write u in terms of a single integral. We proceed by expanding this function in a sum.

$$
\begin{aligned}
\frac{\sinh(s(L-x)/c)}{\sinh(sL/c)} &= \frac{\mathrm{e}^{s(L-x)/c} - \mathrm{e}^{-s(L-x)/c}}{\mathrm{e}^{sL/c} - \mathrm{e}^{-sL/c}} \\
&= \frac{\mathrm{e}^{-sx/c} - \mathrm{e}^{-s(2L-x)/c}}{1 - \mathrm{e}^{-2sL/c}} \\
&= \left(\mathrm{e}^{-sx/c} - \mathrm{e}^{-s(2L-x)/c}\right) \sum_{n=0}^{\infty} \mathrm{e}^{-2nsL/c} \\
&= \sum_{n=0}^{\infty} \mathrm{e}^{-s(2nL+x)/c} - \sum_{n=0}^{\infty} \mathrm{e}^{-s(2(n+1)L-x)/c} \\
&= \sum_{n=0}^{\infty} \mathrm{e}^{-s(2nL+x)/c} - \sum_{n=1}^{\infty} \mathrm{e}^{-s(2nL-x)/c}
\end{aligned}
$$

Now we use the Laplace transform pair:

$$\mathcal{L}[\delta(x-a)] = \mathrm{e}^{-sa}.$$

$$\mathcal{L}^{-1}\left[\frac{\sinh(s(L-x)/c)}{\sinh(sL/c)}\right] = \sum_{n=0}^{\infty} \delta(t-(2nL+x)/c) - \sum_{n=1}^{\infty} \delta(t-(2nL-x)/c)$$

We write $\hat{u}$ in the form,

$$\hat{u}(x,s) = \mathcal{L}[f(t)]\mathcal{L}\left[\sum_{n=0}^{\infty} \delta(t-(2nL+x)/c) - \sum_{n=1}^{\infty} \delta(t-(2nL-x)/c)\right].$$

By the convolution theorem we have

$$u(x,t) = \int_0^t f(\tau)\left(\sum_{n=0}^{\infty} \delta(t-\tau-(2nL+x)/c) - \sum_{n=1}^{\infty} \delta(t-\tau-(2nL-x)/c)\right) d\tau.$$

We can simplify this a bit. First we determine which Dirac delta functions have their singularities in the range $\tau \in (0..t)$. For the first sum, this condition is

$$0 < t - (2nL + x)/c < t.$$

The right inequality is always satisfied. The left inequality becomes

$$(2nL + x)/c < t,$$
$$n < \frac{ct - x}{2L}.$$

For the second sum, the condition is

$$0 < t - (2nL - x)/c < t.$$

Again the right inequality is always satisfied. The left inequality becomes

$$n < \frac{ct + x}{2L}.$$

We change the index range to reflect the nonzero contributions and do the integration.

$$u(x,t) = \int_0^t f(\tau) \left(\sum_{n=0}^{\lfloor \frac{ct-x}{2L} \rfloor} \delta(t - \tau - (2nL + x)/c) \sum_{n=1}^{\lfloor \frac{ct+x}{2L} \rfloor} \delta(t - \tau - (2nL - x)/c) \right) d\tau.$$

$$\boxed{u(x,t) = \sum_{n=0}^{\lfloor \frac{ct-x}{2L} \rfloor} f(t - (2nL + x)/c) \sum_{n=1}^{\lfloor \frac{ct+x}{2L} \rfloor} f(t - (2nL - x)/c)}$$

Solution 45.14

We take the Fourier transform of the partial differential equation and the boundary conditions.

$$-\omega^2 \hat{\phi} + \hat{\phi}_{yy} = 0, \quad \hat{\phi}(\omega, 0) = \frac{1}{2\pi} \mathrm{e}^{-i\omega\xi}, \quad \hat{\phi}(\omega, l) = 0$$

We solve this boundary value problem.

$$\hat{\phi}(\omega, y) = c_1 \cosh(\omega(l - y)) + c_2 \sinh(\omega(l - y))$$
$$\hat{\phi}(\omega, y) = \frac{1}{2\pi} \mathrm{e}^{-i\omega\xi} \frac{\sinh(\omega(l - y))}{\sinh(\omega l)}$$

We take the inverse Fourier transform to obtain an expression for the solution.

$$\boxed{\phi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\omega(x-\xi)} \frac{\sinh(\omega(l - y))}{\sinh(\omega l)} \, d\omega}$$

Chapter 46

Green Functions

46.1 Inhomogeneous Equations and Homogeneous Boundary Conditions

Consider a linear differential equation on the domain Ω subject to homogeneous boundary conditions.

$$L[u(\mathbf{x})] = f(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega, \quad B[u(\mathbf{x})] = 0 \quad \text{for } \mathbf{x} \in \partial\Omega \tag{46.1}$$

For example, $L[u]$ might be

$$L[u] = u_t - \kappa \Delta u, \quad \text{or} \quad L[u] = u_t - c^2 \Delta u.$$

and $B[u]$ might be $u = 0$, or $\nabla u \cdot \hat{n} = 0$.

If we find a Green function $G(\mathbf{x}; \boldsymbol{\xi})$ that satisfies

$$L[G(\mathbf{x}; \boldsymbol{\xi})] = \delta(\mathbf{x} - \boldsymbol{\xi}), \quad B[G(\mathbf{x}; \boldsymbol{\xi})] = 0$$

then the solution to Equation 46.1 is

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x};\boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

We verify that this solution satisfies the equation and boundary condition.

$$\begin{aligned}
L[u(\mathbf{x})] &= \int_\Omega L[G(\mathbf{x};\boldsymbol{\xi})] f(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\
&= \int_\Omega \delta(\mathbf{x}-\boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\
&= f(\mathbf{x}) \\
B[u(\mathbf{x})] &= \int_\Omega B[G(\mathbf{x};\boldsymbol{\xi})] f(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\
&= \int_\Omega 0\, f(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\
&= 0
\end{aligned}$$

46.2 Homogeneous Equations and Inhomogeneous Boundary Conditions

Consider a homogeneous linear differential equation on the domain Ω subject to inhomogeneous boundary conditions,

$$L[u(\mathbf{x})] = 0 \quad \text{for } \mathbf{x} \in \Omega, \qquad B[u(\mathbf{x})] = h(\mathbf{x}) \quad \text{for } \mathbf{x} \in \partial\Omega. \tag{46.2}$$

If we find a Green function $g(\mathbf{x};\boldsymbol{\xi})$ that satisfies

$$L[g(\mathbf{x};\boldsymbol{\xi})] = 0, \qquad B[g(\mathbf{x};\boldsymbol{\xi})] = \delta(\mathbf{x}-\boldsymbol{\xi})$$

then the solution to Equation 46.2 is

$$u(\mathbf{x}) = \int_{\partial\Omega} g(\mathbf{x}; \boldsymbol{\xi}) h(\boldsymbol{\xi}) \, d\boldsymbol{\xi}.$$

We verify that this solution satisfies the equation and boundary condition.

$$\begin{aligned} L[u(\mathbf{x})] &= \int_{\partial\Omega} L[g(\mathbf{x}; \boldsymbol{\xi})] h(\boldsymbol{\xi}) \, d\boldsymbol{\xi} \\ &= \int_{\partial\Omega} 0 \, h(\boldsymbol{\xi}) \, d\boldsymbol{\xi} \\ &= 0 \\ B[u(\mathbf{x})] &= \int_{\partial\Omega} B[g(\mathbf{x}; \boldsymbol{\xi})] h(\boldsymbol{\xi}) \, d\boldsymbol{\xi} \\ &= \int_{\partial\Omega} \delta(\mathbf{x} - \boldsymbol{\xi}) h(\boldsymbol{\xi}) \, d\boldsymbol{\xi} \\ &= h(\mathbf{x}) \end{aligned}$$

Example 46.2.1 Consider the Cauchy problem for the homogeneous heat equation.

$$\begin{gathered} u_t = \kappa u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \\ u(x, 0) = h(x), \quad u(\pm\infty, t) = 0 \end{gathered}$$

We find a Green function that satisfies

$$\begin{gathered} g_t = \kappa g_{xx}, \quad -\infty < x < \infty, \quad t > 0 \\ g(x, 0; \xi) = \delta(x - \xi), \quad g(\pm\infty, t; \xi) = 0. \end{gathered}$$

Then we write the solution

$$u(x, t) = \int_{-\infty}^{\infty} g(x, t; \xi) h(\xi) \, d\xi.$$

To find the Green function for this problem, we apply a Fourier transform to the equation and boundary condition for g.

$$\hat{g}_t = -\kappa\omega^2 \hat{g}, \quad \hat{g}(\omega, 0; \xi) = \mathcal{F}[\delta(x - \xi)]$$
$$\hat{g}(\omega, t; \xi) = \mathcal{F}[\delta(x - \xi)]\,\mathrm{e}^{-\kappa\omega^2 t}$$
$$\hat{g}(\omega, t; \xi) = \mathcal{F}[\delta(x - \xi)]\mathcal{F}\left[\sqrt{\frac{\pi}{\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right)\right]$$

We invert using the convolution theorem.

$$g(x, t; \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\eta - \xi) \sqrt{\frac{\pi}{\kappa t}} \exp\left(-\frac{(x - \eta)^2}{4\kappa t}\right) d\eta$$
$$= \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{(x - \xi)^2}{4\kappa t}\right)$$

The solution of the heat equation is

$$\boxed{u(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x - \xi)^2}{4\kappa t}\right) h(\xi)\, d\xi.}$$

46.3 Eigenfunction Expansions for Elliptic Equations

Consider a Green function problem for an elliptic equation on a finite domain.

$$L[G] = \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \mathbf{x} \in \Omega \tag{46.3}$$
$$B[G] = 0, \quad \mathbf{x} \in \partial\Omega$$

Let the set of functions $\{\phi_\mathbf{n}\}$ be orthonormal and complete on Ω. (Here $\mathbf{n}$ is the multi-index $\mathbf{n} = n_1, \ldots, n_d$.)

$$\int_\Omega \overline{\phi_\mathbf{n}(\mathbf{x})} \phi_\mathbf{m}(\mathbf{x})\, d\mathbf{x} = \delta_{\mathbf{nm}}$$

In addition, let the $\phi_{\mathbf{n}}$ be eigenfunctions of L subject to the homogeneous boundary conditions.

$$L\left[\phi_{\mathbf{n}}\right]=\lambda_{\mathbf{n}}\phi_{\mathbf{n}}, \quad B\left[\phi_{\mathbf{n}}\right]=0$$

We expand the Green function in the eigenfunctions.

$$G=\sum_{\mathbf{n}} g_{\mathbf{n}}\phi_{\mathbf{n}}(\mathbf{x})$$

Then we expand the Dirac Delta function.

$$\delta(\mathbf{x}-\boldsymbol{\xi})=\sum_{\mathbf{n}} d_{\mathbf{n}}\phi_{\mathbf{n}}(\mathbf{x})$$
$$d_{\mathbf{n}}=\int_{\Omega}\overline{\phi_{\mathbf{n}}(\mathbf{x})}\delta(\mathbf{x}-\boldsymbol{\xi})\,d\mathbf{x}$$
$$d_{\mathbf{n}}=\overline{\phi_{\mathbf{n}}(\boldsymbol{\xi})}$$

We substitute the series expansions for the Green function and the Dirac Delta function into Equation 46.3.

$$\sum_{\mathbf{n}} g_{\mathbf{n}}\lambda_{\mathbf{n}}\phi_{\mathbf{n}}(\mathbf{x})=\sum_{\mathbf{n}}\overline{\phi_{\mathbf{n}}(\boldsymbol{\xi})}\phi_{\mathbf{n}}(\mathbf{x})$$

We equate coefficients to solve for the $g_{\mathbf{n}}$ and hence determine the Green function.

$$g_{\mathbf{n}}=\frac{\overline{\phi_{\mathbf{n}}(\boldsymbol{\xi})}}{\lambda_{\mathbf{n}}}$$
$$G(\mathbf{x};\boldsymbol{\xi})=\sum_{\mathbf{n}}\frac{\overline{\phi_{\mathbf{n}}(\boldsymbol{\xi})}\phi_{\mathbf{n}}(\mathbf{x})}{\lambda_{\mathbf{n}}}$$

Example 46.3.1 Consider the Green function for the reduced wave equation, $\Delta u - k^2 u$ in the rectangle, $0 \leq x \leq a$, $0 \leq y \leq b$, and vanishing on the sides.

First we find the eigenfunctions of the operator $L = \Delta - k^2 = 0$. Note that $\phi = X(x)Y(y)$ is an eigenfunction of L if X is an eigenfunction of $\frac{\partial^2}{\partial x^2}$ and Y is an eigenfunction of $\frac{\partial^2}{\partial y^2}$. Thus we consider the two regular Sturm-Liouville eigenvalue problems:

$$X'' = \lambda X, \quad X(0) = X(a) = 0$$
$$Y'' = \lambda Y, \quad Y(0) = Y(b) = 0$$

This leads us to the eigenfunctions

$$\phi_{mn} = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

We use the orthogonality relation

$$\int_0^{2\pi} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\,dx = \frac{a}{2}\delta_{mn}$$

to make the eigenfunctions orthonormal.

$$\phi_{mn} = \frac{2}{\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \quad m, n \in \mathbb{Z}^+$$

The ϕ_{mn} are eigenfunctions of L.

$$L\left[\phi_{mn}\right] = -\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k^2\right)\phi_{mn}$$

By expanding the Green function and the Dirac Delta function in the ϕ_{mn} and substituting into the differential equation we obtain the solution.

$$G = \sum_{m,n=1}^{\infty} \frac{\frac{2}{\sqrt{ab}}\sin\left(\frac{m\pi\xi}{a}\right)\sin\left(\frac{n\pi\eta}{b}\right)\frac{2}{\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}{-\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k^2\right)}$$

$$\boxed{G(x, y; \xi, \eta) = -4ab\sum_{m,n=1}^{\infty}\frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi\xi}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{n\pi\eta}{b}\right)}{(m\pi b)^2 + (n\pi a)^2 + (kab)^2}}$$

Example 46.3.2 Consider the Green function for Laplace's equation, $\Delta u = 0$ in the disk, $|r| < a$, and vanishing at $r = a$.

First we find the eigenfunctions of the operator

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

We will look for eigenfunctions of the form $\phi = \Theta(\theta) R(r)$. We choose the Θ to be eigenfunctions of $\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}$ subject to the periodic boundary conditions in θ.

$$\Theta'' = \lambda \Theta, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)$$
$$\Theta_n = \mathrm{e}^{in\theta}, \quad n \in \mathbb{Z}$$

We determine $R(r)$ by requiring that ϕ be an eigenfunction of Δ.

$$\Delta \phi = \lambda \phi$$
$$(\Theta_n R)_{rr} + \frac{1}{r}(\Theta_n R)_r + \frac{1}{r^2}(\Theta_n R)_{\theta\theta} = \lambda \Theta_n R$$
$$\Theta_n R'' + \frac{1}{r}\Theta_n R' + \frac{1}{r^2}(-n^2)\Theta_n R = \lambda \Theta R$$

For notational convenience, we denote $\lambda = -\mu^2$.

$$R'' + \frac{1}{r}R' + \left(\mu^2 - \frac{n^2}{r^2}\right) R = 0, \quad R(0) \text{ bounded}, \quad R(a) = 0$$

The general solution for R is

$$R = c_1 J_n(\mu r) + c_2 Y_n(\mu r).$$

The left boundary condition demands that $c_2 = 0$. The right boundary condition determines the eigenvalues.

$$R_{nm} = J_n\left(\frac{j_{n,m} r}{a}\right), \quad \mu_{nm} = \frac{j_{n,m}}{a}$$

Here $j_{n,m}$ is the m^{th} positive root of J_n. This leads us to the eigenfunctions

$$\phi_{nm} = \mathrm{e}^{in\theta} J_n\left(\frac{j_{n,m} r}{a}\right)$$

We use the orthogonality relations

$$\int_0^{2\pi} \mathrm{e}^{-im\theta}\, \mathrm{e}^{in\theta}\, d\theta = 2\pi\delta_{mn},$$

$$\int_0^1 r J_\nu(j_{\nu,m} r) J_\nu(j_{\nu,n} r)\, dr = \frac{1}{2}\left(J'_\nu(j_{\nu,n})\right)^2 \delta_{mn}$$

to make the eigenfunctions orthonormal.

$$\phi_{nm} = \frac{1}{\sqrt{\pi} a |J'_n(j_{n,m})|} \mathrm{e}^{in\theta} J_n\left(\frac{j_{n,m} r}{a}\right), \quad n \in \mathbb{Z}, \quad m \in \mathbb{Z}^+$$

The ϕ_{nm} are eigenfunctions of L.

$$\Delta\phi_{nm} = -\left(\frac{j_{n,m}}{a}\right)^2 \phi_{nm}$$

By expanding the Green function and the Dirac Delta function in the ϕ_{nm} and substituting into the differential equation we obtain the solution.

$$G = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{\frac{1}{\sqrt{\pi} a |J'_n(j_{n,m})|} \mathrm{e}^{-in\vartheta} J_n\left(\frac{j_{n,m}\rho}{a}\right) \frac{1}{\sqrt{\pi} a |J'_n(j_{n,m})|} \mathrm{e}^{in\theta} J_n\left(\frac{j_{n,m} r}{a}\right)}{-\left(\frac{j_{n,m}}{a}\right)^2}$$

$$\boxed{G(r,\theta;\rho,\vartheta) = -\sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\pi (j_{n,m} J'_n(j_{n,m}))^2} \mathrm{e}^{in(\theta-\vartheta)} J_n\left(\frac{j_{n,m}\rho}{a}\right) J_n\left(\frac{j_{n,m} r}{a}\right)}$$

46.4 The Method of Images

Consider the problem

$$\begin{aligned}
&\nabla^2 u = f(x, y), \quad -\infty < x < \infty, \quad y > 0 \\
&u(x, 0) = 0, \quad u(x, y) \to 0 \text{ as } (x^2 + y^2) \to \infty.
\end{aligned}$$

The equations for the Green function are

$$\begin{aligned}
&\nabla^2 g = \delta(x - \xi)\delta(y - \eta), \quad -\infty < x < \infty, \quad y > 0 \\
&g(x, 0; \xi, \eta) = 0, \quad g(x, y; \xi, \eta) \to 0 \text{ as } (x^2 + y^2) \to \infty.
\end{aligned}$$

To solve this problem we will use the method of images. We expand the domain to include the lower half plane and solve the problem

$$\begin{aligned}
&\nabla^2 g = \delta(x - \xi)\delta(y - \eta) - \delta(x - \xi)\delta(y + \eta), \quad -\infty < x, \xi, y < \infty, \quad \eta > 0 \\
&g(x, y; \xi, \eta) \to 0 \text{ as } (x^2 + y^2) \to \infty.
\end{aligned}$$

Because of symmetry, g is zero for $y = 0$.

We solve the differential equation using the infinite space Green functions.

$$\begin{aligned}
g &= \frac{1}{4\pi} \log\left[(x - \xi)^2 + (y - \eta)^2\right] - \frac{1}{4\pi} \log\left[(x - \xi)^2 + (y + \eta)^2\right] \\
&= \frac{1}{4\pi} \log\left[\frac{(x - \xi)^2 + (y - \eta)^2}{(x - \xi)^2 + (y + \eta)^2}\right]
\end{aligned}$$

Thus we can write the solution

$$\boxed{u(x, y) = \int_0^\infty \int_{-\infty}^\infty \frac{1}{4\pi} \log\left[\frac{(x - \xi)^2 + (y - \eta)^2}{(x - \xi)^2 + (y + \eta)^2}\right] f(\xi, \eta)\, d\xi d\eta.}$$

46.5 Exercises

Exercise 46.1
Derive the causal Green function for the one dimensional wave equation on $(-\infty..\infty)$. That is, solve

$$G_{tt} - c^2 G_{xx} = \delta(x - \xi)\delta(t - \tau),$$
$$G(x, t; \xi, \tau) = 0 \quad \text{for } t < \tau.$$

Exercise 46.2
By reducing the problem to a series of one dimensional Green function problems, determine $G(\mathbf{x}, \boldsymbol{\xi})$ if

$$\nabla^2 G = \delta(\mathbf{x} - \boldsymbol{\xi})$$

(a) on the rectangle $0 < x < L$, $0 < y < H$ and

$$G(0, y; \xi, \eta) = G_x(L, y; \xi, \eta) = G_y(x, 0; \xi, \eta) = G_y(x, H; \xi, \eta) = 0$$

(b) on the box $0 < x < L$, $0 < y < H$, $0 < z < W$ with $G = 0$ on the boundary.

(c) on the semi-circle $0 < r < a$, $0 < \theta < \pi$ with $G = 0$ on the boundary.

(d) on the quarter-circle $0 < r < a$, $0 < \theta < \pi/2$ with $G = 0$ on the straight sides and $G_r = 0$ at $r = a$.

Exercise 46.3
Using the method of multi-dimensional eigenfunction expansions, determine $G(\mathbf{x}, \mathbf{x}_0)$ if

$$\nabla^2 G = \delta(\mathbf{x} - \mathbf{x}_0)$$

and

(a) on the rectangle $(0 < x < L,\ 0 < y < H)$

$$\text{at } x = 0, \quad G = 0 \qquad \text{at } y = 0, \quad \frac{\partial G}{\partial y} = 0$$

$$\text{at } x = L, \quad \frac{\partial G}{\partial x} = 0 \qquad \text{at } y = H, \quad \frac{\partial G}{\partial y} = 0$$

(b) on the rectangular shaped box $(0 < x < L,\ 0 < y < H,\ 0 < z < W)$ with $G = 0$ on the six sides.

(c) on the semi-circle $(0 < r < a,\ 0 < \theta < \pi)$ with $G = 0$ on the entire boundary.

(d) on the quarter-circle $(0 < r < a,\ 0 < \theta < \pi/2)$ with $G = 0$ on the straight sides and $\partial G/\partial r = 0$ at $r = a$.

Exercise 46.4
Using the method of images solve

$$\nabla^2 G = \delta(\mathbf{x} - \mathbf{x}_0)$$

in the first quadrant ($x \geq 0$ and $y \geq 0$) with $G = 0$ at $x = 0$ and $\partial G/\partial y = 0$ at $y = 0$. Use the Green function to solve in the first quadrant

$$\begin{aligned} \nabla^2 u &= 0 \\ u(0, y) &= g(y) \\ \frac{\partial u}{\partial y}(x, 0) &= h(x). \end{aligned}$$

Exercise 46.5
Consider the wave equation defined on the half-line $x > 0$:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Q(x,t),$$
$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x)$$
$$u(0,t) = h(t)$$

(a) Determine the appropriate Green's function using the method of images.

(b) Solve for $u(x,t)$ if $Q(x,t) = 0$, $f(x) = 0$, and $g(x) = 0$.

(c) For what values of t does $h(t)$ influence $u(x_1,t_1)$. Interpret this result physically.

Exercise 46.6
Derive the Green functions for the one dimensional wave equation on $(-\infty..\infty)$ for non-homogeneous initial conditions. Solve the two problems

$$g_{tt} - c^2 g_{xx} = 0, \quad g(x,0;\xi,\tau) = \delta(x-\xi), \quad g_t(x,0;\xi,\tau) = 0,$$
$$\gamma_{tt} - c^2 \gamma_{xx} = 0, \quad \gamma(x,0;\xi,\tau) = 0, \quad \gamma_t(x,0;\xi,\tau) = \delta(x-\xi),$$

using the Fourier transform.

Exercise 46.7
Use the Green functions from Problem 46.1 and Problem 46.6 to solve

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad x > 0, \quad -\infty < t < \infty$$
$$u(x,0) = p(x), \quad u_t(x,0) = q(x).$$

Use the solution to determine the domain of dependence of the solution.

Exercise 46.8
Show that the Green function for the reduced wave equation, $\Delta u - k^2 u = 0$ in the rectangle, $0 \leq x \leq a$, $0 \leq y \leq b$, and vanishing on the sides is:

$$G(x, y; \xi, \eta) = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sinh(\sigma_n y_<) \sinh(\sigma_n (y_> - b))}{\sigma_n \sinh(\sigma_n b)} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi \xi}{a}\right),$$

where

$$\sigma_n = \sqrt{k^2 + \frac{n^2\pi^2}{a^2}}.$$

Exercise 46.9
Find the Green function for the reduced wave equation $\Delta u - k^2 u = 0$, in the quarter plane: $0 < x < \infty$, $0 < y < \infty$ subject to the mixed boundary conditions:

$$u(x, 0) = 0, \quad u_x(0, y) = 0.$$

Find two distinct integral representations for $G(x, y; \xi, \eta)$.

Exercise 46.10
Show that in polar coordinates the Green function for $\Delta u = 0$ in the infinite sector, $0 < \theta < \alpha$, $0 < r < \infty$, and vanishing on the sides is given by,

$$G(r, \theta, \rho, \vartheta) = \frac{1}{4\pi} \log\left(\frac{\cosh\left(\frac{\pi}{\alpha} \log \frac{r}{\rho}\right) - \cos\left(\frac{\pi}{\alpha}(\theta - \vartheta)\right)}{\cosh\left(\frac{\pi}{\alpha} \log \frac{r}{\rho}\right) - \cos\left(\frac{\pi}{\alpha}(\theta + \vartheta)\right)}\right).$$

Use this to find the harmonic function $u(r, \theta)$ in the given sector which takes on the boundary values:

$$u(r, \theta) = u(r, \alpha) = \begin{cases} 0 & \text{for } r < c \\ 1 & \text{for } r > c. \end{cases}$$

Exercise 46.11
The Green function for the initial value problem,

$$u_t - \kappa u_{xx} = 0, \quad u(x,0) = f(x),$$

on $-\infty < x < \infty$ is

$$G(x,t;\xi) = \frac{1}{\sqrt{4\pi\kappa t}}\,\mathrm{e}^{-(x-\xi)^2/(4\kappa t)}.$$

Use the method of images to find the corresponding Green function for the mixed initial-boundary problems:

$$\begin{aligned} &\text{i)} \quad u_t = \kappa u_{xx}, \quad u(x,0) = f(x) \text{ for } x > 0, \quad u(0,t) = 0, \\ &\text{i)} \quad u_t = \kappa u_{xx}, \quad u(x,0) = f(x) \text{ for } x > 0, \quad u_x(0,t) = 0. \end{aligned}$$

Exercise 46.12
Find the Green function (expansion) for the one dimensional wave equation $u_{tt} - c^2 u_{xx} = 0$ on the interval $0 < x < L$, subject to the boundary conditions:

$$\begin{aligned} &\text{a)} \quad u(0,t) = u_x(L,t) = 0, \\ &\text{b)} \quad u_x(0,t) = u_x(L,t) = 0. \end{aligned}$$

Write the final forms in terms showing the propagation properties of the wave equation, i.e., with arguments $((x \pm \xi) \pm (t - \tau))$.

Exercise 46.13
Solve, using the above determined Green function,

$$\begin{gathered} u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < 1, \quad t > 0, \\ u_x(0,t) = u_x(1,t) = 0, \\ u(x,0) = x^2(1-x)^2, \quad u_t(x,0) = 1. \end{gathered}$$

For $c = 1$, find $u(x,t)$ at $x = 3/4$, $t = 7/2$.

46.6 Hints

Hint 46.1

Hint 46.2
Take a Fourier transform in x. This will give you an ordinary differential equation Green function problem for $\hat{G}$. Find the continuity and jump conditions at $t = \tau$. After solving for $\hat{G}$, do the inverse transform with the aid of a table.

Hint 46.3

Hint 46.4

Hint 46.5

Hint 46.6

Hint 46.7

Hint 46.8
Use Fourier sine and cosine transforms.

Hint 46.9
The the conformal mapping $z = w^{\pi/\alpha}$ to map the sector to the upper half plane. The new problem will be

$$
\begin{aligned}
&G_{xx} + G_{yy} = \delta(x - \xi)\delta(y - \eta), \quad -\infty < x < \infty, \quad 0 < y < \infty, \\
&G(x, 0, \xi, \eta) = 0, \\
&G(x, y, \xi, \eta) \to 0 \text{ as } x, y \to \infty.
\end{aligned}
$$

Solve this problem with the image method.

Hint 46.10

Hint 46.11

Hint 46.12

46.7 Solutions

Solution 46.1

$$G_{tt} - c^2 G_{xx} = \delta(x-\xi)\delta(t-\tau),$$
$$G(x,t;\xi,\tau) = 0 \quad \text{for } t < \tau.$$

We take the Fourier transform in x.

$$\hat{G}_{tt} + c^2\omega^2 G = \mathcal{F}[\delta(x-\xi)]\delta(t-\tau), \quad \hat{G}(\omega,0;\xi,\tau) = \hat{G}_t(\omega,0;\xi,\tau) = 0$$

Now we have an ordinary differential equation Green function problem for $\hat{G}$. We have written the causality condition, the Green function is zero for $t < \tau$, in terms of initial conditions. The homogeneous solutions of the ordinary differential equation are

$$\{\cos(c\omega t), \sin(c\omega t)\}.$$

It will be handy to use the fundamental set of solutions at $t = \tau$:

$$\left\{\cos(c\omega(t-\tau)), \frac{1}{c\omega}\sin(c\omega(t-\tau))\right\}.$$

We write the solution for $\hat{G}$ and invert using the convolution theorem.

$$\hat{G} = \mathcal{F}[\delta(x-\xi)]H(t-\tau)\frac{1}{c\omega}\sin(c\omega(t-\tau))$$
$$\hat{G} = H(t-\tau)\mathcal{F}[\delta(x-\xi)]\mathcal{F}\left[\frac{\pi}{c}H(c(t-\tau)-|x|)\right]$$
$$G = H(t-\tau)\frac{\pi}{c}\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(y-\xi)H(c(t-\tau)-|x-y|)\,dy$$
$$G = \frac{1}{2c}H(t-\tau)H(c(t-\tau)-|x-\xi|)$$
$$\boxed{G = \frac{1}{2c}H(c(t-\tau)-|x-\xi|)}$$

The Green function for $\xi = \tau = 0$ and $c = 1$ is plotted in Figure 46.1 on the domain $x \in (-1..1)$, $t \in (0..1)$. The Green function is a displacement of height $\frac{1}{2c}$ that propagates out from the point $x = \xi$ in both directions with speed c. The Green function shows the *range of influence* of a disturbance at the point $x = \xi$ and time $t = \tau$. The disturbance influences the solution for all $\xi - ct < x < \xi + ct$ and $t > \tau$.

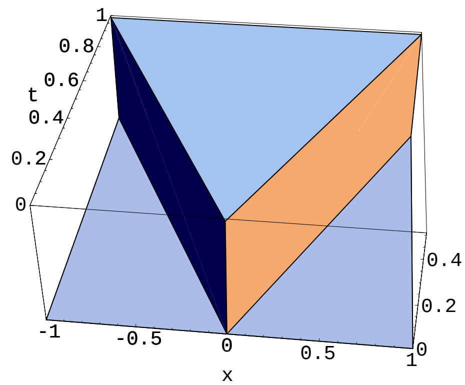

Figure 46.1: Green function for the wave equation.

Solution 46.2

1. We expand the Green function in eigenfunctions in x.

$$G(\mathbf{x}; \boldsymbol{\xi}) = \sum_{n=1}^{\infty} a_n(y) \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

We substitute the expansion into the differential equation.

$$\nabla^2 \sum_{n=1}^{\infty} a_n(y) \sqrt{\frac{2}{L}} \sin\left(\frac{(2n-1)\pi x}{2L}\right) = \delta(x-\xi)\delta(y-\eta)$$

$$\sum_{n=1}^{\infty}\left(a_n''(y)-\left(\frac{(2n-1)\pi}{2L}\right)^2 a_n(y)\right)\sqrt{\frac{2}{L}}\sin\left(\frac{(2n-1)\pi x}{2L}\right)$$
$$=\delta(y-\eta)\sum_{n=1}^{\infty}\sqrt{\frac{2}{L}}\sin\left(\frac{(2n-1)\pi\xi}{2L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

$$a_n''(y)-\left(\frac{(2n-1)\pi}{2L}\right)^2 a_n(y)=\sqrt{\frac{2}{L}}\sin\left(\frac{(2n-1)\pi\xi}{2L}\right)\delta(y-\eta)$$

From the boundary conditions at $y=0$ and $y=H$, we obtain boundary conditions for the $a_n(y)$.

$$a_n'(0)=a_n'(H)=0.$$

The solutions that satisfy the left and right boundary conditions are

$$a_{n1}=\cosh\left(\frac{(2n-1)\pi y}{2L}\right),\quad a_{n2}=\cosh\left(\frac{(2n-1)\pi(H-y)}{2L}\right).$$

The Wronskian of these solutions is

$$W=-\frac{(2n-1)\pi}{2L}\sinh\left(\frac{(2n-1)\pi}{2}\right).$$

Thus the solution for $a_n(y)$ is

$$a_n(y)=\sqrt{\frac{2}{L}}\sin\left(\frac{(2n-1)\pi\xi}{2L}\right)\frac{\cosh\left(\frac{(2n-1)\pi y_<}{2L}\right)\cosh\left(\frac{(2n-1)\pi(H-y_>)}{2L}\right)}{-\frac{(2n-1)\pi}{2L}\sinh\left(\frac{(2n-1)\pi}{2}\right)}$$

$$a_n(y)=-\frac{2\sqrt{2L}}{(2n-1)\pi}\operatorname{csch}\left(\frac{(2n-1)\pi}{2}\right)\cosh\left(\frac{(2n-1)\pi y_<}{2L}\right)$$
$$\cosh\left(\frac{(2n-1)\pi(H-y_>)}{2L}\right)\sin\left(\frac{(2n-1)\pi\xi}{2L}\right).$$

This determines the Green function.

$$G(\mathbf{x};\boldsymbol{\xi}) = -\frac{2\sqrt{2L}}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\operatorname{csch}\left(\frac{(2n-1)\pi}{2}\right)\cosh\left(\frac{(2n-1)\pi y_<}{2L}\right)$$
$$\cosh\left(\frac{(2n-1)\pi(H-y_>)}{2L}\right)\sin\left(\frac{(2n-1)\pi\xi}{2L}\right)\sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

2. We seek a solution of the form

$$G(\mathbf{x};\boldsymbol{\xi}) = \sum_{\substack{m=1\\n=1}}^{\infty} a_{mn}(z)\frac{2}{\sqrt{LH}}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right).$$

We substitute this into the differential equation.

$$\nabla^2\sum_{\substack{m=1\\n=1}}^{\infty} a_{mn}(z)\frac{2}{\sqrt{LH}}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right) = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)$$

$$\sum_{\substack{m=1\\n=1}}^{\infty}\left(a''_{mn}(z) - \left(\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2\right)a_{mn}(z)\right)\frac{2}{\sqrt{LH}}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)$$
$$= \delta(z-\zeta)\sum_{\substack{m=1\\n=1}}^{\infty}\frac{2}{\sqrt{LH}}\sin\left(\frac{m\pi\xi}{L}\right)\sin\left(\frac{n\pi\eta}{H}\right)\frac{2}{\sqrt{LH}}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)$$

$$a''_{mn}(z) - \pi\left(\left(\frac{m}{L}\right)^2 + \left(\frac{n}{H}\right)^2\right)a_{mn}(z) = \frac{2}{\sqrt{LH}}\sin\left(\frac{m\pi\xi}{L}\right)\sin\left(\frac{n\pi\eta}{H}\right)\delta(z-\zeta)$$

From the boundary conditions on G, we obtain boundary conditions for the a_{mn}.

$$a_{mn}(0) = a_{mn}(W) = 0$$

The solutions that satisfy the left and right boundary conditions are

$$a_{mn1} = \sinh\left(\sqrt{\left(\frac{m}{L}\right)^2 + \left(\frac{n}{H}\right)^2}\pi z\right), \quad a_{mn2} = \sinh\left(\sqrt{\left(\frac{m}{L}\right)^2 + \left(\frac{n}{H}\right)^2}\pi(W-z)\right).$$

The Wronskian of these solutions is

$$W = -\sqrt{\left(\frac{m}{L}\right)^2 + \left(\frac{n}{H}\right)^2}\pi \sinh\left(\sqrt{\left(\frac{m}{L}\right)^2 + \left(\frac{n}{H}\right)^2}\pi W\right).$$

Thus the solution for $a_{mn}(z)$ is

$$a_{mn}(z) = \frac{2}{\sqrt{LH}} \sin\left(\frac{m\pi\xi}{L}\right) \sin\left(\frac{n\pi\eta}{H}\right)$$
$$\frac{\sinh\left(\sqrt{\left(\frac{m}{L}\right)^2 + \left(\frac{n}{H}\right)^2}\pi z_<\right) \sinh\left(\sqrt{\left(\frac{m}{L}\right)^2 + \left(\frac{n}{H}\right)^2}\pi(W-z_>)\right)}{-\sqrt{\left(\frac{m}{L}\right)^2 + \left(\frac{n}{H}\right)^2}\pi \sinh\left(\sqrt{\left(\frac{m}{L}\right)^2 + \left(\frac{n}{H}\right)^2}\pi W\right)}$$

$$a_{mn}(z) = -\frac{2}{\pi\lambda_{mn}\sqrt{LH}} \operatorname{csch}\left(\lambda_{mn}\pi W\right) \sin\left(\frac{m\pi\xi}{L}\right) \sin\left(\frac{n\pi\eta}{H}\right)$$
$$\sinh\left(\lambda_{mn}\pi z_<\right) \sinh\left(\lambda_{mn}\pi(W-z_>)\right),$$

where

$$\lambda_{mn} = \sqrt{\left(\frac{m}{L}\right)^2 + \left(\frac{n}{H}\right)^2}.$$

This determines the Green function.

$$G(\mathbf{x};\boldsymbol{\xi}) = -\frac{4}{\pi L H} \sum_{\substack{m=1\\n=1}}^{\infty} \frac{1}{\lambda_{mn}} \operatorname{csch}\left(\lambda_{mn}\pi W\right) \sin\left(\frac{m\pi\xi}{L}\right) \sin\left(\frac{m\pi x}{L}\right)$$
$$\sin\left(\frac{n\pi\eta}{H}\right) \sin\left(\frac{n\pi y}{H}\right) \sinh\left(\lambda_{mn}\pi z_<\right) \sinh\left(\lambda_{mn}\pi(W - z_>)\right)$$

3. First we write the problem in circular coordinates.

$$\nabla^2 G = \delta(\mathbf{x} - \boldsymbol{\xi})$$
$$G_{rr} + \frac{1}{r}G_r + \frac{1}{r^2}G_{\theta\theta} = \frac{1}{r}\delta(r-\rho)\delta(\theta-\vartheta),$$
$$G(r,0;\rho,\vartheta) = G(r,\pi;\rho,\vartheta) = G(0,\theta;\rho,\vartheta) = G(a,\theta;\rho,\vartheta) = 0$$

Because the Green function vanishes at $\theta = 0$ and $\theta = \pi$ we expand it in a series of the form

$$G = \sum_{n=1}^{\infty} g_n(r)\sin(n\theta).$$

We substitute the series into the differential equation.

$$\sum_{n=1}^{\infty}\left(g_n''(r) + \frac{1}{r}g_n'(r) - \frac{n^2}{r^2}g_n(r)\right)\sin(n\theta) = \frac{1}{r}\delta(r-\rho)\sum_{n=1}^{\infty}\frac{2}{\pi}\sin(n\vartheta)\sin(n\theta)$$
$$g_n''(r) + \frac{1}{r}g_n'(r) - \frac{n^2}{r^2}g_n(r) = \frac{2}{\pi r}\sin(n\vartheta)\delta(r-\rho)$$

From the boundary conditions on G, we obtain boundary conditions for the g_n.

$$g_n(0) = g_n(a) = 0$$

The solutions that satisfy the left and right boundary conditions are

$$g_{n1} = r^n, \quad g_{n2} = \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n.$$

The Wronskian of these solutions is

$$W = \frac{2na^n}{r}.$$

Thus the solution for $g_n(r)$ is

$$g_n(r) = \frac{2}{\pi\rho}\sin(n\vartheta)\frac{r_<^n\left(\left(\frac{r_>}{a}\right)^n - \left(\frac{a}{r_>}\right)^n\right)}{\frac{2na^n}{\rho}}$$
$$g_n(r) = \frac{1}{n\pi}\sin(n\vartheta)\left(\frac{r_<}{a}\right)^n\left(\left(\frac{r_>}{a}\right)^n - \left(\frac{a}{r_>}\right)^n\right).$$

This determines the solution.

$$G = \sum_{n=1}^{\infty}\frac{1}{n\pi}\left(\frac{r_<}{a}\right)^n\left(\left(\frac{r_>}{a}\right)^n - \left(\frac{a}{r_>}\right)^n\right)\sin(n\vartheta)\sin(n\theta)$$

4. First we write the problem in circular coordinates.

$$G_{rr} + \frac{1}{r}G_r + \frac{1}{r^2}G_{\theta\theta} = \frac{1}{r}\delta(r-\rho)\delta(\theta-\vartheta),$$
$$G(r,0;\rho,\vartheta) = G(r,\pi/2;\rho,\vartheta) = G(0,\theta;\rho,\vartheta) = G_r(a,\theta;\rho,\vartheta) = 0$$

Because the Green function vanishes at $\theta = 0$ and $\theta = \pi/2$ we expand it in a series of the form

$$G = \sum_{n=1}^{\infty} g_n(r)\sin(2n\theta).$$

We substitute the series into the differential equation.

$$\sum_{n=1}^{\infty} \left(g_n''(r) + \frac{1}{r} g_n'(r) - \frac{4n^2}{r^2} g_n(r) \right) \sin(2n\theta) = \frac{1}{r} \delta(r-\rho) \sum_{n=1}^{\infty} \frac{4}{\pi} \sin(2n\vartheta) \sin(2n\theta)$$

$$g_n''(r) + \frac{1}{r} g_n'(r) - \frac{4n^2}{r^2} g_n(r) = \frac{4}{\pi r} \sin(2n\vartheta) \delta(r-\rho)$$

From the boundary conditions on G, we obtain boundary conditions for the g_n.

$$g_n(0) = g_n'(a) = 0$$

The solutions that satisfy the left and right boundary conditions are

$$g_{n1} = r^{2n}, \quad g_{n2} = \left(\frac{r}{a}\right)^{2n} + \left(\frac{a}{r}\right)^{2n}.$$

The Wronskian of these solutions is

$$W = -\frac{4na^{2n}}{r}.$$

Thus the solution for $g_n(r)$ is

$$g_n(r) = \frac{4}{\pi\rho} \sin(2n\vartheta) \frac{r_<^{2n} \left(\left(\frac{r_>}{a}\right)^{2n} + \left(\frac{a}{r_>}\right)^{2n} \right)}{-\frac{4na^{2n}}{\rho}}$$

$$g_n(r) = -\frac{1}{\pi n} \sin(2n\vartheta) \left(\frac{r_<}{a}\right)^{2n} \left(\left(\frac{r_>}{a}\right)^{2n} + \left(\frac{a}{r_>}\right)^{2n} \right)$$

This determines the solution.

$$G = -\sum_{n=1}^{\infty} \frac{1}{\pi n} \left(\frac{r_<}{a}\right)^{2n} \left(\left(\frac{r_>}{a}\right)^{2n} + \left(\frac{a}{r_>}\right)^{2n} \right) \sin(2n\vartheta) \sin(2n\theta)$$

Solution 46.3

1. The set

$$\{X_n\} = \left\{ \sin\left(\frac{(2m-1)\pi x}{2L} \right) \right\}_{m=1}^{\infty}$$

are eigenfunctions of ∇^2 and satisfy the boundary conditions $X_n(0) = X_n'(L) = 0$. The set

$$\{Y_n\} = \left\{ \cos\left(\frac{n\pi y}{H} \right) \right\}_{n=0}^{\infty}$$

are eigenfunctions of ∇^2 and satisfy the boundary conditions $Y_n'(0) = Y_n'(H) = 0$. The set

$$\left\{ \sin\left(\frac{(2m-1)\pi x}{2L} \right) \cos\left(\frac{n\pi y}{H} \right) \right\}_{m=1,n=0}^{\infty}$$

are eigenfunctions of ∇^2 and satisfy the boundary conditions of this problem. We expand the Green function in a series of these eigenfunctions.

$$G = \sum_{m=1}^{\infty} g_{m0} \sqrt{\frac{2}{LH}} \sin\left(\frac{(2m-1)\pi x}{2L} \right) + \sum_{\substack{m=1\\n=1}}^{\infty} g_{mn} \frac{2}{\sqrt{LH}} \sin\left(\frac{(2m-1)\pi x}{2L} \right) \cos\left(\frac{n\pi y}{H} \right)$$

We substitute the series into the Green function differential equation.

$$\Delta G = \delta(x-\xi)\delta(y-\eta)$$

$$
-\sum_{m=1}^{\infty} g_{m0} \left(\frac{(2m-1)\pi}{2L}\right)^2 \sqrt{\frac{2}{LH}} \sin\left(\frac{(2m-1)\pi x}{2L}\right)
$$

$$
-\sum_{\substack{m=1\\n=1}}^{\infty} g_{mn} \left(\left(\frac{(2m-1)\pi}{2L}\right)^2 + \left(\frac{n\pi y}{H}\right)^2\right) \frac{2}{\sqrt{LH}} \sin\left(\frac{(2m-1)\pi x}{2L}\right) \cos\left(\frac{n\pi y}{H}\right)
$$

$$
= \sum_{m=1}^{\infty} \sqrt{\frac{2}{LH}} \sin\left(\frac{(2m-1)\pi\xi}{2L}\right) \sqrt{\frac{2}{LH}} \sin\left(\frac{(2m-1)\pi x}{2L}\right)
$$

$$
+ \sum_{\substack{m=1\\n=1}}^{\infty} \frac{2}{\sqrt{LH}} \sin\left(\frac{(2m-1)\pi\xi}{2L}\right) \cos\left(\frac{n\pi\eta}{H}\right) \frac{2}{\sqrt{LH}} \sin\left(\frac{(2m-1)\pi x}{2L}\right) \cos\left(\frac{n\pi y}{H}\right)
$$

We equate terms and solve for the coefficients g_{mn}.

$$
g_{m0} = -\sqrt{\frac{2}{LH}} \left(\frac{2L}{(2m-1)\pi}\right)^2 \sin\left(\frac{(2m-1)\pi\xi}{2L}\right)
$$

$$
g_{mn} = -\frac{2}{\sqrt{LH}} \frac{1}{\pi^2\left(\left(\frac{2m-1}{2L}\right)^2 + \left(\frac{n}{H}\right)^2\right)} \sin\left(\frac{(2m-1)\pi\xi}{2L}\right) \cos\left(\frac{n\pi\eta}{H}\right)
$$

This determines the Green function.

2. Note that

$$
\left\{ \sqrt{\frac{8}{LHW}} \sin\left(\frac{k\pi x}{L}\right), \sin\left(\frac{m\pi y}{H}\right), \sin\left(\frac{n\pi z}{W}\right) \ : \ k, m, n \in \mathbb{Z}^+ \right\}
$$

is orthonormal and complete on $(0 \ldots L) \times (0 \ldots H) \times (0 \ldots W)$. The functions are eigenfunctions of ∇^2. We expand the Green function in a series of these eigenfunctions.

$$
G = \sum_{k,m,n=1}^{\infty} g_{kmn} \sqrt{\frac{8}{LHW}} \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) \sin\left(\frac{n\pi z}{W}\right)
$$

We substitute the series into the Green function differential equation.

$$\Delta G = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)$$

$$-\sum_{k,m,n=1}^{\infty} g_{kmn}\left(\left(\frac{k\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2+\left(\frac{n\pi}{W}\right)^2\right)\sqrt{\frac{8}{LHW}}\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{n\pi z}{W}\right)$$
$$=\sum_{k,m,n=1}^{\infty}\sqrt{\frac{8}{LHW}}\sin\left(\frac{k\pi\xi}{L}\right)\sin\left(\frac{m\pi\eta}{H}\right)\sin\left(\frac{n\pi\zeta}{W}\right)$$
$$\sqrt{\frac{8}{LHW}}\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{n\pi z}{W}\right)$$

We equate terms and solve for the coefficients g_{kmn}.

$$g_{kmn} = -\frac{\sqrt{\frac{8}{LHW}}\sin\left(\frac{k\pi\xi}{L}\right)\sin\left(\frac{m\pi\eta}{H}\right)\sin\left(\frac{n\pi\zeta}{W}\right)}{\pi^2\left(\left(\frac{k}{L}\right)^2+\left(\frac{m}{H}\right)^2+\left(\frac{n}{W}\right)^2\right)}$$

This determines the Green function.

3. The Green function problem is

$$\Delta G \equiv G_{rr} + \frac{1}{r}G_r + \frac{1}{r^2}G_{\theta\theta} = \frac{1}{r}\delta(r-\rho)\delta(\theta-\vartheta).$$

We seek a set of functions $\{\Theta_n(\theta)R_{nm}(r)\}$ which are orthogonal and complete on $(0\ldots a)\times(0\ldots\pi)$ and which are eigenfunctions of the laplacian. For the Θ_n we choose eigenfunctions of $\frac{\partial^2}{\partial\theta^2}$.

$$\Theta'' = -\nu^2\Theta, \quad \Theta(0) = \Theta(\pi) = 0$$
$$\nu_n = n, \quad \Theta_n = \sin(n\theta), \quad n \in \mathbb{Z}^+$$

Now we look for eigenfunctions of the laplacian.

$$(R\Theta_n)_{rr} + \frac{1}{r}(R\Theta_n)_r + \frac{1}{r^2}(R\Theta_n)_{\theta\theta} = -\mu^2 R\Theta_n$$

$$R''\Theta_n + \frac{1}{r}R'\Theta_n - \frac{n^2}{r^2}R\Theta_n = -\mu^2 R\Theta_n$$

$$R'' + \frac{1}{r}R' + \left(\mu^2 - \frac{n^2}{r^2}\right) R = 0, \quad R(0) = R(a) = 0$$

The general solution for R is

$$R = c_1 J_n(\mu r) + c_2 Y_n(\mu r).$$

the solution that satisfies the left boundary condition is $R = cJ_n(\mu r)$. We use the right boundary condition to determine the eigenvalues.

$$\mu_m = \frac{j_{n,m}}{a}, \quad R_{nm} = J_n\left(\frac{j_{n,m}r}{a}\right), \quad m, n \in \mathbb{Z}^+$$

here $j_{n,m}$ is the m^{th} root of J_n.

Note that

$$\left\{\sin(n\theta) J_n\left(\frac{j_{n,m}r}{a}\right) \;:\; m, n \in \mathbb{Z}^+\right\}$$

is orthogonal and complete on $(r, \theta) \in (0 \ldots a) \times (0 \ldots \pi)$. We use the identities

$$\int_0^\pi \sin^2(n\theta)\, d\theta = \frac{\pi}{2}, \quad \int_0^1 r J_n^2(j_{n,m}r)\, dr = \frac{1}{2}J_{n+1}^2(j_{n,m})$$

to make the functions orthonormal.

$$\left\{\frac{2}{\sqrt{\pi}\, a |J_{n+1}(j_{n,m})|} \sin(n\theta) J_n\left(\frac{j_{n,m}r}{a}\right) \;:\; m, n \in \mathbb{Z}^+\right\}$$

We expand the Green function in a series of these eigenfunctions.

$$G = \sum_{n,m=1}^{\infty} g_{nm} \frac{2}{\sqrt{\pi}\, a |J_{n+1}(j_{n,m})|} J_n\left(\frac{j_{n,m} r}{a}\right) \sin(n\theta)$$

We substitute the series into the Green function differential equation.

$$G_{rr} + \frac{1}{r} G_r + \frac{1}{r^2} G_{\theta\theta} = \frac{1}{r}\delta(r-\rho)\delta(\theta-\vartheta)$$

$$-\sum_{n,m=1}^{\infty} \left(\frac{j_{n,m}}{a}\right)^2 g_{nm} \frac{2}{\sqrt{\pi}\, a |J_{n+1}(j_{n,m})|} J_n\left(\frac{j_{n,m} r}{a}\right) \sin(n\theta)$$
$$= \sum_{n,m=1}^{\infty} \frac{2}{\sqrt{\pi}\, a |J_{n+1}(j_{n,m})|} J_n\left(\frac{j_{n,m} \rho}{a}\right) \sin(n\vartheta) \frac{2}{\sqrt{\pi}\, a |J_{n+1}(j_{n,m})|} J_n\left(\frac{j_{n,m} r}{a}\right) \sin(n\theta)$$

We equate terms and solve for the coefficients g_{mn}.

$$g_{nm} = -\left(\frac{a}{j_{n,m}}\right)^2 \frac{2}{\sqrt{\pi}\, a |J_{n+1}(j_{n,m})|} J_n\left(\frac{j_{n,m} \rho}{a}\right) \sin(n\vartheta)$$

This determines the green function.

4. The Green function problem is

$$\Delta G \equiv G_{rr} + \frac{1}{r} G_r + \frac{1}{r^2} G_{\theta\theta} = \frac{1}{r}\delta(r-\rho)\delta(\theta-\vartheta).$$

We seek a set of functions $\{\Theta_n(\theta) R_{nm}(r)\}$ which are orthogonal and complete on $(0 \ldots a) \times (0 \ldots \pi/2)$ and which are eigenfunctions of the laplacian. For the Θ_n we choose eigenfunctions of $\frac{\partial^2}{\partial \theta^2}$.

$$\Theta'' = -\nu^2 \Theta, \quad \Theta(0) = \Theta(\pi/2) = 0$$
$$\nu_n = 2n, \quad \Theta_n = \sin(2n\theta), \quad n \in \mathbb{Z}^+$$

Now we look for eigenfunctions of the laplacian.

$$
\begin{gathered}
(R\Theta_n)_{rr} + \frac{1}{r}(R\Theta_n)_r + \frac{1}{r^2}(R\Theta_n)_{\theta\theta} = -\mu^2 R\Theta_n \\
R''\Theta_n + \frac{1}{r}R'\Theta_n - \frac{(2n)^2}{r^2}R\Theta_n = -\mu^2 R\Theta_n \\
R'' + \frac{1}{r}R' + \left(\mu^2 - \frac{(2n)^2}{r^2}\right)R = 0, \quad R(0) = R(a) = 0
\end{gathered}
$$

The general solution for R is

$$
R = c_1 J_{2n}(\mu r) + c_2 Y_{2n}(\mu r).
$$

the solution that satisfies the left boundary condition is $R = cJ_{2n}(\mu r)$. We use the right boundary condition to determine the eigenvalues.

$$
\mu_m = \frac{{j'}_{2n,m}}{a}, \quad R_{nm} = J_{2n}\left(\frac{{j'}_{2n,m} r}{a}\right), \quad m, n \in \mathbb{Z}^+
$$

here ${j'}_{n,m}$ is the m^{th} root of ${J'}_n$.

Note that

$$
\left\{\sin(2n\theta){J'}_{2n}\left(\frac{{j'}_{2n,m} r}{a}\right) \; : \; m, n \in \mathbb{Z}^+\right\}
$$

is orthogonal and complete on $(r, \theta) \in (0 \ldots a) \times (0 \ldots \pi/2)$. We use the identities

$$
\begin{gathered}
\int_0^\pi \sin(m\theta)\sin(n\theta)\, d\theta = \frac{\pi}{2}\delta_{mn}, \\
\int_0^1 r J_\nu({j'}_{\nu,m} r) J_\nu({j'}_{\nu,n} r)\, dr = \frac{{j'}^2_{\nu,n} - \nu^2}{2{j'}^2_{\nu,n}} \left(J_\nu({j'}_{\nu,n})\right)^2 \delta_{mn}
\end{gathered}
$$

to make the functions orthonormal.

$$\left\{ \frac{2 {j'}_{2n,m}}{\sqrt{\pi}\, a \sqrt{{j'}^2_{2n,m} - 4n^2} |J_{2n}({j'}_{2n,m})|} \sin(2n\theta) J_{2n}\left(\frac{{j'}_{2n,m} r}{a}\right) \ :\ m,n \in \mathbb{Z}^+ \right\}$$

We expand the Green function in a series of these eigenfunctions.

$$G = \sum_{n,m=1}^{\infty} g_{nm} \frac{2 {j'}_{2n,m}}{\sqrt{\pi}\, a \sqrt{{j'}^2_{2n,m} - 4n^2} |J_{2n}({j'}_{2n,m})|} J_{2n}\left(\frac{{j'}_{2n,m} r}{a}\right) \sin(2n\theta)$$

We substitute the series into the Green function differential equation.

$$G_{rr} + \frac{1}{r} G_r + \frac{1}{r^2} G_{\theta\theta} = \frac{1}{r} \delta(r-\rho)\delta(\theta-\vartheta)$$

$$-\sum_{n,m=1}^{\infty} \left(\frac{{j'}_{2n,m}}{a}\right)^2 g_{nm} \frac{2 {j'}_{2n,m}}{\sqrt{\pi}\, a \sqrt{{j'}^2_{2n,m} - 4n^2} |J_{2n}({j'}_{2n,m})|} J_{2n}\left(\frac{{j'}_{2n,m} r}{a}\right) \sin(2n\theta)$$
$$= \sum_{n,m=1}^{\infty} \frac{2 {j'}_{2n,m}}{\sqrt{\pi}\, a \sqrt{{j'}^2_{2n,m} - 4n^2} |J_{2n}({j'}_{2n,m})|} J_{2n}\left(\frac{{j'}_{2n,m} \rho}{a}\right) \sin(2n\vartheta)$$
$$\frac{2 {j'}_{2n,m}}{\sqrt{\pi}\, a \sqrt{{j'}^2_{2n,m} - 4n^2} |J_{2n}({j'}_{2n,m})|} J_{2n}\left(\frac{{j'}_{2n,m} r}{a}\right) \sin(2n\theta)$$

We equate terms and solve for the coefficients g_{mn}.

$$g_{nm} = -\left(\frac{a}{{j'}_{2n,m}}\right)^2 \frac{2 {j'}_{2n,m}}{\sqrt{\pi}\, a \sqrt{{j'}^2_{2n,m} - 4n^2} |J_{2n}({j'}_{2n,m})|} J_{2n}\left(\frac{{j'}_{2n,m} \rho}{a}\right) \sin(2n\vartheta)$$

This determines the green function.

Solution 46.4
We start with the equation

$$\nabla^2 G = \delta(x-\xi)\delta(y-\eta).$$

We do an odd reflection across the y axis so that $G(0, y; \xi, \eta) = 0$.

$$\nabla^2 G = \delta(x-\xi)\delta(y-\eta) - \delta(x+\xi)\delta(y-\eta)$$

Then we do an even reflection across the x axis so that $G_y(x, 0; \xi, \eta) = 0$.

$$\nabla^2 G = \delta(x-\xi)\delta(y-\eta) - \delta(x+\xi)\delta(y-\eta) + \delta(x-\xi)\delta(y+\eta) - \delta(x+\xi)\delta(y+\eta)$$

We solve this problem using the infinite space Green function.

$$G = \frac{1}{4\pi}\log\left((x-\xi)^2+(y-\eta)^2\right) - \frac{1}{4\pi}\log\left((x+\xi)^2+(y-\eta)^2\right) \\ + \frac{1}{4\pi}\log\left((x-\xi)^2+(y+\eta)^2\right) - \frac{1}{4\pi}\log\left((x+\xi)^2+(y+\eta)^2\right)$$

$$G = \frac{1}{4\pi}\log\left(\frac{\left((x-\xi)^2+(y-\eta)^2\right)\left((x-\xi)^2+(y+\eta)^2\right)}{\left((x+\xi)^2+(y-\eta)^2\right)\left((x+\xi)^2+(y+\eta)^2\right)}\right)$$

Now we solve the boundary value problem.

$$u(\xi,\eta) = \int_S \left(u(x,y)\frac{\partial G}{\partial n} - G\frac{\partial u(x,y)}{\partial n}\right) dS + \int_V G\Delta u\, dV$$

$$u(\xi,\eta) = \int_\infty^0 u(0,y)(-G_x(0,y;\xi,\eta))\, dy + \int_0^\infty -G(x,0;\xi,\eta)(-u_y(x,0))\, dx$$

$$u(\xi,\eta) = \int_0^\infty g(y)G_x(0,y;\xi,\eta)\, dy + \int_0^\infty G(x,0;\xi,\eta)h(x)\, dx$$

$$u(\xi,\eta) = -\frac{\xi}{\pi}\int_0^\infty \left(\frac{1}{\xi^2+(y-\eta)^2} + \frac{1}{\xi^2+(y+\eta)^2}\right) g(y)\, dy + \frac{1}{2\pi}\int_0^\infty \log\left(\frac{(x-\xi)^2+\eta^2}{(x+\xi)^2+\eta^2}\right) h(x)\, dx$$

$$u(x,y) = -\frac{x}{\pi}\int_0^\infty \left(\frac{1}{x^2+(y-\eta)^2} + \frac{1}{x^2+(y+\eta)^2}\right) g(\eta)\, d\eta + \frac{1}{2\pi}\int_0^\infty \log\left(\frac{(x-\xi)^2+y^2}{(x+\xi)^2+y^2}\right) h(\xi)\, d\xi$$

Solution 46.5
First we find the infinite space Green function.

$$G_{tt} - c^2 G_{xx} = \delta(x-\xi)\delta(t-\tau), \quad G = G_t = 0 \text{ for } t < \tau$$

We solve this problem with the Fourier transform.

$$\hat{G}_{tt} + c^2\omega^2 \hat{G} = \mathcal{F}[\delta(x-\xi)]\delta(t-\tau)$$
$$\hat{G} = \mathcal{F}[\delta(x-\xi)]H(t-\tau)\frac{1}{c\omega}\sin(c\omega(t-\tau))$$
$$\hat{G} = H(t-\tau)\mathcal{F}[\delta(x-\xi)]\mathcal{F}\left[\frac{\pi}{c}H(c(t-\tau)-|x|)\right]$$
$$G = H(t-\tau)\frac{\pi}{c}\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(y-\xi)H(c(t-\tau)-|x-y|)\,dy$$
$$G = \frac{1}{2c}H(t-\tau)H(c(t-\tau)-|x-\xi|)$$
$$G = \frac{1}{2c}H(c(t-\tau)-|x-\xi|)$$

1. So that the Green function vanishes at $x = 0$ we do an odd reflection about that point.

$$G_{tt} - c^2 G_{xx} = \delta(x-\xi)\delta(t-\tau) - \delta(x+\xi)\delta(t-\tau)$$
$$G = \frac{1}{2c}H(c(t-\tau)-|x-\xi|) - \frac{1}{2c}H(c(t-\tau)-|x+\xi|)$$

2. Note that the Green function satisfies the symmetry relation

$$G(x,t;\xi,\tau) = G(\xi,-\tau;x,-t).$$

This implies that

$$G_{xx} = G_{\xi\xi}, \quad G_{tt} = G_{\tau\tau}.$$

We write the Green function problem and the inhomogeneous differential equation for u in terms of ξ and τ.

$$G_{\tau\tau} - c^2 G_{\xi\xi} = \delta(x-\xi)\delta(t-\tau) \tag{46.4}$$

$$u_{\tau\tau} - c^2 u_{\xi\xi} = Q(\xi,\tau) \tag{46.5}$$

We take the difference of u times Equation 46.4 and G times Equation 46.5 and integrate this over the domain $(0,\infty)\times(0,t^+)$.

$$\int_0^{t^+}\int_0^\infty (u\delta(x-\xi)\delta(t-\tau) - GQ)\,d\xi\,d\tau = \int_0^{t^+}\int_0^\infty \left(uG_{\tau\tau} - u_{\tau\tau}G - c^2\left(uG_{\xi\xi} - u_{\xi\xi}G\right)\right)\,d\xi\,d\tau$$

$$u(x,t) = \int_0^{t^+}\int_0^\infty GQ\,d\xi\,d\tau + \int_0^{t^+}\int_0^\infty \left(\frac{\partial}{\partial\tau}\left(uG_\tau - u_\tau G\right) - c^2\frac{\partial}{\partial\xi}\left(uG_\xi - u_\xi G\right)\right)\,d\xi\,d\tau$$

$$u(x,t) = \int_0^{t^+}\int_0^\infty GQ\,d\xi\,d\tau + \int_0^\infty \left[uG_\tau - u_\tau G\right]_0^{t+}\,d\xi - c^2\int_0^{t^+}\left[uG_\xi - u_\xi G\right]_0^\infty\,d\tau$$

$$u(x,t) = \int_0^{t^+}\int_0^\infty GQ\,d\xi\,d\tau - \int_0^\infty \left[uG_\tau - u_\tau G\right]_{\tau=0}\,d\xi + c^2\int_0^{t^+}\left[uG_\xi\right]_{\xi=0}\,d\tau$$

We consider the case $Q(x,t) = f(x) = g(x) = 0$.

$$u(x,t) = c^2\int_0^{t^+} h(\tau) G_\xi(x,t;0,\tau)\,d\tau$$

We calculate G_ξ.

$$G = \frac{1}{2c}\left(H(c(t-\tau) - |x-\xi|) - H(c(t-\tau) - |x+\xi|)\right)$$

$$G_\xi = \frac{1}{2c}\left(\delta(c(t-\tau) - |x-\xi|)(-1)\operatorname{sign}(x-\xi)(-1) - \delta(c(t-\tau) - |x+\xi|)(-1)\operatorname{sign}(x+\xi)\right)$$

$$G_\xi(x,t;0,\eta) = \frac{1}{c}\delta(c(t-\tau) - |x|)\operatorname{sign}(x)$$

We are interested in $x > 0$.

$$G_\xi(x,t;0,\eta) = \frac{1}{c}\delta(c(t-\tau)-x)$$

Now we can calculate the solution u.

$$u(x,t) = c^2 \int_0^{t^+} h(\tau)\frac{1}{c}\delta(c(t-\tau)-x)\,d\tau$$
$$u(x,t) = \int_0^{t^+} h(\tau)\delta\left((t-\tau)-\frac{x}{c}\right)\,d\tau$$
$$u(x,t) = h\left(t-\frac{x}{c}\right)$$

3. The boundary condition influences the solution $u(x_1,t_1)$ only at the point $t = t_1 - x_1/c$. The contribution from the boundary condition $u(0,t) = h(t)$ is a wave moving to the right with speed c.

Solution 46.6

$$g_{tt} - c^2 g_{xx} = 0, \quad g(x,0;\xi,\tau) = \delta(x-\xi), \quad g_t(x,0;\xi,\tau) = 0$$
$$\hat{g}_{tt} + c^2\omega^2 \hat{g}_{xx} = 0, \quad \hat{g}(x,0;\xi,\tau) = \mathcal{F}[\delta(x-\xi)], \quad \hat{g}_t(x,0;\xi,\tau) = 0$$
$$\hat{g} = \mathcal{F}[\delta(x-\xi)]\cos(c\omega t)$$
$$\hat{g} = \mathcal{F}[\delta(x-\xi)]\mathcal{F}[\pi(\delta(x+ct)+\delta(x-ct))]$$
$$g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\eta-\xi)\pi(\delta(x-\eta+ct)+\delta(x-\eta-ct))\,d\eta$$
$$\boxed{g(x,t;\xi) = \frac{1}{2}(\delta(x-\xi+ct)+\delta(x-\xi-ct))}$$

$$\gamma_{tt} - c^2\gamma_{xx} = 0, \quad \gamma(x, 0; \xi, \tau) = 0, \quad \gamma_t(x, 0; \xi, \tau) = \delta(x - \xi)$$
$$\hat{\gamma}_{tt} + c^2\omega^2\hat{\gamma}_{xx} = 0, \quad \hat{\gamma}(x, 0; \xi, \tau) = 0, \quad \hat{\gamma}_t(x, 0; \xi, \tau) = \mathcal{F}[\delta(x - \xi)]$$
$$\hat{\gamma} = \mathcal{F}[\delta(x - \xi)]\frac{1}{c\omega}\sin(c\omega t)$$
$$\hat{\gamma} = \mathcal{F}[\delta(x - \xi)]\mathcal{F}\left[\frac{\pi}{c}(H(x + ct) + H(x - ct))\right]$$
$$\gamma = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\eta - \xi)\frac{\pi}{c}(H(x - \eta + ct) + H(x - \eta - ct))\, d\eta$$
$$\boxed{\gamma(x, t; \xi) = \frac{1}{2c}(H(x - \xi + ct) + H(x - \xi - ct))}$$

Solution 46.7

$$u(x, t) = \int_0^\infty \int_{-\infty}^{\infty} G(x, t; \xi, \tau)f(\xi, \tau)\, d\xi\, d\tau + \int_{-\infty}^{\infty} g(x, t; \xi)p(\xi)\, d\xi + \int_{-\infty}^{\infty} \gamma(x, t; \xi)q(\xi)\, d\xi$$

$$u(x, t) = \frac{1}{2c}\int_0^\infty \int_{-\infty}^{\infty} H(t - \tau)(H(x - \xi + c(t - \tau)) - H(x - \xi - c(t - \tau)))f(\xi, \tau)\, d\xi\, d\tau \\ + \frac{1}{2}\int_{-\infty}^{\infty}(\delta(x - \xi + ct) + \delta(x - \xi - ct))p(\xi)\, d\xi + \frac{1}{2c}\int_{-\infty}^{\infty}(H(x - \xi + ct) + H(x - \xi - ct))q(\xi)\, d\xi$$

$$u(x, t) = \frac{1}{2c}\int_0^t \int_{-\infty}^{\infty}(H(x - \xi + c(t - \tau)) - H(x - \xi - c(t - \tau)))f(\xi, \tau)\, d\xi\, d\tau \\ + \frac{1}{2}(p(x + ct) + p(x - ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} q(\xi)\, d\xi$$

$$
\boxed{u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi\,d\tau + \frac{1}{2}(p(x+ct)+p(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} q(\xi)\,d\xi}
$$

This solution demonstrates the *domain of dependence* of the solution. The first term is an integral over the triangle domain $\{(\xi,\tau) : 0 < \tau < t, x - c\tau < \xi < x + c\tau\}$. The second term involves only the points $(x \pm ct, 0)$. The third term is an integral on the line segment $\{(\xi, 0) : x - ct < \xi < x + ct\}$. In totallity, this is just the triangle domain. This is shown graphically in Figure 46.2.

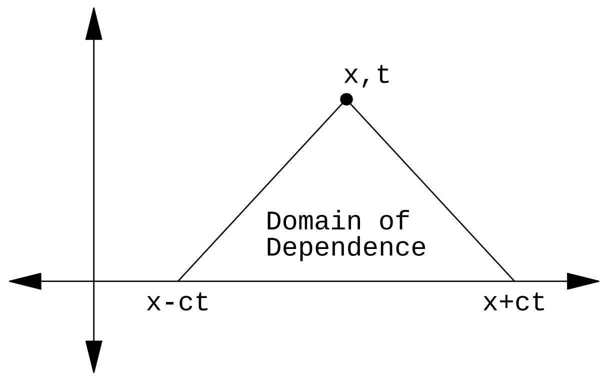

Figure 46.2: Domain of dependence for the wave equation.

Solution 46.8

Single Sum Representation. First we find the eigenfunctions of the homogeneous problem $\Delta u - k^2 u = 0$. We substitute the separation of variables, $u(x,y) = X(x)Y(y)$ into the partial differential equation.

$$
X''Y + XY'' - k^2XY = 0
$$
$$
\frac{X''}{X} = k^2 - \frac{Y''}{Y} = -\lambda^2
$$

We have the regular Sturm-Liouville eigenvalue problem,

$$X'' = -\lambda^2 X, \quad X(0) = X(a) = 0,$$

which has the solutions,

$$\lambda_n = \frac{n\pi}{a}, \quad X_n = \sin\left(\frac{n\pi x}{a}\right), \quad n \in \mathbb{N}.$$

We expand the solution u in a series of these eigenfunctions.

$$G(x, y; \xi, \eta) = \sum_{n=1}^{\infty} c_n(y) \sin\left(\frac{n\pi x}{a}\right)$$

We substitute this series into the partial differential equation to find equations for the $c_n(y)$.

$$\sum_{n=1}^{\infty} \left(-\left(\frac{n\pi}{a}\right)^2 c_n(y) + c_n''(y) - k^2 c_n(y) \right) \sin\left(\frac{n\pi x}{a}\right) = \delta(x-\xi)\delta(y-\eta)$$

The series expansion of the right side is,

$$\delta(x-\xi)\delta(y-\eta) = \sum_{n=1}^{\infty} d_n(y) \sin\left(\frac{n\pi x}{a}\right)$$

$$d_n(y) = \frac{2}{a} \int_0^a \delta(x-\xi)\delta(y-\eta) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$d_n(y) = \frac{2}{a} \sin\left(\frac{n\pi\xi}{a}\right) \delta(y-\eta).$$

The the equations for the $c_n(y)$ are

$$c_n''(y) - \left(k^2 + \left(\frac{n\pi}{a}\right)^2\right) c_n(y) = \frac{2}{a} \sin\left(\frac{n\pi\xi}{a}\right) \delta(y-\eta), \quad c_n(0) = c_n(b) = 0.$$

The homogeneous solutions are $\{\cosh(\sigma_n y), \sinh(\sigma_n y)\}$, where $\sigma_n = \sqrt{k^2(n\pi/a)^2}$. The solutions that satisfy the boundary conditions at $y = 0$ and $y = b$ are, $\sinh(\sigma_n y)$ and $\sinh(\sigma_n(y - b))$, respectively. The Wronskian of these solutions is,

$$\begin{aligned} W(y) &= \begin{vmatrix} \sinh(\sigma_n y) & \sinh(\sigma_n(y-b)) \\ \sigma_n \cosh(\sigma_n y) & \sigma_n \cosh(\sigma_n(y-b)) \end{vmatrix} \\ &= \sigma_n \left(\sinh(\sigma_n y)\cosh(\sigma_n(y-b)) - \sinh(\sigma_n(y-b))\cosh(\sigma_n y)\right) \\ &= \sigma_n \sinh(\sigma_n b). \end{aligned}$$

The solution for $c_n(y)$ is

$$c_n(y) = \frac{2}{a}\sin\left(\frac{n\pi\xi}{a}\right)\frac{\sinh(\sigma_n y_<)\sinh(\sigma_n(y_> - b))}{\sigma_n \sinh(\sigma_n b)}.$$

The Green function for the partial differential equation is

$$\boxed{G(x, y; \xi, \eta) = \frac{2}{a}\sum_{n=1}^{\infty}\frac{\sinh(\sigma_n y_<)\sinh(\sigma_n(y_> - b))}{\sigma_n \sinh(\sigma_n b)}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi\xi}{a}\right).}$$

Solution 46.9

We take the Fourier cosine transform in x of the partial differential equation and the boundary condition along $y = 0$.

$$\begin{gathered} G_{xx} + G_{yy} - k^2 G = \delta(x - \xi)\delta(y - \eta) \\ -\alpha^2 \hat{G}(\alpha, y) - \frac{1}{\pi}\hat{G}_x(0, y) + \hat{G}_{yy}(\alpha, y) - k^2\hat{G}(\alpha, y) = \frac{1}{\pi}\cos(\alpha\xi)\delta(y - \eta) \\ \hat{G}_{yy}(\alpha, y) - (k^2 + \alpha^2)\hat{G}(\alpha, y) == \frac{1}{\pi}\cos(\alpha\xi)\delta(y - \eta), \quad \hat{G}(\alpha, 0) = 0 \end{gathered}$$

Then we take the Fourier sine transform in y.

$$-\beta^2 \hat{\hat{G}}(\alpha,\beta) + \frac{\beta}{\pi}\hat{\hat{G}}(\alpha,0) - (k^2+\alpha^2)\hat{\hat{G}}(\alpha,\beta) = \frac{1}{\pi^2}\cos(\alpha\xi)\sin(\beta\eta)$$
$$\hat{\hat{G}} = -\frac{\cos(\alpha\xi)\sin(\beta\eta)}{\pi^2(k^2+\alpha^2+\beta^2)}$$

We take two inverse transforms to find the solution. For one integral representation of the Green function we take the inverse sine transform followed by the inverse cosine transform.

$$\hat{\hat{G}} = -\cos(\alpha\xi)\frac{\sin(\beta\eta)}{\pi}\frac{1}{\pi(k^2+\alpha^2+\beta^2)}$$
$$\hat{\hat{G}} = -\cos(\alpha\xi)\mathcal{F}_s[\delta(y-\eta)]\mathcal{F}_c\left[\frac{1}{\sqrt{k^2+\alpha^2}}\,\mathrm{e}^{-\sqrt{k^2+\alpha^2}y}\right]$$
$$\hat{G}(\alpha,y) = -\cos(\alpha\xi)\frac{1}{2\pi}\int_0^\infty \delta(z-\eta)\frac{1}{\sqrt{k^2+\alpha^2}}\left(\exp\left(-\sqrt{k^2+\alpha^2}|y-z|\right) - \exp\left(-\sqrt{k^2+\alpha^2}(y+z)\right)\right)\,dz$$
$$\hat{G}(\alpha,y) = -\frac{\cos(\alpha\xi)}{2\pi\sqrt{k^2+\alpha^2}}\left(\exp\left(-\sqrt{k^2+\alpha^2}|y-\eta|\right) - \exp\left(-\sqrt{k^2+\alpha^2}(y+\eta)\right)\right)$$
$$\boxed{G(x,y;\xi,\eta) = -\frac{1}{\pi}\int_0^\infty \frac{\cos(\alpha\xi)}{\sqrt{k^2+\alpha^2}}\left(\exp\left(-\sqrt{k^2+\alpha^2}|y-\eta|\right) - \exp\left(-\sqrt{k^2+\alpha^2}(y+\eta)\right)\right)\,d\alpha}$$

For another integral representation of the Green function, we take the inverse cosine transform followed by the inverse sine transform.

$$\hat{\hat{G}}(\alpha,\beta) = -\sin(\beta\eta)\frac{\cos(\alpha\xi)}{\pi}\frac{1}{\pi(k^2+\alpha^2+\beta^2)}$$

$$\hat{\hat{G}}(\alpha,\beta) = -\sin(\beta\eta)\mathcal{F}_c[\delta(x-\xi)]\mathcal{F}_c\left[\frac{1}{\sqrt{k^2+\beta^2}}\,\mathrm{e}^{-\sqrt{k^2+\beta^2}x}\right]$$

$$\hat{G}(x,\beta) = -\sin(\beta\eta)\frac{1}{2\pi}\int_0^\infty \delta(z-\xi)\frac{1}{\sqrt{k^2+\beta^2}}\left(\mathrm{e}^{-\sqrt{k^2+\beta^2}|x-z|} + \mathrm{e}^{-\sqrt{k^2+\beta^2}(x+z)}\right)\,dz$$

$$\hat{G}(x,\beta) = -\sin(\beta\eta)\frac{1}{2\pi}\frac{1}{\sqrt{k^2+\beta^2}}\left(\mathrm{e}^{-\sqrt{k^2+\beta^2}|x-\xi|} + \mathrm{e}^{-\sqrt{k^2+\beta^2}(x+\xi)}\right)$$

$$\boxed{G(x,y;\xi,\eta) = -\frac{1}{\pi}\int_0^\infty \frac{\sin(\beta y)\sin(\beta\eta)}{\sqrt{k^2+\beta^2}}\left(\mathrm{e}^{-\sqrt{k^2+\beta^2}|x-\xi|} + \mathrm{e}^{-\sqrt{k^2+\beta^2}(x+\xi)}\right)\,d\beta}$$

Solution 46.10

The problem is:

$$\begin{aligned}
&G_{rr} + \frac{1}{r}G_r + \frac{1}{r^2}G_{\theta\theta} = \frac{\delta(r-\rho)\delta(\theta-\vartheta)}{r}, \quad 0 < r < \infty, \quad 0 < \theta < \alpha, \\
&G(r,0,\rho,\vartheta) = G(r,\alpha,\rho,\vartheta) = 0, \\
&G(0,\theta,\rho,\vartheta) = 0 \\
&G(r,\theta,\rho,\vartheta) \to 0 \text{ as } r \to \infty.
\end{aligned}$$

Let $w = r\,\mathrm{e}^{i\theta}$ and $z = x + iy$. We use the conformal mapping, $z = w^{\pi/\alpha}$ to map the sector to the upper half z plane. The problem is (x,y) space is

$$\begin{aligned}
&G_{xx} + G_{yy} = \delta(x-\xi)\delta(y-\eta), \quad -\infty < x < \infty, \quad 0 < y < \infty, \\
&G(x,0,\xi,\eta) = 0, \\
&G(x,y,\xi,\eta) \to 0 \text{ as } x,y \to \infty.
\end{aligned}$$

We will solve this problem with the method of images. Note that the solution of,

$$G_{xx} + G_{yy} = \delta(x-\xi)\delta(y-\eta) - \delta(x-\xi)\delta(y+\eta), \quad -\infty < x < \infty, \quad -\infty < y < \infty,$$
$$G(x, y, \xi, \eta) \to 0 \text{ as } x, y \to \infty,$$

satisfies the condition, $G(x, 0, \xi, \eta) = 0$. Since the infinite space Green function for the Laplacian in two dimensions is

$$\frac{1}{4\pi} \log\left((x-\xi)^2 + (y-\eta)^2\right),$$

the solution of this problem is,

$$\begin{aligned}
G(x, y, \xi, \eta) &= \frac{1}{4\pi} \log\left((x-\xi)^2 + (y-\eta)^2\right) - \frac{1}{4\pi} \log\left((x-\xi)^2 + (y+\eta)^2\right) \\
&= \frac{1}{4\pi} \log\left(\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2}\right).
\end{aligned}$$

Now we solve for x and y in the conformal mapping.

$$z = w^{\pi/\alpha} = (r\,\mathrm{e}^{i\theta})^{\pi/\alpha}$$
$$x + iy = r^{\pi/\alpha}(\cos(\theta\pi/\alpha) + i\sin(\theta\pi/\alpha))$$
$$x = r^{\pi/\alpha}\cos(\theta\pi/\alpha), \qquad y = r^{\pi/\alpha}\sin(\theta\pi/\alpha)$$

We substitute these expressions into $G(x, y, \xi, \eta)$ to obtain $G(r, \theta, \rho, \vartheta)$.

$$\begin{aligned}
G(r, \theta, \rho, \vartheta) &= \frac{1}{4\pi} \log\left(\frac{(r^{\pi/\alpha}\cos(\theta\pi/\alpha) - \rho^{\pi/\alpha}\cos(\vartheta\pi/\alpha))^2 + (r^{\pi/\alpha}\sin(\theta\pi/\alpha) - \rho^{\pi/\alpha}\sin(\vartheta\pi/\alpha))^2}{(r^{\pi/\alpha}\cos(\theta\pi/\alpha) - \rho^{\pi/\alpha}\cos(\vartheta\pi/\alpha))^2 + (r^{\pi/\alpha}\sin(\theta\pi/\alpha) + \rho^{\pi/\alpha}\sin(\vartheta\pi/\alpha))^2}\right) \\
&= \frac{1}{4\pi} \log\left(\frac{r^{2\pi/\alpha} + \rho^{2\pi/\alpha} - 2r^{\pi/\alpha}\rho^{\pi/\alpha}\cos(\pi(\theta-\vartheta)/\alpha)}{r^{2\pi/\alpha} + \rho^{2\pi/\alpha} - 2r^{\pi/\alpha}\rho^{\pi/\alpha}\cos(\pi(\theta+\vartheta)/\alpha)}\right) \\
&= \frac{1}{4\pi} \log\left(\frac{(r/\rho)^{\pi/\alpha}/2 + (\rho/r)^{\pi/\alpha}/2 - \cos(\pi(\theta-\vartheta)/\alpha)}{(r/\rho)^{\pi/\alpha}/2 + (\rho/r)^{\pi/\alpha}/2 - \cos(\pi(\theta+\vartheta)/\alpha)}\right) \\
&= \frac{1}{4\pi} \log\left(\frac{\mathrm{e}^{\pi\log(r/\rho)/\alpha}/2 + \mathrm{e}^{\pi\log(\rho/r)/\alpha}/2 - \cos(\pi(\theta-\vartheta)/\alpha)}{\mathrm{e}^{\pi\log(r/\rho)/\alpha}/2 + \mathrm{e}^{\pi\log(\rho/r)/\alpha}/2 - \cos(\pi(\theta+\vartheta)/\alpha)}\right)
\end{aligned}$$

$$G(r,\theta,\rho,\vartheta) = \frac{1}{4\pi}\log\left(\frac{\cosh\left(\frac{\pi/\alpha}{\log}\frac{r}{\rho}\right) - \cos(\pi(\theta-\vartheta)/\alpha)}{\cosh\left(\frac{\pi/\alpha}{\log}\frac{r}{\rho}\right) - \cos(\pi(\theta+\vartheta)/\alpha)}\right)$$

Now recall that the solution of

$$\Delta u = f(\mathbf{x}),$$

subject to the boundary condition,

$$u(\mathbf{x}) = g(\mathbf{x}),$$

is

$$u(\mathbf{x}) = \int\int f(\boldsymbol{\xi})G(\mathbf{x};\boldsymbol{\xi})\,dA_\xi + \oint g(\boldsymbol{\xi})\nabla_\xi G(\mathbf{x};\boldsymbol{\xi})\cdot\hat{\mathbf{n}}\,ds_\xi.$$

The normal directions along the lower and upper edges of the sector are $-\hat{\theta}$ and $\hat{\theta}$, respectively. The gradient in polar coordinates is

$$\nabla_\xi = \hat{\rho}\frac{\partial}{\partial\rho} + \frac{\hat{\vartheta}}{\rho}\frac{\partial}{\partial\vartheta}.$$

We only need to compute the $\hat{\vartheta}$ component of the gradient of G. This is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}G = \frac{\sin(\pi(\theta-\vartheta)/\alpha)}{4\alpha\rho\left(\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) - \cos(\pi(\theta-\vartheta)/\alpha)\right)} + \frac{\sin(\pi(\theta-\vartheta)/\alpha)}{4\alpha\rho\left(\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) - \cos(\pi(\theta+\vartheta)/\alpha)\right)}$$

Along $\vartheta = 0$, this is

$$\frac{1}{\rho}G_\vartheta(r,\theta,\rho,0) = \frac{\sin(\pi\theta/\alpha)}{2\alpha\rho\left(\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) - \cos(\pi\theta/\alpha)\right)}.$$

Along $\vartheta = \alpha$, this is

$$\frac{1}{\rho} G_\vartheta(r, \theta, \rho, \alpha) = -\frac{\sin(\pi\theta/\alpha)}{2\alpha\rho\left(\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) + \cos(\pi\theta/\alpha)\right)}.$$

The solution of our problem is

$$u(r, \theta) = \int_\infty^c -\frac{\sin(\pi\theta/\alpha)}{2\alpha\rho\left(\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) + \cos(\pi\theta/\alpha)\right)} \, d\rho + \int_c^\infty -\frac{\sin(\pi\theta/\alpha)}{2\alpha\rho\left(\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) - \cos(\pi\theta/\alpha)\right)} \, d\rho$$

$$u(r, \theta) = \int_c^\infty \frac{-\sin(\pi\theta/\alpha)}{2\alpha\rho\left(\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) - \cos(\pi\theta/\alpha)\right)} + \frac{\sin(\pi\theta/\alpha)}{2\alpha\rho\left(\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) + \cos(\pi\theta/\alpha)\right)} \, d\rho$$

$$u(r, \theta) = -\frac{1}{\alpha}\sin\left(\frac{\pi\theta}{\alpha}\right)\cos\left(\frac{\pi\theta}{\alpha}\right)\int_c^\infty \frac{1}{\rho\left(\cosh^2\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) - \cos^2\left(\frac{\pi\theta}{\alpha}\right)\right)} \, d\rho$$

$$u(r, \theta) = -\frac{1}{\alpha}\sin\left(\frac{\pi\theta}{\alpha}\right)\cos\left(\frac{\pi\theta}{\alpha}\right)\int_{\log(c/r)}^\infty \frac{1}{\cosh^2\left(\frac{\pi x}{\alpha}\right) - \cos^2\left(\frac{\pi\theta}{\alpha}\right)} \, dx$$

$$\boxed{u(r, \theta) = -\frac{2}{\alpha}\sin\left(\frac{\pi\theta}{\alpha}\right)\cos\left(\frac{\pi\theta}{\alpha}\right)\int_{\log(c/r)}^\infty \frac{1}{\cosh\left(\frac{2\pi x}{\alpha}\right) - \cos\left(\frac{2\pi\theta}{\alpha}\right)} \, dx}$$

Solution 46.11
First consider the Green function for

$$u_t - \kappa u_{xx} = 0, \quad u(x, 0) = f(x).$$

The differential equation and initial condition is

$$G_t = \kappa G_{xx}, \quad G(x, 0; \xi) = \delta(x - \xi).$$

The Green function is a solution of the homogeneous heat equation for the initial condition of a unit amount of heat concentrated at the point $x = \xi$. You can verify that the Green function is a solution of the heat equation for $t > 0$ and that it has the property:

$$\int_{-\infty}^{\infty} G(x, t; \xi)\, dx = 1, \quad \text{for } t > 0.$$

This property demonstrates that the total amount of heat is the constant 1. At time $t = 0$ the heat is concentrated at the point $x = \xi$. As time increases, the heat diffuses out from this point.

The solution for $u(x, t)$ is the linear combination of the Green functions that satisfies the initial condition $u(x, 0) = f(x)$. This linear combination is

$$u(x, t) = \int_{-\infty}^{\infty} G(x, t; \xi) f(\xi)\, d\xi.$$

$G(x, t; 1)$ and $G(x, t; -1)$ are plotted in Figure 46.3 for the domain $t \in [1/100..1/4]$, $x \in [-2..2]$ and $\kappa = 1$.

Now we consider the problem

$$u_t = \kappa u_{xx}, \quad u(x, 0) = f(x) \text{ for } x > 0, \quad u(0, t) = 0.$$

Note that the solution of

$$G_t = \kappa G_{xx}, \quad x > 0, \quad t > 0,$$
$$G(x, 0; \xi) = \delta(x - \xi) - \delta(x + \xi),$$

satisfies the boundary condition $G(0, t; \xi) = 0$. We write the solution as the difference of infinite space Green functions.

$$\begin{aligned} G(x, t; \xi) &= \frac{1}{\sqrt{4\pi\kappa t}} \mathrm{e}^{-(x-\xi)^2/(4\kappa t)} - \frac{1}{\sqrt{4\pi\kappa t}} \mathrm{e}^{-(x+\xi)^2/(4\kappa t)} \\ &= \frac{1}{\sqrt{4\pi\kappa t}} \left(\mathrm{e}^{-(x-\xi)^2/(4\kappa t)} - \mathrm{e}^{-(x+\xi)^2/(4\kappa t)} \right) \end{aligned}$$

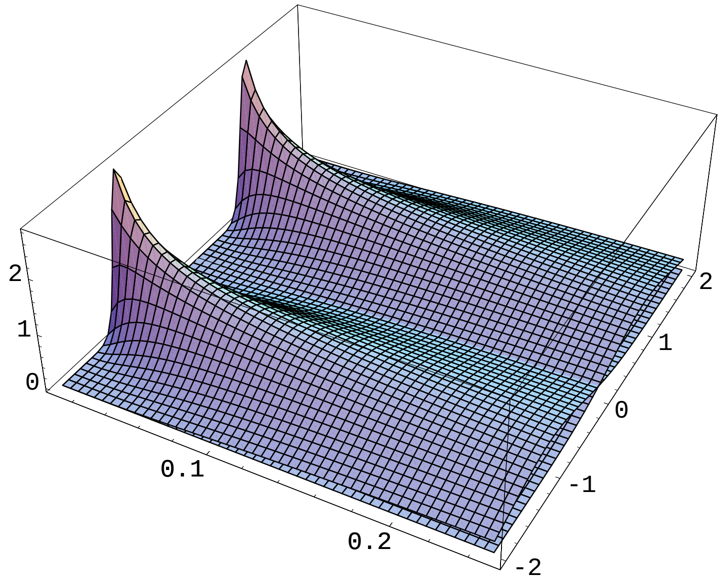

Figure 46.3: $G(x,t;1)$ and $G(x,t;-1)$

$$\boxed{G(x,t;\xi) = \frac{1}{\sqrt{4\pi\kappa t}} \mathrm{e}^{-(x^2+\xi^2)/(4\kappa t)} \sinh\left(\frac{x\xi}{2\kappa t}\right)}$$

Next we consider the problem

$$u_t = \kappa u_{xx}, \quad u(x,0) = f(x) \text{ for } x > 0, \quad u_x(0,t) = 0.$$

Note that the solution of

$$G_t = \kappa G_{xx}, \quad x > 0, \quad t > 0,$$
$$G(x,0;\xi) = \delta(x-\xi) + \delta(x+\xi),$$

satisfies the boundary condition $G_x(0,t;\xi) = 0$. We write the solution as the sum of infinite space Green functions.

$$G(x,t;\xi) = \frac{1}{\sqrt{4\pi\kappa t}} \,\mathrm{e}^{-(x-\xi)^2/(4\kappa t)} + \frac{1}{\sqrt{4\pi\kappa t}} \,\mathrm{e}^{-(x+\xi)^2/(4\kappa t)}$$

$$\boxed{G(x,t;\xi) = \frac{1}{\sqrt{4\pi\kappa t}} \,\mathrm{e}^{-(x^2+\xi^2)/(4\kappa t)} \cosh\left(\frac{x\xi}{2\kappa t}\right)}$$

The Green functions for the two boundary conditions are shown in Figure 46.4.

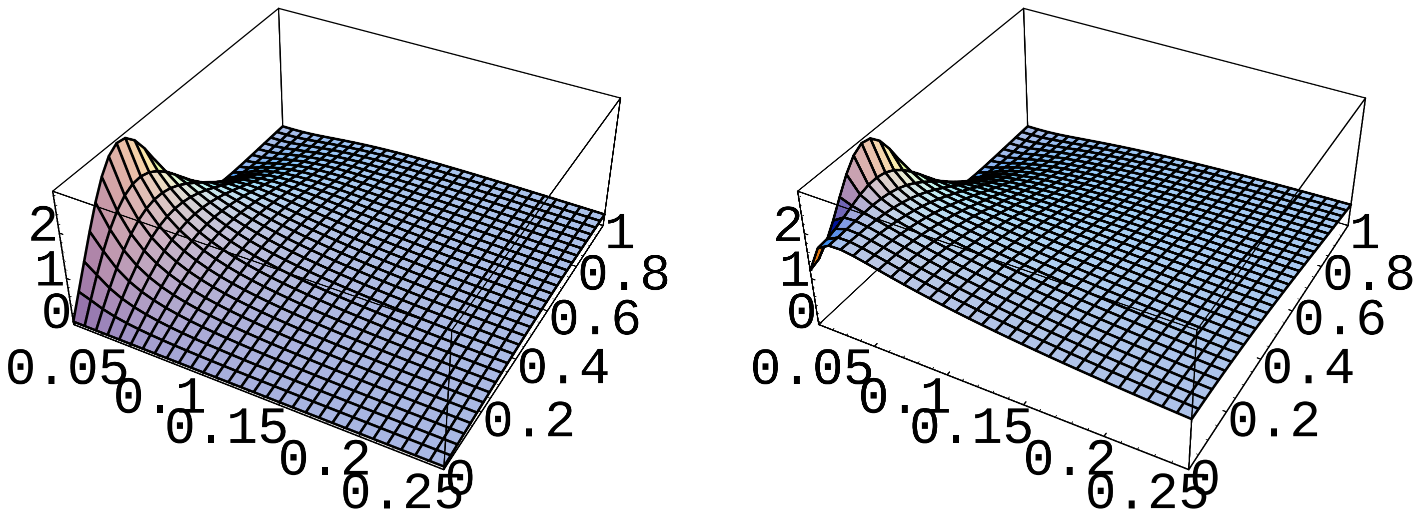

Figure 46.4: Green functions for the boundary conditions $u(0,t) = 0$ and $u_x(0,t) = 0$.

Solution 46.12

a) The Green function problem is

$$G_{tt} - c^2 G_{xx} = \delta(t-\tau)\delta(x-\xi), \quad 0 < x < L, \quad t > 0,$$
$$G(0,t;\xi,\tau) = G_x(L,t;\xi,\tau) = 0,$$
$$G(x,t;\xi,\tau) = 0 \text{ for } t < \tau.$$

The condition that G is zero for $t < \tau$ makes this a *causal* Green function. We solve this problem by expanding G in a series of eigenfunctions of the x variable. The coefficients in the expansion will be functions of t. First we find the eigenfunctions of x in the homogeneous problem. We substitute the separation of variables $u = X(x)T(t)$ into the homogeneous partial differential equation.

$$XT'' = c^2 X'' T$$
$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda^2$$

The eigenvalue problem is

$$X'' = -\lambda^2 X, \quad X(0) = X'(L) = 0,$$

which has the solutions,

$$\lambda_n = \frac{(2n-1)\pi}{2L}, \quad X_n = \sin\left(\frac{(2n-1)\pi x}{2L}\right), \quad n \in \mathbb{N}.$$

The series expansion of the Green function has the form,

$$\boxed{G(x,t;\xi,\tau) = \sum_{n=1}^{\infty} g_n(t) \sin\left(\frac{(2n-1)\pi x}{2L}\right).}$$

We determine the coefficients by substituting the expansion into the Green function differential equation.

$$G_{tt} - c^2 G_{xx} = \delta(x-\xi)\delta(t-\tau)$$
$$\sum_{n=1}^{\infty}\left(g_n''(t) + \left(\frac{(2n-1)\pi c}{2L}\right)^2 g_n(t)\right)\sin\left(\frac{(2n-1)\pi x}{2L}\right) = \delta(x-\xi)\delta(t-\tau)$$

We need to expand the right side of the equation in the sine series

$$\delta(x-\xi)\delta(t-\tau) = \sum_{n=1}^{\infty} d_n(t) \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$
$$d_n(t) = \frac{2}{L}\int_0^L \delta(x-\xi)\delta(t-\tau) \sin\left(\frac{(2n-1)\pi x}{2L}\right) dx$$
$$d_n(t) = \frac{2}{L}\sin\left(\frac{(2n-1)\pi\xi}{2L}\right)\delta(t-\tau)$$

By equating coefficients in the sine series, we obtain ordinary differential equation Green function problems for the g_n's.

$$g_n''(t;\tau) + \left(\frac{(2n-1)\pi c}{2L}\right)^2 g_n(t;\tau) = \frac{2}{L}\sin\left(\frac{(2n-1)\pi\xi}{2L}\right)\delta(t-\tau)$$

From the causality condition for G, we have the causality conditions for the g_n's,

$$g_n(t;\tau) = g_n'(t;\tau) = 0 \text{ for } t < \tau.$$

The continuity and jump conditions for the g_n are

$$g_n(\tau^+;\tau) = 0, \quad g_n'(\tau^+;\tau) = \frac{2}{L}\sin\left(\frac{(2n-1)\pi\xi}{2L}\right).$$

A set of homogeneous solutions of the ordinary differential equation are

$$\left\{\cos\left(\frac{(2n-1)\pi ct}{2L}\right), \sin\left(\frac{(2n-1)\pi ct}{2L}\right)\right\}$$

Since the continuity and jump conditions are given at the point $t = \tau$, a handy set of solutions to use for this problem is the fundamental set of solutions at that point:

$$\left\{\cos\left(\frac{(2n-1)\pi c(t-\tau)}{2L}\right), \frac{2L}{(2n-1)\pi c}\sin\left(\frac{(2n-1)\pi c(t-\tau)}{2L}\right)\right\}$$

The solution that satisfies the causality condition and the continuity and jump conditions is,

$$g_n(t;\tau) = \frac{4}{(2n-1)\pi c} \sin\left(\frac{(2n-1)\pi\xi}{2L}\right) \sin\left(\frac{(2n-1)\pi c(t-\tau)}{2L}\right) H(t-\tau).$$

Substituting this into the sum yields,

$$G(x,t;\xi,\tau) = \frac{4}{\pi c} H(t-\tau) \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi\xi}{2L}\right) \sin\left(\frac{(2n-1)\pi c(t-\tau)}{2L}\right) \sin\left(\frac{(2n-1)\pi x}{2L}\right).$$

We use trigonometric identities to write this in terms of traveling waves.

$$\boxed{\begin{aligned} G(x,t;\xi,\tau) = \frac{1}{\pi c} H(t-\tau) \sum_{n=1}^{\infty} \frac{1}{2n-1} \Bigg(&\sin\left(\frac{(2n-1)\pi((x-\xi)-c(t-\tau))}{2L}\right) \\ + \sin\left(\frac{(2n-1)\pi((x-\xi)+c(t-\tau))}{2L}\right) - &\sin\left(\frac{(2n-1)\pi((x+\xi)-c(t-\tau))}{2L}\right) \\ - &\sin\left(\frac{(2n-1)\pi((x+\xi)+c(t-\tau))}{2L}\right)\Bigg) \end{aligned}}$$

b) Now we consider the Green function with the boundary conditions,

$$u_x(0,t) = u_x(L,t) = 0.$$

First we find the eigenfunctions in x of the homogeneous problem. The eigenvalue problem is

$$X'' = -\lambda^2 X, \quad X'(0) = X'(L) = 0,$$

which has the solutions,

$$\lambda_0 = 0, \quad X_0 = 1,$$
$$\lambda_n = \frac{n\pi}{L}, \quad X_n = \cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots .$$

The series expansion of the Green function for $t > \tau$ has the form,

$$G(x, t; \xi, \tau) = \frac{1}{2} g_0(t) + \sum_{n=1}^{\infty} g_n(t) \cos\left(\frac{n\pi x}{L}\right).$$

(Note the factor of 1/2 in front of $g_0(t)$. With this, the integral formulas for all the coefficients are the same.) We determine the coefficients by substituting the expansion into the partial differential equation.

$$G_{tt} - c^2 G_{xx} = \delta(x - \xi)\delta(t - \tau)$$
$$\frac{1}{2} g_0''(t) + \sum_{n=1}^{\infty} \left(g_n''(t) + \left(\frac{n\pi c}{L}\right)^2 g_n(t) \right) \cos\left(\frac{n\pi x}{L}\right) = \delta(x - \xi)\delta(t - \tau)$$

We expand the right side of the equation in the cosine series.

$$\delta(x - \xi)\delta(t - \tau) = \frac{1}{2} d_0(t) + \sum_{n=1}^{\infty} d_n(t) \cos\left(\frac{n\pi x}{L}\right)$$
$$d_n(t) = \frac{2}{L} \int_0^L \delta(x - \xi)\delta(t - \tau) \cos\left(\frac{n\pi x}{L}\right) dx$$
$$d_n(t) = \frac{2}{L} \cos\left(\frac{n\pi \xi}{L}\right) \delta(t - \tau)$$

By equating coefficients in the cosine series, we obtain ordinary differential equations for the g_n.

$$g_n''(t; \tau) + \left(\frac{n\pi c}{L}\right)^2 g_n(t; \tau) = \frac{2}{L} \cos\left(\frac{n\pi \xi}{L}\right) \delta(t - \tau), \quad n = 0, 1, 2, \ldots$$

From the causality condition for G, we have the causality condiions for the g_n,

$$g_n(t;\tau) = g_n'(t;\tau) = 0 \text{ for } t < \tau.$$

The continuity and jump conditions for the g_n are

$$g_n(\tau^+;\tau) = 0, \quad g_n'(\tau^+;\tau) = \frac{2}{L}\cos\left(\frac{n\pi\xi}{L}\right).$$

The homogeneous solutions of the ordinary differential equation for $n = 0$ and $n > 0$ are respectively,

$$\{1, t\}, \quad \left\{\cos\left(\frac{n\pi ct}{L}\right), \sin\left(\frac{n\pi ct}{L}\right)\right\}.$$

Since the continuity and jump conditions are given at the point $t = \tau$, a handy set of solutions to use for this problem is the fundamental set of solutions at that point:

$$\{1, t-\tau\}, \quad \left\{\cos\left(\frac{n\pi c(t-\tau)}{L}\right), \frac{L}{n\pi c}\sin\left(\frac{n\pi c(t-\tau)}{L}\right)\right\}.$$

The solutions that satisfy the causality condition and the continuity and jump conditions are,

$$g_0(t) = \frac{2}{L}(t-\tau)H(t-\tau),$$
$$g_n(t) = \frac{2}{n\pi c}\cos\left(\frac{n\pi\xi}{L}\right)\sin\left(\frac{n\pi c(t-\tau)}{L}\right)H(t-\tau).$$

Substituting this into the sum yields,

$$G(x,t;\xi,\tau) = H(t-\tau)\left(\frac{t-\tau}{L} + \frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi\xi}{L}\right)\sin\left(\frac{n\pi c(t-\tau)}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\right).$$

We can write this as the sum of traveling waves.

$$G(x,t;\xi,\tau) = \frac{t-\tau}{L} H(t-\tau) + \frac{1}{2\pi c} H(t-\tau) \sum_{n=1}^{\infty} \frac{1}{n} \Bigg(-\sin\left(\frac{n\pi((x-\xi)-c(t-\tau))}{2L}\right)$$
$$+ \sin\left(\frac{n\pi((x-\xi)+c(t-\tau))}{2L}\right) - \sin\left(\frac{n\pi((x+\xi)-c(t-\tau))}{2L}\right)$$
$$+ \sin\left(\frac{n\pi((x+\xi)+c(t-\tau))}{2L}\right)\Bigg)$$

Solution 46.13

First we derive Green's identity for this problem. We consider the integral of $uL[v] - L[u]v$ on the domain $0 < x < 1$, $0 < t < T$.

$$\int_0^T \int_0^1 (uL[v] - L[u]v)\, dx\, dt$$
$$\int_0^T \int_0^1 \left(u(v_{tt} - c^2 v_{xx} - (u_{tt} - c^2 u_{xx})v\right)\, dx\, dt$$
$$\int_0^T \int_0^1 \left(\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) \cdot \left(-c^2(uv_x - u_x v), uv_t - u_t v\right)\right)\, dx\, dt$$

Now we can use the divergence theorem to write this as an integral along the boundary of the domain.

$$\oint_{\partial\Omega} \left(-c^2(uv_x - u_x v), uv_t - u_t v\right) \cdot \mathbf{n}\, ds$$

The domain and the outward normal vectors are shown in Figure 46.5.

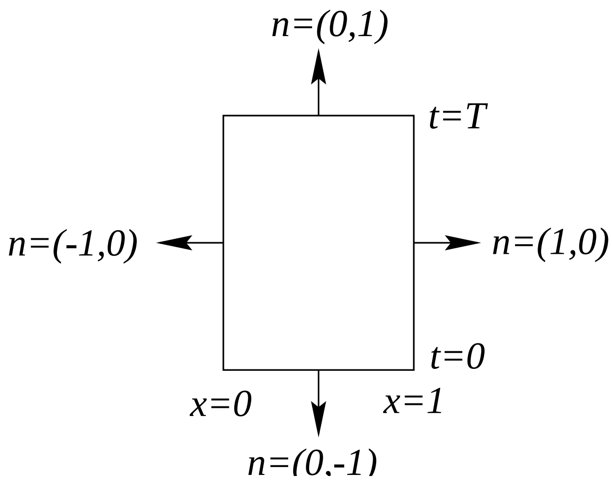

Figure 46.5: Outward normal vectors of the domain.

Writing out the boundary integrals, Green's identity for this problem is,

$$\int_0^T \int_0^1 \left(u(v_{tt} - c^2 v_{xx}) - (u_{tt} - c^2 u_{xx}) v \right) \, dx \, dt = -\int_0^1 (u v_t - u_t v)_{t=0} \, dx$$
$$+ \int_1^0 (u v_t - u_t v)_{t=T} \, dx - c^2 \int_0^T (u v_x - u_x v)_{x=1} \, dt + c^2 \int_T^1 (u v_x - u_x v)_{x=0} \, dt$$

The Green function problem is

$$G_{tt} - c^2 G_{xx} = \delta(x - \xi)\delta(t - \tau), \quad 0 < x, \xi < 1, \quad t, \tau > 0,$$
$$G_x(0, t; \xi, \tau) = G_x(1, t; \xi, \tau) = 0, \quad t > 0, G(x, t; \xi, \tau) = 0 \quad \text{for} \quad t < \tau.$$

If we consider G as a function of (ξ, τ) with (x, t) as parameters, then it satisfies:

$$G_{\tau\tau} - c^2 G_{\xi\xi} = \delta(x - \xi)\delta(t - \tau),$$
$$G_\xi(x, t; 0, \tau) = G_\xi(x, t; 1, \tau) = 0, \quad \tau > 0, G(x, t; \xi, \tau) = 0 \quad \text{for} \quad \tau > t.$$

Now we apply Green's identity for $u = u(\xi, \tau)$, (the solution of the wave equation), and $v = G(x, t; \xi, \tau)$, (the Green function), and integrate in the (ξ, τ) variables. The left side of Green's identity becomes:

$$\int_0^T \int_0^1 \left(u(G_{\tau\tau} - c^2 G_{\xi\xi}) - (u_{\tau\tau} - c^2 u_{\xi\xi})G\right) \, d\xi \, d\tau$$
$$\int_0^T \int_0^1 (u(\delta(x - \xi)\delta(t - \tau)) - (0)G) \, d\xi \, d\tau$$
$$u(x, t).$$

Since the normal derivative of u and G vanish on the sides of the domain, the integrals along $\xi = 0$ and $\xi = 1$ in Green's identity vanish. If we take $T > t$, then G is zero for $\tau = T$ and the integral along $\tau = T$ vanishes. The one remaining integral is

$$-\int_0^1 (u(\xi, 0)G_\tau(x, t; \xi, 0) - u_\tau(\xi, 0)G(x, t; \xi, 0) \, d\xi.$$

Thus Green's identity allows us to write the solution of the inhomogeneous problem.

$$\boxed{u(x, t) = \int_0^1 (u_\tau(\xi, 0)G(x, t; \xi, 0) - u(\xi, 0)G_\tau(x, t; \xi, 0)) \, d\xi.}$$

With the specified initial conditions this becomes

$$u(x, t) = \int_0^1 (G(x, t; \xi, 0) - \xi^2(1 - \xi)^2 G_\tau(x, t; \xi, 0)) \, d\xi.$$

Now we substitute in the Green function that we found in the previous exercise. The Green function and its derivative are,

$$G(x,t;\xi,0) = t + \sum_{n=1}^{\infty} \frac{2}{n\pi c} \cos(n\pi\xi)\sin(n\pi ct)\cos(n\pi x),$$
$$G_\tau(x,t;\xi,0) = -1 - 2\sum_{n=1}^{\infty} \cos(n\pi\xi)\cos(n\pi ct)\cos(n\pi x).$$

The integral of the first term is,

$$\int_0^1 \left(t + \sum_{n=1}^{\infty} \frac{2}{n\pi c} \cos(n\pi\xi)\sin(n\pi ct)\cos(n\pi x) \right) d\xi = t.$$

The integral of the second term is

$$\int_0^1 \xi^2(1-\xi)^2 \left(1 + 2\sum_{n=1}^{\infty} \cos(n\pi\xi)\cos(n\pi ct)\cos(n\pi x) \right) d\xi = \frac{1}{30} - 3\sum_{n=1}^{\infty} \frac{1}{n^4\pi^4} \cos(2n\pi x)\cos(2n\pi ct).$$

Thus the solution is

$$\boxed{u(x,t) = \frac{1}{30} + t - 3\sum_{n=1}^{\infty} \frac{1}{n^4\pi^4} \cos(2n\pi x)\cos(2n\pi ct).}$$

For $c = 1$, the solution at $x = 3/4$, $t = 7/2$ is,

$$u(3/4, 7/2) = \frac{1}{30} + \frac{7}{2} - 3\sum_{n=1}^{\infty} \frac{1}{n^4\pi^4} \cos(3n\pi/2)\cos(7n\pi).$$

Note that the summand is nonzero only for even terms.

$$\begin{aligned} u(3/4, 7/2) &= \frac{53}{15} - \frac{3}{16pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \cos(3n\pi)\cos(14n\pi) \\ &= \frac{53}{15} - \frac{3}{16pi^4} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \\ &= \frac{53}{15} - \frac{3}{16pi^4} \frac{-7\pi^4}{720} \end{aligned}$$

$$\boxed{u(3/4, 7/2) = \frac{12727}{3840}}$$

Chapter 47

Conformal Mapping

47.1 Exercises

Exercise 47.1

$\zeta = \xi + i\eta$ is an analytic function of z, $\zeta = \zeta(z)$. We assume that $\zeta'(z)$ is nonzero on the domain of interest. $u(x, y)$ is an arbitrary smooth function of x and y. When expressed in terms of ξ and η, $u(x, y) = \upsilon(\xi, \eta)$. In Exercise 10.13 we showed that

$$\frac{\partial^2 \upsilon}{\partial \xi^2} + \frac{\partial^2 \upsilon}{\partial \eta^2} = \left| \frac{\mathrm{d}\zeta}{\mathrm{d}z} \right|^{-2} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

1. Show that if u satisfies Laplace's equation in the z-plane,
$$u_{xx} + u_{yy} = 0,$$
then υ satisfies Laplace's equation in the ζ-plane,
$$\upsilon_{\xi\xi} + \upsilon_{\eta\eta} = 0,$$

2. Show that if u satisfies Helmholtz's equation in the z-plane,
$$u_{xx} + u_{yy} = \lambda u,$$
then in the ζ-plane υ satisfies
$$\upsilon_{\xi\xi} + \upsilon_{\eta\eta} = \lambda \left| \frac{\mathrm{d}z}{\mathrm{d}\zeta} \right|^2 \upsilon.$$

3. Show that if u satisfies Poisson's equation in the z-plane,
$$u_{xx} + u_{yy} = f(x, y),$$
then υ satisfies Poisson's equation in the ζ-plane,
$$\upsilon_{\xi\xi} + \upsilon_{\eta\eta} = \left| \frac{\mathrm{d}z}{\mathrm{d}\zeta} \right|^2 \phi(\xi, \eta),$$
where $\phi(\xi, \eta) = f(x, y)$.

4. Show that if in the z-plane, u satisfies the Green function problem,

$$u_{xx} + u_{yy} = \delta(x - x_0)\delta(y - y_0),$$

then in the ζ-plane, υ satisfies the Green function problem,

$$\upsilon_{\xi\xi} + \upsilon_{\eta\eta} = \delta(\xi - \xi_0)\delta(\eta - \eta_0).$$

Exercise 47.2
A semi-circular rod of infinite extent is maintained at temperature $T = 0$ on the flat side and at $T = 1$ on the curved surface:

$$x^2 + y^2 = 1, \quad y > 0.$$

Use the conformal mapping

$$w = \xi + i\eta = \frac{1+z}{1-z}, \quad z = x + iy,$$

to formulate the problem in terms of ξ and η. Solve the problem in terms of these variables. This problem is solved with an eigenfunction expansion in Exercise **??**. Verify that the two solutions agree.

Exercise 47.3
Consider Laplace's equation on the domain $-\infty < x < \infty$, $0 < y < \pi$, subject to the mixed boundary conditions,

$$\begin{aligned} u = 1 \quad & \text{on } y = 0,\ x > 0, \\ u = 0 \quad & \text{on } y = \pi,\ x > 0, \\ u_y = 0 \quad & \text{on } y = 0,\ y = \pi,\ x < 0. \end{aligned}$$

Because of the mixed boundary conditions, (u and u_y are given on separate parts of the same boundary), this problem cannot be solved with separation of variables. Verify that the conformal map,

$$\zeta = \cosh^{-1}(\mathrm{e}^z),$$

with $z = x + iy$, $\zeta = \xi + i\eta$ maps the infinite interval into the semi-infinite interval, $\xi > 0$, $0 < \eta < \pi$. Solve Laplace's equation with the appropriate boundary conditions in the ζ plane by inspection. Write the solution u in terms of x and y.

47.2 Hints

Hint 47.1

Hint 47.2
Show that $w = (1 + z)/(1 - z)$ maps the semi-disc, $0 < r < 1$, $0 < \theta < \pi$ to the first quadrant of the w plane. Solve the problem for $v(\xi, \eta)$ by taking Fourier sine transforms in ξ and η.

To show that the solution for $v(\xi, \eta)$ is equivalent to the series expression for $u(r, \theta)$, first find an analytic function $g(w)$ of which $v(\xi, \eta)$ is the imaginary part. Change variables to z to obtain the analytic function $f(z) = g(w)$. Expand $f(z)$ in a Taylor series and take the imaginary part to show the equivalence of the solutions.

Hint 47.3
To see how the boundary is mapped, consider the map,

$$z = \log(\cosh \zeta).$$

The problem in the ζ plane is,

$$v_{\xi\xi} + v_{\eta\eta} = 0, \quad \xi > 0, \quad 0 < \eta < \pi,$$
$$v_\xi(0, \eta) = 0, \quad v(\xi, 0) = 1, \quad v(\xi, \pi) = 0.$$

To solve this, find a plane that satisfies the boundary conditions.

47.3 Solutions

Solution 47.1

$$\frac{\partial^2 \upsilon}{\partial \xi^2} + \frac{\partial^2 \upsilon}{\partial \eta^2} = \left| \frac{d\zeta}{dz} \right|^{-2} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

1.

$$u_{xx} + u_{yy} = 0$$
$$\left| \frac{d\zeta}{dz} \right|^2 (\upsilon_{\xi\xi} + \upsilon_{\eta\eta}) = 0$$
$$\upsilon_{\xi\xi} + \upsilon_{\eta\eta} = 0$$

2.

$$u_{xx} + u_{yy} = \lambda u$$
$$\left| \frac{d\zeta}{dz} \right|^2 (\upsilon_{\xi\xi} + \upsilon_{\eta\eta}) = \lambda \upsilon$$
$$\upsilon_{\xi\xi} + \upsilon_{\eta\eta} = \lambda \left| \frac{dz}{d\zeta} \right|^2 \upsilon$$

3.

$$u_{xx} + u_{yy} = f(x, y)$$
$$\left| \frac{d\zeta}{dz} \right|^2 (\upsilon_{\xi\xi} + \upsilon_{\eta\eta}) = \phi(\xi, \eta)$$
$$\upsilon_{\xi\xi} + \upsilon_{\eta\eta} = \left| \frac{dz}{d\zeta} \right|^2 \phi(\xi, \eta)$$

4. The Jacobian of the mapping is

$$J = \begin{vmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{vmatrix} = x_\xi y_\eta - x_\eta y_\xi = x_\xi^2 + y_\xi^2.$$

Thus the Dirac delta function on the right side gets mapped to

$$\frac{1}{x_\xi^2 + y_\xi^2}\delta(\xi - \xi_0)\delta(\eta - \eta_0).$$

Next we show that $|\mathrm{d}z/\mathrm{d}\zeta|^2$ has the same value as the Jacobian.

$$\left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2 = (x_\xi + iy_\xi)(x_\xi - iy_\xi) = x_\xi^2 + y_\xi^2$$

Now we transform the Green function problem.

$$u_{xx} + u_{yy} = \delta(x - x_0)\delta(y - y_0)$$

$$\left|\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right|^2 (v_{\xi\xi} + v_{\eta\eta}) = \frac{1}{x_\xi^2 + y_\xi^2}\delta(\xi - \xi_0)\delta(\eta - \eta_0)$$

$$v_{\xi\xi} + v_{\eta\eta} = \delta(\xi - \xi_0)\delta(\eta - \eta_0)$$

Solution 47.2
The mapping,

$$w = \frac{1+z}{1-z},$$

maps the unit semi-disc to the first quadrant of the complex plane.

We write the mapping in terms of r and θ.

$$\xi + i\eta = \frac{1 + r\,\mathrm{e}^{i\theta}}{1 - r\,\mathrm{e}^{i\theta}} = \frac{1 - r^2 + i2r\sin\theta}{1 + r^2 - 2r\cos\theta}$$

$$\xi = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}$$
$$\eta = \frac{2r\sin\theta}{1 + r^2 - 2r\cos\theta}$$

Consider a semi-circle of radius r. The image of this under the conformal mapping is a semi-circle of radius $\frac{2r}{1-r^2}$ and center $\frac{1+r^2}{1-r^2}$ in the first quadrant of the w plane. This semi-circle intersects the ξ axis at $\frac{1-r}{1+r}$ and $\frac{1+r}{1-r}$. As r ranges from zero to one, these semi-circles cover the first quadrant of the w plane. (See Figure 47.1.)

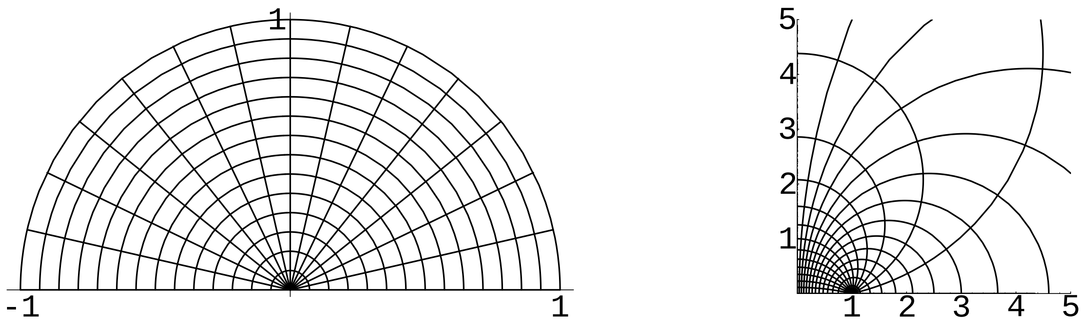

Figure 47.1: The conformal map, $w = \frac{1+z}{1-z}$.

We also note how the boundary of the semi-disc is mapped to the boundary of the first quadrant of the w plane. The line segment $\theta = 0$ is mapped to the real axis $\xi > 1$. The line segment $\theta = \pi$ is mapped to the real axis $0 < \xi < 1$. Finally, the semi-circle $r = 1$ is mapped to the positive imaginary axis.

The problem for $v(\xi, \eta)$ is,

$$v_{\xi\xi} + v_{\eta\eta} = 0, \quad \xi > 0, \quad \eta > 0,$$
$$v(\xi, 0) = 0, \quad v(0, \eta) = 1.$$

We will solve this problem with the Fourier sine transform. We take the Fourier sine transform of the partial differential equation, first in ξ and then in η.

$$-\alpha^2 \hat{v}(\alpha,\eta) + \frac{\alpha}{\pi} v(0,\eta) + \hat{v}(\alpha,\eta) = 0, \quad \hat{v}(\alpha,0) = 0$$

$$-\alpha^2 \hat{v}(\alpha,\eta) + \frac{\alpha}{\pi} + \hat{v}(\alpha,\eta) = 0, \quad \hat{v}(\alpha,0) = 0$$

$$-\alpha^2 \hat{\hat{v}}(\alpha,\beta) + \frac{\alpha}{\pi^2 \beta} - \beta^2 \hat{\hat{v}}(\alpha,\beta) + \frac{\beta}{\pi} \hat{v}(\alpha,0) = 0$$

$$\hat{\hat{v}}(\alpha,\beta) = \frac{\alpha}{\pi^2 \beta(\alpha^2 + \beta^2)}$$

Now we utilize the Fourier sine transform pair,

$$\mathcal{F}_s\left[\mathrm{e}^{-cx}\right] = \frac{\omega/\pi}{\omega^2 + c^2},$$

to take the inverse sine transform in α.

$$\hat{v}(\xi,\beta) = \frac{1}{\pi\beta} \mathrm{e}^{-\beta\xi}$$

With the Fourier sine transform pair,

$$\mathcal{F}_s\left[2\arctan\left(\frac{x}{c}\right)\right] = \frac{1}{\omega} \mathrm{e}^{-c\omega},$$

we take the inverse sine transform in β to obtain the solution.

$$\boxed{v(\xi,\eta) = \frac{2}{\pi} \arctan\left(\frac{\eta}{\xi}\right)}$$

Since v is harmonic, it is the imaginary part of an analytic function $g(w)$. By inspection, we see that this function is

$$g(w) = \frac{2}{\pi} \log(w).$$

We change variables to z, $f(z) = g(w)$.

$$f(z) = \frac{2}{\pi} \log\left(\frac{1+z}{1-z}\right)$$

We expand $f(z)$ in a Taylor series about $z = 0$,

$$f(z) = \frac{4}{\pi} \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{z^n}{n},$$

and write the result in terms of r and θ, $z = r\,\mathrm{e}^{i\theta}$.

$$f(z) = \frac{4}{\pi} \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{r^n\,\mathrm{e}^{i\theta}}{n}$$

$u(r, \theta)$ is the imaginary part of $f(z)$.

$$\boxed{u(r,\theta) = \frac{4}{\pi} \sum_{\substack{n=1\\ \text{odd} n}}^{\infty} \frac{1}{n} r^n \sin(n\theta)}$$

This demonstrates that the solutions obtained with conformal mapping and with an eigenfunction expansion in Exercise **??** agree.

Solution 47.3

Instead of working with the conformal map from the z plane to the ζ plane,

$$\zeta = \cosh^{-1}(\mathrm{e}^z),$$

it will be more convenient to work with the inverse map,

$$z = \log(\cosh\zeta),$$

which maps the semi-infinite strip to the infinite one. We determine how the boundary of the domain is mapped so that we know the appropriate boundary conditions for the semi-infinite strip domain.

$$
\begin{array}{llll}
\text{A} & \{\zeta : \xi > 0,\ \eta = 0\} & \mapsto & \{\log(\cosh(\xi)) : \xi > 0\} = \{z : x > 0,\ y = 0\} \\
\text{B} & \{\zeta : \xi > 0,\ \eta = \pi\} & \mapsto & \{\log(-\cosh(\xi)) : \xi > 0\} = \{z : x > 0,\ y = \pi\} \\
\text{C} & \{\zeta : \xi = 0,\ 0 < \eta < \pi/2\} & \mapsto & \{\log(\cos(\eta)) : 0 < \eta < \pi/2\} = \{z : x < 0,\ y = 0\} \\
\text{D} & \{\zeta : \xi = 0,\ \pi/2 < \eta < \pi\} & \mapsto & \{\log(\cos(\eta)) : \pi/2 < \eta < \pi\} = \{z : x < 0,\ y = \pi\}
\end{array}
$$

From the mapping of the boundary, we see that the solution $v(\xi, \eta) = u(x, y)$, is 1 on the bottom of the semi-infinite strip, 0 on the top. The normal derivative of v vanishes on the vertical boundary. See Figure 47.2.

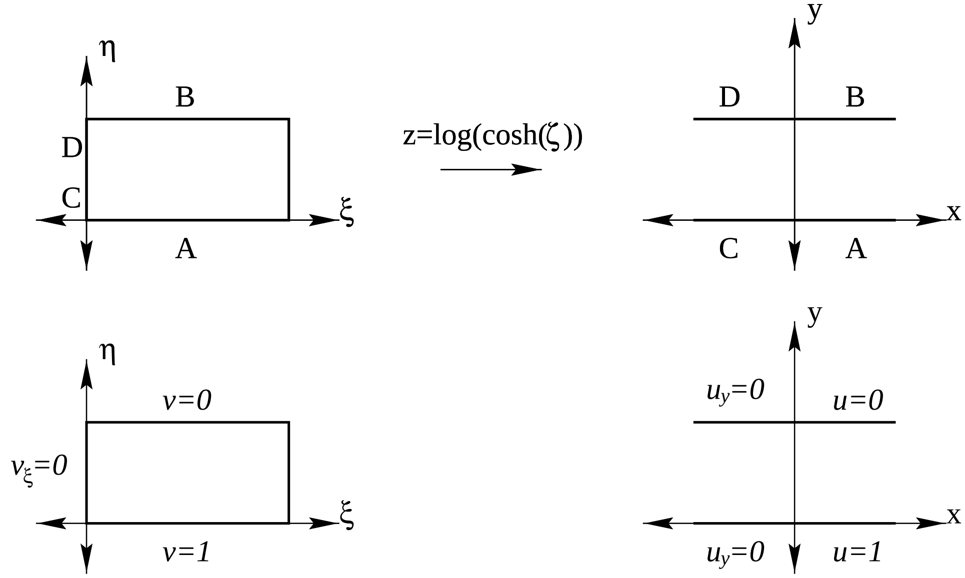

Figure 47.2: The mapping of the boundary conditions.

In the ζ plane, the problem is,

$$v_{\xi\xi} + v_{\eta\eta} = 0, \quad \xi > 0, \quad 0 < \eta < \pi,$$
$$v_\xi(0, \eta) = 0, \quad v(\xi, 0) = 1, \quad v(\xi, \pi) = 0.$$

By inspection, we see that the solution of this problem is,

$$\boxed{v(\xi, \eta) = 1 - \frac{\eta}{\pi}.}$$

The solution in the z plane is

$$u(x, y) = 1 - \frac{1}{\pi} \Im\left(\cosh^{-1}(\mathrm{e}^z)\right),$$

where $z = x + iy$. We will find the imaginary part of $\cosh^{-1}(\mathrm{e}^z)$ in order to write this explicitly in terms of x and y. Recall that we can write the $\cosh^{-1}$ in terms of the logarithm.

$$\begin{aligned}
\cosh^{-1}(w) &= \log\left(w + \sqrt{w^2 - 1}\right) \\
\cosh^{-1}(\mathrm{e}^z) &= \log\left(\mathrm{e}^z + \sqrt{\mathrm{e}^{2z} - 1}\right) \\
&= \log\left(\mathrm{e}^z\left(1 + \sqrt{1 - \mathrm{e}^{-2z}}\right)\right) \\
&= z + \log\left(1 + \sqrt{1 - \mathrm{e}^{-2z}}\right)
\end{aligned}$$

Now we need to find the imaginary part. We'll work from the inside out. First recall,

$$\sqrt{x + iy} = \sqrt{\sqrt{x^2 + y^2}\exp\left(i\tan^{-1}\left(\frac{y}{x}\right)\right)} = \sqrt[4]{x^2 + y^2}\exp\left(\frac{i}{2}\tan^{-1}\left(\frac{y}{x}\right)\right),$$

so that we can write the innermost factor as,

$$\begin{aligned}
\sqrt{1-\mathrm{e}^{-2z}} &= \sqrt{1-\mathrm{e}^{-2x}\cos(2y)+i\,\mathrm{e}^{-2x}\sin(2y)} \\
&= \sqrt[4]{(1-\mathrm{e}^{-2x}\cos(2y))^2+(\mathrm{e}^{-2x}\sin(2y))^2}\exp\left(\frac{i}{2}\tan^{-1}\left(\frac{\mathrm{e}^{-2x}\sin(2y)}{1-\mathrm{e}^{-2x}\cos(2y)}\right)\right) \\
&= \sqrt[4]{1-2\,\mathrm{e}^{-2x}\cos(2y)+\mathrm{e}^{-4x}}\exp\left(\frac{i}{2}\tan^{-1}\left(\frac{\sin(2y)}{\mathrm{e}^{2x}-\cos(2y)}\right)\right)
\end{aligned}$$

We substitute this into the logarithm.

$$\log\left(1+\sqrt{1-\mathrm{e}^{-2z}}\right)=\log\left(1+\sqrt[4]{1-2\,\mathrm{e}^{-2x}\cos(2y)+\mathrm{e}^{-4x}}\exp\left(\frac{i}{2}\tan^{-1}\left(\frac{\sin(2y)}{\mathrm{e}^{2x}-\cos(2y)}\right)\right)\right)$$

Now we can write η.

$$\eta=\Im\left(z+\log\left(1+\sqrt{1-\mathrm{e}^{-2z}}\right)\right)$$

$$\eta=y+\tan^{-1}\left(\frac{\sqrt[4]{1-2\,\mathrm{e}^{-2x}\cos(2y)+\mathrm{e}^{-4x}}\sin\left(\frac{1}{2}\tan^{-1}\left(\frac{\sin(2y)}{\mathrm{e}^{2x}-\cos(2y)}\right)\right)}{1+\sqrt[4]{1-2\,\mathrm{e}^{-2x}\cos(2y)+\mathrm{e}^{-4x}}\cos\left(\frac{1}{2}\tan^{-1}\left(\frac{\sin(2y)}{\mathrm{e}^{2x}-\cos(2y)}\right)\right)}\right)$$

Finally we have the solution, $u(x,y)$.

$$\boxed{u(x,y)=1-\frac{y}{\pi}-\frac{1}{\pi}\tan^{-1}\left(\frac{\sqrt[4]{1-2\,\mathrm{e}^{-2x}\cos(2y)+\mathrm{e}^{-4x}}\sin\left(\frac{1}{2}\tan^{-1}\left(\frac{\sin(2y)}{\mathrm{e}^{2x}-\cos(2y)}\right)\right)}{1+\sqrt[4]{1-2\,\mathrm{e}^{-2x}\cos(2y)+\mathrm{e}^{-4x}}\cos\left(\frac{1}{2}\tan^{-1}\left(\frac{\sin(2y)}{\mathrm{e}^{2x}-\cos(2y)}\right)\right)}\right)}$$

Chapter 48

Non-Cartesian Coordinates

48.1 Spherical Coordinates

Writing rectangular coordinates in terms of spherical coordinates,

$$x = r\cos\theta\sin\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\phi.$$

The Jacobian is

$$
\begin{aligned}
&\begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix} \\
&\quad = r^2\sin\phi \begin{vmatrix} \cos\theta\sin\phi & -\sin\theta & \cos\theta\cos\phi \\ \sin\theta\sin\phi & \cos\theta & \sin\theta\cos\phi \\ \cos\phi & 0 & -\sin\phi \end{vmatrix} \\
&\quad = \left| r^2\sin\phi(-\cos^2\theta\sin^2\phi - \sin^2\theta\cos^2\phi - \cos^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi) \right| \\
&\quad = r^2\sin\phi(\sin^2\phi + \cos^2\phi) \\
&\quad = r^2\sin\phi.
\end{aligned}
$$

Thus we have that

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(r,\theta,\phi) r^2 \sin\phi\,dr\,d\theta\,d\phi.$$

48.2 Laplace's Equation in a Disk

Consider Laplace's equation in polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad 0 \le r \le 1$$

subject to the the boundary conditions

1. $u(1,\theta) = f(\theta)$
2. $u_r(1,\theta) = g(\theta)$.

We separate variables with $u(r,\theta) = R(r)T(\theta)$.

$$\frac{1}{r}(R'T + rR''T) + \frac{1}{r^2}RT'' = 0$$
$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{T''}{T} = \lambda$$

Thus we have the two ordinary differential equations

$$T'' + \lambda T = 0, \qquad T(0) = T(2\pi), \quad T'(0) = T'(2\pi)$$
$$r^2 R'' + rR' - \lambda R = 0, \qquad R(0) < \infty.$$

The eigenvalues and eigenfunctions for the equation in T are

$$\lambda_0 = 0, \qquad T_0 = \frac{1}{2}$$
$$\lambda_n = n^2, \qquad T_n^{(1)} = \cos(n\theta), \quad T_n^{(2)} = \sin(n\theta)$$

(I chose $T_0 = 1/2$ so that all the eigenfunctions have the same norm.)

For $\lambda = 0$ the general solution for R is

$$R = c_1 + c_2 \log r.$$

Requiring that the solution be bounded gives us

$$R_0 = 1.$$

For $\lambda = n^2 > 0$ the general solution for R is

$$R = c_1 r^n + c^2 r^{-n}.$$

Requiring that the solution be bounded gives us

$$R_n = r^n.$$

Thus the general solution for u is

$$\boxed{u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].}$$

For the boundary condition $u(1,\theta) = f(\theta)$ we have the equation

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

If $f(\theta)$ has a Fourier series then the coefficients are

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta.
\end{aligned}$$

For the boundary condition $u_r(1,\theta) = g(\theta)$ we have the equation

$$g(\theta) = \sum_{n=1}^{\infty} n\left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

$g(\theta)$ has a series of this form only if

$$\int_0^{2\pi} g(\theta)\,d\theta = 0.$$

The coefficients are

$$a_n = \frac{1}{n\pi} \int_0^{2\pi} g(\theta) \cos(n\theta)\, d\theta$$
$$b_n = \frac{1}{n\pi} \int_0^{2\pi} g(\theta) \sin(n\theta)\, d\theta.$$

48.3 Laplace's Equation in an Annulus

Consider the problem

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad 0 \leq r < a, \quad -\pi < \theta \leq \pi,$$

with the boundary condition

$$u(a, \theta) = \theta^2.$$

So far this problem only has one boundary condition. By requiring that the solution be finite, we get the boundary condition

$$|u(0, \theta)| < \infty.$$

By specifying that the solution be C^1, (continuous and continuous first derivative) we obtain

$$u(r, -\pi) = u(r, \pi) \qquad \text{and} \qquad \frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi).$$

We will use the method of separation of variables. We seek solutions of the form

$$u(r, \theta) = R(r)\Theta(\theta).$$

Substituting into the partial differential equation,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$
$$R''\Theta + \frac{1}{r}R'\Theta = -\frac{1}{r^2}R\Theta''$$
$$\frac{r^2 R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

Now we have the boundary value problem for Θ,

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \qquad -\pi < \theta \leq \pi,$$

subject to

$$\Theta(-\pi) = \Theta(\pi) \qquad \text{and} \qquad \Theta'(-\pi) = \Theta'(\pi)$$

We consider the following three cases for the eigenvalue, λ,

$\boldsymbol{\lambda < 0}$. No linear combination of the solutions, $\Theta = \exp(\sqrt{-\lambda}\theta), \exp(-\sqrt{-\lambda}\theta)$, can satisfy the boundary conditions. Thus there are no negative eigenvalues.

$\boldsymbol{\lambda = 0}$. The general solution solution is $\Theta = a + b\theta$. By applying the boundary conditions, we get $\Theta = a$. Thus we have the eigenvalue and eigenfunction,

$$\lambda_0 = 0, \qquad A_0 = 1.$$

$\boldsymbol{\lambda > 0}$. The general solution is $\Theta = a\cos(\sqrt{\lambda}\theta) + b\sin(\sqrt{\lambda}\theta)$. Applying the boundary conditions yields the eigenvalues

$$\lambda_n = n^2, \quad n = 1, 2, 3, \ldots$$

with the associated eigenfunctions

$$A_n = \cos(n\theta) \quad \text{and} \quad B_n = \sin(n\theta).$$

The equation for R is

$$r^2 R'' + rR' - \lambda_n R = 0.$$

In the case $\lambda_0 = 0$, this becomes

$$R'' = -\frac{1}{r} R'$$
$$R' = \frac{a}{r}$$
$$R = a \log r + b$$

Requiring that the solution be bounded at $r = 0$ yields (to within a constant multiple)

$$R_0 = 1.$$

For $\lambda_n = n^2$, $n \geq 1$, we have

$$r^2 R'' + rR' - n^2 R = 0$$

Recognizing that this is an Euler equation and making the substitution $R = r^\alpha$,

$$\alpha(\alpha - 1) + \alpha - n^2 = 0$$
$$\alpha = \pm n$$
$$R = ar^n + br^{-n}.$$

requiring that the solution be bounded at $r = 0$ we obtain (to within a constant multiple)

$$R_n = r^n$$

The general solution to the partial differential equation is a linear combination of the eigenfunctions

$$u(r, \theta) = c_0 + \sum_{n=1}^{\infty} \left[c_n r^n \cos n\theta + d_n r^n \sin n\theta \right].$$

We determine the coefficients of the expansion with the boundary condition

$$u(a, \theta) = \theta^2 = c_0 + \sum_{n=1}^{\infty} \left[c_n a^n \cos n\theta + d_n a^n \sin n\theta \right].$$

We note that the eigenfunctions 1, $\cos n\theta$, and $\sin n\theta$ are orthogonal on $-\pi \leq \theta \leq \pi$. Integrating the boundary condition from $-\pi$ to π yields

$$\int_{-\pi}^{\pi} \theta^2 \, d\theta = \int_{-\pi}^{\pi} c_0 \, d\theta$$
$$c_0 = \frac{\pi^2}{3}.$$

Multiplying the boundary condition by $\cos m\theta$ and integrating gives

$$\int_{-\pi}^{\pi} \theta^2 \cos m\theta \, d\theta = c_m a^m \int_{-\pi}^{\pi} \cos^2 m\theta \, d\theta$$
$$c_m = \frac{(-1)^m 8\pi}{m^2 a^m}.$$

We multiply by $\sin m\theta$ and integrate to get

$$\int_{-\pi}^{\pi} \theta^2 \sin m\theta \, d\theta = d_m a^m \int_{-\pi}^{\pi} \sin^2 m\theta \, d\theta$$
$$d_m = 0$$

Thus the solution is

$$\boxed{u(r, \theta) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 8\pi}{n^2 a^n} r^n \cos n\theta.}$$

Part VI

Calculus of Variations

Chapter 49

Calculus of Variations

49.1 Exercises

Exercise 49.1
Discuss the problem of minimizing $\int_0^\alpha ((y')^4 - 6(y')^2)\,dx$, $y(0) = 0$, $y(\alpha) = \beta$. Consider both $C^1[0,\alpha]$ and $C_p^1[0,\alpha]$, and comment (with reasons) on whether your answers are weak or strong minima.

Exercise 49.2
Consider

1. $\int_{x_0}^{x_1} (a(y')^2 + byy' + cy^2)\,dx$, $y(x_0) = y_0$, $y(x_1) = y_1$, $a \neq 0$,
2. $\int_{x_0}^{x_1} (y')^3\,dx$, $y(x_0) = y_0$, $y(x_1) = y_1$.

Can these functionals have broken extremals, and if so, find them.

Exercise 49.3
Discuss finding a weak extremum for the following:

1. $\int_0^1 \left((y'')^2 - 2xy\right)\,dx, \qquad y(0) = y'(0) = 0, \quad y(1) = \frac{1}{120}$
2. $\int_0^1 \left(\frac{1}{2}(y')^2 + yy' + y' + y\right)\,dx$
3. $\int_a^b (y^2 + 2xyy')\,dx, \qquad y(a) = A, \quad y(b) = B$
4. $\int_0^1 (xy + y^2 - 2y^2y')\,dx, \qquad y(0) = 1, \quad y(1) = 2$

Exercise 49.4
Find the natural boundary conditions associated with the following functionals:

1. $\iint_D F(x, y, u, u_x, u_y)\,dx\,dy$

2. $\iint_D \left(p(x,y)(u_x^2+u_y^2) - q(x,y)u^2\right) \, dx \, dy + \int_\Gamma \sigma(x,y) u^2 \, ds$

Here D represents a closed boundary domain with boundary Γ, and ds is the arc-length differential. p and q are known in D, and σ is known on Γ.

Exercise 49.5

The equations for water waves with free surface $y = h(x,t)$ and bottom $y = 0$ are

$$\begin{aligned} &\phi_{xx} + \phi_{yy} = 0 && 0 < y < h(x,t), \\ &\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + gy = 0 && \text{on } y = h(x,t), \\ &h_t + \phi_x h_x - \phi_y = 0, && \text{on } y = h(x,t), \\ &\phi_y = 0 && \text{on } y = 0, \end{aligned}$$

where the fluid motion is described by $\phi(x,y,t)$ and g is the acceleration of gravity. Show that all these equations may be obtained by varying the functions $\phi(x,y,t)$ and $h(x,t)$ in the variational principle

$$\delta \iint_R \left(\int_0^{h(x,t)} \left(\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + gy \right) dy \right) dx \, dt = 0,$$

where R is an arbitrary region in the (x,t) plane.

Exercise 49.6

Extremize the functional $\int_a^b F(x,y,y') \, dx$, $y(a) = A$, $y(b) = B$ given that the admissible curves can not penetrate the interior of a given region R in the (x,y) plane. Apply your results to find the curves which extremize $\int_0^{10} (y')^3 \, dx$, $y(0) = 0$, $y(10) = 0$ given that the admissible curves can not penetrate the interior of the circle $(x-5)^2 + y^2 = 9$.

Exercise 49.7
Consider the functional $\int \sqrt{y}\, ds$ where ds is the arc-length differential ($ds = \sqrt{(dx)^2 + (dy)^2}$). Find the curve or curves from a given vertical line to a given fixed point $B = (x_1, y_1)$ which minimize this functional. Consider both the classes C^1 and C^1_p.

Exercise 49.8
A perfectly flexible uniform rope of length L hangs in equilibrium with one end fixed at (x_1, y_1) so that it passes over a frictionless pin at (x_2, y_2). What is the position of the free end of the rope?

Exercise 49.9
The drag on a supersonic airfoil of chord c and shape $y = y(x)$ is proportional to

$$D = \int_0^c \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 dx.$$

Find the shape for minimum drag if the moment of inertia of the contour with respect to the x-axis is specified; that is, find the shape for minimum drag if

$$\int_0^c y^2\, dx = A, \qquad y(0) = y(c) = 0, \qquad (c, A \text{ given}).$$

Exercise 49.10
The deflection y of a beam executing free (small) vibrations of frequency ω satisfies the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - \rho\omega^2 y = 0,$$

where EI is the flexural rigidity and ρ is the linear mass density. Show that the deflection modes are extremals of the problem

$$\delta\omega^2 \equiv \delta\left(\frac{\int_0^L EI(y'')^2\, dx}{\int_0^L \rho y^2\, dx}\right) = 0, \qquad (L = \text{length of beam})$$

when appropriate homogeneous end conditions are prescribed. Show that stationary values of the ratio are the squares of the natural frequencies.

Exercise 49.11
A boatman wishes to steer his boat so as to minimize the transit time required to cross a river of width l. The path of the boat is given parametrically by

$$x = X(t), \quad y = Y(t),$$

for $0 \leq t \leq T$. The river has no cross currents, so the current velocity is directed downstream in the y-direction. v_0 is the constant boat speed relative to the surrounding water, and $w = w(x, y, t)$ denotes the downstream river current at point (x, y) at time t. Then,

$$\dot{X}(t) = v_0 \cos \alpha(t), \qquad \dot{Y}(t) = v_0 \sin \alpha(t) + w,$$

where $\alpha(t)$ is the steering angle of the boat at time t. Find the steering control function $\alpha(t)$ and the final time T that will transfer the boat from the initial state $(X(0), Y(0)) = (0, 0)$ to the final state at $X(t) = l$ in such a way as to minimize T.

Exercise 49.12
Two particles of equal mass m are connected by an inextensible string which passes through a hole in a smooth horizontal table. The first particle is on the table moving with angular velocity $\omega = \sqrt{g/\alpha}$ in a circular path, of radius α, around the hole. The second particle is suspended vertically and is in equilibrium. At time $t = 0$, the suspended mass is pulled downward a short distance and released while the first mass continues to rotate.

1. If x represents the distance of the second mass below its equilibrium at time t and θ represents the angular position of the first particle at time t, show that the Lagrangian is given by

   $$L = m\left(\dot{x}^2 + \frac{1}{2}(\alpha - x)^2\dot{\theta}^2 + gx\right)$$

   and obtain the equations of motion.

2. In the case where the displacement of the suspended mass from equilibrium is small, show that the suspended mass performs small vertical oscillations and find the period of these oscillations.

Exercise 49.13
A rocket is propelled vertically upward so as to reach a prescribed height h in minimum time while using a given fixed quantity of fuel. The vertical distance $x(t)$ above the surface satisfies,

$$m\ddot{x} = -mg + mU(t), \qquad x(0) = 0, \quad \dot{(x)}(0) = 0,$$

where $U(t)$ is the acceleration provided by engine thrust. We impose the terminal constraint $x(T) = h$, and we wish to find the particular thrust function $U(t)$ which will minimize T assuming that the total thrust of the rocket engine over the entire thrust time is limited by the condition,

$$\int_0^T U^2(t)\,dt = k^2.$$

Here k is a given positive constant which measures the total amount of fuel available.

Exercise 49.14
A space vehicle moves along a straight path in free space. $x(t)$ is the distance to its docking pad, and a, b are its position and speed at time $t = 0$. The equation of motion is

$$\ddot{x} = M \sin V, \qquad x(0) = a, \quad \dot{x}(0) = b,$$

where the control function $V(t)$ is related to the rocket acceleration $U(t)$ by $U = M \sin V$, $M = \text{const}$. We wish to dock the vehicle in minimum time; that is, we seek a thrust function $U(t)$ which will minimize the final time T while bringing the vehicle to rest at the origin with $x(T) = 0$, $\dot{x}(T) = 0$. Find $U(t)$, and in the $(x, \dot{x})$-plane plot the corresponding trajectory which transfers the state of the system from (a, b) to $(0, 0)$. Account for all values of a and b.

Exercise 49.15
Find a minimum for the functional $I(y) = \int_0^m \sqrt{y+h}\sqrt{1+(y')^2}\,dx$ in which $h > 0$, $y(0) = 0$, $y(m) = M > -h$. Discuss the nature of the minimum, (i.e., weak, strong, ...).

Exercise 49.16
Show that for the functional $\int n(x,y)\sqrt{1+(y')^2}\,dx$, where $n(x,y) \geq 0$ in some domain D, the Weierstrass E function $E(x,y,q,y')$ is non-negative for arbitrary finite p and y' at any point of D. What is the implication of this for Fermat's Principle?

Exercise 49.17
Consider the integral $\int \frac{1+y^2}{(y')^2}\,dx$ between fixed limits. Find the extremals, (hyperbolic sines), and discuss the Jacobi, Legendre, and Weierstrass conditions and their implications regarding weak and strong extrema. Also consider the value of the integral on any extremal compared with its value on the illustrated strong variation. Comment!

P_iQ_i are vertical segments, and the lines Q_iP_{i+1} are tangent to the extremal at P_{i+1}.

Exercise 49.18
Consider $I = \int_{x_0}^{x_1} y'(1+x^2y')\,dx$, $y(x_0) = y_0$, $y(x_1) = y_1$. Can you find continuous curves which will minimize I if

$$\begin{array}{lllll} (i) & x_0 = -1, & y_0 = 1, & x_1 = 2, & y_1 = 4, \\ (ii) & x_0 = 1, & y_0 = 3, & x_1 = 2, & y_1 = 5, \\ (iii) & x_0 = -1, & y_0 = 1, & x_1 = 2, & y_1 = 1. \end{array}$$

Exercise 49.19
Starting from

$$\iint_D (Q_x - P_y)\,dx\,dy = \int_\Gamma (P\,dx + Q\,dy)$$

prove that

$$
\begin{aligned}
&\text{(a)} \quad \iint_D \phi\psi_{xx}\,dx\,dy = \iint_D \psi\phi_{xx}\,dx\,dy + \int_\Gamma (\phi\psi_x - \psi\phi_x)\,dy,\\
&\text{(b)} \quad \iint_D \phi\psi_{yy}\,dx\,dy = \iint_D \psi\phi_{yy}\,dx\,dy - \int_\Gamma (\phi\psi_y - \psi\phi_y)\,dx,\\
&\text{(c)} \quad \iint_D \phi\psi_{xy}\,dx\,dy = \iint_D \psi\phi_{xy}\,dx\,dy - \frac{1}{2}\int_\Gamma (\phi\psi_x - \psi\phi_x)\,dx + \frac{1}{2}\int_\Gamma (\phi\psi_y - \psi\phi_y)\,dy.
\end{aligned}
$$

Then, consider

$$I(u) = \int_{t_0}^{t_1} \iint_D \left(-(u_{xx} + u_{yy})^2 + 2(1-\mu)(u_{xx}u_{yy} - u_{xy}^2)\right)\,dx\,dy\,dt.$$

Show that

$$\delta I = \int_{t_0}^{t_1} \iint_D (-\nabla^4 u)\delta u\,dx\,dy\,dt + \int_{t_0}^{t_1} \int_\Gamma \left(P(u)\delta u + M(u)\frac{\partial \delta u}{\partial n}\right)\,ds\,dt,$$

where P and M are the expressions we derived in class for the problem of the vibrating plate.

Exercise 49.20

For the following functionals use the Rayleigh-Ritz method to find an approximate solution of the problem of minimizing the functionals and compare your answers with the exact solutions.

- $$\int_0^1 \left((y')^2 - y^2 - 2xy\right)\,dx, \quad y(0) = 0 = y(1).$$

  For this problem take an approximate solution of the form

  $$y = x(1-x)\left(a_0 + a_1 x + \cdots + a_n x^n\right),$$

  and carry out the solutions for $n = 0$ and $n = 1$.

- $$\int_0^2 \left((y')^2 + y^2 + 2xy\right) \, dx, \quad y(0) = 0 = y(2).$$

- $$\int_1^2 \left(x(y')^2 - \frac{x^2-1}{x} y^2 - 2x^2 y\right) \, dx, \quad y(1) = 0 = y(2)$$

Exercise 49.21
Let $K(x)$ belong to $L_1(-\infty, \infty)$ and define the operator T on $L_2(-\infty, \infty)$ by

$$T f(x) = \int_{-\infty}^{\infty} K(x - y) f(y) \, dy.$$

1. Show that the spectrum of T consists of the range of the Fourier transform $\hat{K}$ of K, (that is, the set of all values $\hat{K}(y)$ with $-\infty < y < \infty$), plus 0 if this is not already in the range. (Note: From the assumption on K it follows that $\hat{K}$ is continuous and approaches zero at $\pm\infty$.)

2. For λ in the spectrum of T, show that λ is an eigenvalue if and only if $\hat{K}$ takes on the value λ on at least some interval of positive length and that every other λ in the spectrum belongs to the continuous spectrum.

3. Find an explicit representation for $(T - \lambda I)^{-1} f$ for λ not in the spectrum, and verify directly that this result agrees with that givenby the Neumann series if λ is large enough.

Exercise 49.22
Let U be the space of twice continuously differentiable functions f on $[-1, 1]$ satisfying $f(-1) = f(1) = 0$, and $W = C[-1, 1]$. Let $L : U \mapsto W$ be the operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. Call λ in the spectrum of L if the following does not occur: There is a bounded linear transformation $T : W \mapsto U$ such that $(L - \lambda I) T f = f$ for all $f \in W$ and $T(L - \lambda I) f = f$ for all $f \in U$. Determine the spectrum of L.

Exercise 49.23
Solve the integral equations

1. $\phi(x) = x + \lambda \int_0^1 \left(x^2 y - y^2\right) \phi(y)\, dy$

2. $\phi(x) = x + \lambda \int_0^x K(x,y)\phi(y)\, dy$

where

$$K(x,y) = \begin{cases} \sin(xy) & \text{for } x \geq 1 \text{ and } y \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

In both cases state for which values of λ the solution obtained is valid.

Exercise 49.24

1. Suppose that $K = L_1 L_2$, where $L_1 L_2 - L_2 L_1 = I$. Show that if x is an eigenvector of K corresponding to the eigenvalue λ, then $L_1 x$ is an eigenvector of K corresponding to the eigenvalue $\lambda - 1$, and $L_2 x$ is an eigenvector corresponding to the eigenvalue $\lambda + 1$.

2. Find the eigenvalues and eigenfunctions of the operator $K \equiv -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{t^2}{4}$ in the space of functions $u \in L_2(-\infty, \infty)$. (Hint: $L_1 = \frac{t}{2} + \frac{\mathrm{d}}{\mathrm{d}t}$, $L_2 = \frac{t}{2} - \frac{\mathrm{d}}{\mathrm{d}t}$. $\mathrm{e}^{-t^2/4}$ is the eigenfunction corresponding to the eigenvalue $1/2$.)

Exercise 49.25
Prove that if the value of $\lambda = \lambda_1$ is in the residual spectrum of T, then $\overline{\lambda_1}$ is in the discrete spectrum of T^*.

Exercise 49.26
Solve

1.

$$u''(t) + \int_0^1 \sin(k(s-t))u(s)\,ds = f(t), \quad u(0) = u'(0) = 0.$$

2.

$$u(x) = \lambda \int_0^\pi K(x,s)u(s)\,ds$$

where

$$K(x,s) = \frac{1}{2}\log\left|\frac{\sin\left(\frac{x+s}{2}\right)}{\sin\left(\frac{x-s}{2}\right)}\right| = \sum_{n=1}^\infty \frac{\sin nx \sin ns}{n}$$

3.

$$\phi(s) = \lambda \int_0^{2\pi} \frac{1}{2\pi} \frac{1-h^2}{1-2h\cos(s-t)+h^2}\phi(t)\,dt, \quad |h| < 1$$

4.

$$\phi(x) = \lambda \int_{-\pi}^{\pi} \cos^n(x-\xi)\phi(\xi)\,d\xi$$

Exercise 49.27
Let $K(x,s) = 2\pi^2 - 6\pi|x-s| + 3(x-s)^2$.

1. Find the eigenvalues and eigenfunctions of

$$\phi(x) = \lambda \int_0^{2\pi} K(x,s)\phi(s)\,ds.$$

(Hint: Try to find an expansion of the form

$$K(x,s) = \sum_{n=-\infty}^{\infty} c_n \, \mathrm{e}^{in(x-s)}.)$$

2. Do the eigenfunctions form a complete set? If not, show that a complete set may be obtained by adding a suitable set of solutions of

$$\int_0^{2\pi} K(x,s)\phi(s)\, ds = 0.$$

3. Find the resolvent kernel $\Gamma(x,s,\lambda)$.

Exercise 49.28

Let $K(x,s)$ be a bounded self-adjoint kernel on the finite interval (a,b), and let T be the integral operator on $L_2(a,b)$ with kernel $K(x,s)$. For a polynomial $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ we define the operator $p(T) = a_0 I + a_1 T + \cdots + a_n T^n$. Prove that the eigenvalues of $p(T)$ are exactly the numbers $p(\lambda)$ with λ an eigenvalue of T.

Exercise 49.29

Show that if $f(x)$ is continuous, the solution of

$$\phi(x) = f(x) + \lambda \int_0^{\infty} \cos(2xs)\phi(s)\, ds$$

is

$$\phi(x) = \frac{f(x) + \lambda \int_0^{\infty} f(s) \cos(2xs)\, ds}{1 - \pi\lambda^2/4}.$$

Exercise 49.30
Consider

$$Lu = 0 \text{ in } D, \quad u = f \text{ on } C,$$

where

$$Lu \equiv u_{xx} + u_{yy} + au_x + bu_y + cu.$$

Here a, b and c are continuous functions of (x, y) on $D + C$. Show that the adjoint L^* is given by

$$L^* v = v_{xx} + v_{yy} - av_x - bv_y + (c - a_x - b_y)v$$

and that

$$\int_D (vLu - uL^* v) = \int_C H(u, v), \tag{49.1}$$

where

$$\begin{aligned} H(u, v) &\equiv (vu_x - uv_x + auv)\, dy - (vu_y - uv_y + buv)\, dx \\ &= \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} + auv\frac{\partial x}{\partial n} + buv\frac{\partial y}{\partial n} \right) ds. \end{aligned}$$

Take v in (49.1) to be the harmonic Green function G given by

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \log\left(\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} \right) + \cdots,$$

and show formally, (use Delta functions), that (49.1) becomes

$$-u(\xi, \eta) - \int_D u(L^* - \Delta)G\, dx\, dy = \int_C H(u, G) \tag{49.2}$$

where u satisfies $Lu = 0$, ($\Delta G = \delta$ in D, $G = 0$ on C). Show that (49.2) can be put into the forms

$$u + \int_D \big((c - a_x - b_y)G - aG_x - bG_y\big) u \, dx \, dy = U \tag{49.3}$$

and

$$u + \int_D (au_x + bu_y + cu) G \, dx \, dy = U, \tag{49.4}$$

where U is the known harmonic function in D with assumes the boundary values prescribed for u. Finally, rigorously show that the integrodifferential equation (49.4) can be solved by successive approximations when the domain D is small enough.

Exercise 49.31
Find the eigenvalues and eigenfunctions of the following kernels on the interval $[0, 1]$.

1.
$$K(x, s) = \min(x, s)$$

2.
$$K(x, s) = \mathrm{e}^{\min(x,s)}$$

(Hint: $\phi'' + \phi' + \lambda \, \mathrm{e}^x \phi = 0$ can be solved in terms of Bessel functions.)

Exercise 49.32
Use Hilbert transforms to evaluate

1. $\displaystyle ⨍_{-\infty}^{\infty} \frac{\sin(kx)\sin(lx)}{x^2 - z^2} \, dx$

2. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(px) - \cos(qx)}{x^2}\, dx$

3. $\displaystyle\int_{-\infty}^{\infty} \frac{-(x^2 - ab)\sin x + (a+b)x\cos x}{x(x^2+a^2)(x^2+b^2)}\, dx$

Exercise 49.33
Show that

$$\int_{-\infty}^{\infty} \frac{(1-t^2)^{1/2}\log(1+t)}{t-x}\, dt = \pi\left(x\log 2 - 1 + (1-x^2)^{1/2}\left(\frac{\pi}{2} - \arcsin(x)\right)\right).$$

Exercise 49.34
Let C be a simple closed contour. Let $g(t)$ be a given function and consider

$$\frac{1}{i\pi} \int_C \frac{f(t)\, dt}{t - t_0} = g(t_0) \qquad (49.5)$$

Note that the left side can be written as $F^+(t_0) + F^-(t_0)$. Define a function $W(z)$ such that $W(z) = F(z)$ for z inside C and $W(z) = -F(z)$ for z outside C. Proceeding in this way, show that the solution of (49.5) is given by

$$f(t_0) = \frac{1}{i\pi} \int_C \frac{g(t)\, dt}{t - t_0}.$$

Exercise 49.35
If C is an arc with endpoints α and β, evaluate

(i) $\displaystyle\frac{1}{i\pi} \int_C \frac{1}{(\tau-\beta)^{1-\gamma}(\tau-\alpha)^{\gamma}(\tau-\zeta)}\, d\tau,$ where $0 < \gamma < 1$

(ii) $\displaystyle\frac{1}{i\pi} \int_C \left(\frac{\tau-\beta}{\tau-\alpha}\right)^{\gamma} \frac{\tau^n}{\tau-\zeta}\, d\tau,$ where $0 < \gamma < 1$, integer $n \geq 0$.

Exercise 49.36
Solve

$$\int_{-1}^{1} \frac{\phi(y)}{y^2 - x^2}\, dy = f(x).$$

Exercise 49.37
Solve

$$\frac{1}{i\pi} \int_{0}^{1} \frac{f(t)}{t - x}\, dt = \lambda f(x), \quad \text{where } -1 < \lambda < 1.$$

Are there any solutions for $\lambda > 1$? (The operator on the left is self-adjoint. Its spectrum is $-1 \leq \lambda \leq 1$.)

Exercise 49.38
Show that the general solution of

$$\frac{\tan(x)}{\pi} \int_{0}^{1} \frac{f(t)}{t - x}\, dt = f(x)$$

is

$$f(x) = \frac{k \sin(x)}{(1 - x)^{1 - x/\pi} x^{x/\pi}}.$$

Exercise 49.39
Show that the general solution of

$$f'(x) + \lambda \int_{C} \frac{f(t)}{t - x}\, dt = 1$$

is given by

$$f(x) = \frac{1}{i\pi\lambda} + k\,\mathrm{e}^{-i\pi\lambda x},$$

(k is a constant). Here C is a simple closed contour, λ a constant and $f(x)$ a differentiable function on C. Generalize the result to the case of an arbitrary function $g(x)$ on the right side, where $g(x)$ is analytic inside C.

Exercise 49.40
Show that the solution of

$$-\!\!\!\!\!\!\int_C \left(\frac{1}{t-x} + P(t-x) \right) f(t)\, dt = g(x)$$

is given by

$$f(t) = -\frac{1}{\pi^2} -\!\!\!\!\!\!\int_C \frac{g(\tau)}{\tau - t}\, d\tau - \frac{1}{\pi^2} \int_C g(\tau) P(\tau - t)\, d\tau.$$

Here C is a simple closed curve, and $P(t)$ is a given entire function of t.

Exercise 49.41
Solve

$$-\!\!\!\!\!\!\int_0^1 \frac{f(t)}{t-x}\, dt + -\!\!\!\!\!\!\int_2^3 \frac{f(t)}{t-x}\, dt = x$$

where this equation is to hold for x in either $(0,1)$ or $(2,3)$.

Exercise 49.42
Solve

$$\int_0^x \frac{f(t)}{\sqrt{x-t}}\, dt + A \int_x^1 \frac{f(t)}{\sqrt{t-x}}\, dt = 1$$

where A is a real positive constant. Outline briefly the appropriate method of A is a function of x.

49.2 Hints

Hint 49.1

Hint 49.2

Hint 49.3

Hint 49.4

Hint 49.5

Hint 49.6

Hint 49.7

Hint 49.8

Hint 49.9

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Hint 49.35

Hint 49.36

Hint 49.37

Hint 49.38

Hint 49.39

Hint 49.40

Hint 49.41

Hint 49.42

49.3 Solutions

Solution 49.1

$C^1[0,\alpha]$ Extremals

Admissible Extremal. First we consider continuously differentiable extremals. Because the Lagrangian is a function of y' alone, we know that the extremals are straight lines. Thus the admissible extremal is

$$\hat{y} = \frac{\beta}{\alpha} x.$$

Legendre Condition.

$$\begin{aligned} \hat{F}_{y'y'} &= 12(\hat{y}')^2 - 12 \\ &= 12\left(\left(\frac{\beta}{\alpha}\right)^2 - 1\right) \\ &\begin{cases} < 0 & \text{for } |\beta/\alpha| < 1 \\ = 0 & \text{for } |\beta/\alpha| = 1 \\ > 0 & \text{for } |\beta/\alpha| > 1 \end{cases} \end{aligned}$$

Thus we see that $\frac{\beta}{\alpha}x$ may be a minimum for $|\beta/\alpha| \geq 1$ and may be a maximum for $|\beta/\alpha| \leq 1$.

Jacobi Condition. Jacobi's accessory equation for this problem is

$$(\hat{F}_{,y'y'} h')' = 0$$

$$\left(12\left(\left(\frac{\beta}{\alpha}\right)^2 - 1\right) h'\right)' = 0$$

$$h'' = 0$$

The problem $h'' = 0$, $h(0) = 0$, $h(c) = 0$ has only the trivial solution for $c > 0$. Thus we see that there are no conjugate points and the admissible extremal satisfies the strengthened Legendre condition.

A Weak Minimum. For $|\beta/\alpha| > 1$ the admissible extremal $\frac{\beta}{\alpha}x$ is a solution of the Euler equation, and satisfies the strengthened Jacobi and Legendre conditions. Thus it is a weak minima. (For $|\beta/\alpha| < 1$ it is a weak maxima for the same reasons.)

Weierstrass Excess Function. The Weierstrass excess function is

$$\begin{aligned}
E(x, \hat{y}, \hat{y}', w) &= F(w) - F(\hat{y}') - (w - \hat{y}')F_{,y'}(\hat{y}') \\
&= w^4 - 6w^2 - (\hat{y}')^4 + 6(\hat{y}')^2 - (w - \hat{y}')(4(\hat{y}')^3 - 12\hat{y}') \\
&= w^4 - 6w^2 - \left(\frac{\beta}{\alpha}\right)^4 + 6\left(\frac{\beta}{\alpha}\right)^2 - (w - \frac{\beta}{\alpha})(4\left(\frac{\beta}{\alpha}\right)^3 - 12\frac{\beta}{\alpha}) \\
&= w^4 - 6w^2 - w\left(4\frac{\beta}{\alpha}\left(\left(\frac{\beta}{\alpha}\right)^2 - 3\right)\right) + 3\left(\frac{\beta}{\alpha}\right)^4 - 6\left(\frac{\beta}{\alpha}\right)^2
\end{aligned}$$

We can find the stationary points of the excess function by examining its derivative. (Let $\lambda = \beta/\alpha$.)

$$E'(w) = 4w^3 - 12w + 4\lambda\left((\lambda)^2 - 3\right) = 0$$

$$w_1 = \lambda, \quad w_2 = \frac{1}{2}\left(-\lambda - \sqrt{4 - \lambda^2}\right) w_3 = \frac{1}{2}\left(-\lambda + \sqrt{4 - \lambda^2}\right)$$

The excess function evaluated at these points is

$$\begin{aligned}
E(w_1) &= 0, \\
E(w_2) &= \frac{3}{2}\left(3\lambda^4 - 6\lambda^2 - 6 - \sqrt{3}\lambda(4 - \lambda^2)^{3/2}\right), \\
E(w_3) &= \frac{3}{2}\left(3\lambda^4 - 6\lambda^2 - 6 + \sqrt{3}\lambda(4 - \lambda^2)^{3/2}\right).
\end{aligned}$$

$E(w_2)$ is negative for $-1 < \lambda < \sqrt{3}$ and $E(w_3)$ is negative for $-\sqrt{3} < \lambda < 1$. This implies that the weak minimum $\hat{y} = \beta x/\alpha$ is not a strong local minimum for $|\lambda| < \sqrt{3}|$. Since $E(w_1) = 0$, we cannot use the Weierstrass excess function to determine if $\hat{y} = \beta x/\alpha$ is a strong local minima for $|\beta/\alpha| > \sqrt{3}$.

Erdmann's Corner Conditions. Erdmann's corner conditions require that

$$\hat{F}_{,y'} = 4(\hat{y}')^3 - 12\hat{y}'$$

and

$$\hat{F} - \hat{y}'\hat{F}_{,y'} = (\hat{y}')^4 - 6(\hat{y}')^2 - \hat{y}'(4(\hat{y}')^3 - 12\hat{y}')$$

are continuous at corners. Thus the quantities

$$(\hat{y}')^3 - 3\hat{y}' \quad \text{and} \quad (\hat{y}')^4 - 2(\hat{y}')^2$$

are continuous. Denoting $p = \hat{y}'_-$ and $q = \hat{y}'_+$, the first condition has the solutions

$$p = q, \quad p = \frac{1}{2}\left(-q \pm \sqrt{3}\sqrt{4 - q^2}\right).$$

The second condition has the solutions,

$$p = \pm q, \quad p = \pm\sqrt{2 - q^2}$$

Combining these, we have

$$p = q, \quad p = \sqrt{3}, q = -\sqrt{3}, \quad p = -\sqrt{3}, q = \sqrt{3}.$$

Thus we see that there can be a corner only when $\hat{y}'_- = \pm\sqrt{3}$ and $\hat{y}'_+ = \mp\sqrt{3}$.

Case 1, $\beta = \pm\sqrt{3}\alpha$. Notice the the Lagrangian is minimized point-wise if $y' = \pm\sqrt{3}$. For this case the unique, strong global minimum is

$$\hat{y} = \sqrt{3}\,\operatorname{sign}(\beta) x.$$

Case 2, $|\beta| < \sqrt{3}|\alpha|$. For this case there are an infinite number of strong minima. Any piecewise linear curve satisfying $y'_-(x) = \pm\sqrt{3}$ and $y'_+(x) = \pm\sqrt{3}$ and $y(0) = 0$, $y(\alpha) = \beta$ is a strong minima.
Case 3, $|\beta| > \sqrt{3}|\alpha|$. First note that the extremal cannot have corners. Thus the unique extremal is $\hat{y} = \frac{\beta}{\alpha}x$. We know that this extremal is a weak local minima.

Solution 49.2

1.

$$\int_{x_0}^{x_1} \left(a(y')^2 + byy' + cy^2\right) dx, \quad y(x_0) = y_0, \quad y(x_1) = y_1, \quad a \neq 0$$

Erdmann's First Corner Condition. $\hat{F}_{y'} = 2a\hat{y}' + b\hat{y}$ must be continuous at a corner. This implies that $\hat{y}$ must be continuous, i.e., there are no corners.

The functional cannot have broken extremals.

2.

$$\int_{x_0}^{x_1} (y')^3\, dx, \quad y(x_0) = y_0, \quad y(x_1) = y_1$$

Erdmann's First Corner Condition. $\hat{F}_{y'} = 3(y')^2$ must be continuous at a corner. This implies that $\hat{y}'_- = \hat{y}'_+$.

Erdmann's Second Corner Condition. $\hat{F} - \hat{y}'\hat{F}_{y'} = (\hat{y}')^3 - \hat{y}'3(\hat{y}')^2 = -2(\hat{y}')^3$ must be continuous at a corner. This implies that $\hat{y}$ is continuous at a corner, i.e. there are no corners.

$$\boxed{\text{The functional cannot have broken extremals.}}$$

Solution 49.3

1.

$$\int_0^1 \left((y'')^2 - 2xy\right)\,dx, \qquad y(0) = y'(0) = 0, \quad y(1) = \frac{1}{120}$$

Euler's Differential Equation. We will consider C^4 extremals which satisfy Euler's DE,

$$(\hat{F}_{,y''})'' - (\hat{F}_{,y'})' + \hat{F}_{,y} = 0.$$

For the given Lagrangian, this is,

$$(2\hat{y}'')'' - 2x = 0.$$

Natural Boundary Condition. The first variation of the performance index is

$$\delta J = \int_0^1 (\hat{F}_{,y}\delta y + \hat{F}_{,y'}\delta y' + \hat{F}_{y''}\delta y'')\,dx.$$

From the given boundary conditions we have $\delta y(0) = \delta y'(0) = \delta y(1) = 0$. Using Euler's DE, we have,

$$\delta J = \int_0^1 ((\hat{F}_{y'} - (\hat{F}_{,y''})')'\delta y + \hat{F}_{,y'}\delta y' + \hat{F}_{y''}\delta y'')\,dx.$$

Now we apply integration by parts.

$$\begin{aligned}
\delta J &= \left[(\hat{F}_{y'} - (\hat{F}_{,y''})')\delta y\right]_0^1 + \int_0^1 (-(\hat{F}_{y'} - (\hat{F}_{,y''})')\delta y' + \hat{F}_{,y'}\delta y' + \hat{F}_{y''}\delta y'')\,dx \\
&= \int_0^1 ((\hat{F}_{,y''})'\delta y' + \hat{F}_{y''}\delta y'')\,dx \\
&= \left[\hat{F}_{,y''}\delta y'\right]_0^1 \\
&= \hat{F}_{,y''}(1)\delta y'(1)
\end{aligned}$$

In order that the first variation vanish, we need the natural boundary condition $\hat{F}_{,y''}(1) = 0$. For the given Lagrangian, this condition is

$$\hat{y}''(1) = 0.$$

The Extremal BVP. The extremal boundary value problem is

$$y'''' = x, \qquad y(0) = y'(0) = y''(1) = 0, \quad y(1) = \frac{1}{120}.$$

The general solution of the differential equation is

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \frac{1}{120}x^5.$$

Applying the boundary conditions, we see that the unique admissible extremal is

$$\hat{y} = \frac{x^2}{120}(x^3 - 5x + 5).$$

This may be a weak extremum for the problem.

Legendre's Condition. Since

$$\hat{F}_{,y''y''} = 2 > 0,$$

the strengthened Legendre condition is satisfied.

Jacobi's Condition. The second variation for $F(x, y, y'')$ is

$$\left.\frac{\mathrm{d}^2 J}{\mathrm{d}\epsilon^2}\right|_{\epsilon=0} = \int_a^b \left(\hat{F}_{,y''y''}(h'')^2 + 2\hat{F}_{,yy''}hh'' + \hat{F}_{,yy}h^2\right)\, dx$$

Jacobi's accessory equation is,

$$(2\hat{F}_{,y''y''}h'' + 2\hat{F}_{,yy''}h)'' + 2\hat{F}_{,yy''}h'' + 2\hat{F}_{,yy}h = 0,$$

$$(h'')'' = 0$$

Since the boundary value problem,

$$h'''' = 0, \qquad h(0) = h'(0) = h(c) = h''(c) = 0,$$

has only the trivial solution for all $c > 0$ the strengthened Jacobi condition is satisfied.

A Weak Minimum. Since the admissible extremal,

$$\hat{y} = \frac{x^2}{120}(x^3 - 5x + 5),$$

satisfies the strengthened Legendre and Jacobi conditions, we conclude that it is a weak minimum.

2.

$$\int_0^1 \left(\frac{1}{2}(y')^2 + yy' + y' + y\right)\, dx$$

Boundary Conditions. Since no boundary conditions are specified, we have the Euler boundary conditions,

$$\hat{F}_{,y'}(0) = 0, \quad \hat{F}_{,y'}(1) = 0.$$

The derivatives of the integrand are,

$$F_{,y} = y' + 1, \qquad F_{,y'} = y' + y + 1.$$

The Euler boundary conditions are then

$$\hat{y}'(0) + \hat{y}(0) + 1 = 0, \qquad \hat{y}'(1) + \hat{y}(1) + 1 = 0.$$

Erdmann's Corner Conditions. Erdmann's first corner condition specifies that

$$\hat{F}_{y'}(x) = \hat{y}'(x) + \hat{y}(x) + 1$$

must be continuous at a corner. This implies that $\hat{y}'(x)$ is continuous at corners, which means that there are no corners.

Euler's Differential Equation. Euler's DE is

$$(F_{,y'})' = F_y,$$

$$y'' + y' = y' + 1,$$

$$y'' = 1.$$

The general solution is

$$y = c_0 + c_1 x + \frac{1}{2}x^2.$$

The boundary conditions give us the constraints,

$$c_0 + c_1 + 1 = 0,$$
$$c_0 + 2c_1 + \frac{5}{2} = 0.$$

The extremal that satisfies the Euler DE and the Euler BC's is

$$\hat{y} = \frac{1}{2}\left(x^2 - 3x + 1\right).$$

Legendre's Condition. Since the strengthened Legendre condition is satisfied,

$$\hat{F}_{,y'y'}(x) = 1 > 0,$$

we conclude that the extremal is a weak local minimum of the problem.

Jacobi's Condition. Jacobi's accessory equation for this problem is,

$$\left(\hat{F}_{,y'y'} h'\right)' - \left(\hat{F}_{,yy} - (\hat{F}_{,yy'})'\right) h = 0, \qquad h(0) = h(c) = 0,$$

$$(h')' - (-(1)')\, h = 0, \qquad h(0) = h(c) = 0,$$

$$h'' = 0, \qquad h(0) = h(c) = 0,$$

Since this has only trivial solutions for $c > 0$ we conclude that there are no conjugate points. The extremal satisfies the strengthened Jacobi condition.

The only admissible extremal,

$$\hat{y} = \frac{1}{2}\left(x^2 - 3x + 1\right),$$

satisfies the strengthened Legendre and Jacobi conditions and is thus a weak extremum.

3.

$$\int_a^b (y^2 + 2xyy')\,dx, \qquad y(a) = A, \quad y(b) = B$$

Euler's Differential Equation. Euler's differential equation,

$$(F_{,y'})' = F_y,$$

$$(2xy)' = 2y + 2xy',$$

$$2y + 2xy' = 2y + 2xy',$$

is trivial. Every C^1 function satisfies the Euler DE.

Erdmann's Corner Conditions. The expressions,

$$\hat{F}_{,y'} = 2xy, \qquad \hat{F} - \hat{y}'\hat{F}_{,y'} = \hat{y}^2 + 2x\hat{y}\hat{y}' - \hat{y}'(2x\hat{h}) = \hat{y}^2$$

are continuous at a corner. The conditions are trivial and do not restrict corners in the extremal.

Extremal. Any piecewise smooth function that satisfies the boundary conditions $\hat{y}(a) = A$, $\hat{y}(b) = B$ is an admissible extremal.

An Exact Derivative. At this point we note that

$$\begin{aligned}\int_a^b (y^2 + 2xyy')\,dx &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}(xy^2)\,dx \\ &= \left[xy^2\right]_a^b \\ &= bB^2 - aA^2.\end{aligned}$$

The integral has the same value for all piecewise smooth functions y that satisfy the boundary conditions.

Since the integral has the same value for all piecewise smooth functions that satisfy the boundary conditions, all such functions are weak extrema.

4.

$$\int_0^1 (xy + y^2 - 2y^2 y')\, dx, \qquad y(0) = 1, \quad y(1) = 2$$

Erdmann's Corner Conditions. Erdmann's first corner condition requires $\hat{F}_{,y'} = -2\hat{y}^2$ to be continuous, which is trivial. Erdmann's second corner condition requires that

$$\hat{F} - \hat{y}'\hat{F}_{,y'} = x\hat{y} + \hat{y}^2 - 2\hat{y}^2\hat{y}' - \hat{y}'(-2\hat{y}^2) = x\hat{y} + \hat{y}^2$$

is continuous. This condition is also trivial. Thus the extremal may have corners at any point.

Euler's Differential Equation. Euler's DE is

$$(F_{,y'})' = F_{,y},$$

$$(-2y^2)' = x + 2y - 4yy'$$

$$y = -\frac{x}{2}$$

Extremal. There is no piecewise smooth function that satisfies Euler's differential equation on its smooth segments and satisfies the boundary conditions $y(0) = 1$, $y(1) = 2$. We conclude that there is no weak extremum.

Solution 49.4

1. We require that the first variation vanishes

$$\iint_D \left(F_u h + F_{u_x} h_x + F_{u_y} h_y\right) \, dx \, dy = 0.$$

We rewrite the integrand as

$$\iint_D \left(F_u h + (F_{u_x} h)_x + (F_{u_y} h)_y - (F_{u_x})_x h - (F_{u_y})_y h\right) \, dx \, dy = 0,$$

$$\iint_D \left(F_u - (F_{u_x})_x - (F_{u_y})_y\right) h \, dx \, dy + \iint_D \left((F_{u_x} h)_x + (F_{u_y} h)_y\right) \, dx \, dy = 0.$$

Using the Divergence theorem, we obtain,

$$\iint_D \left(F_u - (F_{u_x})_x - (F_{u_y})_y\right) h \, dx \, dy + \int_\Gamma (F_{u_x}, F_{u_y}) \cdot \mathbf{n} \, h \, ds = 0.$$

In order that the line integral vanish we have the natural boundary condition,

$$\boxed{(F_{u_x}, F_{u_y}) \cdot \mathbf{n} = 0 \quad \text{for } (x, y) \in \Gamma.}$$

We can also write this as

$$F_{u_x} \frac{\mathrm{d}y}{\mathrm{d}s} - F_{u_y} \frac{\mathrm{d}x}{\mathrm{d}s} = 0 \quad \text{for } (x, y) \in \Gamma.$$

The Euler differential equation for this problem is

$$F_u - (F_{u_x})_x - (F_{u_y})_y = 0.$$

2. We consider the natural boundary conditions for

$$\iint_D F(x, y, u, u_x, u_y)\, dx\, dy + \int_\Gamma G(x, y, u)\, ds.$$

We require that the first variation vanishes.

$$\iint_D \left(F_u - (F_{u_x})_x - (F_{u_y})_y\right) h\, dx\, dy + \int_\Gamma (F_{u_x}, F_{u_y}) \cdot \mathbf{n}\, h\, ds + \int_\Gamma G_u h\, ds = 0,$$

$$\iint_D \left(F_u - (F_{u_x})_x - (F_{u_y})_y\right) h\, dx\, dy + \int_\Gamma \left((F_{u_x}, F_{u_y}) \cdot \mathbf{n} + G_u\right) h\, ds = 0,$$

In order that the line integral vanishes, we have the natural boundary conditions,

$$(F_{u_x}, F_{u_y}) \cdot \mathbf{n} + G_u = 0 \quad \text{for } (x, y) \in \Gamma.$$

For the given integrand this is,

$$(2pu_x, 2pu_y) \cdot \mathbf{n} + 2\sigma u = 0 \quad \text{for } (x, y) \in \Gamma,$$

$$\boxed{p\nabla \mathbf{u} \cdot \mathbf{n} + \sigma u = 0 \quad \text{for } (x, y) \in \Gamma.}$$

We can also denote this as

$$p\frac{\partial u}{\partial n} + \sigma u = 0 \quad \text{for } (x, y) \in \Gamma.$$

Solution 49.5

First we vary ϕ.

$$\psi(\epsilon) = \iint_R \left(\int_0^{h(x,t)} \left(\phi_t + \epsilon\eta_t + \frac{1}{2}(\phi_x + \epsilon\eta_x)^2 + \frac{1}{2}(\phi_y + \epsilon\eta_y)^2 + gy \right) dy \right) dx\, dt$$

$$\psi'(0) = \iint_R \left(\int_0^{h(x,t)} (\eta_t + \phi_x \eta_x + \phi_y \eta_y) \; dy \right) dx\, dt = 0$$

$$\psi'(0) = \iint_R \left(\frac{\partial}{\partial t} \int_0^{h(x,t)} \eta \, dy - [\eta h_t]_{y=h(x,t)} + \frac{\partial}{\partial x} \int_0^{h(x,t)} \phi_x \eta \, dy - [\phi_x \eta h_x]_{y=h(x,t)} - \int_0^{h(x,t)} \phi_{xx} \eta \, dy \right.$$
$$\left. + [\phi_y \eta]_0^{h(x,t)} - \int_0^{h(x,t)} \phi_{yy} \eta \, dy \right) dx\, dt = 0$$

Since η vanishes on the boundary of R, we have

$$\psi'(0) = \iint_R \left(- [(h_t \phi_x h_x - \phi_y)\eta]_{y=h(x,t)} - [\phi_y \eta]_{y=0} - \int_0^{h(x,t)} (\phi_{xx} + \phi_{yy}) \eta \, dy \right) dx\, dt = 0.$$

From the variations η which vanish on $y = 0, h(x,t)$ we have

$$\boxed{\nabla^2 \phi = 0.}$$

This leaves us with

$$\psi'(0) = \iint_R \left(- [(h_t \phi_x h_x - \phi_y)\eta]_{y=h(x,t)} - [\phi_y \eta]_{y=0} \right) dx\, dt = 0.$$

By considering variations η which vanish on $y = 0$ we obtain,

$$\boxed{h_t \phi_x h_x - \phi_y = 0 \quad \text{on } y = h(x,t).}$$

Finally we have

$$\boxed{\phi_y = 0 \quad \text{on } y = 0.}$$

Next we vary $h(x,t)$.

$$\psi(\epsilon) = \iint_R \int_0^{h(x,t)+\epsilon\eta(x,t)} \left(\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + gy \right) dx\, dt$$

$$\psi'(\epsilon) = \iint_R \left[\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + gy \right]_{y=h(x,t)} \eta\, dx\, dt = 0$$

This gives us the boundary condition,

$$\boxed{\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + gy = 0 \quad \text{on } y = h(x,t).}$$

Solution 49.6

The parts of the extremizing curve which lie outside the boundary of the region R must be extremals, (i.e., solutions of Euler's equation) since if we restrict our variations to admissible curves outside of R and its boundary, we immediately obtain Euler's equation. Therefore an extremum can be reached only on curves consisting of arcs of extremals and parts of the boundary of region R.

Thus, our problem is to find the points of transition of the extremal to the boundary of R. Let the boundary of R be given by $\phi(x)$. Consider an extremum that starts at the point (a, A), follows an extremal to the point $(x_0, \phi(x_0))$, follows the ∂R to $(x_1, \phi(x_1))$ then follows an extremal to the point (b, B). We seek *transversality conditions* for the points x_0 and x_1. We will extremize the expression,

$$I(y) = \int_a^{x_0} F(x, y, y')\, dx + \int_{x_0}^{x_1} F(x, \phi, \phi')\, dx + \int_{x_1}^{b} F(x, y, y')\, dx.$$

Let c be any point between x_0 and x_1. Then extremizing $I(y)$ is equivalent to extremizing the two functionals,

$$I_1(y) = \int_a^{x_0} F(x, y, y')\, dx + \int_{x_0}^{c} F(x, \phi, \phi')\, dx,$$

$$I_2(y) = \int_c^{x_1} F(x, \phi, \phi')\, dx + \int_{x_1}^{b} F(x, y, y')\, dx,$$

$$\delta I = 0 \quad \Rightarrow \quad \delta I_1 = \delta I_2 = 0.$$

We will extremize $I_1(y)$ and then use the derived transversality condition on all points where the extremals meet ∂R. The general variation of I_1 is,

$$\delta I_1(y) = \int_a^{x_0} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} \right) dx + [F_{y'} \delta y]_a^{x_0} + [(F - y' F_{y'}) \delta x]_a^{x_0}$$
$$+ [F_{\phi'} \delta \phi(x)]_{x_0}^{c} + [(F - \phi' F_{\phi'}) \delta x]_{x_0}^{c} = 0$$

Note that $\delta x = \delta y = 0$ at $x = a, c$. That is, $x = x_0$ is the only point that varies. Also note that $\delta \phi(x)$ is not independent of δx. $\delta \phi(x) \to \phi'(x) \delta x$. At the point x_0 we have $\delta y \to \phi'(x) \delta x$.

$$\delta I_1(y) = \int_a^{x_0} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} \right) dx + (F_{y'} \phi' \delta x) \Big|_{x_0} + ((F - y' F_{y'}) \delta x) \Big|_{x_0}$$
$$- (F_{\phi'} \phi' \delta x) \Big|_{x_0} - ((F - \phi' F_{\phi'}) \delta x) \Big|_{x_0} = 0$$

$$\delta I_1(y) = \int_a^{x_0} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} \right) dx + ((F(x, y, y') - F(x, \phi, \phi') + (\phi' - y') F_{y'}) \delta x) \Big|_{x_0} = 0$$

Since δI_1 vanishes for those variations satisfying $\delta x_0 = 0$ we obtain the Euler differential equation,

$$F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} = 0.$$

Then we have

$$((F(x, y, y') - F(x, \phi, \phi') + (\phi' - y') F_{y'}) \delta x) \Big|_{x_0} = 0$$

for all variations δx_0. This implies that

$$\left. (F(x, y, y') - F(x, \phi, \phi') + (\phi' - y')F_{y'}) \right|_{x_0} = 0.$$

Two solutions of this equation are

$$y'(x_0) = \phi'(x_0) \quad \text{and} \quad F_{y'} = 0.$$

Transversality condition. If $F_{y'}$ is not identically zero, the extremal must be tangent to ∂R at the points of contact.

Now we apply this result to to find the curves which extremize $\int_0^{10} (y')^3 \, dx$, $y(0) = 0$, $y(10) = 0$ given that the admissible curves can not penetrate the interior of the circle $(x - 5)^2 + y^2 = 9$. Since the Lagrangian is a function of y' alone, the extremals are straight lines.

The Erdmann corner conditions require that

$$F_{y'} = 3(y')^2 \quad \text{and} \quad F - y'F_{y'} = (y')^3 - y'3(y')^2 = -2(y')^3$$

are continuous at corners. This implies that y' is continuous. There are no corners.

We see that the extrema are

$$\boxed{y(x) = \begin{cases} \pm\frac{3}{4}x, & \text{for } 0 \leq x \leq \frac{16}{5}, \\ \pm\sqrt{9 - (x-5)^2}, & \text{for } \frac{16}{5} \leq x \leq \frac{34}{5}, \\ \mp\frac{3}{4}x, & \text{for } \frac{34}{5} \leq x \leq 10. \end{cases}}$$

Note that the extremizing curves neither minimize nor maximize the integral.

Solution 49.7
C^1 Extremals. Without loss of generality, we take the vertical line to be the y axis. We will consider $x_1, y_1 > 1$. With $ds = \sqrt{1 + (y')^2}\, dx$ we extremize the integral,

$$\int_0^{x_1} \sqrt{y}\sqrt{1 + (y')^2}\, dx.$$

Since the Lagrangian is independent of x, we know that the Euler differential equation has a first integral.

$$\frac{\mathrm{d}}{\mathrm{d}x} F_{y'} - F_y = 0$$

$$y' F_{y'y} + y'' F_{y'y'} - F_y = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(y' F_{y'} - F) = 0$$

$$y' F_{y'} - F = \text{const}$$

For the given Lagrangian, this is

$$y' \sqrt{y} \frac{y'}{\sqrt{1 + (y')^2}} - \sqrt{y}\sqrt{1 + (y')^2} = \text{const},$$

$$(y')^2 \sqrt{y} - \sqrt{y}(1 + (y')^2) = \text{const}\sqrt{1 + (y')^2},$$

$$\sqrt{y} = \text{const}\sqrt{1 + (y')^2}$$

$y = \text{const}$ is one solution. To find the others we solve for y' and then solve the differential equation.

$$y = a(1 + (y')^2)$$

$$y' = \pm\sqrt{\frac{y-a}{a}}$$

$$dx = \sqrt{\frac{a}{y-a}}\, dy$$

$$\pm x + b = 2\sqrt{a(y-a)}$$

$$y = \frac{x^2}{4a} \pm \frac{bx}{2a} + \frac{b^2}{4a} + a$$

The natural boundary condition is

$$F_{y'}\big|_{x=0} = \left.\frac{\sqrt{y}y'}{\sqrt{1+(y')^2}}\right|_{x=0} = 0,$$

$$y'(0) = 0$$

The extremal that satisfies this boundary condition is

$$y = \frac{x^2}{4a} + a.$$

Now we apply $y(x_1) = y_1$ to obtain

$$a = \frac{1}{2}\left(y_1 \pm \sqrt{y_1^2 - x_1^2}\right)$$

for $y_1 \geq x_1$. The value of the integral is

$$\int_0^{x_1} \sqrt{\left(\frac{x^2}{4a} + a\right)\left(1 + \left(\frac{x}{2a}\right)^2\right)}\, dx = \frac{x_1(x_1^2 + 12a^2)}{12a^{3/2}}.$$

By denoting $y_1 = cx_1$, $c \geq 1$ we have

$$a = \frac{1}{2}\left(cx_1 \pm x_1\sqrt{c^2 - 1}\right)$$

The values of the integral for these two values of a are

$$\sqrt{2}(x_1)^{3/2}\frac{-1 + 3c^2 \pm 3c\sqrt{c^2 - 1}}{3(c \pm \sqrt{c^2 - 1})^{3/2}}.$$

The values are equal only when $c = 1$. These values, (divided by $\sqrt{x_1}$), are plotted in Figure 49.1 as a function of c. The former and latter are fine and coarse dashed lines, respectively. The extremal with

$$a = \frac{1}{2}\left(y_1 + \sqrt{y_1^2 - x_1^2}\right)$$

has the smaller performance index. The value of the integral is

$$\frac{x_1(x_1^2 + 3(y_1 + \sqrt{y_1^2 - x_1^2})^2}{3\sqrt{2}(y_1 + \sqrt{y_1^2 - x_1^2})^3}.$$

The function $y = y_1$ is an admissible extremal for all x_1. The value of the integral for this extremal is $x_1\sqrt{y_1}$ which is larger than the integral of the quadratic we analyzed before for $y_1 > x_1$.

Thus we see that

$$\boxed{\hat{y} = \frac{x^2}{4a} + a, \qquad a = \frac{1}{2}\left(y_1 + \sqrt{y_1^2 - x_1^2}\right)}$$

is the extremal with the smaller integral and is the minimizing curve in C^1 for $y_1 \geq x^1$. For $y_1 < x_1$ the C^1 extremum is,

$$\boxed{\hat{y} = y_1.}$$

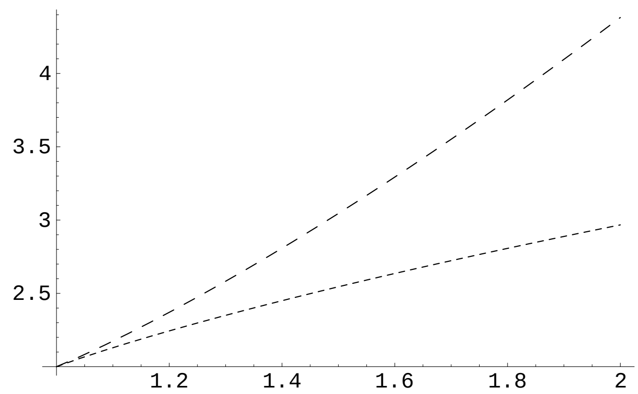

Figure 49.1:

C^1_p Extremals. Consider the parametric form of the Lagrangian.

$$\int_{t_0}^{t_1} \sqrt{y(t)}\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

The Euler differential equations are

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{x'} - f_x = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} f_{y'} - f_y = 0.$$

If one of the equations is satisfied, then the other is automatically satisfied, (or the extremal is straight). With either of these equations we could derive the quadratic extremal and the $y = \text{const}$ extremal that we found previously. We will find one more extremal by considering the first parametric Euler differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{x'} - f_x = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sqrt{y(t)}\, x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} \right) = 0$$

$$\frac{\sqrt{y(t)}x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} = \text{const}$$

Note that $x(t) = \text{const}$ is a solution. Thus the extremals are of the three forms,

$$x = \text{const},$$
$$y = \text{const},$$
$$y = \frac{x^2}{4a} + \frac{bx}{2a} + \frac{b^2}{4a} + a.$$

The Erdmann corner conditions require that

$$F_{y'} = \frac{\sqrt{y}y'}{\sqrt{1 + (y')^2}},$$
$$F - y'F_{y'} = \sqrt{y}\sqrt{1 + (y')^2} - \frac{\sqrt{y}(y')^2}{\sqrt{1 + (y')^2}} = \frac{\sqrt{y}}{\sqrt{1 + (y')^2}}$$

are continuous at corners. There can be corners only if $y = 0$.

Now we piece the three forms together to obtain C_p^1 extremals that satisfy the Erdmann corner conditions. The only possibility that is not C^1 is the extremal that is a horizontal line from $(0, 0)$ to $(x_1, 0)$ and then a vertical line from (x_1, y_1). The value of the integral for this extremal is

$$\int_0^{y_1} \sqrt{t}\, dt = \frac{2}{3}(y_1)^{3/2}.$$

Equating the performance indices of the quadratic extremum and the piecewise smooth extremum,

$$\frac{x_1(x_1^2 + 3(y_1 + \sqrt{y_1^2 - x_1^2})^2}{3\sqrt{2}(y_1 + \sqrt{y_1^2 - x_1^2})^3} = \frac{2}{3}(y_1)^{3/2},$$

$$y_1 = \pm x_1 \frac{\sqrt{3 \pm 2\sqrt{3}}}{\sqrt{3}}.$$

The only real positive solution is

$$y_1 = x_1 \frac{\sqrt{3 + 2\sqrt{3}}}{\sqrt{3}} \approx 1.46789\, x_1.$$

The piecewise smooth extremal has the smaller performance index for y_1 smaller than this value and the quadratic extremal has the smaller performance index for y_1 greater than this value.

The C_p^1 extremum is the piecewise smooth extremal for $y_1 \leq x_1\sqrt{3 + 2\sqrt{3}}/\sqrt{3}$ and is the quadratic extremal for $y_1 \geq x_1\sqrt{3 + 2\sqrt{3}}/\sqrt{3}$.

Solution 49.8

The shape of the rope will be a catenary between x_1 and x_2 and be a vertically hanging segment after that. Let the length of the vertical segment be z. Without loss of generality we take $x_1 = y_2 = 0$. The potential energy, (relative to $y = 0$), of a length of rope ds in $0 \leq x \leq x_2$ is $mgy = \rho g y\, ds$. The total potential energy of the vertically hanging rope is $m(\text{center of mass})g = \rho z(-z/2)g$. Thus we seek to minimize,

$$\rho g \int_0^{x_2} y\, ds - \frac{1}{2}\rho g z^2, \qquad y(0) = y_1, \quad y(x_2) = 0,$$

subject to the isoperimetric constraint,

$$\int_0^{x_2} ds - z = L.$$

Writing the arc-length differential as $ds = \sqrt{1 + (y')^2}\, dx$ we minimize

$$\rho g \int_0^{x_2} y\sqrt{1 + (y')^2}\, ds - \frac{1}{2}\rho g z^2, \qquad y(0) = y_1, \quad y(x_2) = 0,$$

subject to,

$$\int_0^{x_2} \sqrt{1+(y')^2}\, dx - z = L.$$

Consider the more general problem of finding functions $y(x)$ and numbers z which extremize $I \equiv \int_a^b F(x,y,y')\, dx + f(z)$ subject to $J \equiv \int_a^b G(x,y,y')\, dx + g(z) = L$.

Suppose $y(x)$ and z are the desired solutions and form the comparison families, $y(x) + \epsilon_1 \eta_1(x) + \epsilon_2 \eta_2(x)$, $z + \epsilon_1 \zeta_1 + \epsilon_2 \zeta_2$. Then, there exists a constant such that

$$\frac{\partial}{\partial \epsilon_1}(I + \lambda J)\Big|_{\epsilon_1,\epsilon_2=0} = 0$$
$$\frac{\partial}{\partial \epsilon_2}(I + \lambda J)\Big|_{\epsilon_1,\epsilon_2=0} = 0.$$

These equations are

$$\int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x} H_{,y'} - H_y \right) \eta_1\, dx + h'(z)\zeta_1 = 0,$$

and

$$\int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x} H_{,y'} - H_y \right) \eta_2\, dx + h'(z)\zeta_2 = 0,$$

where $H = F + \lambda G$ and $h = f + \lambda g$. From this we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}x} H_{,y'} - H_y = 0, \qquad h'(z) = 0$$

with λ determined by

$$J = \int_a^b G(x,y,y')\, dx + g(z) = L.$$

Now we apply these results to our problem. Since $f(z) = -\frac{1}{2}\rho g z^2$ and $g(z) = -z$ we have

$$-\rho g z - \lambda = 0,$$

$$\boxed{z = -\frac{\lambda}{\rho g}.}$$

It was shown in class that the solution of the Euler differential equation is a family of catenaries,

$$y = -\frac{\lambda}{\rho g} + c_1 \cosh\left(\frac{x - c_2}{c_1}\right).$$

One can find c_1 and c_2 in terms of λ by applying the end conditions $y(0) = y_1$ and $y(x_2) = 0$. Then the expression for $y(x)$ and $z = -\lambda/\rho g$ are substituted into the isoperimetric constraint to determine λ.

Consider the special case that $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$. In this case we can use the fact that $y(0) = y(1)$ to solve for c_2 and write y in the form

$$y = -\frac{\lambda}{\rho g} + c_1 \cosh\left(\frac{x - 1/2}{c_1}\right).$$

Applying the condition $y(0) = 0$ would give us the algebraic-transcendental equation,

$$y(0) = -\frac{\lambda}{\rho g} + c_1 \cosh\left(\frac{1}{2c_1}\right) = 0,$$

which we can't solve in closed form. Since we ran into a dead end in applying the boundary condition, we turn to the isoperimetric constraint.

$$\int_0^1 \sqrt{1 + (y')^2}\, dx - z = L$$

$$\int_0^1 \cosh\left(\frac{x - 1/2}{c_1}\right) dx - z = L$$

$$2c_1 \sinh\left(\frac{1}{2c_1}\right) - z = L$$

With the isoperimetric constraint, the algebraic-transcendental equation and $z = -\lambda/\rho g$ we now have

$$z = -c_1 \cosh\left(\frac{1}{2c_1}\right),$$
$$z = 2c_1 \sinh\left(\frac{1}{2c_1}\right) - L.$$

For any fixed L, we can numerically solve for c_1 and thus obtain z. You can derive that there are no solutions unless L is greater than about 1.9366. If L is smaller than this, the rope would slip off the pin. For $L = 2$, c_1 has the values 0.4265 and 0.7524. The larger value of c_1 gives the smaller potential energy. The position of the end of the rope is $z = -0.9248$.

Solution 49.9

Using the method of Lagrange multipliers, we look for stationary values of $\int_0^c ((y')^2 + \lambda y^2)\, dx$,

$$\delta \int_0^c ((y')^2 + \lambda y^2)\, dx = 0.$$

The Euler differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}x} F_{(,y')} - F_{,y} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(2y') - 2\lambda y = 0.$$

Together with the homogeneous boundary conditions, we have the problem

$$y'' - \lambda y = 0, \qquad y(0) = y(c) = 0,$$

which has the solutions,

$$\lambda_n = -\left(\frac{n\pi}{c}\right)^2, \qquad y_n = a_n \sin\left(\frac{n\pi x}{c}\right), \qquad n \in \mathbb{Z}^+.$$

Now we determine the constants a_n with the moment of inertia constraint.

$$\int_0^c a_n^2 \sin^2\left(\frac{n\pi x}{c}\right) dx = \frac{c a_n^2}{2} = A$$

Thus we have the extremals,

$$y_n = \sqrt{\frac{2A}{c}} \sin\left(\frac{n\pi x}{c}\right), \qquad n \in \mathbb{Z}^+.$$

The drag for these extremals is

$$D = \frac{2A}{c} \int_0^c \left(\frac{n\pi}{c}\right)^2 \cos^2\left(\frac{n\pi x}{c}\right) dx = \frac{A n^2 \pi^2}{c^2}.$$

We see that the drag is minimum for $n = 1$. The shape for minimum drag is

$$\boxed{\hat{y} = \sqrt{\frac{2A}{c}} \sin\left(\frac{n\pi x}{c}\right).}$$

Solution 49.10
Consider the general problem of determining the stationary values of the quantity ω^2 given by

$$\omega^2 = \frac{\int_a^b F(x, y, y', y'')\, dx}{\int_a^b G(x, y, y', y'')\, dx} \equiv \frac{I}{J}.$$

The variation of ω^2 is

$$\begin{aligned} \delta\omega^2 &= \frac{J\delta I - I\delta J}{J^2} \\ &= \frac{1}{J}\left(\delta I - \frac{I}{J}\delta J\right) \\ &= \frac{1}{J}\left(\delta I - \omega^2 \delta J\right). \end{aligned}$$

The the values of y and y' are specified on the boundary, then the variations of I and J are

$$\delta I = \int_a^b \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} F_{,y''} - \frac{\mathrm{d}}{\mathrm{d}x} F_{,y'} + F_{,y}\right) \delta y\, dx, \qquad \delta J = \int_a^b \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} G_{,y''} - \frac{\mathrm{d}}{\mathrm{d}x} G_{,y'} + G_{,y}\right) \delta y\, dx$$

Thus $\delta\omega^2 = 0$ becomes

$$\frac{\int_a^b \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} H_{,y''} - \frac{\mathrm{d}}{\mathrm{d}x} H_{,y'} + H_{,y}\right) \delta y\, dx}{\int_a^b G\, dx} = 0,$$

where $H = F - \omega^2 G$. A necessary condition for an extremum is

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}x^2} H_{,y''} - \frac{\mathrm{d}}{\mathrm{d}x} H_{,y'} + H_{,y} = 0 \quad \text{where } H \equiv F - \omega^2 G.}$$

For our problem we have $F = EI(y'')^2$ and $G = \rho y$ so that the extremals are solutions of

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - \rho\omega^2 y = 0,$$

With homogeneous boundary conditions we have an eigenvalue problem with deflections modes $y_n(x)$ and corresponding natural frequencies ω_n.

Solution 49.11

We assume that $v_0 > w(x, y, t)$ so that the problem has a solution for any end point. The crossing time is

$$T = \int_0^l \left(\dot{X}(t)\right)^{-1} dx = \frac{1}{v_0}\int_0^l \sec\alpha(t)\, dx.$$

Note that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{w + v_0 \sin\alpha}{v_0\cos\alpha} \\ &= \frac{w}{v_0}\sec\alpha + \tan\alpha \\ &= \frac{w}{v_0}\sec\alpha + \sqrt{\sec^2\alpha - 1}.\end{aligned}$$

We solve this relation for $\sec\alpha$.

$$\left(y' - \frac{w}{v_0}\sec\alpha\right)^2 = \sec^2\alpha - 1$$

$$(y')^2 - 2\frac{w}{v_0}y'\sec\alpha + \frac{w^2}{v_0^2}\sec^2\alpha = \sec^2\alpha - 1$$

$$(v_0^2 - w^2)\sec^2\alpha + 2v_0 w y'\sec\alpha - v_0^2((y')^2 + 1) = 0$$

$$\sec\alpha = \frac{-2v_0 w y' \pm \sqrt{4v_0^2 w^2 (y')^2 + 4(v_0^2 - w^2)v_0^2((y')^2 + 1)}}{2(v_0^2 - w^2)}$$

$$\sec\alpha = v_0\frac{-wy' \pm \sqrt{v_0^2((y')^2 + 1) - w^2}}{(v_0^2 - w^2)}$$

Since the steering angle satisfies $-\pi/2 \leq \alpha \leq \pi/2$ only the positive solution is relevant.

$$\sec\alpha = v_0 \frac{-wy' + \sqrt{v_0^2((y')^2+1) - w^2}}{(v_0^2 - w^2)}$$

Time Independent Current. If we make the assumption that $w = w(x, y)$ then we can write the crossing time as an integral of a function of x and y.

$$T(y) = \int_0^l \frac{-wy' + \sqrt{v_0^2((y')^2+1) - w^2}}{(v_0^2 - w^2)}\, dx$$

A necessary condition for a minimum is $\delta T = 0$. The Euler differential equation for this problem is

$$\frac{\mathrm{d}}{\mathrm{d}x} F_{,y'} - F_{,y} = 0$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v_0^2 - w^2}\left(-w + \frac{v_0^2 y'}{\sqrt{v_0^2((y')^2+1) - w^2}}\right)\right) - \frac{w_y}{(v_0^2 - w^2)^2}\left(\frac{w(v^2(1 + 2(y')^2) - w^2)}{\sqrt{v_0^2((y')^2+1) - w^2}} - y'(v_0^2 + w^2)\right)}$$

By solving this second order differential equation subject to the boundary conditions $y(0) = 0$, $y(l) = y_1$ we obtain the path of minimum crossing time.

Current $\mathbf{w} = \mathbf{w}(\mathbf{x})$. If the current is only a function of x, then the Euler differential equation can be integrated to obtain,

$$\frac{1}{v_0^2 - w^2}\left(-w + \frac{v_0^2 y'}{\sqrt{v_0^2((y')^2+1) - w^2}}\right) = c_0.$$

Solving for y',

$$y' = \pm \frac{w + c_0(v_0^2 - w^2)}{v_0\sqrt{1 - 2c_0 w - c_0^2(v_0^2 - w^2)}}.$$

Since $y(0) = 0$, we have

$$\boxed{y(x) = \pm \int_0^x \frac{w(\xi) + c_0(v_0^2 - (w(\xi))^2)}{v_0\sqrt{1 - 2c_0 w(\xi) - c_0^2(v_0^2 - (w(\xi))^2)}}.}$$

For any given $w(x)$ we can use the condition $y(l) = y_1$ to solve for the constant c_0.

Constant Current. If the current is constant then the Lagrangian is a function of y' alone. The admissible extremals are straight lines. The solution is then

$$\boxed{y(x) = \frac{y_1 x}{l}.}$$

Solution 49.12

1. The kinetic energy of the first particle is $\frac{1}{2}m((\alpha - x)\hat{\theta})^2$. Its potential energy, relative to the table top, is zero. The kinetic energy of the second particle is $\frac{1}{2}m\hat{x}^2$. Its potential energy, relative to its equilibrium position is $-mgx$. The Lagrangian is the difference of kinetic and potential energy.

$$\boxed{L = m\left(\dot{x}^2 + \frac{1}{2}(\alpha - x)^2\dot{\theta}^2 + gx\right)}$$

The Euler differential equations are the equations of motion.

$$\frac{\mathrm{d}}{\mathrm{d}t}L_{,\dot{x}} - L_x = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}L_{,\dot{\theta}} - L_\theta = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(2m\dot{x}) + m(\alpha - x)\dot{\theta}^2 - mg = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(m(\alpha - x)^2\dot{\theta}^2\right) = 0$$

$$2\ddot{x} + (\alpha - x)\dot{\theta}^2 - g = 0, \qquad (\alpha - x)^2\dot{\theta}^2 = \text{const}$$

When $x = 0$, $\dot{\theta} = \omega = \sqrt{g/\alpha}$. This determines the constant in the equation of motion for θ.

$$\boxed{\dot{\theta} = \frac{\alpha\sqrt{\alpha g}}{(\alpha - x)^2}}$$

Now we substitute the expression for $\dot{\theta}$ into the equation of motion for x.

$$2\ddot{x} + (\alpha - x)\frac{\alpha^3 g}{(\alpha - x)^4} - g = 0$$

$$2\ddot{x} + \left(\frac{\alpha^3}{(\alpha - x)^3} - 1\right) g = 0$$

$$\boxed{2\ddot{x} + \left(\frac{1}{(1 - x/\alpha)^3} - 1\right) g = 0}$$

2. For small oscillations, $\left|\frac{x}{\alpha}\right| \ll 1$. Recall the binomial expansion,

$$(1 + z)^a = \sum_{n=0}^{\infty} \binom{a}{n} z^n, \quad \text{for } |z| < 1,$$

$$(1 + z)^a \approx 1 + az, \quad \text{for } |z| \ll 1.$$

We make the approximation,

$$\frac{1}{(1 - x/\alpha)^3} \approx 1 + 3\frac{x}{\alpha},$$

to obtain the linearized equation of motion,

$$2\ddot{x} + \frac{3g}{\alpha} x = 0.$$

This is the equation of a harmonic oscillator with solution

$$x = a \sin\left(\sqrt{3g} 2\alpha (t - b)\right).$$

The period of oscillation is,

$$\boxed{T = 2\pi\sqrt{2\alpha} 3g.}$$

Solution 49.13

We write the equation of motion and boundary conditions,

$$\ddot{x} = U(t) - g, \qquad x(0) = \dot{x}(0) = 0, \quad x(T) = h,$$

as the first order system,

$$\begin{aligned} \dot{x} &= 0, & x(0) &= 0, \quad x(T) = h, \\ \dot{y} &= U(t) - g, & y(0) &= 0. \end{aligned}$$

We seek to minimize,

$$T = \int_0^T dt,$$

subject to the constraints,

$$\begin{aligned} &\dot{x} - y = 0, \\ &\dot{y} - U(t) + g = 0, \\ &\int_0^T U^2(t)\, dt = k^2. \end{aligned}$$

Thus we seek extrema of

$$\int_0^T H\,dt \equiv \int_0^T \left(1 + \lambda(t)(\dot{x} - y) + \mu(t)(\dot{y} - U(t) + g) + \nu U^2(t)\right)\,dt.$$

Since y is not specified at $t = T$, we have the natural boundary condition,

$$H_{,\dot{y}}\big|_{t=T} = 0,$$

$$\mu(T) = 0.$$

The first Euler differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{,\dot{x}} - H_{,x} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \lambda(t) = 0.$$

We see that $\lambda(t) = \lambda$ is constant. The next Euler DE is

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{,\dot{y}} - H_{,y} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \mu(t) + \lambda = 0.$$

$$\mu(t) = -\lambda t + \mathrm{const}$$

With the natural boundary condition, $\mu(T) = 0$, we have

$$\mu(t) = \lambda(T - t).$$

The final Euler DE is,

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{,\dot{U}} - H_{,U} = 0,$$

$$\mu(t) - 2\nu U(t) = 0.$$

Thus we have

$$U(t) = \frac{\lambda(T-t)}{2\nu}.$$

This is the required thrust function. We use the constraints to find λ, ν and T.

Substituting $U(t) = \lambda(T-t)/(2\nu)$ into the isoperimetric constraint, $\int_0^T U^2(t)\,dt = k^2$ yields

$$\frac{\lambda^2 T^3}{12\nu^2} = k^2,$$

$$\boxed{U(t) = \frac{\sqrt{3}k}{T^{3/2}}(T-t).}$$

The equation of motion for x is

$$\ddot{x} = U(t) - g = \frac{\sqrt{3}k}{T^{3/2}}(T-t).$$

Integrating and applying the initial conditions $x(0) = \dot{x}(0) = 0$ yields,

$$x(t) = \frac{kt^2(3T-t)}{2\sqrt{3}T^{3/2}} - \frac{1}{2}gt^2.$$

Applying the condition $x(T) = h$ gives us,

$$\frac{k}{\sqrt{3}}T^{3/2} - \frac{1}{2}gT^2 = h,$$

$$\boxed{\frac{1}{4}g^2T^4 - \frac{k}{3}T^3 + ghT^2 + h^2 = 0.}$$

If $k \geq 4\sqrt{2/3}g^{3/2}\sqrt{h}$ then this fourth degree polynomial has positive, real solutions for T. With strict inequality, the minimum time is the smaller of the two positive, real solutions. If $k < 4\sqrt{2/3}g^{3/2}\sqrt{h}$ then there is not enough fuel to reach the target height.

Solution 49.14

We have $\ddot{x} = U(t)$ where $U(t)$ is the acceleration furnished by the thrust of the vehicles engine. In practice, the engine will be designed to operate within certain bounds, say $-M \leq U(t) \leq M$, where $\pm M$ is the maximum forward/backward acceleration. To account for the inequality constraint we write $U = M \sin V(t)$ for some suitable $V(t)$. More generally, if we had $\phi(t) \leq U(t) \leq \psi(t)$, we could write this as $U(t) = \frac{\psi+\phi}{2} + \frac{\psi-\phi}{2}\sin V(t)$.

We write the equation of motion as a first order system,

$$\begin{aligned} \dot{x} &= y, & x(0) &= a, & x(T) &= 0, \\ \dot{y} &= M\sin V, & y(0) &= b, & y(T) &= 0. \end{aligned}$$

Thus we minimize

$$T = \int_0^T dt$$

subject to the constraints,

$$\begin{aligned} \dot{x} - y &= 0 \\ \dot{y} - M\sin V &= 0. \end{aligned}$$

Consider

$$H = 1 + \lambda(t)(\dot{x} - y) + \mu(t)(\dot{y} - M\sin V).$$

The Euler differential equations are

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H_{,\dot{x}} - H_{,x} = 0 \quad &\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = 0 &&\Rightarrow \quad \lambda(t) = \text{const} \\
\frac{\mathrm{d}}{\mathrm{d}t} H_{,\dot{y}} - H_{,y} = 0 \quad &\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\mu(t) + \lambda = 0 &&\Rightarrow \quad \mu(t) = -\lambda t + \text{const} \\
\frac{\mathrm{d}}{\mathrm{d}t} H_{,\dot{V}} - H_{,V} = 0 \quad &\Rightarrow \quad \mu(t) M \cos V(t) = 0 &&\Rightarrow \quad V(t) = \frac{\pi}{2} + n\pi.
\end{aligned}
$$

Thus we see that

$$
U(t) = M \sin\left(\frac{\pi}{2} + n\pi\right) = \pm M.
$$

Therefore, if the rocket is to be transferred from its initial state to is specified final state in minimum time with a limited source of thrust, ($|U| \leq M$), then the engine should operate at full power at all times except possibly for a finite number of switching times. (Indeed, if some power were not being used, we would expect the transfer would be speeded up by using the additional power suitably.)

To see how this "bang-bang" process works, we'll look at the phase plane. The problem

$$
\begin{aligned}
\dot{x} &= y, & x(0) &= c, \\
\dot{y} &= \pm M, & y(0) &= d,
\end{aligned}
$$

has the solution

$$
x(t) = c + dt \pm M\frac{t^2}{2}, \qquad y(t) = d \pm Mt.
$$

We can eliminate t to get

$$
x = \pm\frac{y^2}{2M} + c \mp \frac{d^2}{2M}.
$$

These curves are plotted in Figure 49.2.

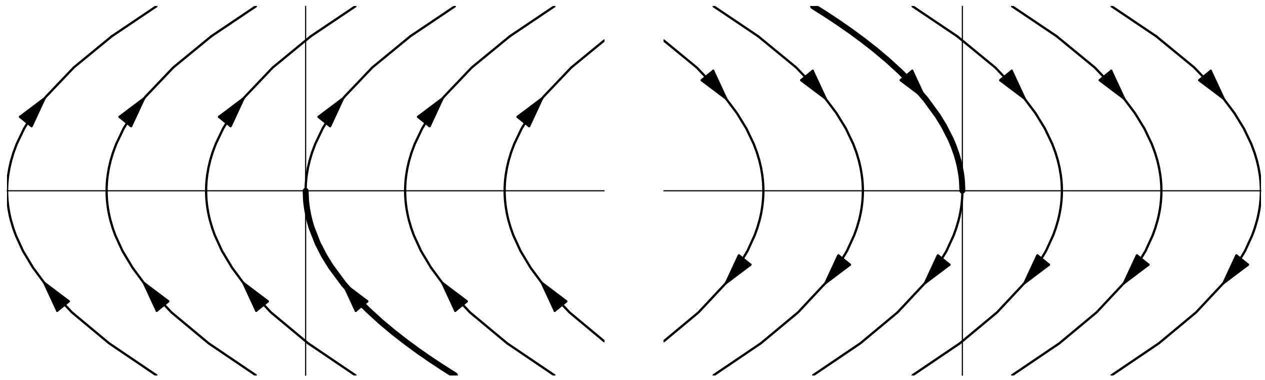

Figure 49.2:

There is only curve in each case which transfers the initial state to the origin. We will denote these curves γ and Γ, respectively. Only if the initial point (a, b) lies on one of these two curves can we transfer the state of the system to the origin along an extremal without switching. If $a = \frac{b^2}{2M}$ and $b < 0$ then this is possible using $U(t) = M$. If $a = -\frac{b^2}{2M}$ and $b > 0$ then this is possible using $U(t) = -M$. Otherwise we follow an extremal that intersects the initial position until this curve intersects γ or Γ. We then follow γ or Γ to the origin.

Solution 49.15

Since the integrand does not explicitly depend on x, the Euler differential equation has the first integral,

$$F - y' F_{y'} = \text{const.}$$

$$\sqrt{y+h}\sqrt{1+(y')^2} - y' \frac{y' \sqrt{y+h}}{\sqrt{1+(y')^2}} = \text{const}$$

$$\frac{\sqrt{y+h}}{\sqrt{1+(y')^2}} = \text{const}$$

$$y + h = c_1^2(1 + (y')^2)$$

$$\sqrt{y + h - c_1^2} = c_1 y'$$

$$\frac{c_1\, dy}{\sqrt{y + h - c_1^2}} = dx$$

$$2c_1\sqrt{y + h - c_1^2} = x - c_2$$

$$4c_1^2(y + h - c_1^2) = (x - c_2)^2$$

Since the extremal passes through the origin, we have

$$4c_1^2(h - c_1^2) = c_2^2.$$

$$4c_1^2 y = x^2 - 2c_2 x \tag{49.6}$$

Introduce as a parameter the slope of the extremal at the origin; that is, $y'(0) = \alpha$. Then differentiating (49.6) at $x = 0$ yields $4c_1^2\alpha = -2c_2$. Together with $c_2^2 = 4c_1^2(h - c_1^2)$ we obtain $c_1^2 = \frac{h}{1+\alpha^2}$ and $c_2 = -\frac{2\alpha h}{1+\alpha^2}$. Thus the equation of the pencil (49.6) will have the form

$$y = \alpha x + \frac{1 + \alpha^2}{4h} x^2. \tag{49.7}$$

To find the envelope of this family we differentiate (49.7) with respect to α to obtain $0 = x + \frac{\alpha}{2h} x^2$ and eliminate α between this and (49.7) to obtain

$$y = -h + \frac{x^2}{4h}.$$

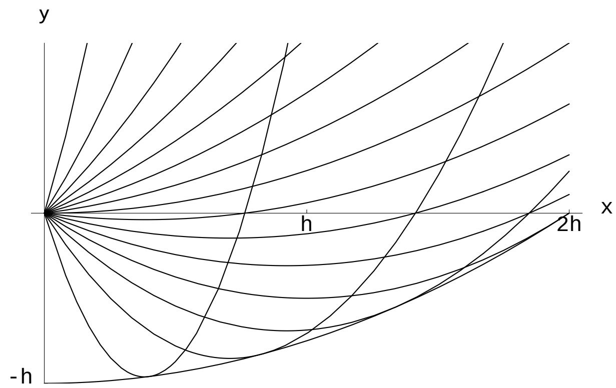

Figure 49.3: Some Extremals and the Envelope.

See Figure 49.3 for a plot of some extremals and the envelope.

All extremals (49.7) lie above the envelope which in ballistics is called the parabola of safety. If (m, M) lies outside the parabola, $M < -h + \frac{m^2}{4h}$, then it cannot be joined to $(0, 0)$ by an extremal. If (m, M) is above the envelope then there are two candidates. Clearly we rule out the one that touches the envelope because of the occurrence of conjugate points. For the other extremal, problem 2 shows that $E \geq 0$ for all y'. Clearly we can embed this extremal in an extremal pencil, so Jacobi's test is satisfied. Therefore the parabola that does not touch the envelope is a strong minimum.

Solution 49.16

$$
\begin{aligned}
E &= F(x,y,y') - F(x,y,p) - (y'-p)F_{y'}(x,y,p) \\
&= n\sqrt{1+(y')^2} - n\sqrt{1+p^2} - (y'-p)\frac{np}{\sqrt{1+p^2}} \\
&= \frac{n}{\sqrt{1+p^2}}\left(\sqrt{1+(y')^2}\sqrt{1+p^2} - (1+p^2) - (y'-p)p\right) \\
&= \frac{n}{\sqrt{1+p^2}}\left(\sqrt{1+(y')^2+p^2+(y')^2p^2-2y'p+2y'p} - (1+py')\right) \\
&= \frac{n}{\sqrt{1+p^2}}\left(\sqrt{(1+py')^2+(y'-p)^2} - (1+py')\right) \\
&\geq 0
\end{aligned}
$$

The speed of light in an inhomogeneous medium is $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{n(x,y}$. The time of transit is then

$$
T = \int_{(a,A)}^{(b,B)} \frac{\mathrm{d}t}{\mathrm{d}s}\,ds = \int_a^b n(x,y)\sqrt{1+(y')^2}\,dx.
$$

Since $E \geq 0$, light traveling on extremals follow the time optimal path as long as the extremals do not intersect.

Solution 49.17

Extremals. Since the integrand does not depend explicitly on x, the Euler differential equation has the first integral,

$$
F - y'F_{,y'} = \text{const.}
$$

$$
\frac{1+y^2}{(y')^2} - y'\frac{-2(1+y^2)}{(y')^3} = \text{const}
$$

$$\frac{dy}{\sqrt{1+(y')^2}} = \text{const}\, dx$$

$$\operatorname{arcsinh}(y) = c_1 x + c_2$$

$$\boxed{y = \sinh(c_1 x + c_2)}$$

Jacobi Test. We can see by inspection that no conjugate points exist. Consider the central field through $(0,0)$, $\sinh(cx)$, (See Figure 49.4).

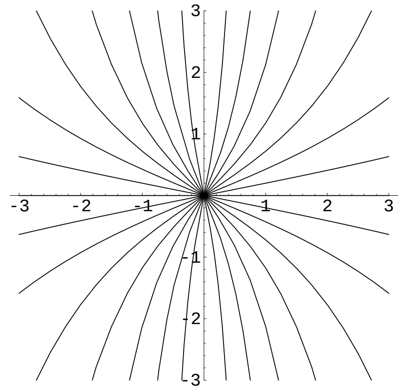

Figure 49.4: $\sinh(cx)$

We can also easily arrive at this conclusion analytically as follows: Solutions u_1 and u_2 of the Jacobi equation are given by

$$u_1 = \frac{\partial y}{\partial c_2} = \cosh(c_1 x + c_2),$$
$$u_2 = \frac{\partial y}{\partial c_1} = x\cosh(c_1 x + c_2).$$

Since $u_2/u_1 = x$ is monotone for all x there are no conjugate points.

Weierstrass Test.

$$\begin{aligned}
E &= F(x,y,y') - F(x,y,p) - (y'-p)F_{,y'}(x,y,p) \\
&= \frac{1+y^2}{(y')^2} - \frac{1+y^2}{p^2} - (y'-p)\frac{-2(1+y^2)}{p^3} \\
&= \frac{1+y^2}{(y')^2p^2}\left(\frac{p^3 - p(y')^2 + 2(y')^3 - 2p(y')^2}{p}\right) \\
&= \frac{1+y^2}{(y')^2p^2}\left(\frac{(p-y')^2(p+2y')}{p}\right)
\end{aligned}$$

For $p = p(x,y)$ bounded away from zero, E is one-signed for values of y' close to p. However, since the factor $(p+2y')$ can have any sign for arbitrary values of y', the conditions for a strong minimum are not satisfied.

Furthermore, since the extremals are $y = \sinh(c_1 x + c_2)$, the slope function $p(x,y)$ will be of one sign only if the range of integration is such that we are on a monotonic piece of the sinh. If we span both an increasing and decreasing section, E changes sign even for weak variations.

Legendre Condition.

$$F_{,y'y'} = \frac{6(1+y^2)}{(y')^4} > 0$$

Note that F cannot be represented in a Taylor series for arbitrary values of y' due to the presence of a discontinuity in F when $y' = 0$. However, $F_{,y'y'} > 0$ on an extremal implies a weak minimum is provided by the extremal.

Strong Variations. Consider $\int \frac{1+y^2}{(y')^2}\,dx$ on both an extremal and on the special piecewise continuous variation in the figure. On PQ we have $y' = \infty$ with implies that $\frac{1+y^2}{(y')^2} = 0$ so that there is no contribution to the integral from PQ.

On QR the value of y' is greater than its value along the extremal PR while the value of y on QR is less than the value of y along PR. Thus on QR the quantity $\frac{1+y^2}{(y')^2}$ is less than it is on the extremal PR.

$$\int_{QR} \frac{1+y^2}{(y')^2}\,dx < \int_{PR} \frac{1+y^2}{(y')^2}\,dx$$

Thus the weak minimum along the extremal can be weakened by a strong variation.

Solution 49.18

The Euler differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}x} F_{,y'} - F_{,y} = 0.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(1 + 2x^2 y') = 0$$

$$1 + 2x^2 y' = \text{const}$$

$$y' = \text{const}\frac{1}{x^2}$$

$$\boxed{y = \frac{c_1}{x} + c_2}$$

(i) No continuous extremal exists in $-1 \leq x \leq 2$ that satisfies $y(-1) = 1$ and $y(2) = 4$.

(ii) The continuous extremal that satisfies the boundary conditions is $y = 7 - \frac{4}{x}$. Since $F_{,y'y'} = 2x^2 \geq 0$ has a Taylor series representation for all y', this extremal provides a strong minimum.

(iii) The continuous extremal that satisfies the boundary conditions is $y = 1$. This is a strong minimum.

Solution 49.19

For identity (a) we take $P = 0$ and $Q = \phi\psi_x - \psi\phi_x$. For identity (b) we take $P = \phi\psi_y - \psi\phi_y$ and $Q = 0$. For identity (c) we take $P = -\frac{1}{2}(\phi\psi_x - \psi\phi_x)$ and $Q = \frac{1}{2}(\phi\psi_y - \psi\phi_y)$.

$$\iint_D \left(\frac{1}{2}(\phi\psi_y - \psi\phi_y)_x - \left(-\frac{1}{2}\right)(\phi\psi_x - \psi\phi_x)_y\right) dx\, dy = \int_\Gamma \left(-\frac{1}{2}(\phi\psi_x - \psi\phi_x)\, dx + \frac{1}{2}(\phi\psi_y - \psi\phi_y)\, dy\right)$$

$$\iint_D \left(\frac{1}{2}(\phi_x\psi_y + \phi\psi_{xy} - \psi_x\phi_y - \psi\phi_{xy}) + \frac{1}{2}(\phi_y\psi_x\phi\psi_{xy} - \psi_y\phi_x - \psi\phi_{xy})\right)\, dx\, dy$$
$$= -\frac{1}{2}\int_\Gamma (\phi\psi_x - \psi\phi_x)\, dx + \frac{1}{2}\int_\Gamma (\phi\psi_y - \psi\phi_y)\, dy$$

$$\iint_D \phi\psi_{xy}\, dx\, dy = \iint_D \psi\phi_{xy}\, dx\, dy - \frac{1}{2}\int_\Gamma (\phi\psi_x - \psi\phi_x)\, dx + \frac{1}{2}\int_\Gamma (\phi\psi_y - \psi\phi_y)\, dy$$

The variation of I is

$$\delta I = \int_{t_0}^{t_1} \iint_D (-2(u_{xx} + u_{yy})(\delta u_{xx} + \delta u_{yy}) + 2(1-\mu)(u_{xx}\delta u_{yy} + u_{yy}\delta u_{xx} - 2u_{xy}\delta u_{xy}))\, dx\, dy\, dt.$$

From (a) we have

$$\iint_D -2(u_{xx} + u_{yy})\delta u_{xx}\, dx\, dy = \iint_D -2(u_{xx} + u_{yy})_{xx}\delta u\, dx\, dy$$
$$+ \int_\Gamma -2((u_{xx} + u_{yy})\delta u_x - (u_{xx} + u_{yy})_x\delta u)\, dy.$$

From (b) we have

$$\iint_D -2(u_{xx} + u_{yy})\delta u_{yy}\, dx\, dy = \iint_D -2(u_{xx} + u_{yy})_{yy}\delta u\, dx\, dy$$
$$- \int_\Gamma -2((u_{xx} + u_{yy})\delta u_y - (u_{xx} + u_{yy})_y\delta u)\, dy.$$

From (a) and (b) we get

$$\iint_D 2(1-\mu)(u_{xx}\delta u_{yy} + u_{yy}\delta u_{xx})\, dx\, dy$$
$$= \iint_D 2(1-\mu)(u_{xxyy} + u_{yyxx})\delta u\, dx\, dy$$
$$+ \int_\Gamma 2(1-\mu)(-(u_{xx}\delta u_y - u_{xxy}\delta u)\, dx + (u_{yy}\delta u_x - u_{yyx}\delta u)\, dy).$$

Using c gives us

$$\iint_D 2(1-\mu)(-2u_{xy}\delta u_{xy})\,dx\,dy = \iint_D 2(1-\mu)(-2u_{xyxy}\delta u)\,dx\,dy$$
$$+\int_\Gamma 2(1-\mu)(u_{xy}\delta u_x - u_{xyx}\delta u)\,dx$$
$$-\int_\Gamma 2(1-\mu)(u_{xy}\delta u_y - u_{xyy}\delta u)\,dy.$$

Note that

$$\frac{\partial u}{\partial n}\,ds = u_x\,dy - u_y\,dx.$$

Using the above results, we obtain

$$\delta I = 2\int_{t_0}^{t_1}\iint_D(-\nabla^4 u)\delta u\,dx\,dy\,dt + 2\int_{t_0}^{t_1}\int_\Gamma\left(\frac{\partial(\nabla^2 u)}{\partial n}\delta u + (\nabla^2 u)\frac{\partial(\delta u)}{\partial n}\right)ds\,dt$$
$$+2(1-\mu)\int_{t_0}^{t_1}\left(\int_\Gamma(u_{yy}\delta u_x - u_{xy}\delta u_y)\,dy + (u_{xy}\delta u_x - u_{xx}\delta u_y)\,dx\right)dt.$$

Solution 49.20

1. **Exact Solution.** The Euler differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}F_{,y'} = F_{,y}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[2y'] = -2y - 2x$$
$$y'' + y = -x.$$

The general solution is

$$y = c_1\cos x + c_2\sin x - x.$$

Applying the boundary conditions we obtain,

$$\boxed{y = \frac{\sin x}{\sin 1} - x.}$$

The value of the integral for this extremal is

$$\boxed{J\left[\frac{\sin x}{\sin 1} - x\right] = \cot(1) - \frac{2}{3} \approx -0.0245741.}$$

n = 0. We consider an approximate solution of the form $y(x) = ax(1-x)$. We substitute this into the functional.

$$J(a) = \int_0^1 \left((y')^2 - y^2 - 2xy\right)\, dx = \frac{3}{10}a^2 - \frac{1}{6}a$$

The only stationary point is

$$J'(a) = \frac{3}{5}a - \frac{1}{6} = 0$$
$$a = \frac{5}{18}.$$

Since

$$J''\left(\frac{5}{18}\right) = \frac{3}{5} > 0,$$

we see that this point is a minimum. The approximate solution is

$$\boxed{y(x) = \frac{5}{18}x(1-x).}$$

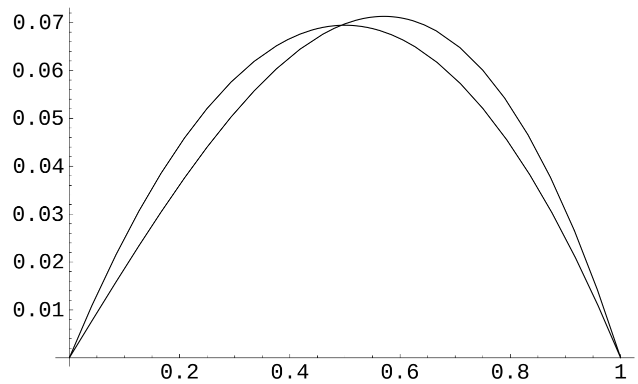

Figure 49.5: One Term Approximation and Exact Solution.

This one term approximation and the exact solution are plotted in Figure 49.5. The value of the functional is

$$\boxed{J = -\frac{5}{216} \approx -0.0231481.}$$

n = 1. We consider an approximate solution of the form $y(x) = x(1-x)(a+bx)$. We substitute this into the functional.

$$J(a,b) = \int_0^1 \left((y')^2 - y^2 - 2xy\right) \, dx = \frac{1}{210}\left(63a^2 + 63ab + 26b^2 - 35a - 21b\right)$$

We find the stationary points.

$$J_a = \frac{1}{30}(18a + 9b - 5) = 0$$
$$J_b = \frac{1}{210}(63a + 52b - 21) = 0$$

$$a = \frac{71}{369}, \quad b = \frac{7}{41}$$

Since the Hessian matrix

$$H = \begin{pmatrix} J_{aa} & J_{ab} \\ J_{ba} & J_{bb} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{3}{10} \\ \frac{3}{10} & \frac{26}{105} \end{pmatrix},$$

is positive definite,

$$\frac{3}{5} > 0, \quad \det(H) = \frac{41}{700},$$

we see that this point is a minimum. The approximate solution is

$$\boxed{y(x) = x(1-x)\left(\frac{71}{369} + \frac{7}{41}x\right).}$$

This two term approximation and the exact solution are plotted in Figure 49.6. The value of the functional is

$$\boxed{J = -\frac{136}{5535} \approx -0.0245709.}$$

2. **Exact Solution.** The Euler differential equation is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} F_{,y'} &= F_{,y} \\ \frac{\mathrm{d}}{\mathrm{d}x}[2y'] &= 2y + 2x \\ y'' - y &= x. \end{aligned}$$

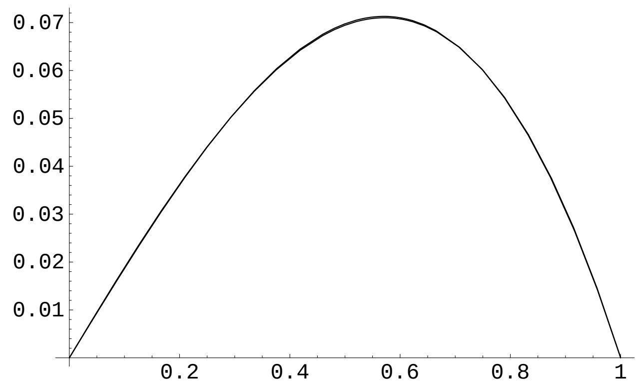

Figure 49.6: Two Term Approximation and Exact Solution.

The general solution is

$$y = c_1 \cosh x + c_2 \sinh x - x.$$

Applying the boundary conditions, we obtain,

$$\boxed{y = \frac{2 \sinh x}{\sinh 2} - x.}$$

The value of the integral for this extremal is

$$\boxed{J = -\frac{2(\mathrm{e}^4 - 13)}{3(\mathrm{e}^4 - 1)} \approx -0.517408.}$$

Polynomial Approximation. Consider an approximate solution of the form

$$y(x) = x(2 - x)(a_0 + a_1 x + \cdots a_n x^n).$$

The one term approximate solution is

$$\boxed{y(x) = -\frac{5}{14}x(2-x).}$$

This one term approximation and the exact solution are plotted in Figure 49.7. The value of the functional is

$$\boxed{J = -\frac{10}{21} \approx -0.47619.}$$

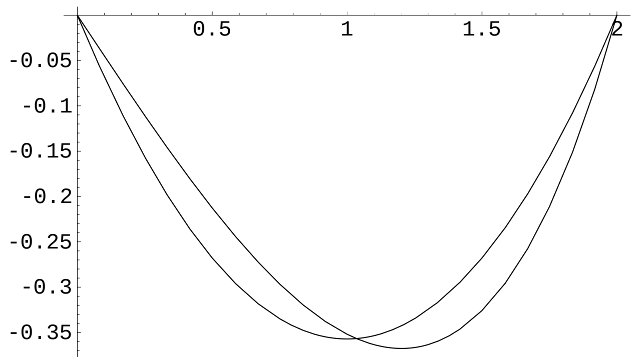

Figure 49.7: One Term Approximation and Exact Solution.

The two term approximate solution is

$$y(x) = x(2-x)\left(-\frac{33}{161} - \frac{7}{46}x\right).$$

This two term approximation and the exact solution are plotted in Figure 49.8. The value of the functional is

$$J = -\frac{416}{805} \approx -0.51677.$$

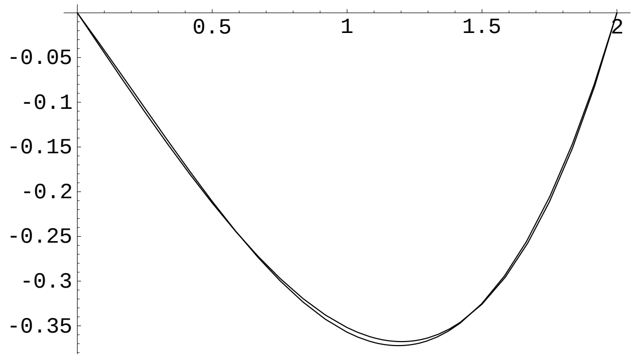

Figure 49.8: Two Term Approximation and Exact Solution.

Sine Series Approximation. Consider an approximate solution of the form

$$y(x) = a_1 \sin\left(\frac{\pi x}{2}\right) + a_2 \sin\left(\pi x\right) + \cdots + a_n \sin\left(n\frac{\pi x}{2}\right).$$

The one term approximate solution is

$$y(x) = -\frac{16}{\pi(\pi^2+4)} \sin\left(\frac{\pi x}{2}\right).$$

This one term approximation and the exact solution are plotted in Figure 49.9. The value of the functional is

$$J = -\frac{64}{\pi^2(\pi^2+4)} \approx -0.467537.$$

The two term approximate solution is

$$y(x) = -\frac{16}{\pi(\pi^2+4)} \sin\left(\frac{\pi x}{2}\right) + \frac{2}{\pi(\pi^2+1)} \sin(\pi x).$$

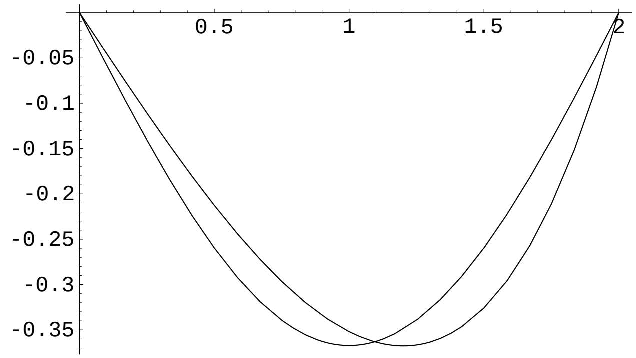

Figure 49.9: One Term Sine Series Approximation and Exact Solution.

This two term approximation and the exact solution are plotted in Figure 49.10. The value of the functional is

$$J = -\frac{4(17\pi^2 + 20)}{\pi^2(\pi^4 + 5\pi^2 + 4)} \approx -0.504823.$$

3. **Exact Solution.** The Euler differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}x} F_{,y'} = F_{,y}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[2xy'] = -2\frac{x^2-1}{x} y - 2x^2$$
$$y'' + \frac{1}{x} y' + \left(1 - \frac{1}{x^2}\right) y = -x$$

The general solution is

$$y = c_1 J_1(x) + c_2 Y_1(x) - x$$

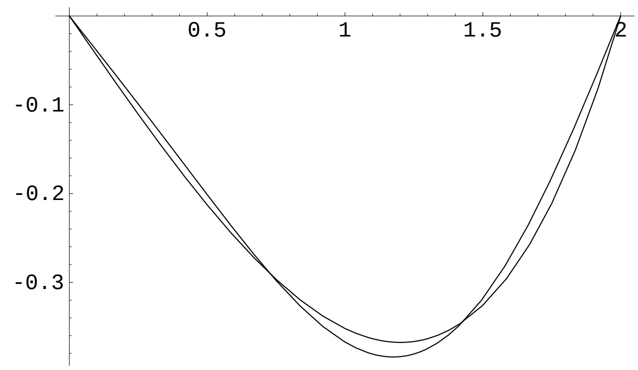

Figure 49.10: Two Term Sine Series Approximation and Exact Solution.

Applying the boundary conditions we obtain,

$$\boxed{y = \frac{(Y_1(2) - 2Y_1(1))J_1(x) + (2J_1(1) - J_1(2))Y_1(x)}{J_1(1)Y_1(2) - Y_1(1)J_1(2)} - x}$$

The value of the integral for this extremal is

$$\boxed{J \approx -0.310947}$$

Polynomial Approximation. Consider an approximate solution of the form

$$y(x) = (x-1)(2-x)(a_0 + a_1 x + \cdots a_n x^n).$$

The one term approximate solution is

$$\boxed{y(x) = (x-1)(2-x)\frac{23}{6(40\log 2 - 23)}}$$

This one term approximation and the exact solution are plotted in Figure 49.11. The one term approximation is a surprisingly close to the exact solution. The value of the functional is

$$\boxed{J = -\frac{529}{360(40\log 2 - 23)} \approx -0.310935.}$$

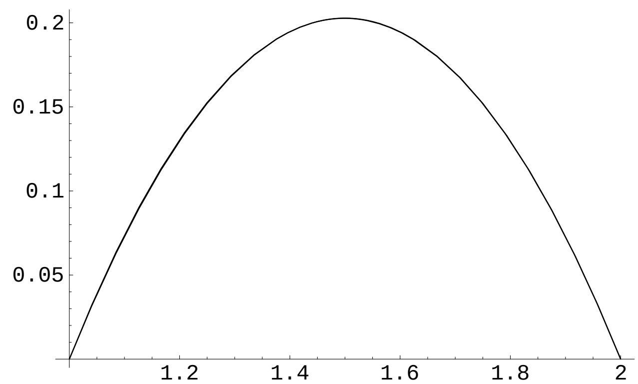

Figure 49.11: One Term Polynomial Approximation and Exact Solution.

Solution 49.21

1. The spectrum of T is the set,

$$\{\lambda : (T - \lambda I) \text{ is not invertible.}\}$$

$$(T - \lambda I) f = g$$

$$\int_{-\infty}^{\infty} K(x - y) f(y) \, dy - \lambda f(x) = g$$

$$\hat{K}(\omega) \hat{f}(\omega) - \lambda \hat{f}(\omega) = \hat{g}(\omega)$$

$$\left(\hat{K}(\omega) - \lambda \right) \hat{f}(\omega) = \hat{g}(\omega)$$

We may not be able to solve for $\hat{f}(\omega)$, (and hence invert $T - \lambda I$), if $\lambda = \hat{K}(\omega)$. Thus all values of $\hat{K}(\omega)$ are in the spectrum. If $\hat{K}(\omega)$ is everywhere nonzero we consider the case $\lambda = 0$. We have the equation,

$$\int_{-\infty}^{\infty} K(x - y) f(y) \, dy = 0$$

Since there are an infinite number of $L_2(-\infty, \infty)$ functions which satisfy this, (those which are nonzero on a set of measure zero), we cannot invert the equation. Thus $\lambda = 0$ is in the spectrum. The spectrum of T is the range of $\hat{K}(\omega)$ plus zero.

2. Let λ be a nonzero eigenvalue with eigenfunction ϕ.

$$(T - \lambda I) \phi = 0, \quad \forall x$$

$$\int_{-\infty}^{\infty} K(x - y) \phi(y) \, dy - \lambda \phi(x) = 0, \quad \forall x$$

Since K is continuous, $T\phi$ is continuous. This implies that the eigenfunction ϕ is continuous. We take the Fourier transform of the above equation.

$$\hat{K}(\omega) \hat{\phi}(\omega) - \lambda \hat{\phi}(\omega) = 0, \quad \forall \omega$$

$$\left(\hat{K}(\omega) - \lambda \right) \hat{\phi}(\omega) = 0, \quad \forall \omega$$

If $\phi(x)$ is absolutely integrable, then $\hat{\phi}(\omega)$ is continous. Since $\phi(x)$ is not identically zero, $\hat{\phi(\omega)}$ is not identically zero. Continuity implies that $\hat{\phi(\omega)}$ is nonzero on some interval of positive length, (a, b). From the above equation we see that $\hat{K}(\omega) = \lambda$ for $\omega \in (a, b)$.

Now assume that $\hat{K}(\omega) = \lambda$ in some interval (a, b). Any function $\hat{\phi}(\omega)$ that is nonzero only for $\omega \in (a, b)$ satisfies

$$\left(\hat{K}(\omega) - \lambda\right) \hat{\phi}(\omega) = 0, \quad \forall \omega.$$

By taking the inverse Fourier transform we obtain an eigenfunction $\phi(x)$ of the eigenvalue λ.

3. First we use the Fourier transform to find an explicit representation of $u = (T - \lambda I)^{-1} f$.

$$u = (T - \lambda I)^{-1} f (T - \lambda I) u = f$$
$$\int_{-\infty}^{\infty} K(x - y) u(y)\, dy - \lambda u = f$$
$$2\pi \hat{K} \hat{u} - \lambda \hat{u} = \hat{f}$$
$$\hat{u} = \frac{\hat{f}}{2\pi \hat{K} - \lambda}$$
$$\hat{u} = -\frac{1}{\lambda} \frac{\hat{f}}{1 - 2\pi \hat{K}/\lambda}$$

For $|\lambda| > |2\pi \hat{K}|$ we can expand the denominator in a geometric series.

$$\hat{u} = -\frac{1}{\lambda} \hat{f} \sum_{n=0}^{\infty} \left(\frac{2\pi \hat{K}}{\lambda}\right)^n$$
$$\boxed{u = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\lambda^n} \int_{-\infty}^{\infty} K_n(x - y) f(y)\, dy}$$

Here K_n is the n^{th} iterated kernel. Now we form the Neumann series expansion.

$$
\begin{aligned}
u &= (T - \lambda I)^{-1} f \\
&= -\frac{1}{\lambda}\left(I - \frac{1}{\lambda}T\right)^{-1} f \\
&= -\frac{1}{\lambda}\sum_{n=0}^{\infty} \frac{1}{\lambda^n} T^n f \\
&= -\frac{1}{\lambda}\sum_{n=0}^{\infty} \frac{1}{\lambda^n} T^n f \\
&= -\frac{1}{\lambda}\sum_{n=0}^{\infty} \frac{1}{\lambda^n} \int_{-\infty}^{\infty} K_n(x-y) f(y)\, dy
\end{aligned}
$$

The Neumann series is the same as the series we derived with the Fourier transform.

Solution 49.22

We seek a transformation T such that

$$(L - \lambda I)Tf = f.$$

We denote $u = Tf$ to obtain a boundary value problem,

$$u'' - \lambda u = f, \qquad u(-1) = u(1) = 0.$$

This problem has a unique solution if and only if the homogeneous adjoint problem has only the trivial solution.

$$u'' - \lambda u = 0, \qquad u(-1) = u(1) = 0.$$

This homogeneous problem has the eigenvalues and eigenfunctions,

$$\lambda_n = -\left(\frac{n\pi}{2}\right)^2, \quad u_n = \sin\left(\frac{n\pi}{2}(x+1)\right), \quad n \in \mathbb{N}.$$

The inhomogeneous problem has the unique solution

$$u(x) = \int_{-1}^{1} G(x, \xi; \lambda) f(\xi)\, d\xi$$

where

$$G(x, \xi; \lambda) = \begin{cases} -\frac{\sin\left(\sqrt{-\lambda}(x_< + 1)\right) \sin\left(\sqrt{-\lambda}(1 - x_>)\right)}{\sqrt{-\lambda} \sin\left(2\sqrt{-\lambda}\right)}, & \lambda < 0, \\ -\frac{1}{2}(x_< + 1)(1 - x_>), & \lambda = 0, \\ -\frac{\sinh\left(\sqrt{\lambda}(x_< + 1)\right) \sinh\left(\sqrt{\lambda}(1 - x_>)\right)}{\sqrt{\lambda} \sinh\left(2\sqrt{\lambda}\right)}, & \lambda > 0, \end{cases}$$

for $\lambda \neq -(n\pi/2)^2$, $n \in \mathbb{N}$. We set

$$Tf = \int_{-1}^{1} G(x, \xi; \lambda) f(\xi)\, d\xi$$

and note that since the kernel is continuous this is a bounded linear transformation. If $f \in W$, then

$$\begin{aligned} (L - \lambda I) T f &= (L - \lambda I) \int_{-1}^{1} G(x, \xi; \lambda) f(\xi)\, d\xi \\ &= \int_{-1}^{1} (L - \lambda I)[G(x, \xi; \lambda)] f(\xi)\, d\xi \\ &= \int_{-1}^{1} \delta(x - \xi) f(\xi)\, d\xi \\ &= f(x). \end{aligned}$$

If $f \in U$ then

$$\begin{aligned}
T(L - \lambda I)f &= \int_{-1}^{1} G(x,\xi;\lambda)\big(f''(\xi) - \lambda f(\xi)\big)\,d\xi \\
&= [G(x,\xi;\lambda)f'(\xi)]_{-1}^{1} - \int_{-1}^{1} G'(x,\xi;\lambda)f'(\xi)\,d\xi - \lambda\int_{-1}^{1} G(x,\xi;\lambda)f(\xi)\,d\xi \\
&= [-G'(x,\xi;\lambda)f(\xi)]_{-1}^{1} + \int_{-1}^{1} G''(x,\xi;\lambda)f(\xi)\,d\xi - \lambda\int_{-1}^{1} G(x,\xi;\lambda)f(\xi)\,d\xi \\
&= \int_{-1}^{1} \big(G''(x,\xi;\lambda) - \lambda G(x,\xi;\lambda)\big)f(\xi)\,d\xi \\
&= \int_{-1}^{1} \delta(x-\xi)f(\xi)\,d\xi \\
&= f(x).
\end{aligned}$$

L has the point spectrum $\lambda_n = -(n\pi/2)^2$, $n \in \mathbb{N}$.

Solution 49.23

1. We see that the solution is of the form $\phi(x) = a + x + bx^2$ for some constants a and b. We substitute this into the integral equation.

$$\begin{gathered}
\phi(x) = x + \lambda\int_0^1 \left(x^2y - y^2\right)\phi(y)\,dy \\
a + x + bx^2 = x + \lambda\int_0^1 \left(x^2y - y^2\right)\left(a + x + bx^2\right)dy \\
a + bx^2 = \frac{\lambda}{60}\left(-(15 + 20a + 12b) + (20 + 30a + 15b)x^2\right)
\end{gathered}$$

By equating the coefficients of x^0 and x^2 we solve for a and b.

$$a = -\frac{\lambda(\lambda+60)}{4(\lambda^2+5\lambda+60)}, \qquad b = -\frac{5\lambda(\lambda-60)}{6(\lambda^2+5\lambda+60)}$$

Thus the solution of the integral equation is

$$\boxed{\phi(x) = x - \frac{\lambda}{\lambda^2 + 5\lambda + 60}\left(\frac{5(\lambda - 24)}{6}x^2 + \frac{\lambda + 60}{4}\right).}$$

2. For $x < 1$ the integral equation reduces to

$$\boxed{\phi(x) = x.}$$

For $x \geq 1$ the integral equation becomes,

$$\phi(x) = x + \lambda \int_0^1 \sin(xy)\phi(y)\, dy.$$

We could solve this problem by writing down the Neumann series. Instead we will use an eigenfunction expansion. Let $\{\lambda_n\}$ and $\{\phi_n\}$ be the eigenvalues and orthonormal eigenfunctions of

$$\phi(x) = \lambda \int_0^1 \sin(xy)\phi(y)\, dy.$$

We expand $\phi(x)$ and x in terms of the eigenfunctions.

$$\phi(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$
$$x = \sum_{n=1}^{\infty} b_n \phi_n(x), \qquad b_n = \langle x, \phi_n(x) \rangle$$

We determine the coefficients a_n by substituting the series expansions into the Fredholm equation and equating coefficients of the eigenfunctions.

$$\phi(x) = x + \lambda \int_0^1 \sin(xy)\phi(y)\,dy$$

$$\sum_{n=1}^{\infty} a_n \phi_n(x) = \sum_{n=1}^{\infty} b_n \phi_n(x) + \lambda \int_0^1 \sin(xy) \sum_{n=1}^{\infty} a_n \phi_n(y)\,dy$$

$$\sum_{n=1}^{\infty} a_n \phi_n(x) = \sum_{n=1}^{\infty} b_n \phi_n(x) + \lambda \sum_{n=1}^{\infty} a_n \frac{1}{\lambda_n} \phi_n(x)$$

$$a_n \left(1 - \frac{\lambda}{\lambda_n}\right) = b_n$$

If λ is not an eigenvalue then we can solve for the a_n to obtain the unique solution.

$$a_n = \frac{b_n}{1 - \lambda/\lambda_n} = \frac{\lambda_n b_n}{\lambda_n - \lambda} = b_n + \frac{\lambda b_n}{\lambda_n - \lambda}$$

$$\boxed{\phi(x) = x + \sum_{n=1}^{\infty} \frac{\lambda b_n}{\lambda_n - \lambda} \phi_n(x), \quad \text{for } x \geq 1.}$$

If $\lambda = \lambda_m$, and $\langle x, \phi_m \rangle = 0$ then there is the one parameter family of solutions,

$$\boxed{\phi(x) = x + c\phi_m(x) + \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{\lambda b_n}{\lambda_n - \lambda} \phi_n(x), \quad \text{for } x \geq 1.}$$

If $\lambda = \lambda_m$, and $\langle x, \phi_m \rangle \neq 0$ then there is no solution.

Solution 49.24

1.

$$Kx = L_1 L_2 x = \lambda x$$

$$\begin{aligned} L_1 L_2 (L_1 x) &= L_1 (L_1 l_2 - I) x \\ &= L_1 (\lambda x - x) \\ &= (\lambda - 1)(L_1 x) \end{aligned}$$

$$\begin{aligned} L_1 L_2 (L_2 x) &= (L_2 L_1 + I) L_2 x \\ &= L_2 \lambda x + L_2 x \\ &= (\lambda + 1)(L_2 x) \end{aligned}$$

2.

$$\begin{aligned} L_1 L_2 - L_2 L_1 &= \left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{t}{2}\right)\left(-\frac{\mathrm{d}}{\mathrm{d}t} + \frac{t}{2}\right) - \left(-\frac{\mathrm{d}}{\mathrm{d}t} + \frac{t}{2}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{t}{2}\right) \\ &= -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{t}{2}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{2}I - \frac{t}{2}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{t^2}{4}I - \left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{t}{2}\frac{\mathrm{d}}{\mathrm{d}t} - \frac{1}{2}I + \frac{t}{2}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{t^2}{4}I\right) \\ &= I \end{aligned}$$

$$L_1 L_2 = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{1}{2}I + \frac{t^2}{4}I = K + \frac{1}{2}I$$

We note that $\mathrm{e}^{-t^2/4}$ is an eigenfunction corresponding to the eigenvalue $\lambda = 1/2$. Since $L_1 \mathrm{e}^{-t^2/4} = 0$ the result of this problem does not produce any negative eigenvalues. However, $L_2^n \mathrm{e}^{-t^2/4}$ is the product of $\mathrm{e}^{-t^2/4}$ and a polynomial of degree n in t. Since this function is square integrable it is and eigenfunction. Thus we have the eigenvalues and eigenfunctions,

$$\boxed{\lambda_n = n - \frac{1}{2}, \qquad \phi_n = \left(\frac{t}{2} - \frac{\mathrm{d}}{\mathrm{d}t}\right)^{n-1} \mathrm{e}^{-t^2/4}, \qquad \text{for } n \in \mathbb{N}.}$$

Solution 49.25

Since λ_1 is in the residual spectrum of T, there exists a nonzero y such that

$$\langle (T - \lambda_1 I)x, y\rangle = 0$$

for all x. Now we apply the definition of the adjoint.

$$\begin{gathered}
\langle x, (T - \lambda_1 I)^* y\rangle = 0, \quad \forall x \\
\langle x, (T^* - \overline{\lambda_1} I) y\rangle = 0, \quad \forall x \\
(T^* - \overline{\lambda_1} I) y = 0
\end{gathered}$$

y is an eigenfunction of T^* corresponding to the eigenvalue $\overline{\lambda_1}$.

Solution 49.26

1.

$$\begin{gathered}
u''(t) + \int_0^1 \sin(k(s-t)) u(s)\, ds = f(t), \quad u(0) = u'(0) = 0 \\
u''(t) + \cos(kt) \int_0^1 \sin(ks) u(s)\, ds - \sin(kt) \int_0^1 \cos(ks) u(s)\, ds = f(t) \\
u''(t) + c_1 \cos(kt) - c_2 \sin(kt) = f(t) \\
u''(t) = f(t) - c_1 \cos(kt) + c_2 \sin(kt)
\end{gathered}$$

The solution of

$$u''(t) = g(t), \quad u(0) = u'(0) = 0$$

using Green functions is

$$u(t) = \int_0^t (t - \tau) g(\tau)\, d\tau.$$

Thus the solution of our problem has the form,

$$u(t) = \int_0^t (t-\tau) f(\tau)\, d\tau - c_1 \int_0^t (t-\tau)\cos(k\tau)\, d\tau + c_2 \int_0^t (t-\tau)\sin(k\tau)\, d\tau$$

$$\boxed{u(t) = \int_0^t (t-\tau) f(\tau)\, d\tau - c_1 \frac{1-\cos(kt)}{k^2} + c_2 \frac{kt - \sin(kt)}{k^2}}$$

We could determine the constants by multiplying in turn by $\cos(kt)$ and $\sin(kt)$ and integrating from 0 to 1. This would yields a set of two linear equations for c_1 and c_2.

2.

$$u(x) = \lambda \int_0^\pi \sum_{n=1}^\infty \frac{\sin nx \sin ns}{n} u(s)\, ds$$

We expand $u(x)$ in a sine series.

$$\sum_{n=1}^\infty a_n \sin nx = \lambda \int_0^\pi \left(\sum_{n=1}^\infty \frac{\sin nx \sin ns}{n} \right) \left(\sum_{m=1}^\infty a_m \sin ms \right) ds$$

$$\sum_{n=1}^\infty a_n \sin nx = \lambda \sum_{n=1}^\infty \frac{\sin nx}{n} \sum_{m=1}^\infty \int_0^\pi a_m \sin ns \sin ms\, ds$$

$$\sum_{n=1}^\infty a_n \sin nx = \lambda \sum_{n=1}^\infty \frac{\sin nx}{n} \sum_{m=1}^\infty \frac{\pi}{2} a_m \delta_{mn}$$

$$\sum_{n=1}^\infty a_n \sin nx = \frac{\pi}{2} \lambda \sum_{n=1}^\infty a_n \frac{\sin nx}{n}$$

The eigenvalues and eigenfunctions are

$$\boxed{\lambda_n = \frac{2n}{\pi}, \qquad u_n = \sin nx, \qquad n \in \mathbb{N}.}$$

3.

$$\phi(\theta) = \lambda \int_0^{2\pi} \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} \phi(t)\, dt, \quad |r| < 1$$

We use Poisson's formula.

$$\phi(\theta) = \lambda u(r, \theta),$$

where $u(r, \theta)$ is harmonic in the unit disk and satisfies, $u(1, \theta) = \phi(\theta)$. For a solution we need $\lambda = 1$ and that $u(r, \theta)$ is independent of r. In this case $u(\theta)$ satisfies

$$u''(\theta) = 0, \quad u(\theta) = \phi(\theta).$$

The solution is $\phi(\theta) = c_1 + c_2\theta$. There is only one eigenvalue and corresponding eigenfunction,

$$\boxed{\lambda = 1, \qquad \phi = c_1 + c_2\theta.}$$

4.

$$\phi(x) = \lambda \int_{-\pi}^{\pi} \cos^n(x - \xi)\phi(\xi)\, d\xi$$

We expand the kernel in a Fourier series. We could find the expansion by integrating to find the Fourier coefficients, but it is easier to expand $\cos^n(x)$ directly.

$$\begin{aligned}
\cos^n(x) &= \left[\frac{1}{2}(\mathrm{e}^{ix} + \mathrm{e}^{-ix})\right]^n \\
&= \frac{1}{2^n}\left[\binom{n}{0}\mathrm{e}^{inx} + \binom{n}{1}\mathrm{e}^{i(n-2)x} + \cdots + \binom{n}{n-1}\mathrm{e}^{-i(n-2)x} + \binom{n}{n}\mathrm{e}^{-inx}\right]
\end{aligned}$$

If n is odd,

$$\begin{aligned}\cos^n(x) &= \frac{1}{2^n}\left[\binom{n}{0}(\,\mathrm{e}^{inx} + \,\mathrm{e}^{-inx}) + \binom{n}{1}(\,\mathrm{e}^{i(n-2)x} + \,\mathrm{e}^{-i(n-2)x}) + \cdots\right.\\ &\qquad\left. + \binom{n}{(n-1)/2}(\,\mathrm{e}^{ix} + \,\mathrm{e}^{-ix})\right]\\ &= \frac{1}{2^n}\left[\binom{n}{0}2\cos(nx) + \binom{n}{1}2\cos((n-2)x) + \cdots + \binom{n}{(n-1)/2}2\cos(x)\right]\\ &= \frac{1}{2^{n-1}}\sum_{m=0}^{(n-1)/2}\binom{n}{m}\cos((n-2m)x)\\ &= \frac{1}{2^{n-1}}\sum_{\substack{k=1\\ \text{odd } k}}^{n}\binom{n}{(n-k)/2}\cos(kx).\end{aligned}$$

If n is even,

$$\cos^n(x) = \frac{1}{2^n}\left[\binom{n}{0}(\mathrm{e}^{inx} + \mathrm{e}^{-inx}) + \binom{n}{1}(\mathrm{e}^{i(n-2)x} + \mathrm{e}^{-i(n-2)x}) + \cdots \right.$$
$$\left. + \binom{n}{n/2-1}(\mathrm{e}^{i2x} + \mathrm{e}^{-i2x}) + \binom{n}{n/2}\right]$$
$$= \frac{1}{2^n}\left[\binom{n}{0}2\cos(nx) + \binom{n}{1}2\cos((n-2)x) + \cdots + \binom{n}{n/2-1}2\cos(2x) + \binom{n}{n/2}\right]$$
$$= \frac{1}{2^n}\binom{n}{n/2} + \frac{1}{2^{n-1}}\sum_{m=0}^{(n-2)/2}\binom{n}{m}\cos((n-2m)x)$$
$$= \frac{1}{2^n}\binom{n}{n/2} + \frac{1}{2^{n-1}}\sum_{\substack{k=2 \\ \text{even } k}}^{n}\binom{n}{(n-k)/2}\cos(kx).$$

We will denote,

$$\cos^n(x-\xi) = \frac{a_0}{2}\sum_{k=1}^{n} a_k \cos(k(x-\xi)),$$

where

$$a_k = \frac{1+(-1)^{n-k}}{2}\frac{1}{2^{n-1}}\binom{n}{(n-k)/2}.$$

We substitute this into the integral equation.

$$\phi(x) = \lambda\int_{-\pi}^{\pi}\left(\frac{a_0}{2}\sum_{k=1}^{n} a_k\cos(k(x-\xi))\right)\phi(\xi)\,d\xi$$

$$\phi(x) = \lambda\frac{a_0}{2}\int_{-\pi}^{\pi}\phi(\xi)\,d\xi + \lambda\sum_{k=1}^{n} a_k\left(\cos(kx)\int_{-\pi}^{\pi}\cos(k\xi)\phi(\xi)\,d\xi + \sin(kx)\int_{-\pi}^{\pi}\sin(k\xi)\phi(\xi)\,d\xi\right)$$

For even n, substituting $\phi(x) = 1$ yields $\lambda = \frac{1}{\pi a_0}$. For n and m both even or odd, substituting $\phi(x) = \cos(mx)$ or $\phi(x) = \sin(mx)$ yields $\lambda = \frac{1}{\pi a_m}$. For even n we have the eigenvalues and eigenvectors,

$$\lambda_0 = \frac{1}{\pi a_0}, \qquad \phi_0 = 1,$$

$$\lambda_m = \frac{1}{\pi a_{2m}}, \qquad \phi_m^{(1)} = \cos(2mx), \quad \phi_m^{(2)} = \sin(2mx), \qquad m = 1, 2, \dots, n/2.$$

For odd n we have the eigenvalues and eigenvectors,

$$\lambda_m = \frac{1}{\pi a_{2m-1}}, \qquad \phi_m^{(1)} = \cos((2m-1)x), \quad \phi_m^{(2)} = \sin((2m-1)x), \qquad m = 1, 2, \dots, (n+1)/2.$$

Solution 49.27

1. First we shift the range of integration to rewrite the kernel.

$$\phi(x) = \lambda \int_0^{2\pi} \left(2\pi^2 - 6\pi|x-s| + 3(x-s)^2\right) \phi(s)\, ds$$
$$\phi(x) = \lambda \int_{-x}^{-x+2\pi} \left(2\pi^2 - 6\pi|y| + 3y^2\right) \phi(x+y)\, dy$$

We expand the kernel in a Fourier series.

$$K(y) = 2\pi^2 - 6\pi|y| + 3y^2 = \sum_{n=-\infty}^{\infty} c_n \, \mathrm{e}^{iny}$$

$$c_n = \frac{1}{2\pi} \int_{-x}^{-x+2\pi} K(y)\, \mathrm{e}^{-iny}\, dy = \begin{cases} \frac{6}{n^2}, & n \neq 0, \\ 0, & n = 0 \end{cases}$$

$$K(y) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{6}{n^2} \mathrm{e}^{iny} = \sum_{n=1}^{\infty} \frac{12}{n^2} \cos(ny)$$

$$K(x,s) = \sum_{n=1}^{\infty} \frac{12}{n^2} \cos(n(x-s)) = \sum_{n=1}^{\infty} \frac{12}{n^2} \big(\cos(nx)\cos(nx) + \sin(nx)\sin(ns) \big)$$

Now we substitute the Fourier series expression for the kernel into the eigenvalue problem.

$$\phi(x) = 12\lambda \int_0^{2\pi} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \big(\cos(nx)\cos(ns) + \sin(nx)\sin(ns) \big) \right) \phi(s)\, ds$$

From this we obtain the eigenvalues and eigenfunctions,

$$\boxed{\lambda_n = \frac{n^2}{12\pi}, \quad \phi_n^{(1)} = \frac{1}{\sqrt{\pi}} \cos(nx), \quad \phi_n^{(2)} = \frac{1}{\sqrt{\pi}} \sin(nx), \quad n \in \mathbb{N}.}$$

2. The set of eigenfunctions do not form a complete set. Only those functions with a vanishing integral on $[0, 2\pi]$ can be represented. We consider the equation

$$\int_0^{2\pi} K(x,s)\phi(s)\, ds = 0$$

$$\int_0^{2\pi} \left(\sum_{n=1}^{\infty} \frac{12}{n^2} \big(\cos(nx)\cos(ns) + \sin(nx)\sin(ns) \big) \right) \phi(s)\, ds = 0$$

This has the solutions $\phi = \text{const}$. The set of eigenfunctions

$$\boxed{\phi_0 = \frac{1}{\sqrt{2\pi}}, \quad \phi_n^{(1)} = \frac{1}{\sqrt{\pi}} \cos(nx), \quad \phi_n^{(2)} = \frac{1}{\sqrt{\pi}} \sin(nx), \quad n \in \mathbb{N},}$$

is a complete set. We can also write the eigenfunctions as

$$\boxed{\phi_n = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{inx}, \quad n \in \mathbb{Z}.}$$

3. We consider the problem

$$u - \lambda T u = f.$$

For $\lambda \neq \lambda$, (λ not an eigenvalue), we can obtain a unique solution for u.

$$u(x) = f(x) + \int_0^{2\pi} \Gamma(x, s, \lambda) f(s) \, ds$$

Since $K(x, s)$ is self-adjoint and $L_2(0, 2\pi)$, we have

$$\begin{aligned}
\Gamma(x, s, \lambda) &= \lambda \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\phi_n(x) \overline{\phi_n(s)}}{\lambda_n - \lambda} \\
&= \lambda \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\frac{1}{2\pi} \mathrm{e}^{inx} \mathrm{e}^{-ins}}{\frac{n^2}{12\pi} - \lambda} \\
&= 6\lambda \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\mathrm{e}^{in(x-s)}}{n^2 - 12\pi\lambda}
\end{aligned}$$

$$\boxed{\Gamma(x, s, \lambda) = 12\lambda \sum_{n=1}^{\infty} \frac{\cos(n(x - s))}{n^2 - 12\pi\lambda}}$$

Solution 49.28
First assume that λ is an eigenvalue of T, $T\phi = \lambda\phi$.

$$
\begin{aligned}
p(T)\phi &= \sum_{k=0}^{n} a_n T^n \phi \\
&= \sum_{k=0}^{n} a_n \lambda^n \phi \\
&= p(\lambda)\phi
\end{aligned}
$$

$p(\lambda)$ is an eigenvalue of $p(T)$.

Now assume that μ is an eigenvalues of $p(T)$, $p(T)\phi = \mu\phi$. We assume that T has a complete, orthonormal set of eigenfunctions, $\{\phi_n\}$ corresponding to the set of eigenvalues $\{\lambda_n\}$. We expand ϕ in these eigenfunctions.

$$
\begin{gathered}
p(T)\phi = \mu\phi \\
p(T)\sum c_n \phi_n = \mu \sum c_n \phi_n \\
\sum c_n p(\lambda_n)\phi_n = \sum c_n \mu \phi_n \\
p(\lambda_n) = \mu, \quad \forall n \text{ such that } c_n \neq 0
\end{gathered}
$$

Thus all eigenvalues of $p(T)$ are of the form $p(\lambda)$ with λ an eigenvalue of T.

Solution 49.29
The Fourier cosine transform is defined,

$$
\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\pi}\int_0^\infty f(x)\cos(\omega x)\,dx, \\
f(x) &= 2\int_0^\infty \hat{f}(\omega)\cos(\omega x)\,d\omega.
\end{aligned}
$$

We can write the integral equation in terms of the Fourier cosine transform.

$$\phi(x) = f(x) + \lambda \int_0^\infty \cos(2xs)\phi(s)\,ds$$

$$\phi(x) = f(x) + \lambda\pi\hat{\phi}(2x) \tag{49.8}$$

We multiply the integral equation by $\frac{1}{\pi}\cos(2xs)$ and integrate.

$$\frac{1}{\pi}\int_0^\infty \cos(2xs)\phi(x)\,dx = \frac{1}{\pi}\int_0^\infty \cos(2xs)f(x)\,dx + \lambda\int_0^\infty \cos(2xs)\hat{\phi}(2x)\,dx$$

$$\hat{\phi}(2s) = \hat{f}(2s) + \frac{\lambda}{2}\int_0^\infty \cos(xs)\hat{\phi}(x)\,dx$$

$$\hat{\phi}(2s) = \hat{f}(2s) + \frac{\lambda}{4}\phi(s)$$

$$\phi(x) = -\frac{4}{\lambda}\hat{f}(2x) + \frac{4}{\lambda}\hat{\phi}(2x) \tag{49.9}$$

We eliminate $\hat{\phi}$ between (49.8) and (49.9).

$$\left(1 - \frac{\pi\lambda^2}{4}\right)\phi(x) = f(x) + \lambda\pi\hat{f}(2x)$$

$$\boxed{\phi(x) = \frac{f(x) + \lambda\int_0^\infty f(s)\cos(2xs)\,ds}{1 - \pi\lambda^2/4}}$$

Solution 49.30

$$
\begin{aligned}
\int_D vLu \, dx \, dy &= \int_D v(u_{xx} + u_{yy} + au_x + bu_y + cu) \, dx \, dy \\
&= \int_D (v\nabla^2 u + avu_x + bvu_y + cuv) \, dx \, dy \\
&= \int_D (u\nabla^2 v + avu_x + bvu_y + cuv) \, dx \, dy + \int_C (v\nabla u - u\nabla v) \cdot n \, ds \\
&= \int_D (u\nabla^2 v - auv_x - buv_y - uva_x - uvb_y + cuv) \, dx \, dy + \int_C \left(auv\frac{\partial x}{\partial n} + buv\frac{\partial y}{\partial n} \right) \, ds + \int_C \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right)
\end{aligned}
$$

Thus we see that

$$\int_D (vLu - uL^* v) \, dx \, dy = \int_C H(u,v) \, ds,$$

where

$$L^* v = v_{xx} + v_{yy} - av_x - bv_y + (c - a_x - b_y)v$$

and

$$H(u,v) = \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} + auv\frac{\partial x}{\partial n} + buv\frac{\partial y}{\partial n} \right).$$

Let G be the harmonic Green function, which satisfies,

$$\Delta G = \delta \text{ in } D, \quad G = 0 \text{ on } C.$$

Let u satisfy $Lu = 0$.

$$
\begin{gathered}
\int_D (GLu - uL^* G)\,dx\,dy = \int_C H(u,G)\,ds \\
-\int_D uL^* G\,dx\,dy = \int_C H(u,G)\,ds \\
-\int_D u\Delta G\,dx\,dy - \int_D u(L^* - \Delta)G\,dx\,dy = \int_C H(u,G)\,ds \\
-\int_D u\delta(x-\xi)\delta(y-\eta)\,dx\,dy - \int_D u(L^* - \Delta)G\,dx\,dy = \int_C H(u,G)\,ds \\
-u(\xi,\eta) - \int_D u(L^* - \Delta)G\,dx\,dy = \int_C H(u,G)\,ds
\end{gathered}
$$

We expand the operators to obtain the first form.

$$
\begin{gathered}
u + \int_D u(-aG_x - bG_y + (c - a_x - b_y)G)\,dx\,dy = -\int_C \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n} + auG\frac{\partial x}{\partial n} + buG\frac{\partial y}{\partial n} \right) ds \\
u + \int_D ((c - a_x - b_y)G - aG_x - bG_y)u\,dx\,dy = \int_C u\frac{\partial G}{\partial n}\,ds \\
u + \int_D ((c - a_x - b_y)G - aG_x - bG_y)u\,dx\,dy = U
\end{gathered}
$$

Here U is the harmonic function that satisfies $U = f$ on C.

We use integration by parts to obtain the second form.

$$u + \int_D (cuG - a_x uG - b_y uG - auG_x - buG_y)\,dx\,dy = U$$

$$u + \int_D (cuG - a_x uG - b_y uG + (au)_x G + (bu)_y G)\,dx\,dy - \int_C \left(auG\frac{\partial y}{\partial n} + buG\frac{\partial x}{\partial n} \right) ds = U$$

$$u + \int_D (cuG - a_x uG - b_y uG + a_x uG + au_x G + b_y uG + bu_y G)\,dx\,dy = U$$

$$\boxed{u + \int_D (au_x + bu_y + cu)G\,dx\,dy = U}$$

Solution 49.31

1. First we differentiate to obtain a differential equation.

$$\phi(x) = \lambda \int_0^1 \min(x,s)\phi(s)\,ds = \lambda \left(\int_0^x \mathrm{e}^s \phi(s)\,ds + \int_x^1 \mathrm{e}^x \phi(s)\,ds \right)$$

$$\phi'(x) = \lambda \left(x\phi(x) + \int_x^1 \phi(s)\,ds - x\phi(x) \right) = \lambda \int_x^1 \phi(s)\,ds$$

$$\phi''(x) = -\lambda\phi(x)$$

We note that that $\phi(x)$ satisfies the constraints,

$$\phi(0) = \lambda \int_0^1 0 \cdot \phi(s)\,ds = 0,$$

$$\phi'(1) = \lambda \int_1^1 \phi(s)\,ds = 0.$$

Thus we have the problem,

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = \phi'(1) = 0.$$

The general solution of the differential equation is

$$\phi(x) = \begin{cases} a + bx & \text{for } \lambda = 0 \\ a\cos\left(\sqrt{\lambda}x\right) + b\sin\left(\sqrt{\lambda}x\right) & \text{for } \lambda > 0 \\ a\cosh\left(\sqrt{-\lambda}x\right) + b\sinh\left(\sqrt{-\lambda}x\right) & \text{for } \lambda < 0 \end{cases}$$

We see that for $\lambda = 0$ and $\lambda < 0$ only the trivial solution satisfies the homogeneous boundary conditions. For positive λ the left boundary condition demands that $a = 0$. The right boundary condition is then

$$b\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) = 0$$

The eigenvalues and eigenfunctions are

$$\boxed{\lambda_n = \left(\frac{(2n-1)\pi}{2}\right)^2, \quad \phi_n(x) = \sin\left(\frac{(2n-1)\pi}{2}x\right), \quad n \in \mathbb{N}}$$

2. First we differentiate the integral equation.

$$\begin{aligned} \phi(x) &= \lambda\left(\int_0^x \mathrm{e}^s\phi(s)\,ds + \int_x^1 \mathrm{e}^x\phi(s)\,ds\right) \\ \phi'(x) &= \lambda\left(\mathrm{e}^x\phi(x) + \mathrm{e}^x\int_x^1 \phi(s)\,ds - \mathrm{e}^x\phi(x)\right) \\ &= \lambda\,\mathrm{e}^x\int_x^1 \phi(s)\,ds \\ \phi''(x) &= \lambda\left(\mathrm{e}^x\int_x^1 \phi(s)\,ds - \mathrm{e}^x\phi(x)\right) \end{aligned}$$

$\phi(x)$ satisfies the differential equation

$$\phi'' - \phi' + \lambda\,\mathrm{e}^x\phi = 0.$$

We note the boundary conditions,

$$\phi(0) - \phi'(0) = 0, \quad \phi'(1) = 0.$$

In self-adjoint form, the problem is

$$\left(\mathrm{e}^{-x}\phi'\right)' + \lambda\phi = 0, \quad \phi(0) - \phi'(0) = 0, \quad \phi'(1) = 0.$$

The Rayleigh quotient is

$$\begin{aligned}
\lambda &= \frac{\left[-\mathrm{e}^{-x}\phi\phi'\right]_0^1 + \int_0^1 \mathrm{e}^{-x}(\phi')^2\,dx}{\int_0^1 \phi^2\,dx} \\
&= \frac{\phi(0)\phi'(0) + \int_0^1 \mathrm{e}^{-x}(\phi')^2\,dx}{\int_0^1 \phi^2\,dx} \\
&= \frac{(\phi(0))^2 + \int_0^1 \mathrm{e}^{-x}(\phi')^2\,dx}{\int_0^1 \phi^2\,dx}
\end{aligned}$$

Thus we see that there are only positive eigenvalues. The differential equation has the general solution

$$\phi(x) = \mathrm{e}^{x/2}\left(aJ_1\left(2\sqrt{\lambda}\,\mathrm{e}^{x/2}\right) + bY_1\left(2\sqrt{\lambda}\,\mathrm{e}^{x/2}\right)\right)$$

We define the functions,

$$u(x;\lambda) = \mathrm{e}^{x/2}J_1\left(2\sqrt{\lambda}\,\mathrm{e}^{x/2}\right), \quad v(x;\lambda) = \mathrm{e}^{x/2}Y_1\left(2\sqrt{\lambda}\,\mathrm{e}^{x/2}\right).$$

We write the solution to automatically satisfy the right boundary condition, $\phi'(1) = 0$,

$$\phi(x) = v'(1;\lambda)u(x;\lambda) - u'(1;\lambda)v(x;\lambda).$$

We determine the eigenvalues from the left boundary condition, $\phi(0) - \phi'(0) = 0$. The first few are

$$\lambda_1 \approx 0.678298$$
$$\lambda_2 \approx 7.27931$$
$$\lambda_3 \approx 24.9302$$
$$\lambda_4 \approx 54.2593$$
$$\lambda_5 \approx 95.3057$$

The eigenfunctions are,

$$\phi_n(x) = v'(1; \lambda_n) u(x; \lambda_n) - u'(1; \lambda_n) v(x; \lambda_n).$$

Solution 49.32

1. First note that

$$\sin(kx)\sin(lx) = \operatorname{sign}(kl)\sin(ax)\sin(bx)$$

where

$$a = \max(|k|, |l|), \quad b = \min(|k|, |l|).$$

Consider the analytic function,

$$\frac{\mathrm{e}^{i(a-b)x} - \mathrm{e}^{i(a+b)}}{2} = \sin(ax)\sin(bx) - i\cos(ax)\sin(bx).$$

$$\begin{aligned}
⨍_{-\infty}^{\infty} \frac{\sin(kx)\sin(lx)}{x^2 - z^2}\,dx &= \operatorname{sign}(kl) ⨍_{-\infty}^{\infty} \frac{\sin(ax)\sin(bx)}{x^2 - z^2}\,dx \\
&= \operatorname{sign}(kl)\frac{1}{2z} ⨍_{-\infty}^{\infty} \left(\frac{\sin(ax)\sin(bx)}{x - z} - \frac{\sin(ax)\sin(bx)}{x + z} \right) dx \\
&= -\pi \operatorname{sign}(kl)\frac{1}{2z}\left(-\cos(az)\sin(bz) + \cos(-az)\sin(-bz)\right)
\end{aligned}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin(kx)\sin(lx)}{x^2 - z^2}\, dx = \operatorname{sign}(kl)\frac{\pi}{z}\cos(az)\sin(bz)}$$

2. Consider the analytic function,

$$\frac{e^{i|p|x} - e^{i|q|x}}{x} = \frac{\cos(|p|x) - \cos(|q|x) + i(\sin(|p|x) - \sin(|q|x))}{x}.$$

$$\begin{aligned}
⨍_{-\infty}^{\infty} \frac{\cos(px) - \cos(qx)}{x^2}\, dx &= ⨍_{-\infty}^{\infty} \frac{\cos(|p|x) - \cos(|q|x)}{x^2}\, dx \\
&= -\pi \lim_{x\to 0} \frac{\sin(|p|x) - \sin(|q|x)}{x}
\end{aligned}$$

$$\boxed{⨍_{-\infty}^{\infty} \frac{\cos(px) - \cos(qx)}{x^2}\, dx = \pi(|q| - |p|)}$$

3. We use the analytic function,

$$\frac{i(x - ia)(x - ib)\, e^{ix}}{(x^2 + a^2)(x^2 + b^2)} = \frac{-(x^2 - ab)\sin x + (a+b)x\cos x + i((x^2 - ab)\cos x + (a+b)x\sin x)}{(x^2 + a^2)(x^2 + b^2)}$$

$$\begin{aligned}
⨍_{-\infty}^{\infty} \frac{-(x^2 - ab)\sin x + (a+b)x\cos x}{x(x^2 + a^2)(x^2 + b^2)} &= -\pi \lim_{x\to 0} \frac{(x^2 - ab)\cos x + (a+b)x\sin x}{(x^2 + a^2)(x^2 + b^2)} \\
&= -\pi\frac{-ab}{a^2b^2}
\end{aligned}$$

$$\boxed{⨍_{-\infty}^{\infty} \frac{-(x^2 - ab)\sin x + (a+b)x\cos x}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{ab}}$$

Solution 49.33
We consider the function

$$G(z) = \left((1-z^2)^{1/2} + iz\right)\log(1+z).$$

For $(1-z^2)^{1/2} = (1-z)^{1/2}(1+z)^{1/2}$ we choose the angles,

$$-\pi < \arg(1-z) < \pi, \quad 0 < \arg(1+z) < 2\pi,$$

so that there is a branch cut on the interval $(-1, 1)$. With this choice of branch, $G(z)$ vanishes at infinity. For the logarithm we choose the principal branch,

$$-\pi < \arg(1+z) < \pi.$$

For $t \in (-1, 1)$,

$$G^+(t) = \left(\sqrt{1-t^2} + it\right)\log(1+t),$$
$$G^-(t) = \left(-\sqrt{1-t^2} + it\right)\log(1+t),$$

$$G^+(t) - G^-(t) = 2\sqrt{1-t^2}\log(1+t),$$
$$\frac{1}{2}\left(G^+(t) + G^-(t)\right) = it\log(1+t).$$

For $t \in (-\infty, -1)$,

$$G^+(t) = i\left(\sqrt{1-t^2} + t\right)\left(\log(-t-1) + i\pi\right),$$
$$G^-(t) = i\left(-\sqrt{1-t^2} + t\right)\left(\log(-t-1) - i\pi\right),$$

$$G^+(t) - G^-(t) = -2\pi\left(\sqrt{t^2-1} + t\right).$$

For $x \in (-1, 1)$ we have

$$
\begin{aligned}
G(x) &= \frac{1}{2}\left(G^+(x) + G^-(x)\right) \\
&= ix \log(1 + x) \\
&= \frac{1}{i2\pi} \int_{-} \infty^{-1} \frac{-2\pi(\sqrt{t^2-1}+t)}{t-x}\, dt + \frac{1}{i2\pi} \int_{-1}^{1} \frac{2\sqrt{1-t^2}\log(1+t)}{t-x}\, dt
\end{aligned}
$$

From this we have

$$
\begin{aligned}
&\int_{-1}^{1} \frac{\sqrt{1-t^2}\log(1+t)}{t-x}\, dt \\
&\quad = -\pi x \log(1+x) + \pi \int_{1}^{\infty} \frac{t-\sqrt{t^2-1}}{t+x}\, dt \\
&\quad = \pi \left(x\log(1+x) - 1 + \frac{\pi}{2}\sqrt{1-x^2} - \sqrt{1-x^2}\arcsin(x) + x\log(2) + x\log(1+x) \right)
\end{aligned}
$$

$$
\boxed{\int_{-1}^{1} \frac{\sqrt{1-t^2}\log(1+t)}{t-x}\, dt = \pi\left(x\log x - 1 + \sqrt{1-x^2}\left(\frac{\pi}{2} - \arcsin(x)\right)\right)}
$$

Solution 49.34

Let $F(z)$ denote the value of the integral.

$$
F(z) = \frac{1}{i\pi} \fint_C \frac{f(t)\, dt}{t-z}
$$

From the Plemelj formula we have,

$$
\begin{gathered}
F^+(t_0) + F^-(t_0) = \frac{1}{i\pi} \fint_C \frac{f(t)}{t-t_0}\, dt, \\
f(t_0) = F^+(t_0) - F^-(t_0).
\end{gathered}
$$

With $W(z)$ defined as above, we have

$$W^+(t_0) + W^-(t_0) = F^+(t_0) - F^-(t_0) = f(t_0),$$

and also

$$\begin{aligned} W^+(t_0) + W^-(t_0) &= \frac{1}{i\pi} \fint_C \frac{W^+(t) - W^-(t)}{t - t_0}\, dt \\ &= \frac{1}{i\pi} \fint_C \frac{F^+(t) + F^-(t)}{t - t_0}\, dt \\ &= \frac{1}{i\pi} \fint_C \frac{g(t)}{t - t_0}\, dt. \end{aligned}$$

Thus the solution of the integral equation is

$$\boxed{f(t_0) = \frac{1}{i\pi} \fint_C \frac{g(t)}{t - t_0}\, dt.}$$

Solution 49.35

(i)

$$G(\tau) = (\tau - \beta)^{-1} \left(\frac{\tau - \beta}{\tau - \alpha} \right)^{\gamma}$$

$$G^+(\zeta) = (\zeta - \beta)^{-1} \left(\frac{\zeta - \beta}{\zeta - \alpha} \right)^{\gamma}$$

$$G^-(\zeta) = \mathrm{e}^{-i2\pi\gamma} G^+(\zeta)$$

$$G^+(\zeta) - G^-(\zeta) = (1 - \mathrm{e}^{-i2\pi\gamma})(\zeta - \beta)^{-1} \left(\frac{\zeta - \beta}{\zeta - \alpha} \right)^{\gamma}$$

$$G^+(\zeta) + G^-(\zeta) = (1 + \mathrm{e}^{-i2\pi\gamma})(\zeta - \beta)^{-1} \left(\frac{\zeta - \beta}{\zeta - \alpha} \right)^{\gamma}$$

$$G^+(\zeta) + G^-(\zeta) = \frac{1}{i\pi} \fint_C \frac{(1 - \mathrm{e}^{-i2\pi\gamma})\, d\tau}{(\tau - \beta)^{1-\gamma}(\tau - \alpha)^{\gamma}(\tau - \zeta)}$$

$$\boxed{\frac{1}{i\pi} \fint_C \frac{d\tau}{(\tau - \beta)^{1-\gamma}(\tau - \alpha)^{\gamma}(\tau - \zeta)} = -i\cot(\pi\gamma) \frac{(\zeta - \beta)^{\gamma-1}}{(\zeta - \alpha)^{\gamma}}}$$

(ii) Consider the branch of

$$\left(\frac{z - \beta}{z - \alpha} \right)^{\gamma}$$

that tends to unity as $z \to \infty$. We find a series expansion of this function about infinity.

$$
\begin{aligned}
\left(\frac{z-\beta}{z-\alpha}\right)^{\gamma} &= \left(1-\frac{\beta}{z}\right)^{\gamma}\left(1-\frac{\alpha}{z}\right)^{-\gamma} \\
&= \left(\sum_{j=0}^{\infty}(-1)^j\binom{\gamma}{j}\left(\frac{\beta}{z}\right)^j\right)\left(\sum_{k=0}^{\infty}(-1)^k\binom{-\gamma}{k}\left(\frac{\alpha}{z}\right)^k\right) \\
&= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}(-1)^j\binom{\gamma}{j-k}\binom{-\gamma}{k}\beta^{j-k}\alpha^k\right)z^{-j}
\end{aligned}
$$

Define the polynomial

$$
Q(z) = \sum_{j=0}^{n}\left(\sum_{k=0}^{j}(-1)^j\binom{\gamma}{j-k}\binom{-\gamma}{k}\beta^{j-k}\alpha^k\right)z^{n-j}.
$$

Then the function

$$
G(z) = \left(\frac{z-\beta}{z-\alpha}\right)^{\gamma} z^n - Q(z)
$$

vanishes at infinity.

$$G^+(\zeta) = \left(\frac{\zeta-\beta}{\zeta-\alpha}\right)^\gamma \zeta^n - Q(\zeta)$$
$$G^-(\zeta) = \mathrm{e}^{-i2\pi\gamma}\left(\frac{\zeta-\beta}{\zeta-\alpha}\right)^\gamma \zeta^n - Q(\zeta)$$
$$G^+(\zeta) - G^-(\zeta) = \left(\frac{\zeta-\beta}{\zeta-\alpha}\right)^\gamma \zeta^n \left(1-\mathrm{e}^{-i2\pi\gamma}\right)$$
$$G^+(\zeta) + G^-(\zeta) = \left(\frac{\zeta-\beta}{\zeta-\alpha}\right)^\gamma \zeta^n \left(1+\mathrm{e}^{-i2\pi\gamma}\right) - 2Q(\zeta)$$
$$\frac{1}{i\pi} ⨍_C \left(\frac{\tau-\beta}{\tau-\alpha}\right)^\gamma \tau^n \left(1-\mathrm{e}^{-i2\pi\gamma}\right)\frac{1}{\tau-\zeta}\,d\tau = \left(\frac{\zeta-\beta}{\zeta-\alpha}\right)^\gamma \zeta^n \left(1+\mathrm{e}^{-i2\pi\gamma}\right) - 2Q(\zeta)$$
$$\frac{1}{i\pi} ⨍_C \left(\frac{\tau-\beta}{\tau-\alpha}\right)^\gamma \frac{\tau^n}{\tau-\zeta}\,d\tau = -i\cot(\pi\gamma)\left(\frac{\zeta-\beta}{\zeta-\alpha}\right)^\gamma \zeta^n - (1-i\cot(\pi\gamma))Q(\zeta)$$
$$\boxed{\frac{1}{i\pi} ⨍_C \left(\frac{\tau-\beta}{\tau-\alpha}\right)^\gamma \frac{\tau^n}{\tau-\zeta}\,d\tau = -i\cot(\pi\gamma)\left(\left(\frac{\zeta-\beta}{\zeta-\alpha}\right)^\gamma \zeta^n - Q(\zeta)\right) - Q(\zeta)}$$

Solution 49.36

$$\begin{aligned}
⨍_{-1}^{1} \frac{\phi(y)}{y^2-x^2}\,dy &= \frac{1}{2x}⨍_{-1}^{1}\frac{\phi(y)}{y-x}\,dy - \frac{1}{2x}⨍_{-1}^{1}\frac{\phi(y)}{y+x}\,dy \\
&= \frac{1}{2x}⨍_{-1}^{1}\frac{\phi(y)}{y-x}\,dy + \frac{1}{2x}⨍_{-1}^{1}\frac{\phi(-y)}{y-x}\,dy \\
&= \frac{1}{2x}⨍_{-1}^{1}\frac{\phi(y)+\phi(-y)}{y-x}\,dy
\end{aligned}$$

$$\frac{1}{2x} \fint_{-1}^{1} \frac{\phi(y) + \phi(-y)}{y - x} \, dy = f(x)$$
$$\frac{1}{i\pi} \fint_{-1}^{1} \frac{\phi(y) + \phi(-y)}{y - x} \, dy = \frac{2x}{i\pi} f(x)$$
$$\phi(x) + \phi(-x) = \frac{1}{i\pi\sqrt{1 - x^2}} \fint_{-1}^{1} \frac{2y}{i\pi} f(y) \sqrt{1 - y^2} \frac{1}{y - x} \, dy + \frac{k}{\sqrt{1 - x^2}}$$
$$\phi(x) + \phi(-x) = -\frac{1}{\pi^2\sqrt{1 - x^2}} \fint_{-1}^{1} \frac{2y f(y) \sqrt{1 - y^2}}{y - x} \, dy + \frac{k}{\sqrt{1 - x^2}}$$
$$\boxed{\phi(x) = -\frac{1}{\pi^2\sqrt{1 - x^2}} \fint_{-1}^{1} \frac{y f(y) \sqrt{1 - y^2}}{y - x} \, dy + \frac{k}{\sqrt{1 - x^2}} + g(x)}$$

Here k is an arbitrary constant and $g(x)$ is an arbitrary odd function.

Solution 49.37
We define

$$F(z) = \frac{1}{i2\pi} \int_{0}^{1} \frac{f(t)}{t - z} \, dt.$$

The Plemelj formulas and the integral equation give us,

$$F^+(x) - F^-(x) = f(x)$$
$$F^+(x) + F^-(x) = \lambda f(x).$$

We solve for F^+ and F^-.

$$F^+(x) = (\lambda + 1) f(x)$$
$$F^-(x) = (\lambda - 1) f(x)$$

By writing

$$\frac{F^+(x)}{F^-(x)} = \frac{\lambda+1}{\lambda-1}$$

we seek to determine F to within a multiplicative constant.

$$\log F^+(x) - \log F^-(x) = \log\left(\frac{\lambda+1}{\lambda-1}\right)$$
$$\log F^+(x) - \log F^-(x) = \log\left(\frac{1+\lambda}{1-\lambda}\right) + i\pi$$
$$\log F^+(x) - \log F^-(x) = \gamma + i\pi$$

We have left off the additive term of $i2\pi n$ in the above equation, which will introduce factors of z^k and $(z-1)^m$ in $F(z)$. We will choose these factors so that $F(z)$ has integrable algebraic singularites and vanishes at infinity. Note that we have defined γ to be the real parameter,

$$\gamma = \log\left(\frac{1+\lambda}{1-\lambda}\right).$$

By the discontinuity theorem,

$$\begin{aligned}\log F(z) &= \frac{1}{i2\pi}\int_0^1 \frac{\gamma+i\pi}{t-z}\,dz \\ &= \left(\frac{1}{2} - i\frac{\gamma}{2\pi}\right)\log\left(\frac{1-z}{-z}\right) \\ &= \log\left(\left(\frac{z-1}{z}\right)^{1/2-i\gamma/(2\pi)}\right)\end{aligned}$$

$$F(z) = \left(\frac{z-1}{z}\right)^{1/2 - i\gamma/(2\pi)} z^k (z-1)^m$$
$$F(z) = \frac{1}{\sqrt{z(z-1)}} \left(\frac{z-1}{z}\right)^{-i\gamma/(2\pi)}$$
$$F^{\pm}(x) = \frac{\mathrm{e}^{\pm i\pi(-i\gamma/(2\pi))}}{\sqrt{x(1-x)}} \left(\frac{1-x}{x}\right)^{-i\gamma/(2\pi)}$$
$$F^{\pm}(x) = \frac{\mathrm{e}^{\pm\gamma/2}}{\sqrt{x(1-x)}} \left(\frac{1-x}{x}\right)^{-i\gamma/(2\pi)}$$

Define

$$f(x) = \frac{1}{\sqrt{x(1-x)}} \left(\frac{1-x}{x}\right)^{-i\gamma/(2\pi)}.$$

We apply the Plemelj formulas.

$$\frac{1}{i\pi} \fint_0^1 \left(\mathrm{e}^{\gamma/2} - \mathrm{e}^{-\gamma/2}\right) \frac{f(t)}{t-x}\, dt = \left(\mathrm{e}^{\gamma/2} + \mathrm{e}^{-\gamma/2}\right) f(x)$$
$$\frac{1}{i\pi} \fint_0^1 \frac{f(t)}{t-x}\, dt = \tanh\left(\frac{\gamma}{2}\right) f(x)$$

Thus we see that the eigenfunctions are

$$\boxed{\phi(x) = \frac{1}{\sqrt{x(1-x)}} \left(\frac{1-x}{x}\right)^{-i\tanh^{-1}(\lambda)/\pi}}$$

for $-1 < \lambda < 1$.

The method used in this problem cannot be used to construct eigenfunctions for $\lambda > 1$. For this case we cannot find an $F(z)$ that has integrable algebraic singularities and vanishes at infinity.

Solution 49.38

$$\frac{1}{i\pi} \int_0^1 \frac{f(t)}{t-x} \, dt = -\frac{i}{\tan(x)} f(x)$$

We define the function,

$$F(z) = \frac{1}{i2\pi} \int_0^1 \frac{f(t)}{t-z} \, dt.$$

The Plemelj formula are,

$$F^+(x) - F^-(x) = f(x)$$
$$F^+(x) + F^-(x) = -\frac{i}{\tan(x)} f(x).$$

We solve for F^+ and F^-.

$$F^\pm(x) = \frac{1}{2}\left(\pm 1 - \frac{i}{\tan(x)}\right) f(x)$$

From this we see

$$\frac{F^+(x)}{F^-(x)} = \frac{1 - i/\tan(x)}{-1 - i/\tan(x)} = \mathrm{e}^{i2x}.$$

We seek to determine $F(z)$ up to a multiplicative constant. Taking the logarithm of this equation yields

$$\log F^+(x) - \log F^-(x) = i2x + i2\pi n.$$

The $i2\pi n$ term will give us the factors $(z-1)^k$ and z^m in the solution for $F(z)$. We will choose the integers k and m so that $F(z)$ has only algebraic singularities and vanishes at infinity. We drop the $i2\pi n$ term for now.

$$\log F(z) = \frac{1}{i2\pi} \int_0^1 \frac{i2t}{t-z} \, dt$$

$$\log F(z) = \frac{1}{\pi} + \frac{z}{\pi} \log\left(\frac{1-z}{-z}\right) F(z) = \mathrm{e}^{1/\pi} \left(\frac{z-1}{z}\right)^{z/\pi}$$

We replace $e^{1/\pi}$ by a multiplicative constant and multiply by $(z-1)^1$ to give $F(z)$ the desired properties.

$$F(z) = \frac{c}{(z-1)^{1-z/\pi} z^{z/\pi}}$$

We evaluate $F(z)$ above and below the branch cut.

$$F^{\pm}(x) = \frac{c}{e^{\pm(i\pi - ix)}(1-x)^{1-x/\pi} x^{x/\pi}} = \frac{c\, e^{\pm ix}}{(1-x)^{1-x/\pi} x^{x/\pi}}$$

Finally we use the Plemelj formulas to determine $f(x)$.

$$\boxed{f(x) = F^+(x) - F^-(x) = \frac{k \sin(x)}{(1-x)^{1-x/\pi} x^{x/\pi}}}$$

Solution 49.39

Consider the equation,

$$f'(z) + \lambda \int_C \frac{f(t)}{t-z}\, dt = 1.$$

Since the integral is an analytic function of z off C we know that $f(z)$ is analytic off C. We use Cauchy's theorem to evaluate the integral and obtain a differential equation for $f(x)$.

$$f'(x) + \lambda \fint_C \frac{f(t)}{t-x}\, dt = 1$$
$$f'(x) + i\lambda\pi f(x) = 1$$
$$\boxed{f(x) = \frac{1}{i\lambda\pi} + c\, e^{-i\lambda\pi x}}$$

Consider the equation,

$$f'(z) + \lambda \int_C \frac{f(t)}{t-z}\, dt = g(z).$$

Since the integral and $g(z)$ are analytic functions inside C we know that $f(z)$ is analytic inside C. We use Cauchy's theorem to evaluate the integral and obtain a differential equation for $f(x)$.

$$f'(x) + \lambda \fint_C \frac{f(t)}{t-x}\, dt = g(x)$$
$$f'(x) + i\lambda\pi f(x) = g(x)$$
$$\boxed{f(x) = \int_{z_0}^{x} \mathrm{e}^{-i\lambda\pi(x-\xi)} g(\xi)\, d\xi + c\,\mathrm{e}^{-i\lambda\pi x}}$$

Here z_0 is any point inside C.

Solution 49.40

$$\fint_C \left(\frac{1}{t-x} + P(t-x) \right) f(t)\, dt = g(x)$$
$$\frac{1}{i\pi} \fint_C \frac{f(t)}{t-x}\, dt = \frac{1}{i\pi} g(x) - \frac{1}{i\pi} \int_C P(t-x) f(t)\, dt$$

We know that if

$$\frac{1}{i\pi} \fint_C \frac{f(\tau)}{\tau-\zeta}\, d\tau = g(\zeta)$$

then

$$f(\zeta) = \frac{1}{i\pi} \fint_C \frac{g(\tau)}{\tau-\zeta}\, d\tau.$$

We apply this theorem to the integral equation.

$$\begin{aligned} f(x) &= -\frac{1}{\pi^2} \fint_C \frac{g(t)}{t-x}\,dt + \frac{1}{\pi^2} \fint_C \left(\int_C P(\tau - t) f(\tau)\,d\tau \right) \frac{1}{t-x}\,dt \\ &= -\frac{1}{\pi^2} \fint_C \frac{g(t)}{t-x}\,dt + \frac{1}{\pi^2} \int_C \left(\fint_C \frac{P(\tau - t)}{t-x}\,dt \right) f(\tau)\,d\tau \\ &= -\frac{1}{\pi^2} \fint_C \frac{g(t)}{t-x}\,dt - \frac{1}{i\pi} \int_C P(t-x) f(t)\,dt \end{aligned}$$

Now we substitute the non-analytic part of $f(t)$ into the integral. (The analytic part integrates to zero.)

$$\begin{aligned} &= -\frac{1}{\pi^2} \fint_C \frac{g(t)}{t-x}\,dt - \frac{1}{i\pi} \int_C P(t-x) \left(-\frac{1}{\pi^2} \fint_C \frac{g(\tau)}{\tau - t}\,d\tau \right) dt \\ &= -\frac{1}{\pi^2} \fint_C \frac{g(t)}{t-x}\,dt - \frac{1}{\pi^2} \int_C \left(-\frac{1}{i\pi} \fint_C \frac{P(t-x)}{\tau - t}\,dt \right) g(\tau)\,d\tau \\ &= -\frac{1}{\pi^2} \fint_C \frac{g(t)}{t-x}\,dt - \frac{1}{\pi^2} \int_C P(\tau - x) g(\tau)\,d\tau \end{aligned}$$

$$\boxed{f(x) = -\frac{1}{\pi^2} \fint_C \frac{g(t)}{t-x}\,dt - \frac{1}{\pi^2} \int_C P(t-x) g(t)\,dt}$$

Solution 49.41

Solution 49.42

Part VII

Nonlinear Differential Equations

1990

Chapter 50

Nonlinear Ordinary Differential Equations

50.1 Exercises

Exercise 50.1
A model set of equations to describe an epidemic, in which $x(t)$ is the number infected, $y(t)$ is the number susceptible, is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rxy - \gamma x, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -rxy + \beta,$$

where $r > 0$, $\beta \geq 0$, $\gamma \geq 0$. Initially $x = x_0$, $y = y_0$ at $t = 0$. Directly from the equations, without using the phase plane:

1. Find the solution, $x(t)$, $y(t)$, in the case $\beta = \gamma = 0$.

2. Show for the case $\beta = 0$, $\gamma \neq 0$ that $x(t)$ first decreases or increases according as $ry_0 < \gamma$ or $ry_0 > \gamma$. Show that $x(t) \to 0$ as $t \to \infty$ in both cases. Find x as a function of y.

3. In the phase plane: Find the position of the singular point and its type when $\beta > 0$, $\gamma > 0$.

Exercise 50.2
Find the singular points and their types for the system

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}x} &= ru + v(1-v)(p-v), \qquad r > 0,\ 0 < p < 1, \\ \frac{\mathrm{d}v}{\mathrm{d}x} &= u, \end{aligned}$$

which comes from one of our nonlinear diffusion problems. Note that there is a solution with

$$u = \alpha(1 - v)$$

for special values of α and r. Find $v(x)$ for this special case.

Exercise 50.3
Check that $r = 1$ is a limit cycle for

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y + x(1 - r^2)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + y(1 - r^2)$$

$(r = x^2 + y^2)$, and that all solution curves spiral into it.

Exercise 50.4
Consider

$$\epsilon \dot{y} = f(y) - x$$
$$\dot{x} = y$$

Introduce new coordinates, R, θ given by

$$x = R\cos\theta$$
$$y = \frac{1}{\sqrt{\epsilon}} R \sin\theta$$

and obtain the exact differential equations for $R(t)$, $\theta(t)$. Show that $R(t)$ continually increases with t when $R \neq 0$. Show that $\theta(t)$ continually decreases when $R > 1$.

Exercise 50.5
One choice of the Lorenz equations is

$$\dot{x} = -10x + 10y$$
$$\dot{y} = Rx - y - xz$$
$$\dot{z} = -\frac{8}{3}z + xy$$

Where R is a positive parameter.

1. Invistigate the nature of the sigular point at $(0,0,0)$ by finding the eigenvalues and their behavior for all $0 < R < \infty$.

2. Find the other singular points when $R > 1$.

3. Show that the appropriate eigenvalues for these other singular points satisfy the cubic

$$3\lambda^3 + 41\lambda^2 + 8(10+R)\lambda + 160(R-1) = 0.$$

4. There is a special value of R, call it R_c, for which the cubic has two pure imaginary roots, $\pm i\mu$ say. Find R_c and μ; then find the third root.

Exercise 50.6
In polar coordinates (r, ϕ), Einstein's equations lead to the equation

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\phi^2} + v = 1 + \epsilon v^2, \quad v = \frac{1}{r},$$

for planetary orbits. For Mercury, $\epsilon = 8 \times 10^{-8}$. When $\epsilon = 0$ (Newtonian theory) the orbit is given by

$$v = 1 + A\cos\phi, \text{ period } 2\pi.$$

Introduce $\theta = \omega\phi$ and use perturbation expansions for $v(\theta)$ and ω in powers of ϵ to find the corrections proportional to ϵ.

[A is not small; ϵ is the small parameter].

Exercise 50.7
Consider the problem

$$\ddot{x} + \omega_0^2 x + \alpha x^2 = 0, \quad x = a, \dot{x} = 0 \text{ at } t = 0$$

Use expansions

$$x = a\cos\theta + a^2 x_2(\theta) + a^3 x_3(\theta) + \cdots\,,\ \theta = \omega t$$
$$\omega = \omega_0 + a^2\omega_2 + \cdots\,,$$

to find a periodic solution and its natural frequency ω.

Note that, with the expansions given, there are no "secular term" troubles in the determination of $x_2(\theta)$, but $x_2(\theta)$ is needed in the subsequent determination of $x_3(\theta)$ and ω.

Show that a term $a\omega_1$ in the expansion for ω would have caused trouble, so ω_1 would have to be taken equal to zero.

Exercise 50.8

Consider the linearized traffic problem

$$\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = \alpha\left[p_{n-1}(t) - p_n(t)\right], \quad n \geq 1,$$
$$p_n(0) = 0, \quad n \geq 1,$$
$$p_0(t) = ae^{i\omega t}, \quad t > 0.$$

(We take the imaginary part of $p_n(t)$ in the final answers.)

1. Find $p_1(t)$ directly from the equation for $n = 1$ and note the behavior as $t \to \infty$.
2. Find the generating function
$$G(s,t) = \sum_{n=1}^{\infty} p_n(t)s^n.$$
3. Deduce that
$$p_n(t) \sim A_n e^{i\omega t}, \quad \text{as } t \to \infty,$$
and find the expression for A_n. Find the imaginary part of this $p_n(t)$.

Exercise 50.9

1. For the equation modified with a reaction time, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}p_n(t+\tau) = \alpha[p_{n-1}(t) - p_n(t)] \quad n \geq 1,$$

   find a solution of the form in 1(c) by direct substitution in the equation. Again take its imaginary part.

2. Find a condition that the disturbance is stable, i.e. $p_n(t)$ remains bounded as $n \to \infty$.

3. In the stable case show that the disturbance is wave-like and find the wave velocity.

50.2 Hints

Hint 50.1

Hint 50.2

Hint 50.3

Hint 50.4

Hint 50.5

Hint 50.6

Hint 50.7

Hint 50.8

Hint 50.9

50.3 Solutions

Solution 50.1

1. When $\beta = \gamma = 0$ the equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rxy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -rxy.$$

Adding these two equations we see that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Integrating and applying the initial conditions $x(0) = x_0$ and $y(0) = y_0$ we obtain

$$x = x_0 + y_0 - y$$

Substituting this into the differential equation for y,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -r(x_0 + y_0 - y)y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -r(x_0 + y_0)y + ry^2.$$

We recognize this as a Bernoulli equation and make the substitution $u = y^{-1}$.

$$-y^{-2}\frac{\mathrm{d}y}{\mathrm{d}t} = r(x_0+y_0)y^{-1} - r$$
$$\frac{\mathrm{d}u}{\mathrm{d}t} = r(x_0+y_0)u - r$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-r(x_0+y_0)t}u\right) = -re^{-r(x_0+y_0)t}$$
$$u = e^{r(x_0+y_0)t}\int^t -re^{-r(x_0+y_0)t}\,dt + ce^{r(x_0+y_0)t}$$
$$u = \frac{1}{x_0+y_0} + ce^{r(x_0+y_0)t}$$
$$y = \left(\frac{1}{x_0+y_0} + ce^{r(x_0+y_0)t}\right)^{-1}$$

Applying the initial condition for y,

$$\left(\frac{1}{x_0+y_0} + c\right)^{-1} = y_0$$
$$c = \frac{1}{y_0} - \frac{1}{x_0+y_0}.$$

The solution for y is then

$$y = \left[\frac{1}{x_0+y_0} + \left(\frac{1}{y_0} - \frac{1}{x_0+y_0}\right)e^{r(x_0+y_0)t}\right]^{-1}$$

Since $x = x_0 + y_0 - y$, the solution to the system of differential equations is

$$\boxed{x = x_0 + y_0 - \left[\frac{1}{y_0} + \frac{1}{x_0+y_0}\left(1 - e^{r(x_0+y_0)t}\right)\right]^{-1}, \quad y = \left[\frac{1}{y_0} + \frac{1}{x_0+y_0}\left(1 - e^{r(x_0+y_0)t}\right)\right]^{-1}.}$$

2. For $\beta = 0$, $\gamma \neq 0$, the equation for x is

$$\dot{x} = rxy - \gamma x.$$

At $t = 0$,

$$\dot{x}(0) = x_0(ry_0 - \gamma).$$

Thus we see that if $ry_0 < \gamma$, x is initially decreasing. If $ry_0 > \gamma$, x is initially increasing.

Now to show that $x(t) \to 0$ as $t \to \infty$. First note that if the initial conditions satisfy $x_0, y_0 > 0$ then $x(t), y(t) > 0$ for all $t \geq 0$ because the axes are a seqaratrix. $y(t)$ is is a strictly decreasing function of time. Thus we see that at some time the quantity $x(ry - \gamma)$ will become negative. Since y is decreasing, this quantity will remain negative. Thus after some time, x will become a strictly decreasing quantity. Finally we see that regardless of the initial conditions, (as long as they are positive), $x(t) \to 0$ as $t \to \infty$.

Taking the ratio of the two differential equations,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -1 + \frac{\gamma}{ry}.$$
$$x = -y + \frac{\gamma}{r} \ln y + c$$

Applying the intial condition,

$$x_0 = -y_0 + \frac{\gamma}{r} \ln y_0 + c$$
$$c = x_0 + y_0 - \frac{\gamma}{r} \ln y_0.$$

Thus the solution for x is

$$\boxed{x = x_0 + (y_0 - y) + \frac{\gamma}{r} \ln\left(\frac{y}{y_0}\right).}$$

3. When $\beta > 0$ and $\gamma > 0$ the system of equations is

$$\dot{x} = rxy - \gamma x$$
$$\dot{y} = -rxy + \beta.$$

The equilibrium solutions occur when

$$x(ry - \gamma) = 0$$
$$\beta - rxy = 0.$$

Thus the singular point is

$$\boxed{x = \frac{\beta}{\gamma}, \qquad y = \frac{\gamma}{r}.}$$

Now to classify the point. We make the substitution $u = (x - \frac{\beta}{\gamma})$, $v = (y - \frac{\gamma}{r})$.

$$\dot{u} = r\left(u + \frac{\beta}{\gamma}\right)\left(v + \frac{\gamma}{r}\right) - \gamma\left(u + \frac{\beta}{\gamma}\right)$$
$$\dot{v} = -r\left(u + \frac{\beta}{\gamma}\right)\left(v + \frac{\gamma}{r}\right) + \beta$$

$$\dot{u} = \frac{r\beta}{\gamma}v + ruv$$
$$\dot{v} = -\gamma u - \frac{r\beta}{\gamma}v - ruv$$

The linearized system is

$$\dot{u} = \frac{r\beta}{\gamma}v$$
$$\dot{v} = -\gamma u - \frac{r\beta}{\gamma}v$$

Finding the eigenvalues of the linearized system,

$$\begin{vmatrix} \lambda & -\frac{r\beta}{\gamma} \\ \gamma & \lambda + \frac{r\beta}{\gamma} \end{vmatrix} = \lambda^2 + \frac{r\beta}{\gamma}\lambda + r\beta = 0$$

$$\lambda = \frac{-\frac{r\beta}{\gamma} \pm \sqrt{(\frac{r\beta}{\gamma})^2 - 4r\beta}}{2}$$

Since both eigenvalues have negative real part, we see that the singular point is asymptotically stable. A plot of the vector field for $r = \gamma = \beta = 1$ is attached. We note that there appears to be a stable singular point at $x = y = 1$ which corroborates the previous results.

Solution 50.2

The singular points are

$$u = 0, v = 0, \qquad u = 0, v = 1, \qquad u = 0, v = p.$$

The point $\mathbf{u = 0, v = 0}$. The linearized system about $u = 0$, $v = 0$ is

$$\frac{du}{dx} = ru$$
$$\frac{dv}{dx} = u.$$

The eigenvalues are

$$\begin{vmatrix} \lambda - r & 0 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - r\lambda = 0.$$

$$\lambda = 0, r.$$

Since there are positive eigenvalues, this point is a source. The critical point is unstable.

The point $\mathbf{u = 0, v = 1}$. Linearizing the system about $u = 0$, $v = 1$, we make the substitution $w = v - 1$.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = ru + (w+1)(-w)(p-1-w)$$
$$\frac{\mathrm{d}w}{\mathrm{d}x} = u$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = ru + (1-p)w$$
$$\frac{\mathrm{d}w}{\mathrm{d}x} = u$$

$$\begin{vmatrix} \lambda - r & (p-1) \\ -1 & \lambda \end{vmatrix} = \lambda^2 - r\lambda + p - 1 = 0$$

$$\lambda = \frac{r \pm \sqrt{r^2 - 4(p-1)}}{2}$$

Thus we see that this point is a saddle point. The critical point is unstable.

The point $\mathbf{u = 0, v = p}$. Linearizing the system about $u = 0$, $v = p$, we make the substitution $w = v - p$.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = ru + (w+p)(1-p-w)(-w)$$
$$\frac{\mathrm{d}w}{\mathrm{d}x} = u$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = ru + p(p-1)w$$
$$\frac{\mathrm{d}w}{\mathrm{d}x} = u$$

$$\begin{vmatrix} \lambda - r & p(1-p) \\ -1 & \lambda \end{vmatrix} = \lambda^2 - r\lambda + p(1-p) = 0$$

$$\lambda = \frac{r \pm \sqrt{r^2 - 4p(1-p)}}{2}$$

Thus we see that this point is a source. The critical point is unstable.

The solution of for special values of $\boldsymbol{\alpha}$ and r. Differentiating $u = \alpha v(1-v)$,

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \alpha - 2\alpha v.$$

Taking the ratio of the two differential equations,

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}v} &= r + \frac{v(1-v)(p-v)}{u} \\ &= r + \frac{v(1-v)(p-v)}{\alpha v(1-v)} \\ &= r + \frac{(p-v)}{\alpha} \end{aligned}$$

Equating these two expressions,

$$\alpha - 2\alpha v = r + \frac{p}{\alpha} - \frac{v}{\alpha}.$$

Equating coefficients of v, we see that $\alpha = \frac{1}{\sqrt{2}}$.

$$\frac{1}{\sqrt{2}} = r + \sqrt{2}p$$

Thus we have the solution $u = \frac{1}{\sqrt{2}} v(1-v)$ when $r = \frac{1}{\sqrt{2}} - \sqrt{2}p$. In this case, the differential equation for v is

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1}{\sqrt{2}} v(1-v) \\ -v^{-2}\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{1}{\sqrt{2}} v^{-1} + \frac{1}{\sqrt{2}} \end{aligned}$$

We make the change of variablles $y = v^{-1}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x/\sqrt{2}}y\right) = \frac{e^{x/\sqrt{2}}}{\sqrt{2}}$$
$$y = e^{-x/\sqrt{2}}\int^x \frac{e^{x/\sqrt{2}}}{\sqrt{2}}\,dx + ce^{-x/\sqrt{2}}$$
$$y = 1 + ce^{-x/\sqrt{2}}$$

The solution for v is

$$\boxed{v(x) = \frac{1}{1 + ce^{-x/\sqrt{2}}}.}$$

Solution 50.3
We make the change of variables

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

Differentiating these expressions with respect to time,

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta.$$

Substituting the new variables into the pair of differential equations,

$$\dot{r}\cos\theta - r\dot{\theta}\sin\theta = -r\sin\theta + r\cos\theta(1 - r^2)$$
$$\dot{r}\sin\theta + r\dot{\theta}\cos\theta = r\cos\theta + r\sin\theta(1 - r^2).$$

Multiplying the equations by $\cos\theta$ and $\sin\theta$ and taking their sum and difference yields

$$\begin{aligned}\dot{r} &= r(1 - r^2) \\ r\dot{\theta} &= r.\end{aligned}$$

We can integrate the second equation.

$$\begin{aligned}\dot{r} &= r(1 - r^2) \\ \theta &= t + \theta_0\end{aligned}$$

At this point we could note that $\dot{r} > 0$ in $(0, 1)$ and $\dot{r} < 0$ in $(1, \infty)$. Thus if r is not initially zero, then the solution tends to $r = 1$.

Alternatively, we can solve the equation for r exactly.

$$\begin{aligned}\dot{r} &= r - r^3 \\ \frac{\dot{r}}{r^3} &= \frac{1}{r^2} - 1\end{aligned}$$

We make the change of variables $u = 1/r^2$.

$$\begin{aligned}-\frac{1}{2}\dot{u} &= u - 1 \\ \dot{u} + 2u &= 2 \\ u &= e^{-2t} \int^t 2e^{2t}\,dt + ce^{-2t} \\ u &= 1 + ce^{-2t} \\ r &= \frac{1}{\sqrt{1 + ce^{-2t}}}\end{aligned}$$

Thus we see that if r is initiall nonzero, the solution tends to 1 as $t \to \infty$.

Solution 50.4
The set of differential equations is

$$\begin{aligned}\epsilon \dot{y} &= f(y) - x \\ \dot{x} &= y.\end{aligned}$$

We make the change of variables

$$\begin{aligned}x &= R\cos\theta \\ y &= \frac{1}{\sqrt{\epsilon}} R\sin\theta\end{aligned}$$

Differentiating x and y,

$$\begin{aligned}\dot{x} &= \dot{R}\cos\theta - R\dot{\theta}\sin\theta \\ \dot{y} &= \frac{1}{\sqrt{\epsilon}}\dot{R}\sin\theta + \frac{1}{\sqrt{\epsilon}} R\dot{\theta}\cos\theta.\end{aligned}$$

The pair of differential equations become

$$\begin{aligned}\sqrt{\epsilon}\dot{R}\sin\theta + \sqrt{\epsilon}R\dot{\theta}\cos\theta &= f\left(\frac{1}{\sqrt{\epsilon}} R\sin\theta\right) - R\cos\theta \\ \dot{R}\cos\theta - R\dot{\theta}\sin\theta &= \frac{1}{\sqrt{\epsilon}} R\sin\theta.\end{aligned}$$

$$\begin{aligned}\dot{R}\sin\theta + R\dot{\theta}\cos\theta &= -\frac{1}{\sqrt{\epsilon}} R\cos\theta \frac{1}{\sqrt{\epsilon}} f\left(\frac{1}{\sqrt{\epsilon}} R\sin\theta\right) \\ \dot{R}\cos\theta - R\dot{\theta}\sin\theta &= \frac{1}{\sqrt{\epsilon}} R\sin\theta.\end{aligned}$$

Multiplying by $\cos\theta$ and $\sin\theta$ and taking the sum and difference of these differential equations yields

$$\begin{aligned}\dot{R} &= \frac{1}{\sqrt{\epsilon}}\sin\theta f\left(\frac{1}{\sqrt{\epsilon}}R\sin\theta\right)\\ R\dot{\theta} &= -\frac{1}{\sqrt{\epsilon}}R + \frac{1}{\sqrt{\epsilon}}\cos\theta f\left(\frac{1}{\sqrt{\epsilon}}R\sin\theta\right).\end{aligned}$$

Dividing by R in the second equation,

$$\begin{aligned}\dot{R} &= \frac{1}{\sqrt{\epsilon}}\sin\theta f\left(\frac{1}{\sqrt{\epsilon}}R\sin\theta\right)\\ \dot{\theta} &= -\frac{1}{\sqrt{\epsilon}} + \frac{1}{\sqrt{\epsilon}}\frac{\cos\theta}{R} f\left(\frac{1}{\sqrt{\epsilon}}R\sin\theta\right).\end{aligned}$$

We make the assumptions that $0 < \epsilon < 1$ and that $f(y)$ is an odd function that is nonnegative for positive y and satisfies $|f(y)| \leq 1$ for all y.

Since $\sin\theta$ is odd,

$$\sin\theta f\left(\frac{1}{\sqrt{\epsilon}}R\sin\theta\right)$$

is nonnegative. Thus $R(t)$ continually increases with t when $R \neq 0$.

If $R > 1$ then

$$\begin{aligned}\left|\frac{\cos\theta}{R} f\left(\frac{1}{\sqrt{\epsilon}}R\sin\theta\right)\right| &\leq \left|f\left(\frac{1}{\sqrt{\epsilon}}R\sin\theta\right)\right|\\ &\leq 1.\end{aligned}$$

Thus the value of $\dot{\theta}$,

$$-\frac{1}{\sqrt{\epsilon}} + \frac{1}{\sqrt{\epsilon}}\frac{\cos\theta}{R} f\left(\frac{1}{\sqrt{\epsilon}}R\sin\theta\right),$$

is always nonpositive. Thus $\theta(t)$ continually decreases with t.

Solution 50.5

1. Linearizing the Lorentz equations about $(0,0,0)$ yields

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ R & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigenvalues of the matrix are

$$\lambda_1 = -\frac{8}{3},$$
$$\lambda_2 = \frac{-11 - \sqrt{81 + 40R}}{2}$$
$$\lambda_3 = \frac{-11 + \sqrt{81 + 40R}}{2}.$$

There are three cases for the eigenvalues of the linearized system.

$\mathbf{R < 1}$. There are three negative, real eigenvalues. In the linearized and also the nonlinear system, the origin is a stable, sink.

$\mathbf{R = 1}$. There are two negative, real eigenvalues and one zero eigenvalue. In the linearized system the origin is stable and has a center manifold plane. The linearized system does not tell us if the nonlinear system is stable or unstable.

$\mathbf{R > 1}$. There are two negative, real eigenvalues, and one positive, real eigenvalue. The origin is a saddle point.

2. The other singular points when $R > 1$ are

$$\left(\pm\sqrt{\frac{8}{3}(R-1)},\ \pm\sqrt{\frac{8}{3}(R-1)},\ R-1 \right).$$

3. Linearizing about the point

$$\left(\sqrt{\frac{8}{3}(R-1)},\ \sqrt{\frac{8}{3}(R-1)},\ R-1\right)$$

yields

$$\begin{pmatrix}\dot{X}\\ \dot{Y}\\ \dot{Z}\end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{\frac{8}{3}(R-1)} \\ \sqrt{\frac{8}{3}(R-1)} & \sqrt{\frac{8}{3}(R-1)} & -\frac{8}{3}\end{pmatrix}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}$$

The characteristic polynomial of the matrix is

$$\lambda^3 + \frac{41}{3}\lambda^2 + \frac{8(10+R)}{3}\lambda + \frac{160}{3}(R-1).$$

Thus the eigenvalues of the matrix satisfy the polynomial,

$$3\lambda^3 + 41\lambda^2 + 8(10+R)\lambda + 160(R-1) = 0.$$

Linearizing about the point

$$\left(-\sqrt{\frac{8}{3}(R-1)},\ -\sqrt{\frac{8}{3}(R-1)},\ R-1\right)$$

yields

$$\begin{pmatrix}\dot{X}\\ \dot{Y}\\ \dot{Z}\end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & \sqrt{\frac{8}{3}(R-1)} \\ -\sqrt{\frac{8}{3}(R-1)} & -\sqrt{\frac{8}{3}(R-1)} & -\frac{8}{3}\end{pmatrix}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}$$

The characteristic polynomial of the matrix is

$$\lambda^3 + \frac{41}{3}\lambda^2 + \frac{8(10+R)}{3}\lambda + \frac{160}{3}(R-1).$$

Thus the eigenvalues of the matrix satisfy the polynomial,

$$3\lambda^3 + 41\lambda^2 + 8(10+R)\lambda + 160(R-1) = 0.$$

4. If the characteristic polynomial has two pure imaginary roots $\pm i\mu$ and one real root, then it has the form

$$(\lambda - r)(\lambda^2 + \mu^2) = \lambda^3 - r\lambda^2 + \mu^2\lambda - r\mu^2.$$

Equating the λ^2 and the λ term with the characteristic polynomial yields

$$r = -\frac{41}{3}, \qquad \mu = \sqrt{\frac{8}{3}(10+R)}.$$

Equating the constant term gives us the equation

$$\frac{41}{3}\frac{8}{3}(10+R_c) = \frac{160}{3}(R_c - 1)$$

which has the solution

$$R_c = \frac{470}{19}.$$

For this critical value of R the characteristic polynomial has the roots

$$\lambda_1 = -\frac{41}{3}$$
$$\lambda_2 = \frac{4}{19}\sqrt{2090}$$
$$\lambda_3 = -\frac{4}{19}\sqrt{2090}.$$

Solution 50.6

The form of the perturbation expansion is

$$v(\theta) = 1 + A\cos\theta + \epsilon u(\theta) + \mathcal{O}(\epsilon^2)$$
$$\theta = (1 + \epsilon\omega_1 + \mathcal{O}(\epsilon^2))\phi.$$

Writing the derivatives in terms of θ,

$$\frac{\mathrm{d}}{\mathrm{d}\phi} = (1 + \epsilon\omega_1 + \cdots)\frac{\mathrm{d}}{\mathrm{d}\theta}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} = (1 + 2\epsilon\omega_1 + \cdots)\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}.$$

Substituting these expressions into the differential equation for $v(\phi)$,

$$\left[1 + 2\epsilon\omega_1 + \mathcal{O}(\epsilon^2)\right]\left[-A\cos\theta + \epsilon u'' + \mathcal{O}(\epsilon^2)\right] + 1 + A\cos\theta + \epsilon u(\theta) + \mathcal{O}(\epsilon^2)$$
$$= 1 + \epsilon\left[1 + 2A\cos\theta + A^2\cos^2\theta + \mathcal{O}(\epsilon)\right]$$

$$\epsilon u'' + \epsilon u - 2\epsilon\omega_1 A\cos\theta = \epsilon + 2\epsilon A\cos\theta + \epsilon A^2\cos^2\theta + \mathcal{O}(\epsilon^2).$$

Equating the coefficient of ϵ,

$$u'' + u = 1 + 2\epsilon(1 + \omega_1)A\cos\theta + \frac{1}{2}A^2(\cos 2\theta + 1)$$
$$u'' + u = (1 + \frac{1}{2}A^2) + 2\epsilon(1 + \omega_1)A\cos\theta + \frac{1}{2}A^2\cos 2\theta.$$

To avoid secular terms, we must have $\omega_1 = -1$. A particular solution for u is

$$u = 1 + \frac{1}{2}A^2 - \frac{1}{6}A^2\cos 2\theta.$$

The the solution for v is

$$\boxed{v(\phi) = 1 + A\cos((1-\epsilon)\phi) + \epsilon\left[1 + \frac{1}{2}A^2 - \frac{1}{6}A^2\cos(2(1-\epsilon)\phi)\right] + \mathcal{O}(\epsilon^2).}$$

Solution 50.7

Substituting the expressions for x and ω into the differential equations yields

$$a^2\left[\omega_0^2\left(\frac{\mathrm{d}^2x_2}{\mathrm{d}\theta^2}+x_2\right)+\alpha\cos^2\theta\right]+a^3\left[\omega_0^2\left(\frac{\mathrm{d}^2x_3}{\mathrm{d}\theta^2}+x3\right)-2\omega_0\omega_2\cos\theta+2\alpha x_2\cos\theta\right]+\mathcal{O}(a^4)=0$$

Equating the coefficient of a^2 gives us the differential equation

$$\frac{\mathrm{d}^2x_2}{\mathrm{d}\theta^2}+x2=-\frac{\alpha}{2\omega_0^2}(1+\cos 2\theta).$$

The solution subject to the initial conditions $x_2(0)=x_2'(0)=0$ is

$$x_2=\frac{\alpha}{6\omega_0^2}(-3+2\cos\theta+\cos 2\theta).$$

Equating the coefficent of a^3 gives us the differential equation

$$\omega_0^2\left(\frac{\mathrm{d}^2x_3}{\mathrm{d}\theta^2}+x_3\right)+\frac{\alpha^2}{3\omega_0^2}-\left(2\omega_0\omega_2+\frac{5\alpha^2}{6\omega_0^2}\right)\cos\theta+\frac{\alpha^2}{3\omega_0^2}\cos 2\theta+\frac{\alpha^2}{6\omega_0^2}\cos 3\theta=0.$$

To avoid secular terms we must have

$$\omega_2=-\frac{5\alpha^2}{12\omega_0}.$$

Solving the differential equation for x_3 subject to the intial conditions $x_3(0)=x_3'(0)=0$,

$$x_3=\frac{\alpha^2}{144\omega_0^4}(-48+29\cos\theta+16\cos 2\theta+3\cos 3\theta).$$

Thus our solution for $x(t)$ is

$$\boxed{x(t)=a\cos\theta+a^2\left[\frac{\alpha}{6\omega_0^2}(-3+2\cos\theta+\cos 2\theta)\right]+a^3\left[\frac{\alpha^2}{144\omega_0^4}(-48+29\cos\theta+16\cos 2\theta+3\cos 3\theta)\right]+\mathcal{O}(a^4)}$$

where $\theta = \left(\omega_0 - a^2 \frac{5\alpha^2}{12\omega_0}\right) t$.

Now to see why we didn't need an $a\omega_1$ term. Assume that

$$x = a\cos\theta + a^2 x_2(\theta) + \mathcal{O}(a^3); \quad \theta = \omega t$$
$$\omega = \omega_0 + a\omega_1 + \mathcal{O}(a^2).$$

Substituting these expressions into the differential equation for x yields

$$a^2 \left[\omega_0^2(x_2'' + x_2) - 2\omega_0\omega_1 \cos\theta + \alpha\cos^2\theta\right] = \mathcal{O}(a^3)$$

$$x_2'' + x_2 = 2\frac{\omega_1}{\omega_0}\cos\theta - \frac{\alpha}{2\omega_0^2}(1 + \cos 2\theta).$$

In order to eliminate secular terms, we need $\omega_1 = 0$.

Solution 50.8

1. The equation for $p_1(t)$ is

$$\frac{\mathrm{d}p_1(t)}{\mathrm{d}t} = \alpha[p_0(t) - p_1(t)].$$
$$\frac{\mathrm{d}p_1(t)}{\mathrm{d}t} = \alpha[a e^{i\omega t} - p_1(t)]$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\alpha t} p_1(t)\right) = \alpha a e^{\alpha t} e^{i\omega t}$$
$$p_1(t) = \frac{\alpha a}{\alpha + i\omega} e^{i\omega t} + c e^{-\alpha t}$$

Applying the initial condition, $p_1(0) = 0$,

$$\boxed{p_1(t) = \frac{\alpha a}{\alpha + i\omega}\left(e^{i\omega t} - e^{-\alpha t}\right)}$$

2. We start with the differential equation for $p_n(t)$.

$$\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = \alpha[p_{n-1}(t) - p_n(t)]$$

Multiply by s^n and sum from $n = 1$ to ∞.

$$\begin{aligned}
\sum_{n=1}^{\infty} p_n'(t)s^n &= \sum_{n=1}^{\infty} \alpha[p_{n-1}(t) - p_n(t)]s^n \\
\frac{\partial G(s,t)}{\partial t} &= \alpha \sum_{n=0}^{\infty} p_n s^{n+1} - \alpha G(s,t) \\
\frac{\partial G(s,t)}{\partial t} &= \alpha s p_0 + \alpha \sum_{n=1}^{\infty} p_n s^{n+1} - \alpha G(s,t) \\
\frac{\partial G(s,t)}{\partial t} &= \alpha a s e^{i\omega t} + \alpha s G(s,t) - \alpha G(s,t) \\
\frac{\partial G(s,t)}{\partial t} &= \alpha a s e^{i\omega t} + \alpha(s-1)G(s,t) \\
\frac{\partial}{\partial t}\left(e^{\alpha(1-s)t}G(s,t)\right) &= \alpha a s e^{\alpha(1-s)t} e^{i\omega t} \\
G(s,t) &= \frac{\alpha a s}{\alpha(1-s) + i\omega} e^{i\omega t} + C(s) e^{\alpha(s-1)t}
\end{aligned}$$

The initial condition is

$$G(s,0) = \sum_{n=1}^{\infty} p_n(0)s^n = 0.$$

The generating function is then

$$\boxed{G(s,t) = \frac{\alpha a s}{\alpha(1-s) + i\omega}\left(\alpha e^{i\omega t} - e^{\alpha(s-1)t}\right).}$$

3. Assume that $|s| < 1$. In the limit $t \to \infty$ we have

$$G(s,t) \sim \frac{\alpha a s}{\alpha(1-s) + i\omega} e^{i\omega t}$$
$$G(s,t) \sim \frac{a s}{1 + i\omega/\alpha - s} e^{i\omega t}$$
$$G(s,t) \sim \frac{a s/(1 + i\omega/\alpha)}{1 - s/(1 + i\omega/\alpha)} e^{i\omega t}$$
$$G(s,t) \sim \frac{a s e^{i\omega t}}{1 + i\omega/\alpha} \sum_{n=0}^{\infty} \left(\frac{s}{1 + i\omega/\alpha} \right)^n$$
$$G(s,t) \sim a e^{i\omega t} \sum_{n=1}^{\infty} \frac{s^n}{(1 + i\omega/\alpha)^n}$$

Thus we have

$$\boxed{p_n(t) \sim \frac{a}{(1 + i\omega/\alpha)^n} e^{i\omega t} \quad \text{as } t \to \infty.}$$

$$\begin{aligned}
\Im(p_n(t)) &\sim \Im\left[\frac{a}{(1 + i\omega/\alpha)^n} e^{i\omega t} \right] \\
&= a \left(\frac{1 - i\omega/\alpha}{1 + (\omega/\alpha)^2} \right)^n [\cos(\omega t) + i \sin(\omega t)] \\
&= \frac{a}{(1 + (\omega/\alpha)^2)^n} \left[\cos(\omega t) \Im[(1 - i\omega/\alpha)^n] + \sin(\omega t) \Re[(1 - i\omega/\alpha)^n] \right] \\
&= \frac{a}{(1 + (\omega/\alpha)^2)^n} \left[\cos(\omega t) \sum_{\substack{j=1 \\ \text{odd } j}}^{n} (-1)^{(j+1)/2} \left(\frac{\omega}{\alpha} \right)^j + \sin(\omega t) \sum_{\substack{j=0 \\ \text{even } j}}^{n} (-1)^{j/2} \left(\frac{\omega}{\alpha} \right)^j \right]
\end{aligned}$$

Solution 50.9

1. Substituting $p_n = A_n e^{i\omega t}$ into the differential equation yields

$$\begin{aligned} A_n i\omega e^{i\omega(t+\tau)} &= \alpha[A_{n-1} e^{i\omega t} - A_n e^{i\omega t}] \\ A_n(\alpha + i\omega e^{i\omega\tau}) &= \alpha A_{n-1} \end{aligned}$$

We make the substitution $A_n = r^n$.

$$\begin{aligned} r^n(\alpha + i\omega e^{i\omega\tau}) &= \alpha r^{n-1} \\ r &= \frac{\alpha}{\alpha + i\omega e^{i\omega\tau}} \end{aligned}$$

Thus we have

$$\boxed{p_n(t) = \left(\frac{1}{1 + i\omega e^{i\omega\tau}/\alpha}\right)^n e^{i\omega t}.}$$

Taking the imaginary part,

$$\begin{aligned}
\Im(p_n(t)) &= \Im\left[\left(\frac{1}{1+i\frac{\omega}{\alpha}e^{i\omega\tau}}\right)^n e^{i\omega t}\right] \\
&= \Im\left[\left(\frac{1-i\frac{\omega}{\alpha}e^{-i\omega\tau}}{1+i\frac{\omega}{\alpha}(e^{i\omega\tau}-e^{-i\omega\tau})+(\frac{\omega}{\alpha})^2}\right)^n \big(\cos(\omega t)+i\sin(\omega t)\big)\right] \\
&= \Im\left[\left(\frac{1-\frac{\omega}{\alpha}\sin(\omega\tau)-i\frac{\omega}{\alpha}\cos(\omega\tau)}{1-2\frac{\omega}{\alpha}\sin(\omega\tau)+(\frac{\omega}{\alpha})^2}\right)^n \big(\cos(\omega t)+i\sin(\omega t)\big)\right] \\
&= \left(\frac{1}{1-2\frac{\omega}{\alpha}\sin(\omega\tau)+(\frac{\omega}{\alpha})^2}\right)^n \left[\cos(\omega t)\Im\left[\left(1-\frac{\omega}{\alpha}\sin(\omega\tau)-i\frac{\omega}{\alpha}\cos(\omega\tau)\right)^n\right]\right. \\
&\qquad \left.+\sin(\omega t)\Re\left[\left(1-\frac{\omega}{\alpha}\sin(\omega\tau)-i\frac{\omega}{\alpha}\cos(\omega\tau)\right)^n\right]\right] \\
&= \left(\frac{1}{1-2\frac{\omega}{\alpha}\sin(\omega\tau)+(\frac{\omega}{\alpha})^2}\right)^n \\
&\qquad \left[\cos(\omega t)\sum_{\substack{j=1 \\ \text{odd } j}}^{n}(-1)^{(j+1)/2}\left[\frac{\omega}{\alpha}\cos(\omega\tau)\right]^j\left[1-\frac{\omega}{\alpha}\sin(\omega\tau)\right]^{n-j}\right. \\
&\qquad \left.+\sin(\omega t)\sum_{\substack{j=0 \\ \text{even } j}}^{n}(-1)^{j/2}\left[\frac{\omega}{\alpha}\cos(\omega\tau)\right]^j\left[1-\frac{\omega}{\alpha}\sin(\omega\tau)\right]^{n-j}\right]
\end{aligned}$$

2. $p_n(t)$ will remain bounded in time as $n \to \infty$ if

$$\left|\frac{1}{1+i\frac{\omega}{\alpha}e^{i\omega\tau}}\right| \leq 1$$

$$\left|1+i\frac{\omega}{\alpha}e^{i\omega\tau}\right|^2 \geq 1$$

$$1-2\frac{\omega}{\alpha}\sin(\omega\tau)+\left(\frac{\omega}{\alpha}\right)^2 \geq 1$$

$$\boxed{\frac{\omega}{\alpha} \geq 2\sin(\omega\tau)}$$

3.

Chapter 51

Nonlinear Partial Differential Equations

2021

51.1 Exercises

Exercise 51.1
Solve the equation

$$\phi_t + (1+x)\phi_x + \phi = 0 \quad \text{in} \quad -\infty < x < \infty,\ t > 0,$$

with initial condition $\phi(x, 0) = f(x)$.

Exercise 51.2
Solve the equation

$$\phi_t + \phi_x + \frac{\alpha\phi}{1+x} = 0$$

in the region $0 < x < \infty$, $t > 0$ with initial condition $\phi(x, 0) = 0$, and boundary condition $\phi(0, t) = g(t)$. [Here α is a positive constant.]

Exercise 51.3
Solve the equation

$$\phi_t + \phi_x + \phi^2 = 0$$

in $-\infty < x < \infty$, $t > 0$ with initial condition $\phi(x, 0) = f(x)$. Note that the solution could become infinite in finite time.

Exercise 51.4
Consider

$$c_t + cc_x + \mu c = 0, \qquad -\infty < x < \infty,\ t > 0.$$

1. Use the method of characteristics to solve the problem with

$$c = F(x) \text{ at } t = 0.$$

(μ is a positive constant.)

2. Find equations for the envelope of characteristics in the case $F'(x) < 0$.

3. Deduce an inequality relating $\max|F'(x)|$ and μ which decides whether breaking does or does not occur.

Exercise 51.5
For water waves in a channel the so-called shallow water equations are

$$h_t + (hv)_x = 0 \tag{51.1}$$

$$(hv)_t + \left(hv^2 + \frac{1}{2}gh^2\right)_x = 0, \ g = \text{constant}. \tag{51.2}$$

Investigate whether there are solutions with $v = V(h)$, where $V(h)$ is not posed in advance but is obtained from requiring consistency between the h equation obtained from (1) and the h equation obtained from (2).

There will be two possible choices for $V(h)$ depending on a choice of sign. Consider each case separately. In each case fix the arbitrary constant that arises in $V(h)$ by stipulating that before the waves arrive, h is equal to the undisturbed depth h_0 and $V(h_0) = 0$.

Find the h equation and the wave speed $c(h)$ in each case.

Exercise 51.6
After a change of variables, the chemical exchange equations can be put in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \sigma}{\partial x} = 0 \tag{51.3}$$

$$\frac{\partial \rho}{\partial t} = \alpha\sigma - \beta\rho - \gamma\rho\sigma; \quad \alpha, \beta, \gamma = \text{positive constants.} \tag{51.4}$$

1. Investigate wave solutions in which $\rho = \rho(X)$, $\sigma = \sigma(X)$, $X = x - Ut$, $U =$ constant, and show that $\rho(X)$ must satisfy an ordinary differential equation of the form

$$\frac{\mathrm{d}\rho}{\mathrm{d}X} = \text{quadratic in } \rho.$$

2. Discuss ths "smooth shock" solution as we did for a different example in class. In particular find the expression for U in terms of the values of ρ as $X \to \pm\infty$, and find the sign of $d\rho/dX$. Check that

$$U = \frac{\sigma_2 - \sigma_1}{\rho_2 - \rho_1}$$

in agreement with the "discontinuous theory."

Exercise 51.7

Find solitary wave solutions for the following equations:

1. $\eta_t + \eta_x + 6\eta\eta_x - \eta_{xxt} = 0$. (Regularized long wave or B.B.M. equation)
2. $u_{tt} - u_{xx} - \left(\frac{3}{2}u^2\right)_{xx} - u_{xxxx} = 0$. ("Boussinesq")
3. $\phi_{tt} - \phi_{xx} + 2\phi_x\phi_{xt} + \phi_{xx}\phi_t - \phi_{xxxx} = 0$. (The solitary wave form is for $u = \phi_x$)
4. $u_t + 30u^2u_1 + 20u_1u_2 + 10uu_3 + u_5 = 0$. (Here the subscripts denote x derivatives.)

51.2 Hints

Hint 51.1

Hint 51.2

Hint 51.3

Hint 51.4

Hint 51.5

Hint 51.6

Hint 51.7

51.3 Solutions

Solution 51.1

The method of characteristics gives us the differential equations

$$x'(t) = (1 + x) \qquad x(0) = \xi$$
$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\phi \qquad \phi(\xi, 0) = f(\xi)$$

Solving the first differential equation,

$$x(t) = ce^t - 1, \qquad x(0) = \xi$$
$$x(t) = (\xi + 1)e^t - 1$$

The second differential equation then becomes

$$\phi(x(t), t) = ce^{-t}, \qquad \phi(\xi, 0) = f(\xi), \qquad \xi = (x + 1)e^{-t} - 1$$
$$\phi(x, t) = f((x + 1)e^{-t} - 1)e^{-t}$$

Thus the solution to the partial differential equation is

$$\boxed{\phi(x, t) = f((x + 1)e^{-t} - 1)e^{-t}.}$$

Solution 51.2

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \phi_t + x'(t)\phi_x = -\frac{\alpha\phi}{1 + x}$$

The characteristic curves $x(t)$ satisfy $x'(t) = 1$, so $x(t) = t + c$. The characteristic curve that separates the region with domain of dependence on the x axis and domain of dependence on the t axis is $x(t) = t$. Thus we consider the two cases $x > t$ and $x < t$.

- $\mathbf{x} > \mathbf{t}$. $x(t) = t + \xi$.
- $\mathbf{x} < \mathbf{t}$. $x(t) = t - \tau$.

Now we solve the differential equation for ϕ in the two domains.

- $\mathbf{x} > \mathbf{t}$.

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\alpha\phi}{1+x}, \qquad \phi(\xi, 0) = 0, \qquad \xi = x - t$$
$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\alpha\phi}{1+t+\xi}$$
$$\phi = c\exp\left(-\alpha\int^{t}\frac{1}{t+\xi+1}\,dt\right)$$
$$\phi = cexp\left(-\alpha\log(t+\xi+1)\right)$$
$$\phi = c(t+\xi+1)^{-\alpha}$$

applying the initial condition, we see that

$$\phi = 0$$

- $\mathbf{x} < \mathbf{t}$.

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\alpha\phi}{1+x}, \qquad \phi(0, \tau) = g(\tau), \qquad \tau = t - x$$
$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\alpha\phi}{1+t-\tau}$$
$$\phi = c(t+1-\tau)^{-\alpha}$$
$$\phi = g(\tau)(t+1-\tau)^{-\alpha}$$
$$\phi = g(t-x)(x+1)^{-\alpha}$$

Thus the solution to the partial differential equation is

$$\boxed{\phi(x,t) = \begin{cases} 0 & \text{for } x > t \\ g(t-x)(x+1)^{-\alpha} & \text{for } x < t. \end{cases}}$$

Solution 51.3
The method of characteristics gives us the differential equations

$$x'(t) = 1 \qquad x(0) = \xi$$
$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\phi^2 \qquad \phi(\xi,0) = f(\xi)$$

Solving the first differential equation,

$$x(t) = t + \xi.$$

The second differential equation is then

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\phi^2, \qquad \phi(\xi,0) = f(\xi), \qquad \xi = x - t$$
$$\phi^{-2} d\phi = -dt$$
$$-\phi^{-1} = -t + c$$
$$\phi = \frac{1}{t-c}$$
$$\phi = \frac{1}{t + 1/f(\xi)}$$
$$\boxed{\phi = \frac{1}{t + 1/f(x-t)}.}$$

Solution 51.4

1. Taking the total derivative of c with respect to t,

$$\frac{\mathrm{d}c}{\mathrm{d}t} = c_t + \frac{\mathrm{d}x}{\mathrm{d}t} c_x.$$

Equating terms with the partial differential equation, we have the system of differential equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= c \\ \frac{\mathrm{d}c}{\mathrm{d}t} &= -\mu c. \end{aligned}$$

subject to the initial conditions

$$x(0) = \xi, \qquad c(\xi, 0) = F(\xi).$$

We can solve the second ODE directly.

$$\begin{aligned} c(\xi, t) &= c_1 e^{-\mu t} \\ c(\xi, t) &= F(\xi) e^{-\mu t} \end{aligned}$$

Substituting this result and solving the first ODE,

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= F(\xi) e^{-\mu t} \\ x(t) &= -\frac{F(\xi)}{\mu} e^{-\mu t} + c_2 \\ x(t) &= \frac{F(\xi)}{\mu}(1 - e^{-\mu t}) + \xi. \end{aligned}$$

The solution to the problem at the point (x, t) is found by first solving

$$x = \frac{F(\xi)}{\mu}(1 - e^{-\mu t}) + \xi$$

for ξ and then using this value to compute

$$c(x,t) = F(\xi)e^{-\mu t}.$$

2. The characteristic lines are given by the equation

$$x(t) = \frac{F(\xi)}{\mu}(1 - e^{-\mu t}) + \xi.$$

The points on the envelope of characteristics also satisfy

$$\frac{\partial x(t)}{\partial \xi} = 0.$$

Thus the points on the envelope satisfy the system of equations

$$x = \frac{F(\xi)}{\mu}(1 - e^{-\mu t}) + \xi$$
$$0 = \frac{F'(\xi)}{\mu}(1 - e^{-\mu t}) + 1.$$

By substituting

$$1 - e^{-\mu t} = -\frac{\mu}{F'(\xi)}$$

into the first equation we can eliminate its t dependence.

$$x = -\frac{F(\xi)}{F'(\xi)} + \xi$$

Now we can solve the second equation in the system for t.

$$e^{-\mu t} = 1 + \frac{\mu}{F'(\xi)}$$
$$t = -\frac{1}{\mu}\log\left(1 + \frac{\mu}{F'(\xi)}\right)$$

Thus the equations that describe the envelope are

$$x = -\frac{F(\xi)}{F'(\xi)} + \xi$$
$$t = -\frac{1}{\mu}\log\left(1 + \frac{\mu}{F'(\xi)}\right).$$

3. The second equation for the envelope has a solution for positive t if there is some x that satisfies

$$-1 < \frac{\mu}{F'(x)} < 0.$$

This is equivalent to

$$-\infty < F'(x) < -\mu.$$

So in the case that $F'(x) < 0$, there will be breaking iff

$$\max|F'(x)| > \mu.$$

Solution 51.5
With the substitution $v = V(h)$, the two equations become

$$h_t + (V + hV')h_x = 0$$
$$(V + hV')h_t + (V^2 + 2hVV' + gh)h_x = 0.$$

We can rewrite the second equation as

$$h_t + \frac{V^2 + 2hVV' + gh}{V + hV'}h_x = 0.$$

Requiring that the two equations be consistent gives us a differential equation for V.

$$V + hV' = \frac{V^2 + 2hVV' + gh}{V + hV'}$$
$$V^2 + 2hVV' + h^2(V')^2 = V^2 + 2hVV' + gh$$
$$(V')^2 = \frac{g}{h}.$$

There are two choices depending on which sign we choose when taking the square root of the above equation.

Positive V′.

$$V' = \sqrt{\frac{g}{h}}$$
$$V = 2\sqrt{gh} + \text{const}$$

We apply the initial condition $V(h_0) = 0$.

$$\boxed{V = 2\sqrt{g}(\sqrt{h} - \sqrt{h_0})}$$

The partial differential equation for h is then

$$h_t + (2\sqrt{g}(\sqrt{h} - \sqrt{h_0})h)_x = 0$$
$$\boxed{h_t + \sqrt{g}(3\sqrt{h} - 2\sqrt{h_0})h_x = 0}$$

The wave speed is

$$\boxed{c(h) = \sqrt{g}(3\sqrt{h} - 2\sqrt{h_0}).}$$

Negative V′.

$$V' = -\sqrt{\frac{g}{h}}$$
$$V = -2\sqrt{gh} + \text{const}$$

We apply the initial condition $V(h_0) = 0$.

$$\boxed{V = 2\sqrt{g}(\sqrt{h_0} - \sqrt{h})}$$

The partial differential equation for h is then

$$\boxed{h_t + \sqrt{g}(2\sqrt{h_0} - 3\sqrt{h})h_x = 0.}$$

The wave speed is

$$\boxed{c(h) = \sqrt{g}(2\sqrt{h_0} - 3\sqrt{h}).}$$

Solution 51.6

1. Making the substitutions, $\rho = \rho(X)$, $\sigma = \sigma(X)$, $X = x - Ut$, the system of partial differential equations becomes

$$-U\rho' + \sigma' = 0$$
$$-U\rho' = \alpha\sigma - \beta\rho - \gamma\rho\sigma.$$

Integrating the first equation yields

$$-U\rho + \sigma = c$$
$$\sigma = c + U\rho.$$

Now we substitute the expression for σ into the second partial differential equation.

$$-U\rho' = \alpha(c + U\rho) - \beta\rho - \gamma\rho(c + U\rho)$$
$$\rho' = -\alpha\left(\rho + \frac{c}{U}\right) + \frac{\beta}{U}\rho + \gamma\rho\left(\rho + \frac{c}{U}\right)$$

Thus $\rho(X)$ satisfies the ordinary differential equation

$$\boxed{\rho' = \gamma\rho^2 + \left(\frac{\gamma c}{U} + \frac{\beta}{U} - \alpha\right)\rho - \frac{\alpha c}{U}.}$$

2. Assume that

$$\begin{gathered} \rho(X) \to \rho_1 \text{ as } X \to +\infty \\ \rho(X) \to \rho_2 \text{ as } X \to -\infty \\ \rho'(X) \to 0 \text{ as } X \to \pm\infty. \end{gathered}$$

Integrating the ordinary differential equation for ρ,

$$X = \int^{\rho} \frac{d\rho}{\gamma\rho^2 + \left(\frac{\gamma c}{U} + \frac{\beta}{U} - \alpha\right)\rho - \frac{\alpha c}{U}}.$$

We see that the roots of the denominator of the integrand must be ρ_1 and ρ_2. Thus we can write the ordinary differential equation for $\rho(X)$ as

$$\rho'(X) = \gamma(\rho - \rho_1)(\rho - \rho_2) = \gamma\rho^2 - \gamma(\rho_1 + \rho_2)\rho + \gamma\rho_1\rho_2.$$

Equating coefficients in the polynomial with the differential equation for part 1, we obtain the two equations

$$-\frac{\alpha c}{U} = \gamma\rho_1\rho_2, \qquad \frac{\gamma c}{U} + \frac{\beta}{U} - \alpha = -\gamma(\rho_1 + \rho_2).$$

Solving the first equation for c,

$$c = -\frac{U\gamma\rho_1\rho_2}{\alpha}.$$

Now we substitute the expression for c into the second equation.

$$
\begin{aligned}
-\frac{\gamma U \gamma \rho_1 \rho_2}{\alpha U} + \frac{\beta}{U} - \alpha = -\gamma(\rho_1 + \rho_2) \\
\frac{\beta}{U} = \alpha + \frac{\gamma^2 \rho_1 \rho_2}{\alpha} - \gamma(\rho_1 + \rho_2)
\end{aligned}
$$

Thus we see that U is

$$
\boxed{U = \frac{\alpha\beta}{\alpha^2 + \gamma^2 \rho_1 \rho_2 - -\alpha\gamma(\rho_1 + \rho_2)}.}
$$

Since the quadratic polynomial in the ordinary differential equation for $\rho(X)$ is convex, it is negative valued between its two roots. Thus we see that

$$
\boxed{\frac{\mathrm{d}\rho}{\mathrm{d}X} < 0.}
$$

Using the expression for σ that we obtained in part 1,

$$
\begin{aligned}
\frac{\sigma_2 - \sigma_1}{\rho_2 - \rho_1} &= \frac{c + U\rho_2 - (c + U\rho_1)}{\rho_2 - \rho_1} \\
&= U \frac{\rho_2 - \rho_1}{\rho_2 - \rho_1} \\
&= U.
\end{aligned}
$$

Now let's return to the ordinary differential equation for $\rho(X)$

$$\rho'(X) = \gamma(\rho - \rho_1)(\rho - \rho_2)$$
$$X = \int^{\rho} \frac{d\rho}{\gamma(\rho - \rho_1)(\rho - \rho_2)}$$
$$X = -\frac{1}{\gamma(\rho_2 - \rho_1)} \int^{\rho} \left(\frac{1}{\rho - \rho_1} + \frac{1}{\rho_2 - \rho} \right) d\rho$$
$$X - X_0 = -\frac{1}{\gamma(\rho_2 - \rho_1)} \ln \left(\frac{\rho - \rho_1}{\rho_2 - \rho} \right)$$
$$-\gamma(\rho_2 - \rho_1)(X - X_0) = \ln \left(\frac{\rho - \rho_1}{\rho_2 - \rho} \right)$$
$$\frac{\rho - \rho_1}{\rho_2 - \rho} = \exp\left(-\gamma(\rho_2 - \rho_1)(X - X_0)\right)$$
$$\rho - \rho_1 = (\rho_2 - \rho) \exp\left(-\gamma(\rho_2 - \rho_1)(X - X_0)\right)$$
$$\rho \left[1 + \exp\left(-\gamma(\rho_2 - \rho_1)(X - X_0)\right)\right] = \rho_1 + \rho_2 \exp\left(-\gamma(\rho_2 - \rho_1)(X - X_0)\right)$$

Thus we obtain a closed form solution for ρ

$$\boxed{\rho = \frac{\rho_1 + \rho_2 \exp\left(-\gamma(\rho_2 - \rho_1)(X - X_0)\right)}{1 + \exp\left(-\gamma(\rho_2 - \rho_1)(X - X_0)\right)}}$$

Solution 51.7

1.

$$\eta_t + \eta_x + 6\eta\eta_x - \eta_{xxt} = 0$$

We make the substitution

$$\eta(x, t) = z(X), \qquad X = x - Ut.$$

$$(1-U)z' + 6zz' + Uz''' = 0$$
$$(1-U)z + 3z^2 + Uz'' = 0$$
$$\frac{1}{2}(1-U)z^2 + z^3 + \frac{1}{2}U(z')^2 = 0$$
$$(z')^2 = \frac{U-1}{U}z^2 - \frac{2}{U}z^3$$
$$z(X) = \frac{U-1}{2}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{U-1}{U}}X\right)$$
$$\eta(x,t) = \frac{U-1}{2}\operatorname{sech}^2\left(\frac{1}{2}\left(\sqrt{\frac{U-1}{U}}x - \sqrt{(U-1)Ut}\right)\right)$$

The linearized equation is

$$\eta_t + \eta_x - \eta_{xxt} = 0.$$

Substituting $\eta = e^{-\alpha x + \beta t}$ into this equation yields

$$\beta - \alpha - \alpha^2\beta = 0$$
$$\beta = \frac{\alpha}{1-\alpha^2}.$$

We set

$$\alpha^2 = \frac{U-1}{U}.$$

β is then

$$\begin{aligned}
\beta &= \frac{\alpha}{1-\alpha^2} \\
&= \frac{\sqrt{(U-1)/U}}{1-(U-1)/U)} \\
&= \frac{\sqrt{(U-1)U}}{U-(U-1)} \\
&= \sqrt{(U-1)U}.
\end{aligned}$$

The solution for η becomes

$$\frac{\alpha\beta}{2} \operatorname{sech}^2\left(\frac{\alpha x - \beta t}{2}\right)$$

where

$$\beta = \frac{\alpha}{1-\alpha^2}.$$

2.

$$u_{tt} - u_{xx} - \left(\frac{3}{2}u^2\right)_{xx} - u_{xxxx} = 0$$

We make the substitution

$$u(x,t) = z(X), \qquad X = x - Ut.$$

$$\begin{gathered}
(U^2-1)z'' - \left(\frac{3}{2}z^2\right)'' - z'''' = 0 \\
(U^2-1)z' - \left(\frac{3}{2}z^2\right)' - z''' = 0 \\
(U^2-1)z - \frac{3}{2}z^2 - z'' = 0
\end{gathered}$$

We multiply by z' and integrate.

$$\frac{1}{2}(U^2-1)z^2 - \frac{1}{2}z^3 - \frac{1}{2}(z')^2 = 0$$
$$(z')^2 = (U^2-1)z^2 - z^3$$
$$z = (U^2-1)\operatorname{sech}^2\left(\frac{1}{2}\sqrt{U^2-1}X\right)$$
$$u(x,t) = (U^2-1)\operatorname{sech}^2\left(\frac{1}{2}\left(\sqrt{U^2-1}x - U\sqrt{U^2-1}t\right)\right)$$

The linearized equation is

$$u_{tt} - u_{xx} - u_{xxxx} = 0.$$

Substituting $u = e^{-\alpha x + \beta t}$ into this equation yields

$$\beta^2 - \alpha^2 - \alpha^4 = 0$$
$$\beta^2 = \alpha^2(\alpha^2+1).$$

We set

$$\alpha = \sqrt{U^2-1}.$$

β is then

$$\beta^2 = \alpha^2(\alpha^2+1)$$
$$= (U^2-1)U^2$$
$$\beta = U\sqrt{U^2-1}.$$

The solution for u becomes

$$u(x,t) = \alpha^2 \operatorname{sech}^2\left(\frac{\alpha x - \beta t}{2}\right)$$

where

$$\beta^2 = \alpha^2(\alpha^2 + 1).$$

3.

$$\phi_{tt} - \phi_{xx} + 2\phi_x\phi_{xt} + \phi_{xx}\phi_t - \phi_{xxxx}$$

We make the substitution

$$\phi(x,t) = z(X), \qquad X = x - Ut.$$

$$\begin{gathered}
(U^2-1)z'' - 2Uz'z'' - Uz''z' - z'''' = 0 \\
(U^2-1)z'' - 3Uz'z'' - z'''' = 0 \\
(U^2-1)z' - \frac{3}{2}(z')^2 - z''' = 0
\end{gathered}$$

Multiply by z'' and integrate.

$$\begin{gathered}
\frac{1}{2}(U^2-1)(z')^2 - \frac{1}{2}(z')^3 - \frac{1}{2}(z'')^2 = 0 \\
(z'')^2 = (U^2-1)(z')^2 - (z')^3 \\
z' = (U^2-1)\operatorname{sech}^2\left(\frac{1}{2}\sqrt{U^2-1}X\right) \\
\phi_x(x,t) = (U^2-1)\operatorname{sech}^2\left(\frac{1}{2}\left(\sqrt{U^2-1}x - U\sqrt{U^2-1}t\right)\right).
\end{gathered}$$

The linearized equation is

$$\phi_{tt} - \phi_{xx} - \phi_{xxxx}$$

Substituting $\phi = e^{-\alpha x + \beta t}$ into this equation yields

$$\beta^2 = \alpha^2(\alpha^2 + 1).$$

The solution for ϕ_x becomes

$$\phi_x = \alpha^2 \operatorname{sech}^2\left(\frac{\alpha x - \beta t}{2}\right)$$

where

$$\beta^2 = \alpha^2(\alpha^2 + 1).$$

4.

$$u_t + 30u^2 u_1 + 20u_1 u_2 + 10uu_3 + u_5 = 0$$

We make the substitution

$$u(x,t) = z(X), \qquad X = x - Ut.$$

$$-Uz' + 30z^2 z' + 20z'z'' + 10zz''' + z^{(5)} = 0$$

Note that $(zz'')' = z'z'' + zz'''$.

$$-Uz' + 30z^2 z' + 10z'z'' + 10(zz'')' + z^{(5)} = 0$$
$$-Uz + 10z^3 + 5(z')^2 + 10zz'' + z^{(4)} = 0$$

Multiply by z' and integrate.

$$-\frac{1}{2}Uz^2 + \frac{5}{2}z^4 + 5z(z')^2 - \frac{1}{2}(z'')^2 + z'z''' = 0$$

Assume that

$$(z')^2 = P(z).$$

Differentiating this relation,

$$\begin{aligned}
2z'z'' &= P'(z)z' \\
z'' &= \frac{1}{2}P'(z) \\
z''' &= \frac{1}{2}P''(z)z' \\
z'''z' &= \frac{1}{2}P''(z)P(z).
\end{aligned}$$

Substituting this expressions into the differential equation for z,

$$\begin{gathered}
-\frac{1}{2}Uz^2 + \frac{5}{2}z^4 + 5zP(z) - \frac{1}{2}\frac{1}{4}(P'(z))^2 + \frac{1}{2}P''(z)P(z) = 0 \\
4Uz^2 + 20z^4 + 40zP(z) - (P'(z))^2 + 4P''(z)P(z) = 0
\end{gathered}$$

Substituting $P(z) = az^3 + bz^2$ yields

$$(20 + 40a + 15a^2)z^4 + (40b + 20ab)z^3 + (4b^2 + 4U)z^2 = 0$$

This equation is satisfied by $b^2 = U$, $a = -2$. Thus we have

$$\begin{aligned}
(z')^2 &= \sqrt{U}z^2 - 2z^3 \\
z &= \frac{\sqrt{U}}{2}\operatorname{sech}^2\left(\frac{1}{2}U^{1/4}X\right) \\
u(x,t) &= \frac{\sqrt{U}}{2}\operatorname{sech}^2\left(\frac{1}{2}(U^{1/4}x - U^{5/4}t)\right)
\end{aligned}$$

The linearized equation is

$$u_t + u_5 = 0.$$

Substituting $u = e^{-\alpha x + \beta t}$ into this equation yields

$$\beta - \alpha^5 = 0.$$

We set

$$\alpha = U^{1/4}.$$

The solution for $u(x, t)$ becomes

$$\frac{\alpha^2}{2} \operatorname{sech}^2 \left(\frac{\alpha x - \beta t}{2} \right)$$

where

$$\beta = \alpha^5.$$

Part VIII

Appendices

Appendix A

Greek Letters

Name	*Lower*	*Upper*
alpha	α	
beta	β	
gamma	γ	Γ
delta	δ	Δ
epsilon	ϵ	
iota	ι	
kappa	κ	
lambda	λ	Λ
mu	μ	
nu	ν	
omicron	o	
pi	π	Π
rho	ρ	
sigma	σ	Σ
tau	τ	
theta	θ	Θ

phi	ϕ	Φ
psi	ψ	Ψ
chi	χ	
omega	ω	Ω
upsilon	υ	Υ
xi	ξ	Ξ
eta	η	
zeta	ζ	

Appendix B

Notation

C	class of continuous functions
C^n	class of n-times continuously differentiable functions
$\mathbb{C}$	set of complex numbers
$\delta(x)$	Dirac delta function
$\mathcal{F}[\cdot]$	Fourier transform
$\mathcal{F}_c[\cdot]$	Fourier cosine transform
$\mathcal{F}_s[\cdot]$	Fourier sine transform
γ	Euler's constant, $\gamma = \int_0^\infty e^{-x} \operatorname{Log} x \, dx$
$\Gamma(\nu)$	Gamma function
$H(x)$	Heaviside function
$H_\nu^{(1)}(x)$	Hankel function of the first kind and order ν
$H_\nu^{(2)}(x)$	Hankel function of the second kind and order ν
$J_\nu(x)$	Bessel function of the first kind and order ν
$K_\nu(x)$	Modified Bessel function of the first kind and order ν
$\mathcal{L}[\cdot]$	Laplace transform
$\mathbb{N}$	set of natural numbers, (positive integers)

$N_\nu(x)$	Modified Bessel function of the second kind and order ν
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of positive real numbers
$\mathbb{R}^-$	set of negative real numbers
$o(z)$	terms smaller than z
$\mathcal{O}(z)$	terms no bigger than z
$⨍$	principal value of the integral
$\psi(\nu)$	digamma function, $\psi(\nu) = \frac{\mathrm{d}}{\mathrm{d}\nu} \log \Gamma(\nu)$
$\psi^{(n)}(\nu)$	polygamma function, $\psi^{(n)}(\nu) = \frac{d^n}{d\nu^n} \psi(\nu)$
$u^{(n)}(x)$	$\frac{\partial^n u}{\partial x^n}$
$u^{(n,m)}(x,y)$	$\frac{\partial^{n+m} u}{\partial x^n \partial y^m}$
$Y_\nu(x)$	Bessel function of the second kind and order ν, Neumann function
$\mathbb{Z}$	set of integers
$\mathbb{Z}^+$	set of positive integers

Appendix C

Formulas from Complex Variables

Analytic Functions. A function $f(z)$ is analytic in a domain if the derivative $f'(z)$ exists in that domain.

If $f(z) = u(x, y) + iv(x, y)$ is defined in some neighborhood of $z_0 = x_0 + iy_0$ and the partial derivatives of u and v are continuous and satisfy the **Cauchy-Riemann equations**

$$u_x = v_y, \qquad u_y = -v_x,$$

then $f'(z_0)$ exists.

Residues. If $f(z)$ has the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

then the residue of $f(z)$ at $z = z_0$ is

$$\text{Res}\,(f(z), z_0) = a_{-1}.$$

Residue Theorem. Let C be a positively oriented, simple, closed contour. If $f(z)$ is analytic in and on C except for isolated singularities at $z_1, z_2, \ldots, z_N$ inside C then

$$\oint_C f(z)\,dz = 2\pi i \sum_{n=1}^{N} \operatorname{Res}(f(z), z_n).$$

If in addition $f(z)$ is analytic outside C in the finite complex plane then

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$

Residues of a pole of order n. If $f(z)$ has a pole of order n at $z = z_0$ then

$$\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} \left(\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right] \right).$$

Jordan's Lemma.

$$\int_0^{\pi} \mathrm{e}^{-R \sin\theta}\,d\theta < \frac{\pi}{R}.$$

Let a be a positive constant. If $f(z)$ vanishes as $|z| \to \infty$ then the integral

$$\int_C f(z)\,\mathrm{e}^{iaz}\,dz$$

along the semi-circle of radius R in the upper half plane vanishes as $R \to \infty$.

Taylor Series. Let $f(z)$ be a function that is analytic and single valued in the disk $|z - z_0| < R$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

The series converges for $|z - z_0| < R$.

Laurent Series. Let $f(z)$ be a function that is analytic and single valued in the annulus $r < |z - z_0| < R$. In this annulus $f(z)$ has the convergent series,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

and the path of integration is any simple, closed, positive contour around z_0 and lying in the annulus. The path of integration is shown in Figure C.1.

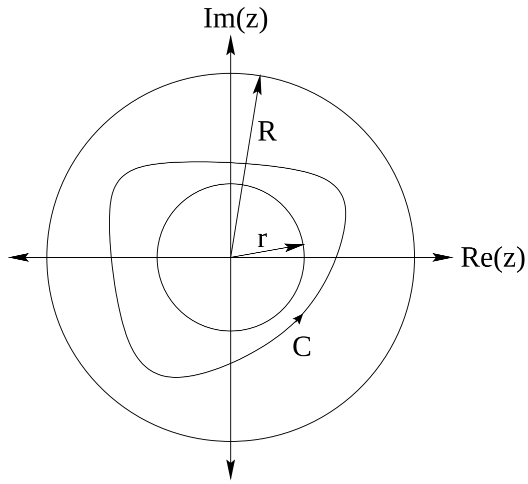

Figure C.1: The Path of Integration.

Appendix D

Table of Derivatives

Note: c denotes a constant and $'$ denotes differentiation.

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = \frac{\mathrm{d}f}{\mathrm{d}x}g + f\frac{\mathrm{d}g}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f^c\right) = cf^{c-1}f'$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g)] = f'(g)g'$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}[f(g)] = f''(g)(g')^2 + f'g''$$

$$\frac{d^n}{dx^n}(fg) = \binom{n}{0}\frac{d^n f}{dx^n}g + \binom{n}{1}\frac{d^{n-1}f}{dx^{n-1}}\frac{\mathrm{d}g}{\mathrm{d}x} + \binom{n}{2}\frac{d^{n-2}f}{dx^{n-2}}\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} + \cdots + \binom{n}{n}f\frac{d^n g}{dx^n}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log x) = \frac{1}{f}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(c^x\right) = c^x \log c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f^g\right) = g f^{g-1} \frac{\mathrm{d}f}{\mathrm{d}x} + f^g \log f \frac{\mathrm{d}g}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\csc x) = -\csc x \cot x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\csc^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \qquad -\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}, \qquad 0 \leq \arccos x \leq \pi$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}, \qquad -\frac{\pi}{2} \leq \arctan x \leq \frac{\pi}{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arccosh} x) = \frac{1}{\sqrt{x^2-1}}, \qquad x > 1, \ \operatorname{arccosh} x > 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arctanh} x) = \frac{1}{1-x^2}, \qquad x^2 < 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_c^x f(\xi)\,d\xi = f(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^c f(\xi)\,d\xi = -f(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_g^h f(\xi,x)\,d\xi = \int_g^h \frac{\partial f(\xi,x)}{\partial x}\,d\xi + f(h,x)h' - f(g,x)g'$$

Appendix E

Table of Integrals

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,dx = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,dx$$

$$\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$$

$$\int \frac{f'(x)}{2\sqrt{f(x)}}\,dx = \sqrt{f(x)}$$

$$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} \qquad \text{for } \alpha \neq = -1$$

$$\int \frac{1}{x}\,dx = \log x$$

$$\int \mathrm{e}^{ax}\,dx = \frac{\mathrm{e}^{ax}}{a}$$

$$\int a^{bx}\,dx = \frac{a^{bx}}{b\log a} \qquad \text{for } a > 0$$

$$\int \log x\,dx = x\log x - x$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}$$

$$\int \frac{1}{x^2-a^2}\,dx = \begin{cases} \frac{1}{2a}\log\frac{a-x}{a+x} & \text{for } x^2 < a^2 \\ \frac{1}{2a}\log\frac{x-a}{x+a} & \text{for } x^2 > a^2 \end{cases}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{|a|} = -\arccos\frac{x}{|a|} \qquad \text{for } x^2 < a^2$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \log(x + \sqrt{x^2 \pm a^2})$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{|a|}\sec^{-1}\frac{x}{a}$$

$$\int \frac{1}{x\sqrt{a^2 \pm x^2}}\,dx = -\frac{1}{a}\log\left(\frac{a+\sqrt{a^2 \pm x^2}}{x}\right)$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax)$$

$$\int \tan(ax)\,dx = -\frac{1}{a}\log\cos(ax)$$

$$\int \csc(ax)\,dx = \frac{1}{a}\log\tan\frac{ax}{2}$$

$$\int \sec(ax)\,dx = \frac{1}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

$$\int \cot(ax)\,dx = \frac{1}{a}\log\sin(ax)$$

$$\int \sinh(ax)\,dx = \frac{1}{a}\cosh(ax)$$

$$\int \cosh(ax)\,dx = \frac{1}{a}\sinh(ax)$$

$$\int \tanh(ax)\,dx = \frac{1}{a}\log\cosh(ax)$$

$$\int \operatorname{csch}(ax)\,dx = \frac{1}{a}\log\tanh\frac{ax}{2}$$

$$\int \operatorname{sech}(ax)\,dx = \frac{i}{a}\log\tanh\left(\frac{i\pi}{4}+\frac{ax}{2}\right)$$

$$\int \coth(ax)\,dx = \frac{1}{a}\log\sinh(ax)$$

$$\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$$

$$\int x^2 \sin ax \, dx = \frac{2x}{a^2} \sin ax - \frac{a^2x^2 - 2}{a^3} \cos ax$$

$$\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$$

$$\int x^2 \cos ax \, dx = \frac{2x \cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3} \sin ax$$

Appendix F

Definite Integrals

Integrals from $-\infty$ to ∞. Let $f(z)$ be analytic except for isolated singularities, none of which lie on the real axis. Let $a_1, \ldots, a_m$ be the singularities of $f(z)$ in the upper half plane; and C_R be the semi-circle from R to $-R$ in the upper half plane. If

$$\lim_{R \to \infty} \left(R \max_{z \in C_R} |f(z)| \right) = 0$$

then

$$\int_{-\infty}^{\infty} f(x) \, dx = i2\pi \sum_{j=1}^{m} \operatorname{Res} \left(f(z), a_j \right).$$

Let $b_1, \ldots, b_n$ be the singularities of $f(z)$ in the lower half plane. Let C_R be the semi-circle from R to $-R$ in the lower half plane. If

$$\lim_{R \to \infty} \left(R \max_{z \in C_R} |f(z)| \right) = 0$$

then

$$\int_{-\infty}^{\infty} f(x)\,dx = -i2\pi \sum_{j=1}^{n} \operatorname{Res}\left(f(z), b_j\right).$$

Integrals from 0 to ∞. Let $f(z)$ be analytic except for isolated singularities, none of which lie on the positive real axis, $[0, \infty)$. Let $z_1, \ldots, z_n$ be the singularities of $f(z)$. If $f(z) \ll z^\alpha$ as $z \to 0$ for some $\alpha > -1$ and $f(z) \ll z^\beta$ as $z \to \infty$ for some $\beta < -1$ then

$$\int_{0}^{\infty} f(x)\,dx = -\sum_{k=1}^{n} \operatorname{Res}\left(f(z)\log z, z_k\right).$$

$$\int_{0}^{\infty} f(x)\log\,dx = -\frac{1}{2}\sum_{k=1}^{n} \operatorname{Res}\left(f(z)\log^2 z, z_k\right) + i\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z)\log z, z_k\right)$$

Assume that a is not an integer. If $z^a f(z) \ll z^\alpha$ as $z \to 0$ for some $\alpha > -1$ and $z^a f(z) \ll z^\beta$ as $z \to \infty$ for some $\beta < -1$ then

$$\int_{0}^{\infty} x^a f(x)\,dx = \frac{i2\pi}{1 - \mathrm{e}^{i2\pi a}} \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z), z_k\right).$$

$$\int_{0}^{\infty} x^a f(x)\log x\,dx = \frac{i2\pi}{1 - \mathrm{e}^{i2\pi a}} \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z)\log z, z_k\right), + \frac{\pi^2 a}{\sin^2(\pi a)} \sum_{k=1}^{n} \operatorname{Res}\left(z^a f(z), z_k\right)$$

Fourier Integrals. Let $f(z)$ be analytic except for isolated singularities, none of which lie on the real axis. Suppose that $f(z)$ vanishes as $|z| \to \infty$. If ω is a positive real number then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{i\omega x}\,dx = i2\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z)\,\mathrm{e}^{i\omega z}, z_k\right),$$

where $z_1, \dots, z_n$ are the singularities of $f(z)$ in the upper half plane. If ω is a negative real number then

$$\int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{i\omega x}\, dx = -i2\pi \sum_{k=1}^{n} \operatorname{Res}\left(f(z)\, \mathrm{e}^{i\omega z}, z_k\right),$$

where $z_1, \dots, z_n$ are the singularities of $f(z)$ in the lower half plane.

Appendix G

Table of Sums

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}, \quad \text{for } |r| < 1$$

$$\sum_{n=1}^{N} r^n = \frac{r - r^{N+1}}{1-r}$$

$$\sum_{n=a}^{b} n = \frac{(a+b)(b+1-a)}{2}$$

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

$$\sum_{n=a}^{b} n^2 = \frac{b(b+1)(2b+1) - a(a-1)(2a-1)}{6}$$

$$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log(2)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} = \frac{3\zeta(3)}{4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^5} = \zeta(5)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} = \frac{15\zeta(5)}{16}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} = \frac{31\pi^6}{30240}$$

Appendix H

Table of Taylor Series

$$(1-z)^{-1} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

$$(1-z)^{-2} = \sum_{n=0}^{\infty} (n+1) z^n \qquad |z| < 1$$

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \qquad |z| < 1$$

$$\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad |z| < \infty$$

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \qquad |z| < 1$$

$$\log\left(\frac{1+z}{1-z}\right) = 2\sum_{n=1}^{\infty}\frac{z^{2n-1}}{2n-1} \qquad |z| < 1$$

$$\cos z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!} \qquad |z| < \infty$$

$$\sin z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad |z| < \infty$$

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \cdots \qquad |z| < \tfrac{\pi}{2}$$

$$\cos^{-1} z = \frac{\pi}{2} - \left(z + \frac{z^3}{2\cdot 3} + \frac{1\cdot 3z^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5z^7}{2\cdot 4\cdot 6\cdot 7} + \cdots\right) \qquad |z| < 1$$

$$\sin^{-1} z = z + \frac{z^3}{2\cdot 3} + \frac{1\cdot 3z^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5z^7}{2\cdot 4\cdot 6\cdot 7} + \cdots \qquad |z| < 1$$

$$\tan^{-1} z = \sum_{n=1}^{\infty}\frac{(-1)^{n+1} z^{2n-1}}{2n-1} \qquad |z| < 1$$

$$\cosh z = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!} \qquad |z| < \infty$$

$$\sinh z = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!} \qquad |z| < \infty$$

$$\tanh z = z - \frac{z^3}{3} + \frac{2z^5}{15} - \frac{17z^7}{315} + \cdots \qquad |z| < \tfrac{\pi}{2}$$

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n} \qquad |z| < \infty$$

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n} \qquad |z| < \infty$$

Appendix I

Table of Laplace Transforms

Let $f(t)$ be piecewise continuous and of exponential order α. Unless otherwise noted, the transform is defined for $s > 0$.

$\mathbf{f(t)}$	$\int_{\mathbf{0}}^{\infty} \mathrm{e}^{-\mathbf{st}} \mathbf{f(t)}\, \mathbf{dt}$	
$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathrm{e}^{ts} F(s)\, ds$	$F(s)$	
$af(t) + bg(t)$	$aF(s) + bG(s)$	
$\mathrm{e}^{ct} f(t)$	$F(s-c)$	$s > c + \alpha$
$f(t+c)$	$F(s-c)$	$s > c + \alpha$
$tf(t)$	$-\frac{\mathrm{d}}{\mathrm{d}s}[F(s)]$	

$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n}[F(s)]$
$\dfrac{f(t)}{t}, \quad \displaystyle\int_0^1 \dfrac{f(t)}{t}\,dt$ exists	$\displaystyle\int_s^\infty F(t)\,dt$
$\displaystyle\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$
$\displaystyle\int_0^t \int_0^\tau f(s)\,ds\,d\tau$	$\dfrac{F(s)}{s^2}$
$\dfrac{\mathrm{d}}{\mathrm{d}t} f(t)$	$sF(s) - f(0)$
$\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} f(t)$	$s^2 F(s) - sf(0) - f'(0)$
$\dfrac{d^n}{dt^n} f(t)$	$s^n F(s) - s^{n-1} f(0)$ $- s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
$\displaystyle\int_0^t f(\tau) g(t-\tau)\,d\tau, \quad f, g \in C^0$	$F(s)G(s)$
$\dfrac{1}{c} f(t/c), \quad c > 0$	$F(cs)$
$f(t), \quad f(t+T) = f(t)$	$\dfrac{\int_0^T \mathrm{e}^{-st} f(t)\,dt}{1 - \mathrm{e}^{-sT}}$

$f(t), \quad f(t+T) = -f(t)$	$\dfrac{\int_0^T e^{-st} f(t)\,dt}{1 + e^{-sT}}$	
$H(t)$	$\dfrac{1}{s}$	
$tH(t)$	$\dfrac{1}{s^2}$	
$t^n H(t)$, for $n = 0, 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$	
$t^{1/2} H(t)$	$\dfrac{\sqrt{\pi}}{2} s^{-3/2}$	
$t^{-1/2} H(t)$	$\sqrt{\pi} s^{-1/2}$	
$t^{n-1/2} H(t), \quad n \in \mathbb{Z}^+$	$\dfrac{(1)(3)(5)\cdots(2n-1)\sqrt{\pi}}{2^n} s^{-n-1/2}$	
$t^\nu H(t), \quad \Re(\nu) > -1$	$\dfrac{\Gamma(\nu+1)}{s^{n+1}}$	
$\operatorname{Log} t H(t)$	$\dfrac{-\gamma - \operatorname{Log} s}{s}$	
$t^\nu \operatorname{Log} t H(t), \quad \Re(\nu) > -1$	$\dfrac{\Gamma(\nu+1)}{s^{n+1}} (\psi(\nu+1) - \operatorname{Log} s)$	
$\delta(t)$	1	$s > 0$
$\delta^{(n)}(t), \quad n \in \mathbb{Z}^{0+}$	s^n	$s > 0$

$e^{ct}H(t)$	$\frac{1}{s-c}$	$s > c$
$t\,e^{ct}H(t)$	$\frac{1}{(s-c)^2}$	$s > c$
$\frac{t^{n-1}\,e^{ct}}{(n-1)!}H(t), n \in \mathbb{Z}^+$	$\frac{1}{(s-c)^n}$	$s > c$
$\sin(ct)H(t)$	$\frac{c}{s^2+c^2}$	
$\cos(ct)H(t)$	$\frac{s}{s^2+c^2}$	
$\sinh(ct)H(t)$	$\frac{c}{s^2-c^2}$	$s > \lvert c \rvert$
$\cosh(ct)H(t)$	$\frac{s}{s^2-c^2}$	$s > \lvert c \rvert$
$t\sin(ct)H(t)$	$\frac{2cs}{(s^2+c^2)^2}$	
$t\cos(ct)H(t)$	$\frac{s^2-c^2}{(s^2+c^2)^2}$	
$t^n\,e^{ct}H(t), \quad n \in \mathbb{Z}^+$	$\frac{n!}{(s-c)^{n+1}}$	
$e^{dt}\sin(ct)H(t)$	$\frac{c}{(s-d)^2+c^2}$	$s > d$

$e^{dt}\cos(ct)H(t)$	$\dfrac{s-d}{(s-d)^2+c^2}$	$s>d$
$\delta(t-c)$	$\begin{cases}0 & \text{for } c<0\\ e^{-sc} & \text{for } c>0\end{cases}$	
$H(t-c)=\begin{cases}0 & \text{for } t<c\\ 1 & \text{for } t>c\end{cases}$	$\dfrac{1}{s}e^{-cs}$	
$J_\nu(ct)H(t)$	$\dfrac{c^n}{\sqrt{s^2+c^2}\left(s+\sqrt{s^2+c^2}\right)^\nu}$	$\nu>-1$
$I_\nu(ct)H(t)$	$\dfrac{c^n}{\sqrt{s^2-c^2}\left(s-\sqrt{s^2+c^2}\right)^\nu}$	$\Re(s)>c$, $\nu>-1$

Appendix J

Table of Fourier Transforms

$\mathbf{f(x)}$	$\dfrac{\mathbf{1}}{\mathbf{2\pi}}\displaystyle\int_{-\infty}^{\infty}\mathbf{f(x)}\,\mathrm{e}^{-\mathbf{i}\boldsymbol{\omega}\mathbf{x}}\,\mathbf{dx}$
$\displaystyle\int_{-\infty}^{\infty}\mathbf{F}(\boldsymbol{\omega})\,\mathrm{e}^{\mathbf{i}\boldsymbol{\omega}\mathbf{x}}\,\mathbf{d}\boldsymbol{\omega}$	$\mathbf{F}(\omega)$
$af(x)+bg(x)$	$aF(\omega)+bG(\omega)$
$f^{(n)}(x)$	$(i\omega)^n F(\omega)$
$x^n f(x)$	$i^n F^{(n)}(\omega)$
$f(x+c)$	$\mathrm{e}^{i\omega c}F(\omega)$
$\mathrm{e}^{-icx}f(x)$	$F(\omega+c)$
$f(cx)$	$\|c\|^{-1}F(\omega/c)$

$f(x)g(x)$	$F * G(\omega) = \int_{-\infty}^{\infty} F(\eta)G(\omega-\eta)\,d\eta$
$\frac{1}{2\pi} f * g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi$	$F(\omega)G(\omega)$
$\mathrm{e}^{-cx^2}, \quad c>0$	$\frac{1}{\sqrt{4\pi c}}\,\mathrm{e}^{-\omega^2/4c}$
$\mathrm{e}^{-c\lvert x\rvert}, \quad c>0$	$\frac{c/\pi}{\omega^2+c^2}$
$\frac{2c}{x^2+c^2}, \quad c>0$	$\mathrm{e}^{-c\lvert\omega\rvert}$
$\frac{1}{x-i\alpha}, \quad \alpha>0$	$\begin{cases} 0 & \text{for } \omega>0 \\ i\,\mathrm{e}^{\alpha\omega} & \text{for } \omega<0 \end{cases}$
$\frac{1}{x-i\alpha}, \quad \alpha<0$	$\begin{cases} i\,\mathrm{e}^{\alpha\omega} & \text{for } \omega>0 \\ 0 & \text{for } \omega<0 \end{cases}$
$\frac{1}{x}$	$-\frac{i}{2}\,\mathrm{sign}\,(\omega)$
$H(x-c) = \begin{cases} 0 & \text{for } x<c \\ 1 & \text{for } x>c \end{cases}$	$\frac{1}{2\pi i\omega}\,\mathrm{e}^{-ic\omega}$

$e^{-cx}H(x), \quad \Re(c) > 0$	$\dfrac{1}{2\pi(c+i\omega)}$
$e^{cx}H(-x), \quad \Re(c) > 0$	$\dfrac{1}{2\pi(c-i\omega)}$
1	$\delta(\omega)$
$\delta(x-\xi)$	$\dfrac{1}{2\pi}\, e^{-i\omega\xi}$
$\pi(\delta(x+\xi)+\delta(x-\xi))$	$\cos(\omega\xi)$
$-i\pi(\delta(x+\xi)-\delta(x-\xi))$	$\sin(\omega\xi)$
$H(c-\lvert x\rvert) = \begin{cases} 1 & \text{for } \lvert x\rvert < c \\ 0 & \text{for } \lvert x\rvert > c \end{cases}, \; c > 0$	$\dfrac{\sin(c\omega)}{\pi\omega}$

Appendix K

Table of Fourier Transforms in n Dimensions

$\mathbf{f(x)}$	$\dfrac{\mathbf{1}}{(\mathbf{2}\boldsymbol{\pi})^{\mathbf{n}}}\displaystyle\int_{\mathbb{R}^{\mathbf{n}}} \mathbf{f(x)}\, \mathrm{e}^{-\mathbf{i}\boldsymbol{\omega}\mathbf{x}}\, \mathbf{dx}$
$\displaystyle\int_{\mathbb{R}^{\mathbf{n}}} \mathbf{F}(\boldsymbol{\omega})\, \mathrm{e}^{\mathbf{i}\boldsymbol{\omega}\mathbf{x}}\, \mathbf{d}\boldsymbol{\omega}$	$\mathbf{F}(\omega)$
$af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
$\left(\dfrac{\pi}{c}\right)^{n/2} \mathrm{e}^{-nx^2/4c}$	$\mathrm{e}^{-c\omega^2}$

Appendix L

Table of Fourier Cosine Transforms

$\mathbf{f(x)}$	$\mathbf{\frac{1}{\pi}\int_0^\infty f(x)\cos(\omega x)\,dx}$
$\mathbf{2\int_0^\infty C(\omega)\cos(\omega x)\,d\omega}$	$\mathbf{C(\omega)}$
$f'(x)$	$\omega S(\omega) - \frac{1}{\pi} f(0)$
$f''(x)$	$-\omega^2 C(\omega) - \frac{1}{\pi} f'(0)$
$x f(x)$	$\frac{\partial}{\partial \omega} \mathcal{F}_s[f(x)]$
$f(cx), \quad c > 0$	$\frac{1}{c} C\left(\frac{\omega}{c}\right)$

$\dfrac{2c}{x^2 + c^2}$	$\mathrm{e}^{-c\omega}$
e^{-cx}	$\dfrac{c/\pi}{\omega^2 + c^2}$
e^{-cx^2}	$\dfrac{1}{\sqrt{4\pi c}}\,\mathrm{e}^{-\omega^2/(4c)}$
$\sqrt{\dfrac{\pi}{c}}\,\mathrm{e}^{-x^2/(4c)}$	$\mathrm{e}^{-c\omega^2}$

Appendix M

Table of Fourier Sine Transforms

$\mathbf{f(x)}$	$\boldsymbol{\frac{1}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx}$
$\boldsymbol{2\int_0^\infty S(\omega)\sin(\omega x)\,d\omega}$	$\boldsymbol{S(\omega)}$
$f'(x)$	$-\omega C(\omega)$
$f''(x)$	$-\omega^2 S(\omega) + \frac{1}{\pi}\omega f(0)$
$xf(x)$	$-\frac{\partial}{\partial\omega}\mathcal{F}_c[f(x)]$
$f(cx), \quad c > 0$	$\frac{1}{c}S\left(\frac{\omega}{c}\right)$

$\dfrac{2x}{x^2+c^2}$	$\mathrm{e}^{-c\omega}$
e^{-cx}	$\dfrac{\omega/\pi}{\omega^2+c^2}$
$2\arctan\left(\dfrac{x}{c}\right)$	$\dfrac{1}{\omega}\,\mathrm{e}^{-c\omega}$
$\dfrac{1}{x}\,\mathrm{e}^{-cx}$	$\dfrac{1}{\pi}\arctan\left(\dfrac{\omega}{c}\right)$
1	$\dfrac{1}{\pi\omega}$
$\dfrac{2}{x}$	1
$x\,\mathrm{e}^{-cx^2}$	$\dfrac{\omega}{4c^{3/2}\sqrt{\pi}}\,\mathrm{e}^{-\omega^2/(4c)}$
$\dfrac{\sqrt{\pi}x}{2c^{3/2}}\,\mathrm{e}^{-x^2/(4c)}$	$\omega\,\mathrm{e}^{-c\omega^2}$

Appendix N

Table of Wronskians

$W[x-a, x-b]$	$b-a$
$W\left[\mathrm{e}^{ax}, \mathrm{e}^{bx}\right]$	$(b-a)\,\mathrm{e}^{(a+b)x}$
$W[\cos(ax), \sin(ax)]$	a
$W[\cosh(ax), \sinh(ax)]$	a
$W[\mathrm{e}^{ax}\cos(bx), \mathrm{e}^{ax}\sin(bx)]$	$b\,\mathrm{e}^{2ax}$
$W[\mathrm{e}^{ax}\cosh(bx), \mathrm{e}^{ax}\sinh(bx)]$	$b\,\mathrm{e}^{2ax}$
$W[\sin(c(x-a)), \sin(c(x-b))]$	$c\sin(c(b-a))$
$W[\cos(c(x-a)), \cos(c(x-b))]$	$c\sin(c(b-a))$
$W[\sin(c(x-a)), \cos(c(x-b))]$	$-c\cos(c(b-a))$

$W\left[\sinh(c(x-a)), \sinh(c(x-b))\right]$	$c\sinh(c(b-a))$
$W\left[\cosh(c(x-a)), \cosh(c(x-b))\right]$	$c\cosh(c(b-a))$
$W\left[\sinh(c(x-a)), \cosh(c(x-b))\right]$	$-c\cosh(c(b-a))$
$W\left[\mathrm{e}^{dx}\sin(c(x-a)),\ \mathrm{e}^{dx}\sin(c(x-b))\right]$	$c\,\mathrm{e}^{2dx}\sin(c(b-a))$
$W\left[\mathrm{e}^{dx}\cos(c(x-a)),\ \mathrm{e}^{dx}\cos(c(x-b))\right]$	$c\,\mathrm{e}^{2dx}\sin(c(b-a))$
$W\left[\mathrm{e}^{dx}\sin(c(x-a)),\ \mathrm{e}^{dx}\cos(c(x-b))\right]$	$-c\,\mathrm{e}^{2dx}\cos(c(b-a))$
$W\left[\mathrm{e}^{dx}\sinh(c(x-a)),\ \mathrm{e}^{dx}\sinh(c(x-b))\right]$	$c\,\mathrm{e}^{2dx}\sinh(c(b-a))$
$W\left[\mathrm{e}^{dx}\cosh(c(x-a)),\ \mathrm{e}^{dx}\cosh(c(x-b))\right]$	$-c\,\mathrm{e}^{2dx}\sinh(c(b-a))$
$W\left[\mathrm{e}^{dx}\sinh(c(x-a)),\ \mathrm{e}^{dx}\cosh(c(x-b))\right]$	$-c\,\mathrm{e}^{2dx}\cosh(c(b-a))$
$W\left[(x-a)\,\mathrm{e}^{cx}, (x-b)\,\mathrm{e}^{cx}\right]$	$(b-a)\,\mathrm{e}^{2cx}$

Appendix O

Sturm-Liouville Eigenvalue Problems

- $y'' + \lambda^2 y = 0$, $y(a) = y(b) = 0$

$$\lambda_n = \frac{n\pi}{b-a}, \quad y_n = \sin\left(\frac{n\pi(x-a)}{b-a}\right), \quad n \in \mathbb{N}$$

$$\langle y_n, y_n \rangle = \frac{b-a}{2}$$

- $y'' + \lambda^2 y = 0$, $y(a) = y'(b) = 0$

$$\lambda_n = \frac{(2n-1)\pi}{2(b-a)}, \quad y_n = \sin\left(\frac{(2n-1)\pi(x-a)}{2(b-a)}\right), \quad n \in \mathbb{N}$$

$$\langle y_n, y_n \rangle = \frac{b-a}{2}$$

- $y'' + \lambda^2 y = 0$, $y'(a) = y(b) = 0$

$$\lambda_n = \frac{(2n-1)\pi}{2(b-a)}, \quad y_n = \cos\left(\frac{(2n-1)\pi(x-a)}{2(b-a)}\right), \quad n \in \mathbb{N}$$

$$\langle y_n, y_n \rangle = \frac{b-a}{2}$$

- $y'' + \lambda^2 y = 0$, $y'(a) = y'(b) = 0$

$$\lambda_n = \frac{n\pi}{b-a}, \quad y_n = \cos\left(\frac{n\pi(x-a)}{b-a}\right), \quad n = 0, 1, 2, \dots$$

$$\langle y_0, y_0 \rangle = b - a, \quad \langle y_n, y_n \rangle = \frac{b-a}{2} \text{ for } n \in \mathbb{N}$$

Appendix P

Green Functions for Ordinary Differential Equations

- $G' + p(x)G = \delta(x - \xi)$, $G(\xi^- : \xi) = 0$

$$G(x|\xi) = \exp\left(-\int_\xi^x p(t)\,\mathrm{d}t\right) H(x - \xi)$$

- $y'' = 0$, $y(a) = y(b) = 0$

$$G(x|\xi) = \frac{(x_< - a)(x_> - b)}{b - a}$$

- $y'' = 0$, $y(a) = y'(b) = 0$

$$G(x|\xi) = a - x_<$$

- $y'' = 0$, $y'(a) = y(b) = 0$

$$G(x|\xi) = x_> - b$$

- $y'' - c^2 y = 0$, $y(a) = y(b) = 0$

$$G(x|\xi) = \frac{\sinh(c(x_< - a)) \sinh(c(x_> - b))}{c \sinh(c(b-a))}$$

- $y'' - c^2 y = 0$, $y(a) = y'(b) = 0$

$$G(x|\xi) = -\frac{\sinh(c(x_< - a)) \cosh(c(x_> - b))}{c \cosh(c(b-a))}$$

- $y'' - c^2 y = 0$, $y'(a) = y(b) = 0$

$$G(x|\xi) = \frac{\cosh(c(x_< - a)) \sinh(c(x_> - b))}{c \cosh(c(b-a))}$$

- $y'' + c^2 y = 0$, $y(a) = y(b) = 0$, $c \neq \frac{npi}{b-a}$, $n \in \mathbb{N}$

$$G(x|\xi) = \frac{\sin(c(x_< - a)) \sin(c(x_> - b))}{c \sin(c(b-a))}$$

- $y'' + c^2 y = 0$, $y(a) = y'(b) = 0$, $c \neq \frac{(2n-1)pi}{2(b-a)}$, $n \in \mathbb{N}$

$$G(x|\xi) = -\frac{\sin(c(x_< - a)) \cos(c(x_> - b))}{c \cos(c(b-a))}$$

- $y'' + c^2 y = 0$, $y'(a) = y(b) = 0$, $c \neq \frac{(2n-1)pi}{2(b-a)}$, $n \in \mathbb{N}$

$$G(x|\xi) = \frac{\cos(c(x_< - a)) \sin(c(x_> - b))}{c \cos(c(b-a))}$$

- $y'' + 2cy' + dy = 0,\ y(a) = y(b) = 0,\ c^2 > d$

$$G(x|\xi) = \frac{\mathrm{e}^{-cx_<} \sinh(\sqrt{c^2 - d}(x_< - a)) \, \mathrm{e}^{-cx_<} \sinh(\sqrt{c^2 - d}(x_> - b))}{\sqrt{c^2 - d}\, \mathrm{e}^{-2c\xi} \sinh(\sqrt{c^2 - d}(b - a))}$$

- $y'' + 2cy' + dy = 0,\ y(a) = y(b) = 0,\ c^2 < d,\ \sqrt{d - c^2} \neq \frac{n\pi}{b-a},\ n \in \mathbb{N}$

$$G(x|\xi) = \frac{\mathrm{e}^{-cx_<} \sin(\sqrt{d - c^2}(x_< - a)) \, \mathrm{e}^{-cx_<} \sin(\sqrt{d - c^2}(x_> - b))}{\sqrt{d - c^2}\, \mathrm{e}^{-2c\xi} \sin(\sqrt{d - c^2}(b - a))}$$

- $y'' + 2cy' + dy = 0,\ y(a) = y(b) = 0,\ c^2 = d$

$$G(x|\xi) = \frac{(x_< - a)\, \mathrm{e}^{-cx_<} (x_> - b)\, \mathrm{e}^{-cx_<}}{(b - a)\, \mathrm{e}^{-2c\xi}}$$

Appendix Q

Trigonometric Identities

Q.1 Circular Functions

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1, \qquad 1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x$$

Angle Sum and Difference Identities

$$\begin{aligned}
\sin(x + y) &= \sin x \cos y + \cos x \sin y \\
\sin(x - y) &= \sin x \cos y - \cos x \sin y \\
\cos(x + y) &= \cos x \cos y - \sin x \sin y \\
\cos(x - y) &= \cos x \cos y + \sin x \sin y
\end{aligned}$$

Function Sum and Difference Identities

$$\sin x + \sin y = 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$$
$$\sin x - \sin y = 2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$$
$$\cos x + \cos y = 2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$$
$$\cos x - \cos y = -2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$$

Double Angle Identities

$$\sin 2x = 2\sin x\cos x, \qquad \cos 2x = \cos^2 x - \sin^2 x$$

Half Angle Identities

$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}, \qquad \cos^2\frac{x}{2} = \frac{1+\cos x}{2}$$

Function Product Identities

$$\sin x\sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$
$$\cos x\cos y = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$
$$\sin x\cos y = \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y)$$
$$\cos x\sin y = \frac{1}{2}\sin(x+y) - \frac{1}{2}\sin(x-y)$$

Exponential Identities

$$e^{ix} = \cos x + i \sin x, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Q.2 Hyperbolic Functions

Exponential Identities

$$\sinh x = \frac{e^{x} - e^{-x}}{2}, \qquad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

Reciprocal Identities

$$\operatorname{csch} x = \frac{1}{\sinh x}, \qquad \operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}$$

Pythagorean Identities

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1$$

Relation to Circular Functions

$$\begin{aligned} \sinh ix &= i \sin x & \sinh x &= -i \sin ix \\ \cosh ix &= \cos x & \cosh x &= \cos ix \\ \tanh ix &= i \tan x & \tanh x &= -i \tan ix \end{aligned}$$

Angle Sum and Difference Identities

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} = \frac{\sinh 2x \pm \sinh 2y}{\cosh 2x \pm \cosh 2y}$$
$$\coth(x \pm y) = \frac{1 \pm \coth x \coth y}{\coth x \pm \coth y} = \frac{\sinh 2x \mp \sinh 2y}{\cosh 2x - \cosh 2y}$$

Function Sum and Difference Identities

$$\sinh x \pm \sinh y = 2 \sinh \frac{1}{2}(x \pm y) \cosh \frac{1}{2}(x \mp y)$$
$$\cosh x + \cosh y = 2 \cosh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$$
$$\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)$$
$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}$$
$$\coth x \pm \coth y = \frac{\sinh(x \pm y)}{\sinh x \sinh y}$$

Double Angle Identities

$$\sinh 2x = 2 \sinh x \cosh x, \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

Half Angle Identities

$$\sinh^2 \frac{x}{2} = \frac{\cosh x - 1}{2}, \qquad \cosh^2 \frac{x}{2} = \frac{\cosh x + 1}{2}$$

Function Product Identities

$$\sinh x \sinh y = \frac{1}{2}\cosh(x+y) - \frac{1}{2}\cosh(x-y)$$
$$\cosh x \cosh y = \frac{1}{2}\cosh(x+y) + \frac{1}{2}\cosh(x-y)$$
$$\sinh x \cosh y = \frac{1}{2}\sinh(x+y) + \frac{1}{2}\sinh(x-y)$$

See Figure Q.1 for plots of the hyperbolic circular functions.

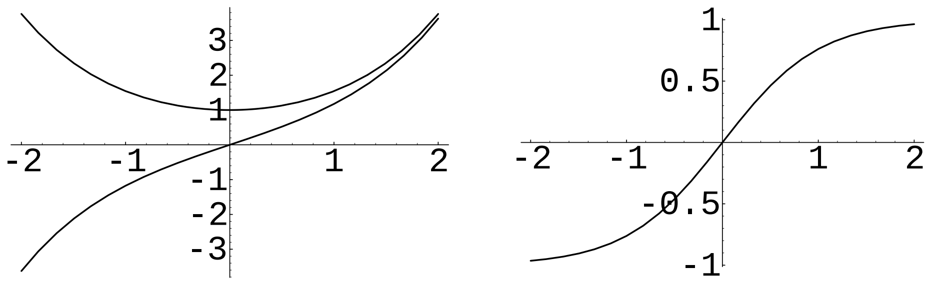

Figure Q.1: $\cosh x$, $\sinh x$ and then $\tanh x$

Appendix R

Bessel Functions

R.1 Definite Integrals

Let $\nu > -1$.

$$\int_0^1 r J_\nu(j_{\nu,m} r) J_\nu(j_{\nu,n} r) \, dr = \frac{1}{2} \left(J'_\nu(j_{\nu,n}) \right)^2 \delta_{mn}$$

$$\int_0^1 r J_\nu(j'_{\nu,m} r) J_\nu(j'_{\nu,n} r) \, dr = \frac{j'^2_{\nu,n} - \nu^2}{2 j'^2_{\nu,n}} \left(J_\nu(j'_{\nu,n}) \right)^2 \delta_{mn}$$

$$\int_0^1 r J_\nu(\alpha_m r) J_\nu(\alpha_n r) \, dr = \frac{1}{2\alpha_n^2} \left(\frac{a^2}{b^2} + \alpha_n^2 - \nu^2 \right) \left(J_\nu(\alpha_n) \right)^2 \delta_{mn}$$

Here α_n is the n^{th} positive root of $a J_\nu(r) + b r J'_\nu(r)$, where $a, b \in \mathbb{R}$.

Appendix S

Formulas from Linear Algebra

Kramer's Rule. Consider the matrix equation

$$A\vec{x} = \vec{b}.$$

This equation has a unique solution if and only if $\det(A) \neq 0$. If the determinant vanishes then there are either no solutions or an infinite number of solutions. If the determinant is nonzero, the solution for each x_j can be written

$$x_j = \frac{\det A_j}{\det A}$$

where A_j is the matrix formed by replacing the j^{th} column of A with b.

Example S.0.1 The matrix equation

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix},$$

has the solution

$$x_1 = \frac{\begin{vmatrix} 5 & 2 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{8}{-2} = -4, \qquad x_2 = \frac{\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{-9}{-2} = \frac{9}{2}.$$

Appendix T

Vector Analysis

Rectangular Coordinates

$$f = f(x, y, z), \qquad \vec{g} = g_x\mathbf{i} + g_y\mathbf{j} + g_z\mathbf{k}$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\nabla \cdot \vec{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}$$

$$\nabla \times \vec{g} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_x & g_y & g_z \end{vmatrix}$$

$$\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi$$

$$f = f(r,\theta,\phi), \qquad \vec{g} = g_r\mathbf{r} + g_\theta\boldsymbol{\theta} + g_\phi\boldsymbol{\phi}$$

Divergence Theorem.

$$\iint \nabla\cdot\mathbf{u}\,dx\,dy = \oint \mathbf{u}\cdot\mathbf{n}\,ds$$

Stoke's Theorem.

$$\iint (\nabla\times\mathbf{u})\cdot\,d\mathbf{s} = \oint \mathbf{u}\cdot\,d\mathbf{r}$$

Appendix U

Partial Fractions

A proper rational function

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x-a)^n r(x)}$$

Can be written in the form

$$\frac{p(x)}{(x-\alpha)^n r(x)} = \left(\frac{a_0}{(x-\alpha)^n} + \frac{a_1}{(x-\alpha)^{n-1}} + \cdots + \frac{a_{n-1}}{x-\alpha} \right) + (\cdots)$$

where the a_k's are constants and the last ellipses represents the partial fractions expansion of the roots of $r(x)$. The coefficients are

$$a_k = \frac{1}{k!} \frac{d^k}{dx^k} \left(\frac{p(x)}{r(x)} \right) \bigg|_{x=\alpha}.$$

Example U.0.2 Consider the partial fraction expansion of

$$\frac{1 + x + x^2}{(x-1)^3}.$$

The expansion has the form

$$\frac{a_0}{(x-1)^3} + \frac{a_1}{(x-1)^2} + \frac{a_2}{x-1}.$$

The coefficients are

$$\begin{aligned}
a_0 &= \frac{1}{0!}(1+x+x^2)|_{x=1} = 3, \\
a_1 &= \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}x}(1+x+x^2)|_{x=1} = (1+2x)|_{x=1} = 3, \\
a_2 &= \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(1+x+x^2)|_{x=1} = \frac{1}{2}(2)|_{x=1} = 1.
\end{aligned}$$

Thus we have

$$\frac{1+x+x^2}{(x-1)^3} = \frac{3}{(x-1)^3} + \frac{3}{(x-1)^2} + \frac{1}{x-1}.$$

Example U.0.3 Consider the partial fraction expansion of

$$\frac{1+x+x^2}{x^2(x-1)^2}.$$

The expansion has the form

$$\frac{a_0}{x^2} + \frac{a_1}{x} + \frac{b_0}{(x-1)^2} + \frac{b_1}{x-1}.$$

The coefficients are

$$\begin{aligned}
a_0 &= \frac{1}{0!}\left(\frac{1+x+x^2}{(x-1)^2}\right)\bigg|_{x=0} = 1,\\
a_1 &= \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+x+x^2}{(x-1)^2}\right)\bigg|_{x=0} = \left(\frac{1+2x}{(x-1)^2} - \frac{2(1+x+x^2)}{(x-1)^3}\right)\bigg|_{x=0} = 3,\\
b_0 &= \frac{1}{0!}\left(\frac{1+x+x^2}{x^2}\right)\bigg|_{x=1} = 3,\\
b_1 &= \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+x+x^2}{x^2}\right)\bigg|_{x=1} = \left(\frac{1+2x}{x^2} - \frac{2(1+x+x^2)}{x^3}\right)\bigg|_{x=1} = -3,
\end{aligned}$$

Thus we have

$$\frac{1+x+x^2}{x^2(x-1)^2} = \frac{1}{x^2} + \frac{3}{x} + \frac{3}{(x-1)^2} - \frac{3}{x-1}.$$

If the rational function has real coefficients and the denominator has complex roots, then you can reduce the work in finding the partial fraction expansion with the following trick: Let α and $\overline{\alpha}$ be complex conjugate pairs of roots of the denominator.

$$\begin{aligned}
\frac{p(x)}{(x-\alpha)^n(x-\overline{\alpha})^n r(x)} &= \left(\frac{a_0}{(x-\alpha)^n} + \frac{a_1}{(x-\alpha)^{n-1}} + \cdots + \frac{a_{n-1}}{x-\alpha}\right)\\
&\quad + \left(\frac{\overline{a_0}}{(x-\overline{\alpha})^n} + \frac{\overline{a_1}}{(x-\overline{\alpha})^{n-1}} + \cdots + \frac{\overline{a_{n-1}}}{x-\overline{\alpha}}\right) + (\cdots)
\end{aligned}$$

Thus we don't have to calculate the coefficients for the root at $\overline{\alpha}$. We just take the complex conjugate of the coefficients for α.

Example U.0.4 Consider the partial fraction expansion of

$$\frac{1+x}{x^2+1}.$$

The expansion has the form

$$\frac{a_0}{x-i} + \frac{\overline{a_0}}{x+i}$$

The coefficients are

$$a_0 = \frac{1}{0!}\left(\frac{1+x}{x+i}\right)\bigg|_{x=i} = \frac{1}{2}(1-i),$$
$$\overline{a_0} = \overline{\frac{1}{2}(1-i)} = \frac{1}{2}(1+i)$$

Thus we have

$$\frac{1+x}{x^2+1} = \frac{1-i}{2(x-i)} + \frac{1+i}{2(x+i)}.$$

Appendix V

Finite Math

Newton's Binomial Formula.

$$(a+b)^n = \sum_{k=0}^{n} \binom{k}{n} a^{n-k} b^k$$
$$= a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n,$$

The *binomial coefficients* are,

$$\binom{k}{n} = \frac{n!}{k!(n-k)!}.$$

Appendix W

Probability

W.1 Independent Events

Once upon a time I was talking with the father of one of my colleagues at Caltech. He was an educated man. I think that he had studied Russian literature and language back when he was in college. We were discussing gambling. He told me that he had a scheme for winning money at the game of 21. I was familiar with counting cards. Being a mathematician, I was not interested in hearing about conditional probability from a literature major, but I said nothing and prepared to hear about his particular technique. I was quite surprised with his "method": He said that when he was on a winning streak he would bet more and when he was on a losing streak he would bet less. He conceded that he lost more hands than he won, but since he bet more when he was winning, he made money in the end.

I respectfully and thoroughly explained to him the concept of an independent event. Also, if one is not counting cards then each hand in 21 is essentially an independent event. The outcome of the previous hand has no bearing on the current. Throughout the explanation he nodded his head and agreed with my reasoning. When I was finished he replied, "Yes, that's true. But you see, I have a method. When I'm on my winning streak I bet more and when I'm on my losing streak I bet less."

I pretended that I understood. I didn't want to be rude. After all, he had taken the time to explain the concept of a winning streak to me. And everyone knows that mathematicians often do not easily understand practical matters, particularly games of chance.

Never explain mathematics to the layperson.

W.2 Playing the Odds

Years ago in a classroom not so far away, your author was being subjected to a presentation of a lengthy proof. About five minutes into the lecture, the entire class was hopelessly lost. At the forty-five minute mark the professor had a combinatorial expression that covered most of a chalk board. From his previous queries the professor knew that none of the students had a clue what was going on. This pleased him and he had became more animated as the lecture had progressed. He gestured to the board with a smirk and asked for the value of the expression. Without a moment's hesitation, I nonchalantly replied, "zero". The professor was taken aback. He was clearly impressed that I was able to evaluate the expression, especially because I had done it in my head and so quickly. He enquired as to my method. "Probability", I replied. "Professors often present difficult problems that have simple, elegant solutions. Zero is the most elegant of numerical answers and thus most likely to be the correct answer. My second guess would have been one." The professor was not amused.

Whenever a professor asks the class a question which has a numeric answer, immediately respond, "zero". If you are asked about your method, casually say something vague about symmetry. Speak with confidence and give non-verbal cues that you consider the problem to be elementary. This tactic will usually suffice. It's quite likely that some kind of symmetry is involved. And if it isn't your response will puzzle the professor. They may continue with the next topic, not wanting to admit that they don't see the "symmetry" in such an elementary problem. If they press further, start mumbling to yourself. Pretend that you are lost in thought, perhaps considering some generalization of the result. They may be a little irked that you are ignoring them, but it's better than divulging your true method.

Appendix X

Economics

There are two important concepts in economics. The first is "Buy low, sell high", which is self-explanitory. The second is *opportunity cost*, the highest valued alternative that must be sacrificed to attain something or otherwise satisfy a want. I discovered this concept as an undergraduate at Caltech. I was never very in to computer games, but I found myself randomly playing tetris. Out of the blue I was struck by a revelation: "I could be having sex right now." I haven't played a computer game since.

Appendix Y

Glossary

Phrases often have different meanings in mathematics than in everyday usage. Here I have collected definitions of some mathematical terms which might confuse the novice.

beyond the scope of this text Beyond the comprehension of the author.

difficult Essentially impossible. Note that mathematicians never refer to problems they have solved as being difficult. This would either be boastful, (claiming that you can solve difficult problems), or self-deprecating, (admitting that you found the problem to be difficult).

interesting This word is grossly overused in math and science. It is often used to describe any work that the author has done, regardless of the work's significance or novelty. It may also be used as a synonym for difficult. It has a completely different meaning when used by the non-mathematician. When I tell people that I am a mathematician they typically respond with, "That must be interesting.", which means, "I don't know anything about math or what mathematicians do." I typically answer, "No. Not really."

non-obvious or **non-trivial** Real fuckin' hard.

one can prove that ... The "one" that proved it was a genius like Gauss. The phrase literally means "you haven't got a chance in hell of proving that ... "

simple Mathematicians communicate their prowess to colleagues and students by referring to all problems as simple or trivial. If you ever become a math professor, introduce every example as being "really quite trivial." [1]

[1]For even more fun say it in your best Elmer Fudd accent. "This next pwobwem is weawy quite twiviaw".

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